Non-spectral problem for a class of planar self-affine measures

Jian-Lin Li

College of Mathematics and Information Science, Shaanxi Normal University, Xi’an 710062, PR China

Received 21 February 2008; accepted 2 April 2008

Available online 8 May 2008

Communicated by L. Gross

Abstract

The self-affine measure $\mu_{M,D}$ corresponding to an expanding matrix $M \in M_n(\mathbb{R})$ and a finite subset $D \subset \mathbb{R}^n$ is supported on the attractor (or invariant set) of the iterated function system \{$\phi_d(x) = M^{-1}(x + d)$\}$_{d \in D}$. The spectral and non-spectral problems on $\mu_{M,D}$, including the spectrum-tiling problem implied in them, have received much attention in recent years. One of the non-spectral problem on $\mu_{M,D}$ is to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them. In the present paper we show that if $a, b, c \in \mathbb{Z}$, $|a| > 1$, $|c| > 1$ and $ac \in \mathbb{Z} \setminus (3\mathbb{Z})$,

$$M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

then there exist at most 3 mutually orthogonal exponentials in $L^2(\mu_{M,D})$, and the number 3 is the best. This extends several known conclusions. The proof of such result depends on the characterization of the zero set of the Fourier transform $\hat{\mu}_{M,D}$, and provides a way of dealing with the non-spectral problem.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Iterated function system; Self-affine measure; Orthogonal exponentials; Spectral measure

1. Introduction

Invariant measures, such as self-similar measures, have recently found wide use in the theory of fractals, in dynamics, in harmonic analysis and in quasicrystals (cf. [1,6]). A measure $\mu$ is self-
similar if it is a convex combination of a given set $S$ of transformations applied to the measure itself. In the literature, one usually restricts attention to the case where the set $S$ is finite. Then, an iterated function system (IFS) results, and varying $S$ yields a rich family of measures $\mu$. To get a manageable problem, further restrictions are placed on the transformations from $S$. E.g., that they are contractive, and that they fall in a definite class, such as conformal maps (giving equilibrium measures on Julia sets), or affine mappings. Here the affine case is considered. Our IFS $\{\phi_d(x)\}_{d \in D}$ consists of the following affine maps on $\mathbb{R}^n$,

$$\phi_d(x) = M^{-1}(x + d) \quad (x \in \mathbb{R}^n),$$

where $M \in M_n(\mathbb{R})$ is an $n \times n$ expanding real matrix (that is, all the eigenvalues of the real matrix $M$ have moduli $> 1$), and $D \subset \mathbb{R}^n$ is a finite subset of the cardinality $|D|$. We denote the corresponding measure by $\mu_{M,D}$, which is a unique probability measure $\mu := \mu_{M,D}$ satisfying

$$\mu = \frac{1}{|D|} \sum_{d \in D} \mu \circ \phi_d^{-1}. \quad (1.1)$$

Such a measure $\mu_{M,D}$ is supported on the attractor (or invariant set) $T(M, D)$ of the affine IFS $\{\phi_d(x)\}_{d \in D}$ (cf. [7,12]), and is called a self-affine measure.

Since this affine case includes restrictions of $n$-dimensional Lebesgue measure, Cantor measures, and IFS fractal measures, say on Sierpinski gaskets, it is natural to ask for Fourier duality. Can one get some kind of Fourier representation for $\mu_{M,D}$? We know from prior research on $L^2(\mu_{M,D})$ that a naive notion of orthogonal Fourier series is not feasible in general for affine IFSs. For example, the familiar middle 3rd Cantor set $T(M, D)$ corresponding to $M = 3$ and $D = \{0, 2\}$, Jorgensen and Pedersen [18, Theorem 6.1] proved that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 2 elements. In the case when $M = p$, $p > 1$, is odd and $D = \{0, 1\}$, Dutkay and Jorgensen [4, Theorem 5.1(i)] proved that there are no 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. In this paper we will explore planar affine IFS-examples when the obstruction to getting a Fourier basis is extreme.

Recall that for a probability measure $\mu$ of compact support on $\mathbb{R}^n$, we call $\mu$ a spectral measure if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that the exponential function system $E_\Lambda := \{e^{2\pi i \langle \lambda, x \rangle}; \lambda \in \Lambda\}$ forms an orthogonal basis (Fourier basis) for $L^2(\mu)$. The set $\Lambda$ is then called a spectrum for $\mu$; we also say that $(\mu, \Lambda)$ is a spectral pair (cf. [19]).

Spectral measure is a natural generalization of spectral set introduced by Fuglede [10] whose famous spectrum-tiling conjecture and its related problems have received much attention in recent years (cf. [6,24,25]). The spectral self-affine measure problem at the present day consists in determining conditions under which $\mu_{M,D}$ is a spectral measure, and has been studied in the papers [2–6,18,23,25,27,28,32] (see also [33,34] for the main goal). In the opposite direction, the non-spectral Lebesgue measure problem has been studied in the papers [10,11,15–17,22,26] and [13,14] where the conjecture that the disk has no more than 3 orthogonal exponentials is still unsolved. Correspondingly, the non-spectral problem on the self-affine measure consists of the following two classes:

(I) There are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, that is, $\mu_{M,D}$-orthogonal exponentials contain at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them (cf. [4,29]).
(II) There are natural infinite families of orthogonal exponentials, but non of them forms an orthogonal basis in \(L^2(\mu_{M,D})\). The main question is whether some of theses families can be combined to form larger collections of orthogonal exponentials. The other questions concerning this class can be found in [21].

Except the case that there might be no more than two orthogonal exponentials, the problem on a non-spectral measure \(\mu_{M,D}\) in fact falls into one of the above two classes. Nevertheless, the first problem we meet is how to determine a measure \(\mu_{M,D}\) is non-spectral. There are some results in this direction, such as [4, Theorem 3.1], but we are still far from settling this problem. Relating to the questions of the class (I), we first recall the following results.

(i) The plane Sierpinski gasket \(T(M,D)\) corresponding to

\[
M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},
\]

(1.2)

Li [25, p. 65], [28, Example 1] proved that \(\mu_{M,D}\)-orthogonal exponentials contain at most 3 elements and found such 3 elements-orthogonal exponentials. Dutkay and Jorgensen [4, Theorem 5.1(ii)] proved that if \(p \in \mathbb{Z}\) is not a multiple of 3 and

\[
M = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \quad (p \geq 2) \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},
\]

(1.3)

then there are no 4 mutually orthogonal exponential functions in \(L^2(\mu_{M,D})\).

(ii) The generalized plane Sierpinski gasket \(T(M,D)\) corresponding to

\[
M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},
\]

(1.4)

see [4, Fig. 3 and Example 3.1], by applying [4, Theorem 3.1], Dutkay and Jorgensen [4] obtained that \(\mu_{M,D}\)-orthogonal exponentials contain at most 7 elements. More recently, Li [29] proved that for the self-affine measure \(\mu_{M,D}\) corresponding to

\[
M = \begin{bmatrix} 2 & b \\ 0 & 2 \end{bmatrix} \quad (b \in \mathbb{Z}) \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},
\]

(1.5)

there are at most 3 mutually orthogonal exponential functions in \(L^2(\mu_{M,D})\), and the number 3 is the best.

In the plane, the above set \(D\) (usually called the digit set) which consists of the canonical vectors in \(\mathbb{R}^2\) is fundamental, many digit sets can be obtained from this set. From (1.2)–(1.4) and (1.5), we see that the condition \(3 \nmid \det(M)\) is always satisfied or assumed. In fact, the condition \(|D| \in W(m)\) is necessary for the discussion of \(\mu_{M,D}\)-orthogonality in the integer case, where \(|\det(M)| = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r} (p_1 < p_2 < \cdots < p_r \text{ are prime numbers}, b_j > 0)\) is the standard prime factorization and \(W(m)\) denotes the non-negative integer combination of \(p_1, p_2, \ldots, p_r\) (cf. [25, Section 4.2], [28, Section 3]). The known results provide some supportive evidence that the following Conjecture 1 should be true, although we cannot prove it.
Conjecture 1. For an expanding integer matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$, if $|D| \notin W(m)$, then $\mu_{M,D}$ is a non-spectral measure and the non-spectral problem on this $\mu_{M,D}$ falls in the class (I).

Motivated by the previous research, especially the above Conjecture 1, in the present paper we will consider the non-spectral problem for a class of more general planar self-affine measures. The main result of the paper is the following.

**Theorem.** Let $a, b, c \in \mathbb{Z}$, $|a| > 1$, $|c| > 1$ and $ac \in \mathbb{Z} \setminus (3\mathbb{Z})$. For the self-affine measure $\mu_{M,D}$ corresponding to

$$M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 3 is the best.

This generalizes the above-mentioned results on the non-spectral self-affine measure problem. The proof of Theorem depends mainly on the characterization of the zero set $Z(\hat{\mu}_{M,D})$ of the Fourier transform $\hat{\mu}_{M,D}$. It is different from the previous research that we find more inclusion relations inside the zero set $Z(\hat{\mu}_{M,D})$. Some facts concerning this zero set are given in Section 2. Based on these established facts, we prove Theorem in Section 3. Finally we give a concluding remark on a related question.

