Hasse–Schmidt derivations and coefficient fields in positive characteristics ☆

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Abstract

We show how to express any Hasse–Schmidt derivation of an algebra in terms of a finite number of them under natural hypothesis. As an application, we obtain coefficient fields of the completion of a regular local ring of positive characteristic in terms of Hasse–Schmidt derivations.
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Introduction

Let $k \to A$ be a ring homomorphism. Hasse–Schmidt derivations of A over k are generalizations of usual derivations, but they do not carry an A-module structure. Nevertheless, Hasse–Schmidt derivations have a non-Abelian group structure lifting the addition of derivations.

In this paper we show how to express any Hasse–Schmidt derivation in terms of a finite number of them under very reasonable conditions. In proving our result, we find a natural way of producing "non-linear combinations" of Hasse–Schmidt derivations which, to some extent, could play the role of the A-module structure of derivations.

As an application, we express coefficient fields of the completion of a regular local ring of positive characteristic in terms of Hasse–Schmidt derivations, generalizing a similar result in characteristic zero.

Let us now comment on the content of this paper.

In Section 1 we recall the notions of Hasse–Schmidt derivation and differential operator.

Section 2 deals with the main result of this paper, namely that any Hasse–Schmidt derivation $\underline{\mathfrak{D}}$ can be expressed as "non-linear combination" of a finite number of them $\underline{D}^1, \ldots, \underline{D}^n$ whenever their degree 1 components $D_1^1, \ldots, D_1^n$ generate the module of usual derivations.

In Section 3 we apply our main result to generalize a well-known theorem of Nomura and to obtain coefficient fields of the completion of Noetherian local regular rings in the positive characteristics case.

Our results seem related to some results in [8]. We hope to return to this relationship in a future work.

1. Preliminaries and notations

All rings and algebras considered in this paper are assumed to be commutative with unit element.

1.1. Hasse–Schmidt derivations (cf. [2] and [6, Section 27])

Let $k \xrightarrow{f} A \xrightarrow{g} B$ be ring homomorphisms. Let t be an indeterminate over B, and set $B_m = B[t]/(t^{m+1})$ for $m \geqslant 0$ and $B_\infty = B[\![t]\!]$. We can view B_m as a k-algebra in a natural way (for $m \leqslant \infty$).

A *Hasse–Schmidt derivation* (over k) of length $m \geqslant 1$ (respectively of length ∞) from A to B, is a sequence $\underline{D} = (D_0, D_1, \ldots, D_m)$ (respectively $\underline{D} = (D_0, D_1, \ldots)$) of k-linear maps $D_i : A \to B$, satisfying the conditions:

$$D_0 = g, \qquad D_i(xy) = \sum_{r+s=i} D_r(x) D_s(y)$$

for all $x, y \in A$ and all $i > 0$. In particular, the first component D_1 is a k-derivation from A to B. Moreover, D_i vanishes on $f(k)$ for all $i > 0$.

Any Hasse–Schmidt derivation $\underline{D}$ is determined by a ring homomorphism

$$E : x \in A \mapsto \sum_{i=0}^{m} D_i(x) t^i \in B_m$$

with $E(x) \equiv g(x) \mod t$.

When $A = B$ and $g = 1_A$, we simply say that $\underline{D}$ is a Hasse–Schmidt derivation of A (over k). We write $\mathrm{HS}_k(A, B; m)$ for the set of all Hasse–Schmidt derivations (over k) of length m from A to B, $\mathrm{HS}_k(A, B) = \mathrm{HS}_k(A, B; \infty)$, $\mathrm{HS}_k(A; m) = \mathrm{HS}_k(A, A; m)$, and $\mathrm{HS}_k(A) = \mathrm{HS}_k(A, A; \infty)$.

We say that a k-derivation $\delta : A \to B$ is *integrable* [5] if there is a Hasse–Schmidt derivation $\underline{D} \in \mathrm{HS}_k(A, B)$ such that $D_1 = \delta$. The set of integrable k-derivations from A to B, denoted by $\mathrm{IDer}_k(A, B)$, is a submodule of the k-derivations B-module $\mathrm{Der}_k(A, B)$.

