Homology of Classical Lie Groups Made Discrete. II. $H_2, H_3$, and Relations with Scissors Congruences*

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The present work extends a number of earlier results; see [5, 7, 15–18]. For example, the following fundamental exact sequence (essentially due to Bloch and Wigner in somewhat different form, but not published by them) can be found in Dupont and Sah [7]:

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(SL(2, \mathbb{C})) \to \mathcal{P}(\mathbb{C}) \to A^2_\mathbb{C}(\mathbb{C}^\times) \to K_2(\mathbb{C}) \to 0. \quad (2.12)$$

The group $\mathcal{P}(\mathbb{C})$ is defined in (1.4) (1.8) and is isomorphic to the group $\mathcal{R}_\mathbb{C}$ considered in Dupont and Sah [7]. Also, the study of the scissors congruence problem had led to two exact sequences in Dupont [5, Theorems 1.3 and 1.4] that roughly correspond to the $\pm$-eigenspaces of (2.12) under the action of complex conjugation:

$$0 \to A \to H_3(SU(2)) \to \mathcal{P} S^3/\mathbb{Z} \to \mathbb{R} \otimes (\mathbb{R}/\mathbb{Z}) \to H_2(SU(2)) \to 0; \quad (0.1)$$

$$0 \to B \to H_3(SL(2, \mathbb{C}))^- \to \mathcal{P} \mathcal{W}^3 \to \mathbb{R} \otimes (\mathbb{R}/\mathbb{Z}) \to H_2(SL(2, \mathbb{C}))^- \to 0. \quad (0.2)$$

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Here $A, B$ denote suitable abelian groups annihilated by $2^N$ for some integer $N \geq 0$. In the unpublished work of Bloch and Wigner, similar 2-torsion problems appeared. One of the purposes of the present work is to settle all these 2-torsion problems and make the correspondences more precise. This involves a number of delicate results on $H_2, H_3$ of $SU(2), SL(2, F), F = \mathbb{R}, \mathbb{C},$ and $\mathbb{H}$. It is well known that $K_2(\mathbb{C}) \cong H_2(SL(n, \mathbb{C})), n \geq 2$. Thus there is a natural map relating the negative eigenspaces of (2.12) and the exact sequence (0.2); see DuPont and Sah [7]. It turns out that the Hopf map from $S^3$ to $S^2$ provides the less obvious relation between the positive eigenspaces of (2.12) and the exact sequence (0.1).

In this connection, another of our main results is

**Theorem 4.1.** Let $G$ be a simple, connected, simply-connected nonabelian Lie group such that its Lie algebra is absolutely simple and not among 10 exceptional ones of type $E$ and $F$ (3 compact, 7 noncompact, and all are not $\mathbb{R}$-split). Let $\rho : G \to SL(n, \mathbb{C})$ denote any nontrivial Lie group homomorphism. Then $\rho_* : H_2(G) \to H_2(SL(n, \mathbb{C}))$ is injective and the image is $K_2(\mathbb{C})^+$, where $H_2(SL(n, \mathbb{C})) \cong K_2(\mathbb{C})$ for $n \geq 2$. In particular, $H_2(SL(n, \mathbb{H})) \to H_2(SL(n + 1, \mathbb{H}))$ is bijective for $n \geq 1$ and $K_2(\mathbb{H}) \cong K_2(\mathbb{C})^+$.

Aside from the 10 exceptions, Theorem 4.1 affirmatively answers a question posed by Milnor in a conversation with one of us. If $H_2(SL(n, \mathbb{C}))$ is replaced by its stable value $K_2(\mathbb{C})$ (when $n \geq 3$), then $\rho_*$ may be described as multiplication by a suitable positive integer (depending on $\rho$) on the $\mathbb{Q}$-vector space $K_2(\mathbb{C})^+$. For example, if $G = SU(2)$ and $n = 3$, then the integer associated to the inclusion map through $SL(2, \mathbb{C})$ is 2 (the integer is 1 when $n = 2$ and $\rho$ is the inclusion map of $SU(2)$ into $SL(2, \mathbb{C})$).