2. Characterization of the zero set $Z(\hat{\mu}_{M,D})$

For a general expanding matrix $M \in M_n(\mathbb{R})$ and a finite subset $D \subset \mathbb{R}^n$, the Fourier transform of the self-affine measure $\mu_{M,D}$ is

$$\hat{\mu}_{M,D}(\xi) = \int e^{2\pi i \langle \xi, t \rangle} d\mu_{M,D}(t) \quad (\xi \in \mathbb{R}^n).$$

From (1.1), we have

$$\hat{\mu}_{M,D}(\xi) = m_D(M^{\ast-1}\xi)\hat{\mu}_{M,D}(M^{\ast-1}\xi) \quad (\xi \in \mathbb{R}^n),$$

which yields

$$\hat{\mu}_{M,D}(\xi) = \prod_{j=1}^{\infty} m_D(M^{\ast-j}\xi),$$

by iteration, where

$$m_D(t) := \frac{1}{|D|} \sum_{d \in D} e^{2\pi i \langle d, t \rangle},$$
and $M^*$ denotes the conjugate transpose of $M$, in fact $M^* = M'$. The infinite product (2.2) converges absolutely for all $\xi \in \mathbb{R}^n$. It also converges uniformly on compact subsets of $\mathbb{R}^n$.

The self-affine measure $\mu_{M,D}$ and its Fourier transform $\hat{\mu}_{M,D}$ given by (2.2) play an important role in analysis and geometry. Previous research on such measure and its Fourier transform revealed some surprising connections with a number of areas in mathematics, such as harmonic analysis, dynamical systems, number theory, and others, see [8,9,20,30,31] and references cited therein. Here we are interested in the zero set $Z(\hat{\mu}_{M,D})$ of $\hat{\mu}_{M,D}$ which is highly important to the spectral and non-spectral problems on the self-affine measures.

For any $\lambda_1, \lambda_2 \in \mathbb{R}^n$, $\lambda_1 \neq \lambda_2$, the orthogonality condition

$$\langle e^{2\pi i \langle \lambda_1, x \rangle}, e^{2\pi i \langle \lambda_2, x \rangle} \rangle_{L^2(\mu_{M,D})} = \int e^{2\pi i \langle \lambda_1 - \lambda_2, x \rangle} d\mu_{M,D} = \hat{\mu}_{M,D}(\lambda_1 - \lambda_2) = 0$$

(2.4)

directly relates to the zero set $Z(\hat{\mu}_{M,D})$ of $\hat{\mu}_{M,D}$. From (2.2), we have

$$Z(\hat{\mu}_{M,D}) = \{ \xi \in \mathbb{R}^n : \exists j \in \mathbb{N} \text{ such that } m_D(M^* - j \xi) = 0 \}.$$  

(2.5)

Furthermore, if we let $\Theta_j = \{ \xi \in \mathbb{R}^n : m_D(M^* - j \xi) = 0 \} (j = 1, 2, \ldots)$, we have the following properties:

(i$'$) $Z(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} \Theta_j$;
(ii$'$) $\xi_0 \in Z(\hat{\mu}_{M,D}) \iff -\xi_0 \in Z(\hat{\mu}_{M,D})$ or $\xi_0 \in \Theta_j \iff -\xi_0 \in \Theta_j$ for $j = 1, 2, \ldots$;
(iii$'$) $\Theta_{j+1} = M^*(\Theta_j)$ for $j = 1, 2, \ldots$.

In the following, we will restrict our discussion on the special $M$ and $D$ given by (1.6), and find out some characteristic properties on the set $Z(\hat{\mu}_{M,D})$.

For the given $M$ and $D$ in (1.6), we first have

$$M^j = \begin{bmatrix} a^j & bq(j) \\ 0 & c^j \end{bmatrix} (j = 1, 2, \ldots)$$

and

$$m_D(M^* - j \xi) = \frac{1}{3} \left\{ 1 + e^{2\pi i \xi_1 a^{-j}} + e^{2\pi i \frac{a^j \xi_2 - bq(j) \xi_1}{a^j c^j}} \right\},$$

(2.6)

where $\xi = (\xi_1, \xi_2)' \in \mathbb{R}^2$ and

$$q(j) = \sum_{k=0}^{j-1} a^k c^{j-k-1}.$$  

Then, we get from (2.5) and (2.6) that

$$Z(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} (Z_j \cup \tilde{Z}_j),$$

(2.7)
where
\[
Z_j = \left\{ \left( \frac{a^j}{3} \right) + \left( \frac{a^j k_1}{bq(j) k_1 + c^j k_2} \right) : k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2
\]  
(2.8)
and
\[
\tilde{Z}_j = \left\{ \left( \frac{2a^j}{3} \right) + \left( \frac{a^j k_1}{bq(j) \tilde{k}_1 + c^j \tilde{k}_2} \right) : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2.
\]  
(2.9)

From (2.8) and (2.9), we first have the following.

**Proposition 1.** The sets \(Z_j\) and \(\tilde{Z}_j\) given by (2.8) and (2.9) satisfy the following properties:

1. \((x, y)^t \in Z_j \iff (-x, -y)^t \in \tilde{Z}_j\), that is, \(Z_j = -\tilde{Z}_j\) or \(\tilde{Z}_j = -Z_j\) \((j = 1, 2, \ldots)\);
2. \(Z_j - Z_j \subseteq \mathbb{Z}^2\) and \(\tilde{Z}_j - \tilde{Z}_j \subseteq \mathbb{Z}^2\) \((j = 1, 2, \ldots)\);
3. \(Z_j + Z_j \subseteq \tilde{Z}_j\) and \(\tilde{Z}_j + \tilde{Z}_j \subseteq Z_j\) \((j = 1, 2, \ldots)\).

In order to find more relations inside the zero set \(Z(\hat{\mu}_{M,D})\), we will reduce the fractional expressions in (2.8) and (2.9) to their lowest terms. The denominator of all such fractional expressions is the number 3. So we consider the integers \(a, b\) and \(c\) according to the residue class modulo-3 where these integers belong. The condition \(b \in \mathbb{Z}\) can be divided into the following three cases:

\[
b = 3g \quad (g \in \mathbb{Z}); \quad b = 3g + 1 \quad (g \in \mathbb{Z}); \quad b = 3g + 2 \quad (g \in \mathbb{Z}).
\]  
(2.10)

The assumption that \(a, c \in \mathbb{Z}, |a| > 1, |c| > 1\) and \(ac \in \mathbb{Z} \setminus (3\mathbb{Z})\) implies that \(a\) and \(c\) satisfy one of the following four cases:

\(A\) \quad \(a = 3l + 1 \quad (l \in \mathbb{Z} \setminus \{0\})\) and \(c = 3l' + 1 \quad (l' \in \mathbb{Z} \setminus \{0\})\);

\(B\) \quad \(a = 3l + 1 \quad (l \in \mathbb{Z} \setminus \{0\})\) and \(c = 3l' + 2 \quad (l' \in \mathbb{Z} \setminus \{-1\})\);

\(C\) \quad \(a = 3l + 2 \quad (l \in \mathbb{Z} \setminus \{-1\})\) and \(c = 3l' + 1 \quad (l' \in \mathbb{Z} \setminus \{0\})\);

\(D\) \quad \(a = 3l + 2 \quad (l \in \mathbb{Z} \setminus \{-1\})\) and \(c = 3l' + 2 \quad (l' \in \mathbb{Z} \setminus \{-1\})\).

We therefore divide our discussion into the following three sections according to (2.10). Section 2.1 deals with the case \(b = 3g \quad (g \in \mathbb{Z})\), Section 2.2 is the case \(b = 3g + 1 \quad (g \in \mathbb{Z})\) and Section 2.3 is the case \(b = 3g + 2 \quad (g \in \mathbb{Z})\). In each section, we will discuss \(Z_j\) and \(\tilde{Z}_j\) according to the above four cases (A), (B), (C) and (D). The main goal of each section is to simplify the expression of the zero set \(Z(\hat{\mu}_{M,D})\) in (2.7). The detailed process is given in the first Section 2.1, the other two sections are presented briefly.

**2.1. The case \(b = 3g \quad (g \in \mathbb{Z})\)**

In the case when \(b = 3g \quad (g \in \mathbb{Z})\), we can rewrite \(Z_j\) and \(\tilde{Z}_j\) in (2.8) and (2.9) as

\[
Z_j = \left\{ \left( \frac{a^j}{3} \right) + \left( \frac{a^j k_1}{gq(j) + 3gq(j) k_1 + c^j k_2} \right) : k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2
\]  
(2.11)
and
\[ \tilde{Z}_j = \left\{ \left( \frac{2a^j}{3}, \frac{c^j}{3} \right) + \left( \frac{a^j \tilde{k}_1}{2gq(j) + 3gq(j) \tilde{k}_1 + c^j \tilde{k}_2} \right) : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2 \]  
(2.12)
respectively. Furthermore, if \( a \) and \( c \) satisfy one of the four conditions (A), (B), (C) and (D), we will find some interesting inclusion relations between \( Z_j \) and \( \tilde{Z}_j \). Therefore, we further divide our discussion into the following four subsections according to (A), (B), (C) and (D).