1.2. Differential operators (cf. [1, Sections 16, 16.8])

Let $f : k \to A$ be a ring homomorphism.

For all $i \geqslant 0$, we inductively define the subsets $\mathcal{D}_{A/k}^{(i)} \subseteq \mathrm{End}_k(A)$ in the following way:

$$\mathcal{D}_{A/k}^{(0)} := A \subseteq \mathrm{End}_k(A), \qquad \mathcal{D}_{A/k}^{(i+1)} := \left\{ \varphi \in \mathrm{End}_k(A) \mid [\varphi, a] \in \mathcal{D}_{A/k}^{(i)}, \ \forall a \in A \right\}.$$

The elements of $\mathcal{D}_{A/k} := \bigcup_{i \geqslant 0} \mathcal{D}_{A/k}^{(i)}$ (respectively of $\mathcal{D}_{A/k}^{(i)}$) are called *linear differential operators* (respectively *linear differential operators of order* $\leqslant i$) of A/k. The family $\{\mathcal{D}_{A/k}^{(i)}\}_{i \geqslant 0}$ is an increasing sequence of (A, A)-bimodules of $\mathrm{End}_k(A)$ satisfying

$$\mathcal{D}_{A/k}^{(1)} = A \oplus \mathrm{Der}_k(A), \qquad \mathcal{D}_{A/k}^{(i)} \circ \mathcal{D}_{A/k}^{(j)} \subset \mathcal{D}_{A/k}^{(i+j)},$$

and $[P, Q] \in \mathcal{D}_{A/k}^{(i+j-1)}$ for all $P \in \mathcal{D}_{A/k}^{(i)}$, $Q \in \mathcal{D}_{A/k}^{(j)}$. Hence, $\mathcal{D}_{A/k}$ is a filtered subring of $\mathrm{End}_k(A)$. Moreover, linear differential operators of A/k are I-continuous for any I-adic topology. In particular, for any linear differential operator P of A/k, there is a unique extension $\widehat{P} \in \mathcal{D}_{\widehat{A}/k}$ to the completion $\widehat{A}$ of A for some separated I-adic topology.

For each $\underline{D} \in \mathrm{HS}_k(A; m)$, one easily proves that $D_i \in \mathcal{D}_{A/k}^{(i)}$ and then there is a unique extension $\widehat{\underline{D}} \in \mathrm{HS}_k(\widehat{A}; m)$.

In a similar way, if $S \subset A$ is a multiplicatively closed subset, any Hasse–Schmidt derivation of A/k extends uniquely to a Hasse–Schmidt derivation of $S^{-1}A/k$.

1.3. Taylor expansions (cf. [7])

Let $n \geqslant 1$ be an integer. We write $\boldsymbol{X} = (X_1, \ldots, X_n)$, $\boldsymbol{T} = (T_1, \ldots, T_n)$, $\boldsymbol{X} + \boldsymbol{T} = (X_1 + T_1, \ldots, X_n + T_n)$ and, for $\alpha, \beta \in \mathbb{N}^n$, $\boldsymbol{X}^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, and $\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}$.

We consider the usual partial ordering in $\mathbb{N}^n$: $\beta \geqslant \alpha$ means $\beta_1 \geqslant \alpha_1, \ldots, \beta_n \geqslant \alpha_n$. We write $\beta > \alpha$ if $\beta \geqslant \alpha$ and $\beta \neq \alpha$.

Let A be the formal power series ring $k[\![\boldsymbol{X}]\!]$ (or the polynomial ring $A = k[\boldsymbol{X}]$). For any $f(\boldsymbol{X}) = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \boldsymbol{X}^\alpha \in A$ we define $\Delta^{(\alpha)}(f(\boldsymbol{X}))$ by $f(\boldsymbol{X} + \boldsymbol{T}) = \sum_{\alpha \in \mathbb{N}^n} \Delta^{(\alpha)}(f(\boldsymbol{X})) \boldsymbol{T}^\alpha$. It is well known that (cf. [1, Sections 16, 16.11]):

$$\Delta^{(\alpha)} \in \mathcal{D}_{A/k}^{(|\alpha|)}, \quad \Delta^{(\alpha)}(f \cdot g) = \sum_{\beta + \sigma = \alpha} \Delta^{(\beta)}(f) \Delta^{(\sigma)}(g),$$

$$\alpha! \Delta^{(\alpha)} = \left(\frac{\partial}{\partial X_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial X_n} \right)^{\alpha_n} \quad \text{and} \quad \mathcal{D}_{A/k}^{(i)} = \bigoplus_{|\alpha| \leqslant i} A \cdot \Delta^{(\alpha)} = \bigoplus_{|\alpha| \leqslant i} \Delta^{(\alpha)} \cdot A.$$