In the case of complex Lie groups and complex Lie group homomorphisms, the analogue of Theorem 4.1 is classical ($\rho_*$ is then a bijection of $K_2(\mathbb{C})$ and the exceptional cases may be included).

By using Theorem 4.1 and a number of other results related to its proof, the relevant 2-torsion problems can be settled, namely $A = 0 = B$. We state our results in the following forms (see Section 5 for details).

**Theorem 5.2.** $\mathfrak{P}(\mathbb{C})^- \cong \mathfrak{P}H^3$ and $K_2(\mathbb{C})^-$ is the cokernel of the hyperbolic Dehn invariant map in dimension 3. $\mathfrak{P}H^3$ is divisible and has no 2-torsion.

**Theorem 5.10.** $K_2(\mathbb{C})^+ \cong H_3(SU(2))$ is isomorphic to the cokernel of the spherical Dehn invariant map (reduced or not) in dimension 3. $H_3(SU(2))$ is a 2-divisible group, and $D(S^3)$ is a $\mathbb{Q}$-vector space.

The only new result in Theorem 5.2 is the assertion on 2-torsion. The
scissors congruence group $\mathcal{P}_3$ in hyperbolic 3-spaces is conjectured to be torsion-free. This conjecture will be verified in Sah [19]. The above intermediate result about 2-torsion appears to be needed for the proof of the stronger result. In fact, the 2-torsion assertion follows from the result:

**Proposition 3.4.** $\mathcal{P}(C)$ is uniquely 2-divisible.

The divisibility of $\mathcal{P}(C)$ was proved in Dupont and Sah [7]; see (1.19). Theorem 5.10 settles several questions left open in Dupont [5]. The 2-divisibility of $H_3(SU(2))$ plays a role in the later determination of the group $H_3(SL(2, \mathbb{H}))$. All these results are used in Sah [19].

We briefly outline the organization of the present work.

Section 1 reviews the classical concept of cross-ratios of ordered sets of 4-tuples of distinct points of a projective line coordinatized by a division ring $F$. It also reviews some of the relevant results from earlier works.

Section 2 begins with a review of some basic facts about real hyperbolic spaces. In particular, we call attention to the existence of "exotic" isomorphisms between the isometry groups in low dimensions and other classical groups. Some of the earlier results reviewed in Section 1 are then rephrased in terms of hyperbolic spaces. The discussion after Remark 2.14 then reviews the various technical arguments to be used in extending our earlier results. A few of these technical arguments are used to clarify the relevant higher differentials in various spectral sequences and enable us to pinpoint the 2-torsion problems. Their roles become more substantial in a later work. The second half of Section 2 then outlines the problems to be attacked in the later sections.

Section 3 addresses some of the principal results in the present work. It is fairly technical and employs many results obtained in earlier works (these are either summarized or discussed again in the present setting).

Section 4 is devoted to the determination of the Schur multipliers of the classical groups. It also discusses very briefly some open problems as well as some possible alternate approaches. For ease of future reference, a number of the results are collected at the end of these discussions.

Section 5 is devoted to the connection of the results in earlier section with the scissors congruence problem in dimension 3 for hyperbolic and spherical spaces. As indicated already, the Hopf map is used to connect the exact sequence (0.1) with the positive eigenspaces in (2.12). This connection is not quite as precise as that of the negative eigenspaces. We obtain the surjectivity of the map from $H_3(SU(2))$ to $H_3(SL(2, \mathbb{C}))^+$ with an unknown kernel (which is conjectured to be 0). In this respect, we have reached the same stage as in Sah and Wagoner [20] for the map from $H_3(SU(2))$ to $H_3(SL(2, \mathbb{C}))^+$. It should be noted that one of the assertions in Theorem 2.20 (repeated in Theorem 4.11) involves the 2-divisibility of
$H_3(SU(2))$ proved in Theorem 5.10. These earlier assertions are not used in the proof of Theorem 5.10 and their inclusion in the earlier is done solely to avoid a messy looking temporary statement. A few open problems are also discussed in Section 5. Basic facts in algebraic topology and algebraic groups can be found in [21, 22].