2.1.1. The cases \( b = 3g \ (g \in \mathbb{Z}) \) and (A)

Under the conditions \( b = 3g \ (g \in \mathbb{Z}) \) and (A), that is
\[ b = 3g \ (g \in \mathbb{Z}), \quad a = 3l + 1 \ (l \in \mathbb{Z} \setminus \{0\}) \quad \text{and} \quad c = 3l' + 1 \ (l' \in \mathbb{Z} \setminus \{0\}), \]  
(2.13)
we can rewrite \( Z_j \) in (2.11) as
\[ Z_j = \left\{ \left( \frac{1}{3}, \frac{2}{3} \right) + \left( x(l, l', g, j; k_1, k_2) \right) : k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \]  
(2.14)
where
\[ x(l, l', g, j; k_1, k_2) = \frac{(3l + 1)^j - 1}{3} + (3l + 1)^j k_1 \in \mathbb{Z}, \]  
(2.15)
\[ y(l, l', g, j; k_1, k_2) = \frac{2((3l' + 1)^j - 1)}{3} + gq(j) + 3gq(j)k_1 + (3l' + 1)^j k_2 \in \mathbb{Z}, \]  
(2.16)
and \( q(j) = \sum_{k=0}^{j-1} (3l + 1)^k (3l' + 1)^{j-k-1} \).

The case \( j = 1 \) plays an important role in (2.15) and (2.16). In fact, we find, from (2.15) and (2.16), that there exist \( k'_1 \in \mathbb{Z}, k'_2 \in \mathbb{Z} \) such that
\[ x(l, l', g, j; k_1, k_2) = x(l, l', g, 1; k'_1, k'_2), \]  
(2.17)
\[ y(l, l', g, j; k_1, k_2) = y(l, l', g, 1; k'_1, k'_2). \]  
(2.17)
This shows that
\[ Z_j \subseteq Z_1 \quad \text{for} \ j \geq 1. \]  
(2.18)

In the same way, we can rewrite \( \tilde{Z}_j \) in (2.12) as
\[ \tilde{Z}_j = \left\{ \left( \frac{2}{3}, \frac{1}{3} \right) + \left( \tilde{x}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) \right) : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \]  
(2.19)
where
\[ \tilde{x}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) = \frac{2((3l + 1)^j - 1)}{3} + (3l + 1)^j \tilde{k}_1 \in \mathbb{Z}, \quad (2.20) \]

\[ \tilde{y}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) = \frac{(3l' + 1)^j - 1}{3} + 2gq(j) + 3gq(j)\tilde{k}_1 + (3l' + 1)^j \tilde{k}_2 \in \mathbb{Z}, \quad (2.21) \]

and \( q(j) = \sum_{k=0}^{j-1} (3l + 1)^k(3l' + 1)^{j-k-1} \). Then, one can verify that there exist \( \tilde{k}'_1 \in \mathbb{Z}, \tilde{k}'_2 \in \mathbb{Z} \) such that

\[ \tilde{x}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) = \tilde{x}(l, l', g, 1; \tilde{k}'_1, \tilde{k}'_2), \]
\[ \tilde{y}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) = \tilde{y}(l, l', g, 1; \tilde{k}'_1, \tilde{k}'_2). \quad (2.22) \]

This also shows that

\[ \tilde{Z}_j \subseteq \tilde{Z}_1 \quad \text{for} \quad j \geq 1. \quad (2.23) \]

Hence, from (2.7), (2.18) and (2.23), we have the following.

**Proposition 2.** Let \( b = 3g \ (g \in \mathbb{Z}), \ a = 3l + 1 \ (l \in \mathbb{Z} \setminus \{0\}) \) and \( c = 3l' + 1 \ (l' \in \mathbb{Z} \setminus \{0\}) \). For the self-affine measure \( \mu \) corresponding to (1.6), the zero set \( Z(\hat{\mu}) \) is given by

\[ Z(\hat{\mu}) = Z_1 \cup \tilde{Z}_1 \quad (2.24) \]

with

\[ Z_1 \cap \tilde{Z}_1 = (Z_1 \cup \tilde{Z}_1) \cap \mathbb{Z}^2 = \emptyset, \quad (2.25) \]

where \( Z_1 \) and \( \tilde{Z}_1 \) are given by (2.14) and (2.19) respectively.

**2.1.2. The cases \( b = 3g \ (g \in \mathbb{Z}) \) and (B)**

Under the conditions \( b = 3g \ (g \in \mathbb{Z}) \) and (B), that is

\[ b = 3g \quad (g \in \mathbb{Z}), \quad a = 3l + 1 \quad (l \in \mathbb{Z} \setminus \{0\}) \quad \text{and} \quad c = 3l' + 2 \quad (l' \in \mathbb{Z} \setminus \{-1\}), \quad (2.26) \]

we can rewrite \( Z_j \) in (2.11) as

\[ Z_j = \left\{ \left( \frac{1/3}{2j + 1/3} \right) + \left( \begin{array}{c}
  x(l, l', g, j; k_1, k_2) \\
  y(l, l', g, j; k_1, k_2)
\end{array} \right) : k_1, k_2 \in \mathbb{Z} \right\} \subseteq \mathbb{R}^2, \quad (2.27) \]

where

\[ x(l, l', g, j; k_1, k_2) = \frac{(3l + 1)^j - 1}{3} + (3l + 1)^j k_1 \in \mathbb{Z}, \quad (2.28) \]

\[ y(l, l', g, j; k_1, k_2) = \frac{2(3l' + 2)^j - 2^j}{3} + gq(j) + 3gq(j)k_1 + (3l' + 2)^j k_2 \in \mathbb{Z}, \quad (2.29) \]

and \( q(j) = \sum_{k=0}^{j-1} (3l + 1)^k(3l' + 2)^{j-k-1} \).
A little difference from the above case, we find, from (2.28) and (2.29), that there exist \( k'_1 \in \mathbb{Z}, k'_2 \in \mathbb{Z} \) such that

\[
x(l, l', g, j + 2; k_1, k_2) = x(l, l', g, j; k'_1, k'_2).
\]

(2.30)

This shows that

\[
Z_{j+2} \subseteq Z_j \quad \text{for } j \geq 1.
\]

(2.31)

In the same way, we can rewrite \( \tilde{Z}_j \) in (2.12) as

\[
\tilde{Z}_j = \left\{ \left( \frac{2}{3} j / 3 \right) + \left( \frac{\tilde{x}(l, l', g, j; \tilde{k}_1, \tilde{k}_2)}{\tilde{y}(l, l', g, j; \tilde{k}_1, \tilde{k}_2)} : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right) \right\} \subseteq \mathbb{R}^2,
\]

(2.32)

where

\[
\tilde{x}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) = \frac{2((3l + 1)^j - 1)}{3} + (3l + 1)^j \tilde{k}_1 \in \mathbb{Z},
\]

(2.33)

\[
\tilde{y}(l, l', g, j; \tilde{k}_1, \tilde{k}_2)
= \frac{(3l' + 2)^j - 2^j}{3} + 2gq(j) + 3gq(j)\tilde{k}_1 + (3l' + 2)^j \tilde{k}_2 \in \mathbb{Z},
\]

(2.34)

and \( q(j) = \sum_{k=0}^{j-1} (3l + 1)^k (3l' + 2)^{j-k-1} \). Then, one can verify that there exist \( \tilde{k}_1 \in \mathbb{Z}, \tilde{k}_2 \in \mathbb{Z} \) such that

\[
\tilde{x}(l, l', g, j + 2; \tilde{k}_1, \tilde{k}_2) = \tilde{x}(l, l', g, j; \tilde{k}'_1, \tilde{k}'_2),
\]

(2.35)

\[
2^j + \tilde{y}(l, l', g, j + 2; \tilde{k}_1, \tilde{k}_2) = \tilde{y}(l, l', g, j; \tilde{k}'_1, \tilde{k}'_2).
\]

This also shows that

\[
\tilde{Z}_{j+2} \subseteq \tilde{Z}_j \quad \text{for } j \geq 1.
\]

(2.36)

Hence, from (2.7), (2.31) and (2.36), we have the following.

**Proposition 3.** Let \( b = 3g \ (g \in \mathbb{Z}), a = 3l + 1 \ (l \in \mathbb{Z} \setminus \{0\}) \) and \( c = 3l' + 2 \ (l' \in \mathbb{Z} \setminus \{-1\}) \). For the self-affine measure \( \mu_{M,D} \) corresponding to (1.6), the zero set \( Z(\hat{\mu}_{M,D}) \) is given by

\[
Z(\hat{\mu}_{M,D}) = Z_1 \cup Z_2 \cup \tilde{Z}_1 \cup \tilde{Z}_2
\]

(2.37)

with

\[
Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2 \text{ are mutually disjoint and } \bigcup_{j=1}^{2}(Z_j \cup \tilde{Z}_j) \cap \mathbb{Z}^2 = \emptyset,
\]

(2.38)

where \( Z_1 \) and \( Z_2 \) are given by (2.27), \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) are given by (2.32).
2.1.3. The cases $b = 3g$ ($g \in \mathbb{Z}$) and (C)

Under the conditions $b = 3g$ ($g \in \mathbb{Z}$) and (C), that is

$$b = 3g \ (g \in \mathbb{Z}), \quad a = 3l + 2 \ (l \in \mathbb{Z} \setminus \{-1\}) \quad \text{and} \quad c = 3l' + 1 \ (l' \in \mathbb{Z} \setminus \{0\}),$$

(2.39)
as in the above Section 2.1.2, we first rewrite $Z_j$ in (2.11) and $\tilde{Z}_j$ in (2.12) as

$$Z_j = \left\{ \left( \frac{2^{j}/3}{2/3} \right) + y(l, l', g, j; k_1, k_2) : k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \tag{2.40}$$

$$\tilde{Z}_j = \left\{ \left( \frac{2^{j+1}/3}{1/3} \right) + \tilde{y}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \tag{2.41}$$

where

$$x(l, l', g, j; k_1, k_2) = \frac{(3l + 2)^j - 2^j}{3} + (3l + 2)^j k_1 \in \mathbb{Z},$$

(2.42)

$$y(l, l', g, j; k_1, k_2) = \frac{2((3l' + 1)^j - 1)}{3} + gq(j) + 3gq(j)k_1 + (3l' + 1)^j k_2 \in \mathbb{Z},$$

(2.43)

$$\tilde{x}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) = \frac{2((3l + 2)^j - 2^j)}{3} + (3l + 2)^j \tilde{k}_1 \in \mathbb{Z},$$

(2.44)

$$\tilde{y}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) = \frac{(3l' + 1)^j - 1}{3} + 2gq(j) + 3gq(j)\tilde{k}_1 + (3l' + 1)^j \tilde{k}_2 \in \mathbb{Z},$$

(2.45)

and $q(j) = \sum_{k=0}^{j-1}(3l + 2)^k(3l' + 1)^{j-k-1}$.