1.4. If we denote $\Delta^{(0,\ldots,\overset{(j)}{i},\ldots,0)} = \Delta_i^j$, then $\underline{\Delta}^j = (1_A, \Delta_1^j, \Delta_2^j, \ldots) \in \mathrm{HS}_k(A)$.

Finally, let us recall the notion of quasi-coefficient field of a local ring.

1.5. Definition (cf. [4, 38.F]). Let $(R, \mathfrak{m})$ be a local ring, $K = R/\mathfrak{m}$, and k_0 a subfield of R. We say that k_0 is a *quasi-coefficient field* of R, if the extension K/k_0 is formally étale (cf. [1, 17.1.1] and [4, 38.E]).

In the case of characteristic 0, a field extension is formally étale if and only if it is separably algebraic (cf. [4, 38.E]). On the other hand, any extension of perfect fields of characteristic $p > 0$ is formally étale.

The following proposition is well known.

1.6. Proposition. *Let* $k_0 \to k \xrightarrow{f} A \xrightarrow{g} B$ *be ring homomorphisms and let us suppose that the extension* $k_0 \to k$ *is formally étale. Then,* $\mathrm{HS}_{k_0}(A, B; m) = \mathrm{HS}_k(A, B; m)$ *for any integer* $m \geqslant 1$ *or* $m = \infty$.

2. Generating Hasse–Schmidt derivations

Throughout this section, let $k \xrightarrow{f} A$ be a ring homomorphism.

We consider the following partial ordering in $\mathbb{N}^n$: $\beta \succcurlyeq \alpha$ means that $\beta \geqslant \alpha$ and if $\alpha_i = 0$ then $\beta_i = 0$.

Let us denote by $\mathbb{N}_+$ de set of strictly positive integer numbers.

Let $N \geqslant 2$ be an integer and $\underline{\mathfrak{D}}, \underline{D}^1, \ldots, \underline{D}^n \in \mathrm{HS}_k(A; N)$. For each $\mu \in \mathbb{N}^n$ we write $D_\mu = D_{\mu_1}^1 \circ \cdots \circ D_{\mu_n}^n$. Let C_{ld} be elements in A, $1 \leqslant d \leqslant n$, $1 \leqslant l \leqslant N - 1$, such that

$$\mathfrak{D}_i = \sum_{m=1}^{i} \left(\sum_{\substack{|\lambda|=i \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) D_\mu \tag{1}$$

for all $i = 1, \ldots, N - 1$, where we write

$$\sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} = 1 \quad \text{if } \mu_d = \lambda_d = 0. \tag{2}$$

Observe that the set $\mathbb{N}_+^0$ ($= \mathbb{N}_+^\emptyset$) has only one element and convention (2) follows by defining $|l| = 0$ for $l \in \mathbb{N}_+^0$. Then we have $|l| \succcurlyeq r$ for any $r \geqslant 0$ and any $l \in \mathbb{N}_+^r$.

2.7. Proposition. *Under the above hypothesis, the k-linear map*

$$\delta = \mathfrak{D}_N - \sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N\\|\mu|=m\\\lambda\succcurlyeq\mu}}\prod_{d=1}^{n}\sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\|l|=\lambda_d}}\prod_{q=1}^{\mu_d}C_{l_qd}\right)D_\mu$$

is a k-derivation of A.