We wish to acknowledge our deep gratitude to the referee for a patient and thorough reading of the various drafts of the present work. In particular, we briefly sketched an alternate (but ad hoc) solution to Problem 4.9 for $n = 2$ in an earlier draft. The referee then suggested an extension (again ad hoc) to the case of $n = 3$. The idea is to combine a combinatorial topological argument with a volume estimate. A similar combinatorial topological argument had also been suggested by Wu-Chung Hsiang earlier. Both of these ideas involve messy bookkeeping problems. These can be absorbed by a systematic use of an appropriate filtration on suitable spectral sequences. The essential point then becomes one of skeletal induction and modification. In the case of $n = 3$, the volume estimate enables the argument to reach the desired conclusion. Unfortunately, the method does not seem to work for $n > 3$. The method presented by us has the slight advantage that the only analysis involves polynomials. As a result, our method works for fields where the concept of volume is no longer applicable. Since the alternate procedure does not provide much saving in the length of the argument, we have omitted it from the write-up. Nevertheless, both Hsiang and the referee deserve credit for the alternate ideas.

1. Review of Cross-ratios and Related Results

In classical projective geometry, the cross-ratio of four collinear points plays an important role. It is often defined by a formula through the choice of a coordinate system. Depending on the source, questions concerning the ordering of the points, the possibility of duplication of the points, and the commutativity of the underlying division rings receive varying degrees of attention. The notational conventions are not always consistent. A careful treatment of these questions can be found in Baer [2, pp. 71–94]. Unfortunately, the notation used in Baer [2] is not convenient for our purposes. We will therefore review the relevant results. The readers should bear in mind that the cases of interest to us are the three classical real division algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

Let $F$ be any division ring. An abstract projective line over $F$ is understood to be a transformation space $(GL_F(V), \mathcal{P}(V))$, where $\mathcal{P}(V)$ is the set of all 1-dimensional $F$-subspaces of the right $F$-vector space $V$ of dimension 2 and $GL_F(V)$ is the group of all invertible $F$-linear maps of $V$. An abstract cross-ratio is understood to be any $GL_F(V)$-equivalence class.
of ordered sets of four distinct points of $\mathbb{P}(V)$. For this to be of use, it is best to introduce coordinates. $V$ can then be identified with the right $\mathbb{F}$-vector space $\mathbb{F}^2$ formed by the set of all column vectors with two entries from $\mathbb{F}$ and $GL_2(\mathbb{F})$ is then identified with the group of all $2 \times 2$ invertible matrices over $\mathbb{F}$ so that it acts on $\mathbb{F}^2$ from the left by means of matrix multiplication. Let $B$ be the subgroup of $GL(2, \mathbb{F})$ formed by the upper triangular matrices. $\mathbb{P}(V)$ is then identified with the coset space $\mathbb{P}(2, \mathbb{F}) = GL(2, \mathbb{F})/B$. In the usual notation, $B$ corresponds to $\infty$ while the coset $(\infty, 0)B$ corresponds to $x \in \mathbb{F}$ so that $\mathbb{P}(2, \mathbb{F}) = \mathbb{F} \cup \{\infty\}$. With this choice of coordinates, the stability subgroup of the ordered triple $(\infty, 0, 1)$ is $\mathbb{F} \cdot I_2$. The effective group $PGL(2, \mathbb{F})$ (or $GL(2, \mathbb{F})/center$) is therefore exactly 3-transitive on $\mathbb{P}(2, \mathbb{F})$ if and only if $\mathbb{F}$ is commutative. Each cross-ratio is represented as the $GL(2, \mathbb{F})$-orbit of $(\infty, 0, 1, x)$, $x \in \mathbb{F} - \{0, 1\} = \mathbb{P}(2, \mathbb{F}) - \{\infty, 0, 1\}$, where $x$ is unique up to inner automorphisms of $\mathbb{F}^\times$. We define $\{x\}$ to be the $GL(2, \mathbb{F})$-orbit of $(\infty, 0, 1, x)$, $x \in \mathbb{F} - \{0, 1\}$, and call it the cross-ratio symbol of the corresponding $GL(2, \mathbb{F})$-orbit. It follows from our definition that

$$\{x\} = \{\beta\} \text{ if and only if } \beta = yxy^{-1} \text{ holds for some } y \in \mathbb{F}^\times. \quad (1.1)$$