Then, one can verify that

$$Z_{j+2} \subseteq Z_j \quad \text{and} \quad \tilde{Z}_{j+2} \subseteq \tilde{Z}_j \quad \text{for} \ j \geq 1. \tag{2.46}$$

Hence, from (2.7) and (2.46), we have the following.

**Proposition 4.** Let $b = 3g$ ($g \in \mathbb{Z}$), $a = 3l + 2$ ($l \in \mathbb{Z} \setminus \{-1\}$) and $c = 3l' + 1$ ($l' \in \mathbb{Z} \setminus \{0\}$). For the self-affine measure $\mu_{M,D}$ corresponding to (1.6), the zero set $Z(\hat{\mu}_{M,D})$ satisfies (2.37) and (2.38), where $Z_1$ and $Z_2$ are given by (2.40), $\tilde{Z}_1$ and $\tilde{Z}_2$ are given by (2.41).

2.1.4. The cases $b = 3g$ ($g \in \mathbb{Z}$) and (D)

Under the conditions $b = 3g$ ($g \in \mathbb{Z}$) and (D), that is

$$b = 3g \ (g \in \mathbb{Z}), \quad a = 3l + 2 \ (l \in \mathbb{Z} \setminus \{-1\}) \quad \text{and} \quad c = 3l' + 2 \ (l' \in \mathbb{Z} \setminus \{-1\}),$$

(2.47)
as in the above Section 2.1.1, we first rewrite $Z_j$ in (2.11) and $\tilde{Z}_j$ in (2.12) as
\begin{equation}
Z_j = \left\{ \left( \frac{2^j}{3} \right) + \left( \frac{x(l, l', g, j; k_1, k_2)}{y(l, l', g, j; k_1, k_2)} \right) : k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \tag{2.48}
\end{equation}
\begin{equation}
\tilde{Z}_j = \left\{ \left( \frac{2^{j+1}}{3} \right) + \left( \frac{\tilde{x}(l, l', g, j; \tilde{k}_1, \tilde{k}_2)}{\tilde{y}(l, l', g, j; \tilde{k}_1, \tilde{k}_2)} \right) : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \tag{2.49}
\end{equation}
where
\begin{align*}
x(l, l', g, j; k_1, k_2) &= \frac{(3l + 2)^j - 2^j}{3} + (3l + 2)^j k_1 \in \mathbb{Z}, \tag{2.50} \\
y(l, l', g, j; k_1, k_2) &= \frac{2((3l' + 2)^j - 2^j)}{3} + gq(j) \\
\quad + 3gq(j)k_1 + (3l' + 2)^j k_2 \in \mathbb{Z}, \tag{2.51} \\
\tilde{x}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) &= \frac{(3l + 2)^j - 2^j}{3} + (3l + 2)^j \tilde{k}_1 \in \mathbb{Z}, \tag{2.52} \\
\tilde{y}(l, l', g, j; \tilde{k}_1, \tilde{k}_2) &= \frac{(3l' + 2)^j - 2^j}{3} + 2gq(j) \\
\quad + 3gq(j)\tilde{k}_1 + (3l' + 2)^j \tilde{k}_2 \in \mathbb{Z}, \tag{2.53}
\end{align*}
and $q(j) = \sum_{k=0}^{j-1} (3l + 2)^k (3l' + 2)^j k^{j-k-1}$.

Then, we find, with a little difference from the above cases, that the following inclusion relations
\begin{equation}
Z_{j+1} \subseteq \tilde{Z}_j \quad \text{and} \quad \tilde{Z}_{j+1} \subseteq Z_j \quad \text{for} \quad j \geq 1 \tag{2.54}
\end{equation}
hold. Hence, from (2.7) and (2.54), we have the following.

**Proposition 5.** Let $b = 3g$ ($g \in \mathbb{Z}$), $a = 3l + 2$ ($l \in \mathbb{Z} \setminus \{-1\}$) and $c = 3l' + 2$ ($l' \in \mathbb{Z} \setminus \{-1\}$). For the self-affine measure $\mu_{M,D}$ corresponding to (1.6), the zero set $Z(\mu_{M,D})$ satisfies (2.24) and (2.25), where $Z_1$ and $\tilde{Z}_1$ are given by (2.48) and (2.49) respectively.

2.2. The case $b = 3g + 1$ ($g \in \mathbb{Z}$)

In the case when $b = 3g + 1$ ($g \in \mathbb{Z}$), we can rewrite the planar sets $Z_j$ and $\tilde{Z}_j$ in (2.8) and (2.9) as
\begin{equation}
Z_j = \left\{ \left( \frac{a^j}{3} \right) + \left( \frac{a^j k_1}{(q(j) + 2c^j)/3} \right) : k_1, k_2 \in \mathbb{Z} \right\} \tag{2.55}
\end{equation}
and
\begin{equation}
\tilde{Z}_j = \left\{ \left( \frac{2a^j}{3} \right) + \left( \frac{a^j \tilde{k}_1}{(2q(j) + c^j)/3} \right) : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \tag{2.56}
\end{equation}
respectively, where \( q(j) = \sum_{k=0}^{j-1} a^k c^{j-k-1} \). Furthermore, if \( a \) and \( c \) satisfy one of the four conditions (A), (B), (C) and (D), we will find certain inclusion relations between \( Z_j \) and \( \tilde{Z}_j \) (a little difference from Section 2.1) by applying the same technique as Section 2.1.

2.2.1. The cases \( b = 3g + 1 \ (g \in \mathbb{Z}) \) and (A)

Under the conditions \( b = 3g + 1 \ (g \in \mathbb{Z}) \) and (A), that is

\[
b = 3g + 1 \quad (g \in \mathbb{Z}), \quad a = 3l + 1 \quad (l \in \mathbb{Z} \setminus \{0\}) \quad \text{and} \quad c = 3l' + 1 \quad (l' \in \mathbb{Z} \setminus \{0\}),
\]

we find that the following inclusion relations

\[
Z_{j+3} \subseteq Z_j \quad \text{and} \quad \tilde{Z}_{j+3} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots)
\]

(2.57)

hold. Hence, from (2.7) and (2.58), we have the following.

**Proposition 6.** Let \( b = 3g + 1 \ (g \in \mathbb{Z}) \), \( a = 3l + 1 \ (l \in \mathbb{Z} \setminus \{0\}) \) and \( c = 3l' + 1 \ (l' \in \mathbb{Z} \setminus \{0\}) \).
For the self-affine measure \( \mu_{M,D} \) corresponding to (1.6), the zero set \( Z(\hat{\mu}_{M,D}) \) is given by

\[
Z(\hat{\mu}_{M,D}) = Z_1 \cup Z_2 \cup Z_3 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3
\]

(2.59)

with

\[
Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3 \text{ are mutually disjoint and } \bigcup_{j=1}^{3} (Z_j \cup \tilde{Z}_j) \cap \mathbb{Z}^2 = \emptyset,
\]

(2.60)

where \( Z_j \) and \( \tilde{Z}_j \ (j = 1, 2, 3) \) are given by (2.55) and (2.56) respectively with \( a, b, c \) given by (2.57).

2.2.2. The cases \( b = 3g + 1 \ (g \in \mathbb{Z}) \) and (B)

Under the conditions \( b = 3g + 1 \ (g \in \mathbb{Z}) \) and (B), that is

\[
b = 3g + 1 \quad (g \in \mathbb{Z}), \quad a = 3l + 1 \quad (l \in \mathbb{Z} \setminus \{0\}) \quad \text{and} \quad c = 3l' + 2 \quad (l' \in \mathbb{Z} \setminus \{-1\}),
\]

(2.61)

we find that the following inclusion relations

\[
Z_{j+1} \subseteq Z_j \quad \text{and} \quad \tilde{Z}_{j+1} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots)
\]

(2.62)

hold. Hence, from (2.7) and (2.62), we have the following.