Proof. Let us take $a, b \in A$. Since

$$D_\mu(ab) = \sum_{\rho+\sigma=\mu}D_\rho(a)D_\sigma(b),$$

we obtain

$$\begin{aligned}\delta(ab) &= \mathfrak{D}_N(a)b + a\mathfrak{D}_N(b) + \sum_{\substack{v+w=N\\1\leqslant v,w\leqslant N-1}}\mathfrak{D}_v(a)\mathfrak{D}_w(b)\\ &\quad -\sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N\\|\mu|=m\\\lambda\succcurlyeq\mu}}\prod_{d=1}^{n}\sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\|l|=\lambda_d}}\prod_{q=1}^{\mu_d}C_{l_qd}\right)\cdot\left(\sum_{\rho+\sigma=\mu}D_\rho(a)D_\sigma(b)\right)\\ &= \mathfrak{D}_N(a)b + a\mathfrak{D}_N(b) + \sum_{\substack{v+w=N\\1\leqslant v,w\leqslant N-1}}\mathfrak{D}_v(a)\mathfrak{D}_w(b)\\ &\quad -\sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N\\|\mu|=m\\\lambda\succcurlyeq\mu}}\prod_{d=1}^{n}\sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\|l|=\lambda_d}}\prod_{q=1}^{\mu_d}C_{l_qd}\right)\cdot\left(D_\mu(a)b + aD_\mu(b)\right)\\ &\quad -\sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N\\|\mu|=m\\\lambda\succcurlyeq\mu}}\prod_{d=1}^{n}\sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\|l|=\lambda_d}}\prod_{q=1}^{\mu_d}C_{l_qd}\right)\cdot\left(\sum_{\substack{\rho+\sigma=\mu\\\rho,\sigma>0}}D_\rho(a)D_\sigma(b)\right)\\ &= \delta(a)b + a\delta(b) + \sum_{\substack{v+w=N\\1\leqslant v,w\leqslant N-1}}\mathfrak{D}_v(a)\mathfrak{D}_w(b)\\ &\quad -\sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N\\|\mu|=m\\\lambda\succcurlyeq\mu}}\prod_{d=1}^{n}\sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\|l|=\lambda_d}}\prod_{q=1}^{\mu_d}C_{l_qd}\right)\cdot\left(\sum_{\substack{\rho+\sigma=\mu\\\rho,\sigma>0}}D_\rho(a)D_\sigma(b)\right).\end{aligned}$$

We need to prove that

$$\sum_{\substack{v+w=N \\ 1 \leqslant v,w \leqslant N-1}} \mathfrak{D}_v(a)\mathfrak{D}_w(b) = \sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d}\right) \cdot \left(\sum_{\substack{\rho+\sigma=\mu \\ \rho,\sigma>0}} D_\rho(a)D_\sigma(b)\right).$$

But

$$\sum_{\substack{v+w=N \\ 1 \leqslant v,w \leqslant N-1}} \mathfrak{D}_v(a)\mathfrak{D}_w(b)$$

$$= \sum_{\substack{v+w=N \\ 1 \leqslant v,w \leqslant N-1}} \left[\sum_{r=1}^{v}\left(\sum_{\substack{|\tau|=v \\ |\rho|=r \\ \tau \succcurlyeq \rho}} \prod_{d=1}^{n} \sum_{\substack{l' \in \mathbb{N}_+^{\rho_d} \\ |l'|=\tau_d}} \prod_{q=1}^{\rho_d} C_{l'_q d}\right) D_\rho(a)\right]$$

$$\times \left[\sum_{s=1}^{w}\left(\sum_{\substack{|\omega|=w \\ |\sigma|=s \\ \omega \succcurlyeq \sigma}} \prod_{d=1}^{n} \sum_{\substack{l'' \in \mathbb{N}_+^{\sigma_d} \\ |l''|=\omega_d}} \prod_{q=1}^{\sigma_d} C_{l''_q d}\right) D_\sigma(b)\right]$$

$$= \sum_{\substack{v+w=N \\ 1 \leqslant v,w \leqslant N-1}} \sum_{m=2}^{N} \sum_{\substack{r+s=m \\ 1 \leqslant r \leqslant v \\ 1 \leqslant s \leqslant w}} \sum_{\substack{|\tau|=v,\ |\omega|=w \\ |\rho|=r,\ |\sigma|=s \\ \tau \succcurlyeq \rho,\ \omega \succcurlyeq \sigma}} \left(\prod_{d=1}^{n} \sum_{\substack{l' \in \mathbb{N}_+^{\rho_d} \\ |l'|=\tau_d}} \prod_{q=1}^{\rho_d} C_{l'_q d} D_\rho(a)\right)$$