We emphasize that these concrete symbols as well as our later computational arguments all depend on the choice of a coordinate system that includes the choice of $\infty$, 0, and 1. By direct computation, we obtain:

Suppose $(p_0, p_1, p_2, p_3)$ has the cross-ratio symbol $\{x\}$.

Then $(p_1, p_0, p_2, p_3)$, $(p_0, p_2, p_1, p_3)$, $(p_0, p_1, p_3, p_2)$

have respective cross-ratio symbols $\{x^{-1}\}$, $\{1 - x\}$, $\{x^{-1}\}$. \quad (1.2)

Since the permutation group of four objects is generated by the three "adjacent" transpositions and has the Klein 4-group as a normal subgroup, (1.2) allows us to compute the six possible cross-ratio symbols associated to the subset $\{p_0, p_1, p_2, p_3\}$ of $\mathbb{P}(2, \mathbb{F})$. Further collapses of these cross-ratio symbols may occur under special circumstances.

The automorphisms of $GL(2, \mathbb{F})$ and $PGL(2, \mathbb{F})$ are known from a theorem of Hua [27, Supplement]. All of them can be extended to equivariant automorphisms of $(GL(2, \mathbb{F}), \mathbb{P}(2, \mathbb{F}))$. They are made up of four types. The first of these are central automorphisms (made up from homomorphisms of $GL(2, \mathbb{F})$ into the center of $GL(2, \mathbb{F})$). These induce the identity on the model $GL(2, \mathbb{F})/B$. The second of these are the inner automorphisms of $GL(2, \mathbb{F})$. These induce the identity on the model based on abstract cross-ratios. To make sense on the level of the concrete symbols, it is necessary to assume that they fix $\infty$, 0, 1 and then use (1.2). The third of these are induced by the automorphisms of $\mathbb{F}$. They induce the obvious actions on the model $GL(2, \mathbb{F})/B$ as well as the model based on
cross-ratio symbols. The last of these arise from the antiautomorphisms of $\mathbb{F}$. Each such antiautomorphism $\ast$ can be used to define an automorphism $\sigma = \sigma(\ast)$ of $GL(2, \mathbb{F})$ by the formula

$$\sigma(A) = w \cdot (A^\ast)^{-1} \cdot w^{-1}, \quad A \in GL(2, \mathbb{F}), \; w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (1.3)$$

Direct calculation shows that $\sigma$ stabilizes $B$ so that it extends equivariantly to $(GL(2, \mathbb{F}), \mathbb{P}^1(\mathbb{F}))$. As such, $\sigma$ fixes $\infty, 0, 1$ and carries $x$ to $x^\ast$. When $\ast$ has order dividing 2 ($\ast$ is then called an involution), $\sigma$ has order 2 exactly when $\mathbb{F}$ has more than two elements. Evidently, $\sigma(\{x\}) = \{x^\ast\}$ holds for the cross-ratio symbols. It should be noted that Hua's theorem shows that $\mathbb{P}^1(\mathbb{F})$ has an "intrinsic" definition in terms of either $GL(2, \mathbb{F})$ or $PGL(2, \mathbb{F})$ while the incidence structure on $\mathbb{P}^1(\mathbb{F})$ is trivial and does not determine $PGL(2, \mathbb{F})$.