**Proposition 7.** Let \( b = 3g + 1 \ (g \in \mathbb{Z}) \), \( a = 3l + 1 \ (l \in \mathbb{Z} \setminus \{0\}) \) and \( c = 3l' + 2 \ (l' \in \mathbb{Z} \setminus \{-1\}) \).
For the self-affine measure \( \mu_{M,D} \) corresponding to (1.6), the zero set \( Z(\hat{\mu}_{M,D}) \) satisfies (2.24) and (2.25), where \( Z_1 \) and \( \tilde{Z}_1 \) are given by (2.55) and (2.56) respectively with \( a, b, c \) given by (2.61).
2.2.3. The cases \( b = 3g + 1 \) (\( g \in \mathbb{Z} \)) and (C)

Under the conditions \( b = 3g + 1 \) (\( g \in \mathbb{Z} \)) and (C), that is

\[
b = 3g + 1 \quad (g \in \mathbb{Z}), \quad a = 3l + 2 \quad (l \in \mathbb{Z} \setminus \{-1\}) \quad \text{and} \quad c = 3l' + 1 \quad (l' \in \mathbb{Z} \setminus \{0\}),
\]

we find that the following inclusion relations

\[
Z_{j+2} \subseteq Z_j \quad \text{and} \quad \tilde{Z}_{j+2} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots)
\]

hold. Hence, from (2.7) and (2.64), we have the following.

**Proposition 8.** Let \( b = 3g + 1 \) (\( g \in \mathbb{Z} \)), \( a = 3l + 2 \) (\( l \in \mathbb{Z} \setminus \{-1\}) \) and \( c = 3l' + 1 \) (\( l' \in \mathbb{Z} \setminus \{0\})\). For the self-affine measure \( \mu_{M,D} \) corresponding to (1.6), the zero set \( Z(\hat{\mu}_{M,D}) \) satisfies (2.37) and (2.38), where \( Z_j \) and \( \tilde{Z}_j \) \((j = 1, 2)\) are given by (2.55) and (2.56) respectively with \( a, b, c \) given by (2.63).

2.2.4. The cases \( b = 3g + 1 \) (\( g \in \mathbb{Z} \)) and (D)

Under the conditions \( b = 3g + 1 \) (\( g \in \mathbb{Z} \)) and (D), that is

\[
b = 3g + 1 \quad (g \in \mathbb{Z}), \quad a = 3l + 2 \quad (l \in \mathbb{Z} \setminus \{-1\}) \quad \text{and} \quad c = 3l' + 2 \quad (l' \in \mathbb{Z} \setminus \{-1\}),
\]

we find that the following inclusion relations

\[
Z_{j+3} \subseteq \tilde{Z}_j \quad \text{and} \quad \tilde{Z}_{j+3} \subseteq Z_j \quad (j = 1, 2, \ldots)
\]

hold. Hence, from (2.7) and (2.66), we have the following.

**Proposition 9.** Let \( b = 3g + 1 \) (\( g \in \mathbb{Z} \)), \( a = 3l + 2 \) (\( l \in \mathbb{Z} \setminus \{-1\}) \) and \( c = 3l' + 2 \) (\( l' \in \mathbb{Z} \setminus \{-1\})\). For the self-affine measure \( \mu_{M,D} \) corresponding to (1.6), the zero set \( Z(\hat{\mu}_{M,D}) \) satisfies (2.59) and (2.60), where \( Z_j \) and \( \tilde{Z}_j \) \((j = 1, 2, 3)\) are given by (2.55) and (2.56) respectively with \( a, b, c \) given by (2.65).

2.3. The case \( b = 3g + 2 \) (\( g \in \mathbb{Z} \))

In the case when \( b = 3g + 2 \) (\( g \in \mathbb{Z} \)), we can rewrite the planar sets \( Z_j \) and \( \tilde{Z}_j \) in (2.8) and (2.9) as

\[
Z_j = \left\{ \left( \frac{a^j / 3}{(2q(j) + 2c^j) / 3} + \frac{a^j k_1}{gq(j) + (3g + 2)q(j)k_1 + c^j k_2} : k_1, k_2 \in \mathbb{Z} \right) \right\} \quad (2.67)
\]

and

\[
\tilde{Z}_j = \left\{ \left( \frac{2a^j / 3}{(q(j) + c^j) / 3} + \frac{a^j \tilde{k}_1}{(2g + 1)q(j) + (3g + 2)q(j)\tilde{k}_1 + c^j \tilde{k}_2} : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right) \right\} \quad (2.68)
\]
respectively, where \( q(j) = \sum_{k=0}^{j-1} a^k c^{j-k-1} \). Furthermore, if \( a \) and \( c \) satisfy one of the four conditions (A), (B), (C) and (D), we will find certain inclusion relations between \( Z_j \) and \( \tilde{Z}_j \) (a little difference from those in Sections 2.1 and 2.2) by applying the same method as above.

2.3.1. The cases \( b = 3g + 2 \ (g \in \mathbb{Z}) \) and (A)

Under the conditions \( b = 3g + 2 \ (g \in \mathbb{Z}) \) and (A), that is
\[
b = 3g + 2 \ (g \in \mathbb{Z}), \quad a = 3l + 1 \ (l \in \mathbb{Z} \setminus \{0\}) \quad \text{and} \quad c = 3l' + 1 \ (l' \in \mathbb{Z} \setminus \{0\}).
\]
(2.69)
we find that the following inclusion relations
\[
Z_{j+3} \subseteq Z_j \quad \text{and} \quad \tilde{Z}_{j+3} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots)
\]
(2.70)
hold. Hence, from (2.7) and (2.70), we have the following.

**Proposition 10.** Let \( b = 3g + 2 \ (g \in \mathbb{Z}) \), \( a = 3l + 1 \ (l \in \mathbb{Z} \setminus \{0\}) \) and \( c = 3l' + 1 \ (l' \in \mathbb{Z} \setminus \{0\}) \). For the self-affine measure \( \mu_{M,D} \) corresponding to (1.6), the zero set \( Z(\hat{\mu}_{M,D}) \) satisfies (2.59) and (2.60), where \( Z_j \) and \( \tilde{Z}_j \) \( (j = 1, 2, 3) \) are given by (2.67) and (2.68) respectively with \( a, b, c \) given by (2.69).

2.3.2. The cases \( b = 3g + 2 \ (g \in \mathbb{Z}) \) and (B)

Under the conditions \( b = 3g + 2 \ (g \in \mathbb{Z}) \) and (B), that is
\[
b = 3g + 2 \ (g \in \mathbb{Z}), \quad a = 3l + 1 \ (l \in \mathbb{Z} \setminus \{0\}) \quad \text{and} \quad c = 3l' + 2 \ (l' \in \mathbb{Z} \setminus \{1\}).
\]
(2.71)
we find that the following inclusion relations
\[
Z_{j+2} \subseteq Z_j \quad \text{and} \quad \tilde{Z}_{j+2} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots)
\]
(2.72)
hold. Hence, from (2.7) and (2.72), we have the following.

**Proposition 11.** Let \( b = 3g + 2 \ (g \in \mathbb{Z}) \), \( a = 3l + 1 \ (l \in \mathbb{Z} \setminus \{0\}) \) and \( c = 3l' + 2 \ (l' \in \mathbb{Z} \setminus \{-1\}) \). For the self-affine measure \( \mu_{M,D} \) corresponding to (1.6), the zero set \( Z(\hat{\mu}_{M,D}) \) satisfies (2.37) and (2.38), where \( Z_j \) and \( \tilde{Z}_j \) \( (j = 1, 2) \) are given by (2.67) and (2.68) respectively with \( a, b, c \) given by (2.71).

2.3.3. The cases \( b = 3g + 2 \ (g \in \mathbb{Z}) \) and (C)

Under the conditions \( b = 3g + 2 \ (g \in \mathbb{Z}) \) and (C), that is
\[
b = 3g + 2 \ (g \in \mathbb{Z}), \quad a = 3l + 2 \ (l \in \mathbb{Z} \setminus \{-1\}) \quad \text{and} \quad c = 3l' + 1 \ (l' \in \mathbb{Z} \setminus \{0\}).
\]
(2.73)
we find that the following inclusion relations

\[ Z_{j+1} \subseteq \tilde{Z}_j \quad \text{and} \quad \tilde{Z}_{j+1} \subseteq Z_j \quad (j = 1, 2, \ldots) \quad (2.74) \]

hold. Hence, from (2.7) and (2.74), we have the following.

**Proposition 12.** Let \( b = 3g + 2 \) \((g \in \mathbb{Z})\), \( a = 3l + 2 \) \((l \in \mathbb{Z} \setminus \{-1\})\) and \( c = 3l' + 1 \) \((l' \in \mathbb{Z} \setminus \{0\})\). For the self-affine measure \( \mu_{M,D} \) corresponding to (1.6), the zero set \( Z(\hat{\mu}_{M,D}) \) satisfies (2.24) and (2.25), where \( Z_1 \) and \( \tilde{Z}_1 \) are given by (2.67) and (2.68) respectively with \( a, b, c \) given by (2.73).

2.3.4. The cases \( b = 3g + 2 \) \((g \in \mathbb{Z})\) and (D)

Under the conditions \( b = 3g + 2 \) \((g \in \mathbb{Z})\) and (D), that is

\[ b = 3g + 2 \quad (g \in \mathbb{Z}), \quad a = 3l + 2 \quad (l \in \mathbb{Z} \setminus \{-1\}) \quad \text{and} \quad c = 3l' + 2 \quad (l' \in \mathbb{Z} \setminus \{-1\}). \quad (2.75) \]

we find that the following inclusion relations

\[ Z_{j+3} \subseteq \tilde{Z}_j \quad \text{and} \quad \tilde{Z}_{j+3} \subseteq Z_j \quad (j = 1, 2, \ldots) \quad (2.76) \]

hold. Hence, from (2.7) and (2.76), we have the following.