$$\times \left(\prod_{d=1}^{n} \sum_{\substack{l'' \in \mathbb{N}_+^{\sigma_d} \\ |l''|=\omega_d}} \prod_{q=1}^{\sigma_d} C_{l''_q d} D_\sigma(b)\right)$$

$$= \sum_{m=2}^{N} \sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \sum_{\substack{\rho+\sigma=\mu \\ \rho,\sigma>0}} \sum_{\substack{\tau+\omega=\lambda \\ \tau \succcurlyeq \rho,\omega \succcurlyeq \sigma}} \left(\prod_{d=1}^{n} \sum_{\substack{l' \in \mathbb{N}_+^{\rho_d} \\ |l'|=\tau_d}} \prod_{q=1}^{\rho_d} C_{l'_q d} D_\rho(a)\right)$$

$$\times \left(\prod_{d=1}^{n} \sum_{\substack{l'' \in \mathbb{N}_+^{\omega_d} \\ |l''|=\omega_d}} \prod_{q=1}^{\sigma_d} C_{l''_q d} D_\sigma(b)\right)$$

$$= \sum_{m=2}^{N} \sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \sum_{\substack{\rho+\sigma=\mu \\ \rho,\sigma>0}} \sum_{\substack{\tau+\omega=\lambda \\ \tau \succcurlyeq \rho,\omega \succcurlyeq \sigma}} \prod_{d=1}^{n} \sum_{\substack{l' \in \mathbb{N}_+^{\rho_d},\ |l'|=\tau_d \\ l'' \in \mathbb{N}_+^{\sigma_d},\ |l''|=\omega_d}} \left(\prod_{q=1}^{\rho_d} C_{l'_q d} \cdot D_\rho(a)\right)\left(\prod_{q=1}^{\sigma_d} C_{l''_q d} \cdot D_\sigma(b)\right)$$

$$= \sum_{m=2}^{N} \sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \sum_{\substack{\rho+\sigma=\mu \\ \rho,\sigma>0}} \prod_{d=1}^{n} \sum_{\substack{\tau_d+\omega_d=\lambda_d \\ \tau_d \succcurlyeq \rho_d,\omega_d \succcurlyeq \sigma_d}} \sum_{\substack{l' \in \mathbb{N}_+^{\rho_d},\ |l'|=\tau_d \\ l'' \in \mathbb{N}_+^{\sigma_d},\ |l''|=\omega_d}} \left(\prod_{q=1}^{\rho_d} C_{l'_q d}\right)\left(\prod_{q=1}^{\sigma_d} C_{l''_q d}\right) D_\rho(a)D_\sigma(b)$$

$$= \sum_{\substack{m=2 \\ }}^{N} \sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \sum_{\substack{\rho+\sigma=\mu \\ \rho,\sigma>0}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d}=\mathbb{N}_+^{\rho_d}\times\mathbb{N}_+^{\sigma_d} \\ |l|=\lambda_d \\ [l=(l',l'')]}} \left(\prod_{q=1}^{\mu_d} C_{l_q d} \right) D_\rho(a) D_\sigma(b)$$

$$= \sum_{m=2}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) \cdot \left(\sum_{\substack{\rho+\sigma=\mu \\ \rho,\sigma>0}} D_\rho(a) D_\sigma(b) \right),$$

and the proposition is proved. □

2.8. Theorem. *Let $m \geqslant 1$ be an integer or $m = \infty$. Let $\underline{D}^1, \ldots, \underline{D}^n \in \mathrm{HS}_k(A; m)$ be Hasse–Schmidt derivations of A/k such that their components of degree 1, $D_1^1, \ldots, D_1^n$, form a system of generators of the A-module $\mathrm{Der}_k(A)$. Then, for any Hasse–Schmidt derivation $\underline{\mathfrak{D}} \in \mathrm{HS}_k(A; m)$ there exist $C_{ld} \in A$, $1 \leqslant d \leqslant n$, $1 \leqslant l < m+1$, such that Eq. (1) holds for all $i \geqslant 1$. Moreover, if $\{D_1^1, \ldots, D_1^n\}$ is a A-basis of $\mathrm{Der}_k(A)$, then the $\{C_{ld}\}$ are unique.*