For the remaining part of this section, we are concerned with some abelian groups that arose in the study of low-dimensional homology groups of $SL(2, \mathbb{C})$ and $SL(2, \mathbb{R})$; see Dupont and Sah [7], Parry and Sah [15], and Dupont [6]. For the purpose of the present work, it is necessary to consider the relation of these groups with a similarly defined group for $SL(2, \mathbb{H})$. In the previous works, special properties of $\mathbb{C}$ and $\mathbb{R}$ together with some ad hoc arguments were used. In the present work, we employ a modified approach that will allow us to give a more uniform treatment at the beginning while the special features of $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$ are used at a later stage. The advantage of this modified approach will become clear in a later work, see [19], when we examine the action of various groups on $\mathbb{P}^2(\mathbb{F})$.

Let $\mathbb{F}$ be a division ring. Define the abelian group $\mathcal{P}_E$ by using the generators $[x], x \in \mathbb{F} - \{0, 1\}$, and defining relations

$$[x] = [y x y^{-1}], \quad y \in \mathbb{F}^\times; \quad (1.4)$$

$$[x] - [\beta] + [x^{-1} \beta] - [(x - 1)^{-1}(\beta - 1)] + [(x^{-1} - 1)^{-1}(\beta^{-1} - 1)] = 0,$$

where $x \neq \beta$ range over $\mathbb{F} - \{0, 1\}$. \quad (1.5)

When $\mathbb{F}$ has two or three elements, $\mathcal{P}_E$ is respectively 0 or $\mathbb{Z}$. When $\mathbb{F}$ is commutative, $\mathcal{P}_E$ was studied in Dupont and Sah [7] by manipulations of (1.5). Under appropriate interpretations, (1.5) gives us functional equations of the "dilogarithm" function when $\mathbb{F} = \mathbb{C}$ and $\mathbb{R}$; see [7, 15]. In the present setting, it is important to note that $x$ and $\beta$ do not have to commute when $\mathbb{F}$ is not commutative.

Let $G$ be any group acting on $\mathbb{P}^1(\mathbb{F})$ by means of a homomorphism of $G$ into $PGL(2, \mathbb{F})$. Let $C_\ast$ denote the normalized Eilenberg–MacLane chain complex based on the nonempty set $\mathbb{P}^1(\mathbb{F})$. $C_\ast$ is therefore the free abelian group based on the set of all ordered $(t + 1)$-tuples $(v_0, \ldots, v_t)$ of points of
$\mathbb{P}^1(\mathbb{F})$ such that $(v_0, ..., v_i)$ is set equal to 0 if $v_i = v_{i-1}$ holds for at least one $i$. The boundary operator is the usual one. $C_\ast$ is then acyclic with augmentation $\mathbb{Z}$. $G$ acts on $C_\ast$ through its action on $\mathbb{P}^1(\mathbb{F})$. The homology of $G$ can be described through a transposed spectral sequence associated with $C_\ast$ with $"E_{ij}^1 \cong H_i(G, C_j)$. $C_\ast$ has a subcomplex $C^{\text{gen}}_\ast$ spanned by all the cells whose vertices are in general position. In the case of $\mathbb{P}^1(\mathbb{F})$, this simply means that all the vertices are distinct. The readers should note that the concept of "general position" is not "intrinsic." The present definition is based on the algebraic dimension with respect to $\mathbb{F}$. When $\mathbb{F}$ is infinite, $C^{\text{gen}}_\ast$ is also acyclic with augmentation $\mathbb{Z}$ so that it can be used in place of $C_\ast$ in the computation of the homology of $G$. When $\mathbb{F}$ is finite, $C^{\text{gen}}_\ast$ is only $(|\mathbb{F}| - 1)$-acyclic with augmentation $\mathbb{Z}$. We must introduce more terms (for example, we can define $C^{\text{gen}}_{|\mathbb{F}| + 1} = \ker \partial_{|\mathbb{F}|}$) to get an acyclic chain complex. For low-dimensional homology groups, we can still use $C^{\text{gen}}_\ast$ as long as $\mathbb{F}$ is sufficiently large.