**Proposition 13.** Let \( b = 3g + 2 \) \((g \in \mathbb{Z})\), \( a = 3l + 2 \) \((l \in \mathbb{Z} \setminus \{-1\})\) and \( c = 3l' + 2 \) \((l' \in \mathbb{Z} \setminus \{-1\})\). For the self-affine measure \( \mu_{M,D} \) corresponding to (1.6), the zero set \( Z(\hat{\mu}_{M,D}) \) satisfies (2.59) and (2.60), where \( Z_j \) and \( \tilde{Z}_j \) \((j = 1, 2, 3)\) are given by (2.67) and (2.68) respectively with \( a, b, c \) given by (2.75).

2.4. Summary of the above three cases (twelve subcases)

The above discussion involves the three cases: \( b = 3g \) \((g \in \mathbb{Z})\) (Section 2.1), \( b = 3g + 1 \) \((g \in \mathbb{Z})\) (Section 2.2) and \( b = 3g + 2 \) \((g \in \mathbb{Z})\) (Section 2.3). Each section contains four subsections. Propositions 2–13 correspond to the twelve subsections. These established propositions characterize the zero set \( Z(\hat{\mu}_{M,D}) \). They can be divided into three types:

**Type 1.** Propositions 2, 5, 7 and 12 illustrate that \( Z(\hat{\mu}_{M,D}) \) satisfies (2.24) and (2.25).

**Type 2.** Propositions 3, 4, 8 and 11 illustrate that \( Z(\hat{\mu}_{M,D}) \) satisfies (2.37) and (2.38).

**Type 3.** Propositions 6, 9, 10 and 13 illustrate that \( Z(\hat{\mu}_{M,D}) \) satisfies (2.59) and (2.60).

The above three types correspond to three kinds of representations for \( Z(\hat{\mu}_{M,D}) \) which will help us to prove Theorem in the next section.
3. Proof of Theorem

If \( \lambda_j \ (j = 1, 2, 3, 4) \in \mathbb{R}^2 \) are such that the exponential functions

\[
e^{2\pi i \langle \lambda_1, x \rangle}, \quad e^{2\pi i \langle \lambda_2, x \rangle}, \quad e^{2\pi i \langle \lambda_3, x \rangle}, \quad e^{2\pi i \langle \lambda_4, x \rangle},
\]

are mutually orthogonal in \( L^2(\mu_{M,D}) \), then the differences \( \lambda_j - \lambda_k \ (1 \leq j \neq k \leq 4) \) are in the zero set \( Z(\hat{\mu}_{M,D}) \). That is, we have

\[
\lambda_j - \lambda_k \in Z(\hat{\mu}_{M,D}) \quad (1 \leq j \neq k \leq 4). \tag{3.1}
\]

We will use the above established facts on the zero set \( Z(\hat{\mu}_{M,D}) \) to deduce a contradiction. The proof will divide into three sections according to Types 1–3.

3.1. The case of Type 1

In the case of Type 1, we obtain from (2.24) and (3.1) that

\[
\lambda_j - \lambda_k \in Z_1 \cup \tilde{Z}_1 \quad (1 \leq j \neq k \leq 4) \tag{3.2}
\]

and (2.25) hold. In particular, the following three differences

\[
\lambda_1 - \lambda_2, \quad \lambda_1 - \lambda_3, \quad \lambda_1 - \lambda_4 \tag{3.3}
\]

are in \( Z_1 \cup \tilde{Z}_1 \). The well-known pigeon hole principle, combined with (2.25), (3.2) and Proposition 1(2), immediately deduces a contradiction, since any two of three differences in (3.3) cannot belong to the same set \( Z_1 \) or \( \tilde{Z}_1 \). For example, if \( \lambda_1 - \lambda_2 \in \tilde{Z}_1 \) and \( \lambda_1 - \lambda_4 \in \tilde{Z}_1 \), then, by Proposition 1(2),

\[
\lambda_4 - \lambda_2 = (\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_4) \in \tilde{Z}_1 - \tilde{Z}_1 \subseteq Z^2
\]

which contradicts (2.25) and (3.2). Hence any set of \( \mu_{M,D} \)-orthogonal exponentials contains at most 3 elements. One can obtain many such orthogonal systems which contain three elements. For instance, the exponential function system \( E_\Lambda \) with \( \Lambda \) given by

\[
\Lambda = \{0, s_1, s_2\} \subset \mathbb{R}^2 \quad \text{for each} \ s_1 \in Z_1 \text{ and } s_2 \in \tilde{Z}_1 \tag{3.4}
\]

is a three-elements orthogonal system in \( L^2(\mu_{M,D}) \). This shows that the number 3 is the best.

3.2. The case of Type 2

In the case of Type 2, we obtain from (2.37) and (3.1) that

\[
\lambda_j - \lambda_k \in Z_1 \cup Z_2 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \quad (1 \leq j \neq k \leq 4) \tag{3.5}
\]

and (2.38) hold. We will use Proposition 1, (2.38) and (3.5) to deduce a contradiction.
Observe that the following six differences:

\[
\lambda_1 - \lambda_2, \quad \lambda_1 - \lambda_3, \quad \lambda_1 - \lambda_4, \\
\lambda_2 - \lambda_3, \quad \lambda_2 - \lambda_4, \quad \lambda_3 - \lambda_4,
\]

belong to the four sets \(Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2\). By Proposition 1 and (2.38), the elements (or differences) in each row of (3.6) (except the final row where there is only one element \(\lambda_3 - \lambda_4\)) and the elements (or differences) in each column of (3.6) (except the first column where there is only one element \(\lambda_1 - \lambda_2\)) cannot belong to the same set. In particular, the following three elements

\[
\lambda_1 - \lambda_2, \quad \lambda_1 - \lambda_3, \quad \lambda_1 - \lambda_4
\]

in the first row will be in the three different sets of the four sets \(Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2\). There are 24 distribution methods. We only consider the following typical case.

**Typical case.** \(\lambda_1 - \lambda_2 \in Z_1, \lambda_1 - \lambda_3 \in Z_2, \lambda_1 - \lambda_4 \in \tilde{Z}_1\).

The other cases (by applying Proposition 1(1)) can be proved in the same manner.

By Propositions 1(1), 1(3), we see that in this Typical case, each set contains elements (or differences) in the following Box 1:

<table>
<thead>
<tr>
<th>(Z_1)</th>
<th>(Z_2)</th>
<th>(\tilde{Z}_1)</th>
<th>(\tilde{Z}_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_1 - \lambda_2)</td>
<td>(\lambda_1 - \lambda_3)</td>
<td>(\lambda_1 - \lambda_4)</td>
<td>(\lambda_3 - \lambda_1)</td>
</tr>
<tr>
<td>(\lambda_2 - \lambda_4)</td>
<td>(\lambda_2 - \lambda_1)</td>
<td>(\lambda_4 - \lambda_2)</td>
<td></td>
</tr>
</tbody>
</table>

Box 1

The other elements in (3.6), that is, the elements \(\lambda_2 - \lambda_3\) and \(\lambda_3 - \lambda_4\), are also in certain small boxes of Box 1. Firstly, we have the following fact that

\[
\lambda_2 - \lambda_3 \text{ cannot belong to the sets (or small boxes) } Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2.
\]

(3.7)

The reason is as follows.

(i) If \(\lambda_2 - \lambda_3 \in Z_1\), then, by Proposition 1(2),

\[
\lambda_4 - \lambda_3 = (\lambda_2 - \lambda_3) - (\lambda_2 - \lambda_4) \in Z_1 - Z_1 \subseteq \mathbb{Z}^2
\]

which contradicts (2.38) and (3.5). Also, if \(\lambda_2 - \lambda_3 \in Z_1\), then, by Proposition 1(3),

\[
\lambda_1 - \lambda_3 = (\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_3) \in Z_1 + Z_1 \subseteq \tilde{Z}_1,
\]

which contradicts (2.38) and \(\lambda_1 - \lambda_3 \notin \tilde{Z}_1\).

(ii) If \(\lambda_2 - \lambda_3 \in Z_2\), then, by Proposition 1(2),

\[
\lambda_2 - \lambda_1 = (\lambda_2 - \lambda_3) - (\lambda_1 - \lambda_3) \in Z_2 - Z_2 \subseteq \mathbb{Z}^2,
\]

which contradicts (2.38) and (3.5) or contradicts (2.38) and \(\lambda_2 - \lambda_1 \in \tilde{Z}_1\).
(iii) if \( \lambda_2 - \lambda_3 \in \tilde{Z}_2 \), then, by Proposition 1(3),

\[
\lambda_2 - \lambda_1 = (\lambda_2 - \lambda_3) + (\lambda_3 - \lambda_1) \in \tilde{Z}_2 + \tilde{Z}_2 \subseteq Z_2, \tag{3.11}
\]

which contradicts (2.38) and \( \lambda_2 - \lambda_1 \in \tilde{Z}_1 \). Hence (3.7) holds.

Similarly, we have the following facts that

\[
\lambda_3 - \lambda_4 \text{ cannot belong to the sets (or small boxes)} \ Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2, \tag{3.12}
\]

The established facts (3.7) and (3.12) clearly contradict (3.5). This proves the Typical case. The other cases can be proved in the same manner.