Proof. We proceed by induction on i. For $i = 1$, $\mathfrak{D}_1$ is a derivation and so there exist $C_{11}, \ldots, C_{1n} \in A$ such that

$$\mathfrak{D}_1 = C_{11} D_1^1 + \cdots + C_{1n} D_1^n.$$

Let N be an integer $\geqslant 2$ and suppose we have elements $C_{ld} \in A$, $1 \leqslant l \leqslant N-1$, $1 \leqslant d \leqslant n$, such that relation (1) is true for $1 \leqslant i \leqslant N-1$.

By Proposition 2.7, the k-linear application

$$\delta = \mathfrak{D}_N - \sum_{m=2}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) D_\mu$$

is a k-derivation of A. Then, there exist $C_{Nd} \in A$, $1 \leqslant d \leqslant n$, s.t.

$$\delta = C_{N1} D_1^1 + \cdots + C_{Nn} D_1^n,$$

$$\mathfrak{D}_N = \delta + \sum_{m=2}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) D_\mu = \sum_{m=1}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) D_\mu,$$

and Eq. (1) holds for $i = N$.

Obviously, the C_{ld} are unique if $\{D_1^1, \ldots, D_1^n\}$ is an A-basis of $\mathrm{Der}_k(A)$. □

2.9. Remark. The C_{ld} in Theorem 2.8 depend on the order of the $\underline{D}^1, \ldots, \underline{D}^n$.

3. Applications: coefficient fields in positive characteristics

3.10. Let $(R, \mathfrak{m}, k)$ be a Noetherian regular local ring of dimension $n \geqslant 1$ containing a field, $X_1, \ldots, X_n \in \mathfrak{m}$ a regular system of parameters of R, $k_0 \subset R$ a quasi-coefficient field and $\widehat{R}$ the completion of R. We can identify $\widehat{R} = k[\![X_1, \ldots, X_n]\!]$ by means of a canonical k_0-isomorphism. Let $\partial/\partial X_1, \ldots, \partial/\partial X_n$ be the usual basis of $\mathrm{Der}_k(\widehat{R})$ and $\underline{\Delta}^1, \ldots, \underline{\Delta}^n$ the Hasse–Schmidt derivations of $\widehat{R}$ over k defined in Section 1.4.

Let us recall the following result of M. Nomura [3, Theorem 2.3], [6, Theorem 30.6].

3.11. Theorem. *Under the hypothesis above, the following conditions are equivalent*:

(1) $\partial(R)/\partial X_i \subset R$ *for all* $i = 1, \ldots, n$.
(2) *There exist* $D_i \in \mathrm{Der}_{k_0}(R)$ *and* $a_i \in R$, $i = 1, \ldots, n$, *such that* $D_i(a_j) = \delta_{ij}$.
(3) *There exist* $D_i \in \mathrm{Der}_{k_0}(R)$ *and* $a_i \in R$, $i = 1, \ldots, n$, *such that* $\det(D_i(a_j)) \notin \mathfrak{m}$.
(4) $\mathrm{Der}_{k_0}(R)$ *is a free* R-*module of rank* n *(and* $\{D_1, \ldots, D_n\}$ *is a basis).*
(5) $\mathrm{rank}\,\mathrm{Der}_{k_0}(R) = n$.

The proof of the following corollaries is straightforward.

3.12. Corollary. *Under hypothesis and equivalent conditions of Theorem* 3.11, *for any basis* $D_1, \ldots, D_n \in \mathrm{Der}_{k_0}(R)$, *their extensions* $\widehat{D}_1, \ldots, \widehat{D}_n$ *to* $\widehat{R}$ *form a basis of* $\mathrm{Der}_k(\widehat{R})$. *Moreover, the restrictions* $\partial/\partial X_i|_R : R \to R$, $i = 1, \ldots, n$, *form a basis of* $\mathrm{Der}_{k_0}(R)$.