Suppose that $G$ maps surjectively onto $\text{PGL}(2, \mathbb{F})$. The discussion on cross-ratios shows that $C^{\text{gen}}_{\text{gen}} \otimes_G \mathbb{Z}$ is the free abelian group based on the set of all cross-ratio symbols. Since $\text{PGL}(2, \mathbb{F})$ is 3-transitive on $\mathbb{P}^1(\mathbb{F})$, this is also the group of all 3-cycles in $C^{\text{gen}}_{\text{gen}} \otimes_G \mathbb{Z}$. In this setting, the relations (1.5) arise from setting the boundary of $(\infty, 0, 1, \alpha, \beta)$ equal to 0, where $\alpha \neq \beta$ range over $\mathbb{P}^1(\mathbb{F}) - \{ \infty, 0, 1 \}$. It is now clear that we have

$$\text{If } G \text{ maps onto } \text{PGL}(2, \mathbb{F}), \text{ then } \mathcal{P}_c \cong H_3(C^{\text{gen}}_{\text{gen}} \otimes_G \mathbb{Z}) \cong "E_{0,3}^{2,\text{gen}}. \quad (1.6)$$

In Dupont and Sah [7], the higher differentials starting from $"E_{0,3}^{2,\text{gen}}$ were determined for the case $G = \text{PGL}(2, \mathbb{F})$ and $\mathbb{F}$ an infinite field. In fact, the argument is formal and works for any division ring $\mathbb{F}$. When $|\mathbb{F}|$ is too small, we have a small problem of interpretation of $"E_{ij}^{x,y,\text{gen}}$.

As noted already, $C_\ast$ is always acyclic with augmentation $\mathbb{Z}$. With (1.6) in mind, we define a second abelian group $\mathcal{P}(\mathbb{F})$ by using generators $[\alpha]$, $\alpha \in \mathbb{P}^1(\mathbb{F}) - \{ \infty, 0, 1 \}$, and supplement the relations (1.4) and (1.5) by

$$[\alpha] + [\alpha^{-1}] = 0; \quad (1.7)$$

$$[\alpha] + [1 - \alpha] \text{ is a constant. \quad (1.8)}$$

There is an obvious surjective homomorphism from $\mathcal{P}_c$ to $\mathcal{P}(\mathbb{F})$. Part of the arguments in Dupont and Sah [7] amounts to showing that this natural map is bijective when $\mathbb{F} = \mathbb{C}$. This result is best viewed in terms of $C_\ast$.

Assume that $G$ maps into $\text{PGL}(2, \mathbb{F})$. The following exact sequence of $G$-chain complexes splits as $G$-modules (i.e., "$G$-mod-split"): $0 \to C^{\text{gen}}_\ast \to C_\ast \to Q_\ast \to 0$. \quad (1.9)

Here $Q_\ast$ is $\mathbb{Z}$-free and has a free $\mathbb{Z}$-basis consisting of cells with at least one
pair of identical but nonadjacent vertices. These basis elements are permuted by $G$. Since (1.9) is $G$-mod-split, we have the $\mathbb{Z}$-split exact sequence of chain complexes:

$$
0 \to H_i(G, C_\text{gen}^*) \to H_i(G, C_\ast) \to H_i(G, Q_\ast) \to 0, \quad i \geq 0. \quad (1.10)
$$

Each of the three terms of (1.10) may be identified with the $i$th column of the $"E^1"$-terms of the transposed spectral sequence associated to the corresponding complex displayed in (1.9). We display some of the $"E^1"$-terms associated to $Q_\ast$ below:

$$
\begin{align*}
Q_4 \otimes_G \mathbb{Z} &\quad H_1(G, Q_4) &\quad H_2(G, Q_4) \\
Q_3 \otimes_G \mathbb{Z} &\quad H_1(G, Q_3) &\quad H_2(G, Q_3) \\
Q_2 \otimes_G \mathbb{Z} &\quad H_1(G, Q_2) &\quad H_2(G, Q_2) &\quad \downarrow "d^1" \\
0 &\quad 0 &\quad 0 &\quad 0 &\quad 0 &\quad 0 &\quad 0
\end{align*}
$$