Therefore, any set of \( \mu_{M,D} \)-orthogonal exponentials contains at most 3 elements. One can obtain many such orthogonal systems which contain 3 elements. For example, the exponential function systems \( E_\Lambda \) with \( \Lambda \) given by (3.4) or with \( \Lambda \) given by

\[
\Lambda = \{0, s_1, s_2\} \subset \mathbb{R}^2 \quad \text{for each} \ s_1 \in Z_2 \text{ and } s_2 \in \tilde{Z}_2 \tag{3.13}
\]

are also the three-elements orthogonal system in \( L^2(\mu_{M,D}) \). This shows that the number 3 is the best.

3.3. The case of Type 3

In the case of Type 3, we obtain from (2.59) and (3.1) that

\[
\lambda_j - \lambda_k \in Z_1 \cup Z_2 \cup Z_3 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3 \quad (1 \leq j \neq k \leq 4) \tag{3.14}
\]

and (2.60) hold. We will use Proposition 1, (2.60) and (3.14) to deduce a contradiction.

In this case, the six differences in (3.6) belong to the six sets \( Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3 \). By Proposition 1 and (2.60), the elements (or differences) in each row of (3.6) (except the final row where there is only one element \( \lambda_3 - \lambda_4 \)) and the elements (or differences) in each column of (3.6) (except the first column where there is only one element \( \lambda_1 - \lambda_2 \)) cannot belong to the same set. In particular, the following three elements in the first row of (3.6)

\[
\lambda_1 - \lambda_2, \quad \lambda_1 - \lambda_3, \quad \lambda_1 - \lambda_4
\]

will be in the three different sets of the six sets \( Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3 \). There are 120 distribution methods. One can use the method presented in [29] to deal with each case. For completeness, we use this method to deal with the following three typical cases:

Case 1. \( \lambda_1 - \lambda_2 \in \tilde{Z}_1, \lambda_1 - \lambda_3 \in \tilde{Z}_2, \lambda_1 - \lambda_4 \in \tilde{Z}_3 \).
Case 2. \( \lambda_1 - \lambda_2 \in Z_1, \lambda_1 - \lambda_3 \in Z_2, \lambda_1 - \lambda_4 \in \tilde{Z}_3 \).
Case 3. \( \lambda_1 - \lambda_2 \in Z_1, \lambda_1 - \lambda_3 \in Z_2, \lambda_1 - \lambda_4 \in \tilde{Z}_1 \).

The other cases (by applying Proposition 1(1)) can be proved in the same manner.

Case 1. By Proposition 1(1), we see that in this case, each set contains elements (or differences) in the following Box 2:
The other elements in (3.6) are also in certain small boxes of Box 2. Firstly, we have the following fact that
\[
\begin{align*}
\lambda_2 - \lambda_3 & \text{ cannot belong to the sets (or small boxes) } Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2. \tag{3.15}
\end{align*}
\]
The reason is as follows.
(i) If \( \lambda_2 - \lambda_3 \in Z_1 \), then, by Proposition 1(2),
\[
\lambda_3 - \lambda_1 = (\lambda_2 - \lambda_1) - (\lambda_2 - \lambda_3) \in Z_1 - Z_1 \subseteq \mathbb{Z}^2, \tag{3.16}
\]
which contradicts (2.60) and \( \lambda_3 - \lambda_1 \in Z_2 \). The same reason shows that \( \lambda_2 - \lambda_3 \notin \tilde{Z}_2 \).
(ii) If \( \lambda_2 - \lambda_3 \in Z_2 \), then, by Proposition 1(3),
\[
\lambda_2 - \lambda_1 = (\lambda_2 - \lambda_3) + (\lambda_3 - \lambda_1) \in Z_2 + Z_2 \subseteq \tilde{Z}_2, \tag{3.17}
\]
which contradicts (2.60) and \( \lambda_2 - \lambda_1 \in Z_1 \). The same reason shows that \( \lambda_2 - \lambda_3 \notin \tilde{Z}_1 \).
Similarly, we have the following facts that:
\[
\begin{align*}
\lambda_2 - \lambda_4 & \text{ cannot belong to the sets (or small boxes) } Z_1, Z_3, \tilde{Z}_1, \tilde{Z}_3; \tag{3.18}
\end{align*}
\[
\begin{align*}
\lambda_3 - \lambda_4 & \text{ cannot belong to the sets (or small boxes) } Z_2, Z_3, \tilde{Z}_2, \tilde{Z}_3. \tag{3.19}
\end{align*}
\]
Hence, from (3.15), (3.18) and (3.19), we have
\[
\begin{align*}
\lambda_2 - \lambda_3 & \in Z_3 \text{ or } \tilde{Z}_3; & \lambda_2 - \lambda_4 & \in Z_2 \text{ or } \tilde{Z}_2; & \lambda_3 - \lambda_4 & \in Z_1 \text{ or } \tilde{Z}_1 \tag{3.20}
\end{align*}
\]
which is impossible. To see this, we only consider the following two typical cases:
(i’) If
\[
\begin{align*}
\lambda_2 - \lambda_3 & \in Z_3, \quad \lambda_2 - \lambda_4 \in Z_2, \quad \lambda_3 - \lambda_4 \in Z_1,
\end{align*}
\]
then, by Proposition 1(1), the above Box 2 becomes the following Box 3:
We apply Propositions 1(2), 1(3) to the elements of sets $Z_1$ and $Z_3$ (or $\tilde{Z}_1$ and $\tilde{Z}_3$) respectively.

Since

$$(\lambda_2 - \lambda_1) - (\lambda_3 - \lambda_4) = (\lambda_4 - \lambda_1) + (\lambda_2 - \lambda_3),$$

the left-hand side is in $Z_1 - Z_1 \subseteq \mathbb{Z}^2$ and the right-hand side is in $Z_3 + Z_3 \subseteq \tilde{Z}_3$, which leads to a contradiction by (2.60). Also, by Proposition 1(3), the elements in $Z_1$ and $Z_2$ (or in $\tilde{Z}_1$ and $\tilde{Z}_2$) have the character that

$$(\lambda_2 - \lambda_1) + (\lambda_3 - \lambda_4) = (\lambda_3 - \lambda_1) + (\lambda_2 - \lambda_4) \in \tilde{Z}_1 \cap \tilde{Z}_2,$$

which contradicts (2.60).

(ii') If

$$\lambda_2 - \lambda_3 \in \tilde{Z}_3, \quad \lambda_2 - \lambda_4 \in \tilde{Z}_2, \quad \lambda_3 - \lambda_4 \in Z_1,$$

then, by Proposition 1(1), the above Box 2 becomes the following Box 4:

<table>
<thead>
<tr>
<th>$Z_1$</th>
<th>$Z_2$</th>
<th>$Z_3$</th>
<th>$\tilde{Z}_1$</th>
<th>$\tilde{Z}_2$</th>
<th>$\tilde{Z}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_2 - \lambda_1$</td>
<td>$\lambda_3 - \lambda_4$</td>
<td>$\lambda_4 - \lambda_1$</td>
<td>$\lambda_1 - \lambda_2$</td>
<td>$\lambda_1 - \lambda_3$</td>
<td>$\lambda_1 - \lambda_4$</td>
</tr>
<tr>
<td>$\lambda_3 - \lambda_4$</td>
<td>$\lambda_4 - \lambda_2$</td>
<td>$\lambda_3 - \lambda_2$</td>
<td>$\lambda_4 - \lambda_3$</td>
<td>$\lambda_2 - \lambda_4$</td>
<td>$\lambda_2 - \lambda_3$</td>
</tr>
</tbody>
</table>

Box 4

By Proposition 1(3), the elements in $Z_2$ and $Z_3$ (or in $\tilde{Z}_2$ and $\tilde{Z}_3$) have the character that

$$(\lambda_3 - \lambda_1) + (\lambda_4 - \lambda_2) = (\lambda_4 - \lambda_1) + (\lambda_3 - \lambda_2) \in \tilde{Z}_2 \cap \tilde{Z}_3,$$

which contradicts (2.60). Another way to deduce a contradiction is to apply Propositions 1(2), 1(3) on the elements of sets $Z_1$ and $Z_2$ (or $\tilde{Z}_1$ and $\tilde{Z}_2$) respectively. Since

$$(\lambda_2 - \lambda_1) + (\lambda_3 - \lambda_4) = (\lambda_3 - \lambda_1) - (\lambda_4 - \lambda_2),$$

the left-hand side is in $Z_1 + Z_1 \subseteq \tilde{Z}_1$ and the right-hand side is in $Z_2 - Z_2 \subseteq \mathbb{Z}^2$, which also leads to a contradiction by (2.60). This completes the proof of Case 1.

**Case 2.** By Proposition 1(1), we see that in this case, each set contains elements (or differences) in the following Box 5:

<table>
<thead>
<tr>
<th>$Z_1$</th>
<th>$Z_2$</th>
<th>$Z_3$</th>
<th>$\tilde{Z}_1$</th>
<th>$\tilde{Z}_2$</th>
<th>$\tilde{Z}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1 - \lambda_2$</td>
<td>$\lambda_1 - \lambda_3$</td>
<td>$\lambda_4 - \lambda_1$</td>
<td>$\lambda_2 - \lambda_1$</td>
<td>$\lambda_3 - \lambda_1$</td>
<td>$\lambda_1 - \lambda_4$</td>
</tr>
</tbody>
</table>

Box 5
The other elements in (3.6) are also in certain small boxes of Box 5. As in Case 1, we have the following facts that:

\[
\begin{align*}
\lambda_2 - \lambda_3 & \text{ cannot belong to the sets (or small boxes) } Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2; \\
\lambda_2 - \lambda_4 & \text{ cannot belong to the sets (or small boxes) } Z_1, Z_3, \tilde{Z}_1, \tilde{Z}_3; \\
\lambda_3 - \lambda_4 & \text{ cannot belong to the sets (or small boxes) } Z_2, Z_3, \tilde{Z}_2, \tilde{Z}_3. 
\end{align*}
\] (3.6)

Hence, from (3.21)–(3.23), we have

\[
\begin{align*}
\lambda_2 - \lambda_3 & \in Z_3 \text{ or } \tilde{Z}_3; \\
\lambda_2 - \lambda_4 & \in Z_2 \text{ or } \tilde{Z}_2; \\
\lambda_3 - \lambda_4 & \in Z_1 \text{ or } \tilde{Z}_1
\end{align*}
\] (3.24)

which is impossible. The reason is the same as in Case 1. This completes the proof of Case 2.