3.13. Corollary. *Under hypothesis of Corollary* 3.12, *let us suppose that* k_0 *is a field of characteristic* 0. *Then, the set* $\{a \in \widehat{R} \mid \widehat{D}_j(a) = 0 \ \forall j = 1, \ldots, n\}$ *is a coefficient field of* $\widehat{R}$ *(the only one containing* k_0).

The following theorem is an improvement of Theorem 3.11 and is based on the results of Section 2.

3.14. Theorem. *Under the hypothesis of Section* 3.10, *the following conditions are equivalent*:

(1) $\Delta_i^j(R) \subset R$, *for all* $j = 1, \ldots, n$, $i \geqslant 0$.
(2) *There exist* $\underline{D}^i \in \mathrm{HS}_{k_0}(R)$ *and* $a_i \in R$, $i = 1, \ldots, n$, *such that*

$$D_i^j(a_l) = \begin{cases} 0 & i \geqslant 2, \ \forall j, \\ \delta_{jl} & i = 1, \ \forall j, l. \end{cases}$$

(3) *There exist* $D_i \in \mathrm{IDer}_{k_0}(R)$ *and* $a_i \in R$, $i = 1, \ldots, n$, *such that* $\det(D_j(a_l)) \notin \mathfrak{m}$.

(4) $\mathrm{Der}_{k_0}(R)$ *is a free R-module of rank n and* $\mathrm{IDer}_{k_0}(R) = \mathrm{Der}_{k_0}(R)$ *(and* $\{D_1, \ldots, D_n\}$ *is a basis).*

(5) $\mathrm{rank}\, \mathrm{IDer}_{k_0}(R) = n$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) are straightforward. (3) $\Rightarrow$ (4) comes from theorem 3.11.

(5) $\Rightarrow$ (1). Let $\underline{D}^1, \ldots, \underline{D}^n \in \mathrm{HS}_{k_0}(R)$ such that $D_1^1, \ldots, D_1^n$ are linear independent over R. Let us consider the extensions $\widehat{\underline{D}}^1, \ldots, \widehat{\underline{D}}^n \in \mathrm{HS}_k(\widehat{R})$, whose degree 1 components $\widehat{D}_1^1, \ldots, \widehat{D}_1^n$ are also linear independent over $\widehat{R}$.

Following the lines of the proof of Theorem 2.8, we are going to prove the following result.

For any $j = 1, \ldots, n$, there exist $C_{ld}^j \in K = \mathrm{Q}(R)$, $1 \leqslant d \leqslant n$, $1 \leqslant l < +\infty$, such that

$$\Delta_i^j = \sum_{m=1}^{i} \left(\sum_{\substack{|\lambda|=i \\ |\mu|=m \\ \lambda \gtrsim \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d}^j \right) \widehat{D}_\mu \quad \text{for all } i \geqslant 1. \tag{3}$$

For $i = 1$, the Δ_1^j are k-derivations of $\widehat{R}$ and then there exist $C_{1d}^j \in L = \mathrm{Q}(\widehat{R})$ such that[1]

$$\Delta_1^j = \sum_{d=1}^{n} C_{1d}^j \widehat{D}_1^d.$$

Then, for any $m = 1, \ldots, n$ we have

$$\delta_{jm} = \Delta_1^j(X_m) = \sum_{d=1}^{n} C_{1d}^j \widehat{D}_1^d(X_m)$$

and the matrix $(\widehat{D}_1^j(X_m))$ with entries in R has a non-zero determinant. In particular, $C_{1d}^j \in K$.

Let us suppose that for $N \geqslant 2$ there exist $C_{ld}^j \in K$, $1 \leqslant d, j \leqslant n$, $1 \leqslant l \leqslant N-1$, such that Eq. (3) holds for $i = 1, \ldots, N-1$. We can consider $\mathrm{HS}_k(\widehat{R}) \subset \mathrm{HS}_k(L)$. By Proposition 2.7, for any $j = 1, \ldots, n$ the k-linear map

$$\delta^j = \Delta_N^j - \sum_{m=2}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \gtrsim \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d}^j \right) \widehat{D}_\mu \tag{4}$$

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[1] This is the same argument as used in the proof of Theorem 3.11.