From now on, $\mathbb{F}$ is assumed to be infinite (in fact, $|\mathbb{F}| > 4$ is enough), and $G$ is assumed to be mapped onto $\text{PGL}(2, \mathbb{F})$. By using the long homology sequence associated to (1.9), $Q_\ast$ is acyclic with augmentation 0. It follows that (1.11) converges to $0 = H_*(G, 0)$. As a consequence, $Q_\ast \otimes_G \mathbb{Z}$ must be 3-acyclic and we have $H_4(Q_\ast \otimes_G \mathbb{Z}) \cong "E^2_4\ast\mathbb{Z}"$. To interpret $\mathcal{P}(\mathbb{F})$, we can take $G = \text{GL}(2, \mathbb{F})$. Now $Q_2 \otimes_G \mathbb{Z} = \mathbb{Z} \cdot (\infty, 0, \infty)$ and $Q_3 \otimes_G \mathbb{Z}$ is $\mathbb{Z}$-free with basis $(\infty, 0, 1, \infty)$, $(\infty, 0, 1, 0) + (\infty, 0, \infty, 1)$, and $(\infty, 0, \infty, 1)$. The first three are boundaries and hence cycles while the last is mapped onto $-(\infty, 0, \infty)$ by the boundary map. The stability subgroup $G_{(\infty, 0, 1)}$ of any cell with three distinct vertices $\infty, 0, 1$ is $\mathbb{F} \times I_2$, where some of these vertices may be repeated. Similarly, $G_{(\infty, 0)}$ is the diagonal subgroup $\mathbb{F} \times \mathbb{F}$ of $\text{GL}(2, \mathbb{F})$. By Shapiro's lemma, we can determine $H_1(G, Q_3)$ and $H_1(G, Q_2)$ as well as $"d^1\ast\mathbb{Z}"$. We note that the Weyl group element $(0 1 \ 0)$ exchanges $\infty$ and 0 so that it exchanges the factors of $G_{(\infty, 0)}$. With this remark as a guide, we can then show that

$$
H_4(Q_\ast \otimes_G \mathbb{Z}) \cong H_1(\mathbb{F} \times) / 2H_1(\mathbb{F} \times) \cong \mathbb{F} \times / (\mathbb{F} \times)^2, \quad G = \text{GL}(2, \mathbb{F}), \mathbb{F} \text{ infinite.} \quad (1.12)
$$

The long homology sequence associated to (1.10) with $i = 0$ yields the exact sequence

$$
H_4(Q_\ast \otimes_G \mathbb{Z}) \to H_3(C_\text{gen}^* \otimes_G \mathbb{Z}) \to H_3(C_\ast \otimes_G \mathbb{Z}) \to 0, \quad (1.13)
$$

where $G$ maps onto $\text{PGL}(2, \mathbb{F})$ and $\mathbb{F}$ is infinite.
In analogy with (1.6), it is not difficult to see that

\[ H_3(C_\ast \otimes_G \mathbb{Z}) \cong \mathcal{P}(F) \] when \( G \) maps onto \( PGL(2, F) \) and \( F \) is infinite. In fact, the constant in (1.8) is represented by 
\(- (\infty, 0, 1, \infty)\) or by \((\infty, 0, \infty, 1) + (\infty, 0, 1, 0)\) and has order dividing 3. It is 0 when \( X^2 + X + 1 = 0 \) has a solution in \( F \).

(1.14)

The exact sequence (1.13) becomes the exact sequence

\[ \mathbb{F}^\times / (\mathbb{F}^\times)^2 \rightarrow \mathcal{P}_e \rightarrow \mathcal{P}(F) \rightarrow 0, \]

\( \mathbb{F} \) is infinite, where the first map sends \( z \) onto \([z] + [z^{-1}], z \in \mathbb{F}^\times - \{1\} \).