**Case 3.** By Proposition 1(1), 1(3), we see that in this case, each set contains elements (or differences) in the following Box 6:

<table>
<thead>
<tr>
<th>(Z_1)</th>
<th>(Z_2)</th>
<th>(Z_3)</th>
<th>(\tilde{Z}_1)</th>
<th>(\tilde{Z}_2)</th>
<th>(\tilde{Z}_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_1 - \lambda_2)</td>
<td>(\lambda_1 - \lambda_3)</td>
<td>(\lambda_1 - \lambda_4)</td>
<td>(\lambda_2 - \lambda_1)</td>
<td>(\lambda_3 - \lambda_1)</td>
<td>(\lambda_4 - \lambda_2)</td>
</tr>
<tr>
<td>(\lambda_4 - \lambda_1)</td>
<td>(\lambda_2 - \lambda_4)</td>
<td>(\lambda_2 - \lambda_1)</td>
<td>(\lambda_3 - \lambda_1)</td>
<td>(\lambda_4 - \lambda_2)</td>
<td></td>
</tr>
<tr>
<td>(\lambda_2 - \lambda_4)</td>
<td>(\lambda_4 - \lambda_2)</td>
<td>(\lambda_2 - \lambda_1)</td>
<td>(\lambda_3 - \lambda_1)</td>
<td>(\lambda_4 - \lambda_2)</td>
<td></td>
</tr>
</tbody>
</table>

Box 6

The other elements in (3.6) are also in certain small boxes of Box 6. As in Case 1, we have the following facts that:

\[
\begin{align*}
\lambda_2 - \lambda_3 & \text{ cannot belong to the sets (or small boxes) } Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2; \\
\lambda_3 - \lambda_4 & \text{ cannot belong to the sets (or small boxes) } Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2.
\end{align*}
\] (3.25)

Hence, from (3.25) and (3.26), we have

\[
\begin{align*}
\lambda_2 - \lambda_3 & \in Z_3 \text{ or } \tilde{Z}_3; \\
\lambda_3 - \lambda_4 & \in Z_3 \text{ or } \tilde{Z}_3
\end{align*}
\] (3.27)

which is impossible. The reason is the same as in Case 1. This completes the proof of Case 3.

Hence any set of \(\mu_{M,D}\)-orthogonal exponentials contains at most 3 elements. One can obtain many such orthogonal systems which contain 3 elements. For example, the exponential function systems \(E_\Lambda\) with \(\Lambda\) given by (3.4) or with \(\Lambda\) given by

\[
\Lambda = \{0, s_1, s_2\} \subseteq \mathbb{R}^2 \text{ for each } s_1 \in Z_2 \text{ and } s_2 \in \tilde{Z}_2
\] (3.28)

or with \(\Lambda\) given by

\[
\Lambda = \{0, s_1, s_2\} \subseteq \mathbb{R}^2 \text{ for each } s_1 \in Z_3 \text{ and } s_2 \in \tilde{Z}_3
\] (3.29)

are also the three elements orthogonal systems in \(L^2(\mu_{M,D})\) (note that in each type, \(Z_j\) and \(\tilde{Z}_j\) have different representations according to the corresponding Proposition 2–13 or twelve subcases). This shows that the number 3 is the best. The proof of Theorem is complete.
4. A concluding remark

The non-spectral self-affine measure problem mentioned in Section 1 depends fundamentally on the characterization of the zero set $Z(\hat{\mu}_{M,D})$. For any finite set $D \subset \mathbb{R}^n$ of the cardinality $|D| = 3$ or $4$, one can obtain the certain expression for the set $Z(\hat{\mu}_{M,D})$ similar to (2.7). But it is more difficult to obtain some characteristic properties on this set. On the other hand, the non-spectral self-affine measure problem mentioned in Section 1 depends fundamentally on the non-spectral self-affine measure problem in the case when $ac - bd \notin 3\mathbb{Z}$ in Theorem is just $|D| \notin W(m)$. Following Conjecture 1, it is reasonable to extend Theorem further as shown in Conjecture 2 below.

**Conjecture 2.** For an expanding integer matrix $M$ and the three elements digit set $D$ given by

$$M = \begin{bmatrix} a & b \\ d & c \end{bmatrix} \in M_2(\mathbb{Z}) \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad (4.1)$$

if $ac - bd \notin 3\mathbb{Z}$, then there exist at most 3 mutually orthogonal exponentials in $L^2(\mu_{M,D})$, and the number 3 is the best.

We have proved Conjecture 2 in the case when $d = 0$ or $b = 0$. It should be pointed out that Conjecture 2 also holds if $a + c = 0$. In fact, in this interesting case, we can write $\hat{\mu}_{M,D}(\xi)$ as

$$\hat{\mu}_{M,D}(\xi) = \hat{\mu}_{\tilde{M},D}(\xi) \cdot m_D(M_{\ast}^{-1} \xi) \cdot \hat{\mu}_{\tilde{M},D}(M_{\ast}^{-1} \xi), \quad (4.2)$$

where $\tilde{M} = (a^2 + bd)I_2$ and $I_2$ is the $2 \times 2$ identity matrix. Since the zero set $Z(\hat{\mu}_{\tilde{M},D}) = Z_1 \cup \tilde{Z}_1$ with $Z_1 \cap \tilde{Z}_1 = (Z_1 \cup \tilde{Z}_1) \cap \mathbb{Z}^2 = \emptyset$ for two concrete sets $Z_1$ and $\tilde{Z}_1$ (as in Sections 2.1.1 and 2.1.4), we have

$$\left\{ \xi \in \mathbb{R}^2 : \hat{\mu}_{\tilde{M},D}(M_{\ast}^{-1} \xi) = 0 \right\} = M^\ast(Z_1 \cup \tilde{Z}_1) \subseteq \left\{ \xi \in \mathbb{R}^2 : m_D(M_{\ast}^{-1} \xi) = 0 \right\}. \quad (4.3)$$

The set $\{ \xi \in \mathbb{R}^2 : m_D(M_{\ast}^{-1} \xi) = 0 \}$ can be expressed as two disjoint sets, say $Q_1$ and $\tilde{Q}_1$. Then $Z(\hat{\mu}_{M,D}) = Z_1 \cup \tilde{Z}_1 \cup Q_1 \cup \tilde{Q}_1$. Such expression on the zero set $Z(\hat{\mu}_{M,D})$ can be further simplified as in Section 2. Hence, the same method shows that Conjecture 2 holds in the case when $a + c = 0$. Here the number 3 matches the cardinality of $|D|$, and we need not divide $|\det(M)|$ or $|D|$ into the two cases: $|D| < |\det(M)|$ and $|D| > |\det(M)|$. The all known results on the non-spectral self-affine measure problem are in the case $|D| < |\det(M)|$. In the IFS $\{\phi_d\}_{d \in D}$, the condition $|D| \geq |\det(M)|$ is necessary for $T(M, D)$ to have positive Lebesgue measure. For the integral self-affine tile $T(M, D)$, there are infinite families of orthogonal exponentials in $L^2(\mu_{M,D})$ (cf. [28, p. 636]). However this conclusion does not hold in the case when $|D| > |\det(M)|$, even if $T(M, D)$ has positive Lebesgue measure. For example, consider the pair $(M, D)$ given by

$$M = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{Z}) \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}; \quad (4.4)$$

we see that $T(M, D)$ has positive Lebesgue measure and $|D| > |\det(M)|$, but there are at most 3 mutually orthogonal exponentials in $L^2(\mu_{M,D})$, and the number 3 is the best.
Finally we would like to point out that for any $2 \times 2$ expanding matrix $M_1 \in M_2(\mathbb{R})$ and any digit sets $D_1 = \{0, d_1, d_2\} \subset \mathbb{R}^2$ (not necessarily an integer matrix and an integer set), if $P = [d_1, d_2]$ is an invertible $2 \times 2$ matrix (whose column vectors are $d_1$ and $d_2$) such that

$$P^{-1} M_1 P = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

with $a, b, c \in \mathbb{Z}$ and $ac \in \mathbb{Z} \setminus (3\mathbb{Z})$, (4.5)

then $\mu_{M_1, D_1}$-orthogonal exponentials contain at most 3 elements and the number 3 is the best. This follows from the fact that

$$Z(\tilde{\mu}_{M, D}) = P^* \left( Z(\tilde{\mu}_{M_1, D_1}) \right) \quad \text{(see [27, p. 208])}$$

and the above Theorem.

Acknowledgments

The author would like to thank the anonymous referees and Professor K.S. Lau for their valuable suggestions. The present research is partially supported by the Key Project of Chinese Ministry of Education (No. 108117).

References