is a k-derivation of L. Let $a \in R$ be a common denominator for the C_{ld}^j, $1 \leqslant d, j \leqslant n$, $1 \leqslant l \leqslant N-1$. Then,

$$a^N \delta^j = a^N \Delta_N^j - \sum_{m=2}^N \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succcurlyeq \mu}} \prod_{d=1}^n \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} (a^{l_q} C_{l_q d}^j) \right) \widehat{D}_\mu$$

maps $\widehat{R}$ into $\widehat{R}$ and $a^N \delta^j \in \mathrm{Der}_k(\widehat{R})$. There exist $\overline{C}_{Nd}^j \in L$, $1 \leqslant d, j \leqslant n$, such that

$$a^N \delta^j = \sum_{d=1}^n \overline{C}_{Nd}^j \widehat{D}_1^d.$$

Since $(a^N \delta^j)(X_m) \in R$ and the matrix $(\widehat{D}_1^j(X_m))$ with entries in R has a non-zero determinant (notice that $\Delta_N^j(X_m) = 0$), we deduce that $\overline{C}_{Nd}^j \in K$. By setting $C_{Nd}^j = a^{-N} \overline{C}_{Nd}^j \in K$, we obtain Eq. (3) for $i = N$.

From Eq. (3) we deduce that

$$\Delta_i^j(R) \subset K \cap \widehat{R} = R$$

for all $j = 1, \ldots, n$, $i \geqslant 0$, and (1) is proved. □

3.15. Remark. As noticed in [4, p. 289] for Theorem 3.11, Theorem 3.14 also holds for $\mathrm{HS}_k(\widehat{R}) \cap \mathrm{HS}(R)$ instead of $\mathrm{HS}_{k_0}(R)$, $\mathrm{Der}_k(\widehat{R}) \cap \mathrm{Der}(R)$ instead of $\mathrm{Der}_{k_0}(R)$, and $\{\delta \in \mathrm{Der}(R) \mid \exists \underline{D} \in \mathrm{HS}_k(\widehat{R}) \cap \mathrm{HS}(R) \text{ s.t. } D_1 = \delta\}$ instead of $\mathrm{IDer}_{k_0}(R)$, and the mention of a quasi-coefficient field can be avoided.

3.16. Remark. We do not know any example of a Noetherian regular local ring $(R, \mathfrak{m}, k)$ containing a quasi-coefficient field k_0 (of positive characteristic) such that $\mathrm{IDer}_{k_0}(R) \neq \mathrm{Der}_{k_0}(R)$.

The following theorem generalizes Corollary 3.13 to the case of characteristic $p \geqslant 0$.

3.17. Theorem. *Under the hypothesis of Section* 3.10, *let* $\underline{D}^1, \ldots, \underline{D}^n \in \mathrm{HS}_{k_0}(R)$ *such that their degree* 1 *components* $\{D_1^1, \ldots, D_1^n\}$ *form a basis of* $\mathrm{Der}_{k_0}(R)$. *Let* $\widehat{\underline{D}}^1, \ldots, \widehat{\underline{D}}^n$ *be the extensions of* $\underline{D}^1, \ldots, \underline{D}^n$ *to* $\widehat{R}$. *Then, the set* $\{a \in \widehat{R} \mid \widehat{D}_i^j(a) = 0 \ \forall j = 1, \ldots, n, \ i \geqslant 1\}$ *is a coefficient field of* $\widehat{R}$ (*the only one containing* k_0).

Proof. Since $\widehat{R} = k[\![X_1, \ldots, X_n]\!]$, we have $k = \{a \in \widehat{R} \mid \Delta_i^j(a) = 0, \ j = 1, \ldots, n; \ i \geqslant 1\}$. By Corollary 3.12, we deduce that $\{\widehat{D}_1^1, \ldots, \widehat{D}_1^n\}$ is a $\widehat{R}$-basis of $\mathrm{Der}_k(\widehat{R})$, and from Theorem 2.8 we can express the Δ_i^j in terms of $\widehat{D}_i^j$. In particular,

$$\{a \in \widehat{R} \mid \widehat{D}_i^j(a) = 0 \ \forall j = 1, \ldots, n, \ i \geqslant 1\} \subset k.$$

The opposite inclusion comes from Proposition 1.6.   □

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