As shown in Dupont and Sah [7], when \( \mathbb{F} = \mathbb{C} \), we can set \([z] = 0 \) for \( z = \infty, 0, 1 \) and remove the restrictions in (1.5) with the understanding that meaningless expressions are 0. We summarize this convention as

\[ \mathcal{P}_e \cong \mathcal{P}(\mathbb{C}), \quad \mathcal{P}_H \cong \mathcal{P}(\mathbb{H}), \quad [z] = 0 \quad \text{for } z = \infty, 0, 1 \text{ in these cases.} \]

(1.16)

When \( \mathbb{F} = \mathbb{R} \), it is known from Parry and Sah [15] that the map from \( \mathcal{P}_e \) to \( \mathcal{P}(\mathbb{R}) \) must have kernel of order at least 2 because \([-1] \) has order 4 in \( \mathcal{P}_e \) while its image in \( \mathcal{P}(\mathbb{R}) \) has order dividing 2 by (1.7). It follows that

\[ 0 \rightarrow \mathbb{F}^\times / (\mathbb{F}^\times)^2 \rightarrow \mathcal{P}_e \rightarrow \mathcal{P}(F) \rightarrow 0 \] is exact and

\[ H_4(C_\ast ^{gen} \otimes_G \mathbb{Z}) \rightarrow H_4(C_\ast \otimes_G \mathbb{Z}) \] is surjective when \( \mathbb{F} = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H} \) and \( G \) maps onto \( PGL(2, \mathbb{F}) \).

(1.17)

Remark 1.18. Equation (2.40) and the paragraph following it in Parry and Sah [15] are faulty. The correct statement is \( \text{im } d^2 = K_2^\ast(\mathbb{R}) \) and has been proved earlier in that work. Since the correct statement was used throughout, the results in [15] are correct as long as one ignores (2.40) and the succeeding paragraph. It would be of some interest to know if (1.17) is valid in general.

One of the main results in Dupont and Sah [7] is the following:

Let \( z \) and \( \omega = \exp(2\pi i/n) \in \mathbb{C} \). Then \([z^n] = n \cdot \sum_{0 \leq j < n} [\omega^j z] \)

holds in \( \mathcal{P}_e \cong \mathcal{P}(\mathbb{C}) \). In particular, \( \mathcal{P}(\mathbb{C}) \) is divisible.

The preceding result is called the distribution formula. With appropriate
modification of the meaning of $\omega$, (1.19) is valid for any algebraically closed field $F$. For $F = \mathbb{R}$ or any real closed field, we have the weaker result:

$\mathcal{R}_x$ is generated by $[r]$, $r > 0$, and $[r^2] = 2[r] + 2[-r] + 2[-1]$ so that both $\mathcal{R}_x$ and $\mathcal{P}(\mathbb{R})$ are 2-divisible.  

(1.20)

By using Roger's $L$-function, it was shown in Parry and Sah [15] that $3[2] = [-1]$ has order 4 in $\mathcal{R}_x$. In a later work [19], $\mathcal{P}_x$ will be shown to be uniquely divisible while $\mathcal{R}_x$ and $\mathcal{P}(\mathbb{R})$ will both be shown to be the direct sums of $\mathbb{Q}/\mathbb{Z}$ and a suitable uniquely divisible group. The $\mathbb{Q}/\mathbb{Z}$ part is known in $\mathcal{R}_x$ and consists of the "rational" multiples of $[2]$. In fact, this $\mathbb{Q}/\mathbb{Z}$ part of $\mathcal{R}_x$ turns out to be the precise kernel of the obvious map from $\mathcal{R}_x$ to $\mathcal{P}_x$. These later results depend on Suslin's resolution of the Lichtenbaum-Quillen conjecture [23-25].

One of the basic goals in the present work is to show that the group $\mathcal{R}_x \cong \mathcal{P}(\mathbb{H})$ is isomorphic to the $\mathbb{Q}$-vector space $\mathbb{A}^\infty(\mathbb{R}^+)$, where $\mathbb{R}^+$ denotes the multiplicative group of positive real numbers.

In general, let $\ast$ be an involution of $F$. We can then use the automorphism $\sigma$ of $GL(2, F)$ in (1.3) to form the semidirect product $GL(2, F) \cdot \langle \sigma \rangle$. We can let this group act on $\mathbb{P}^1(F)$ so that $GL(2, F) \cdot \langle \sigma \rangle / \text{center}$ plays the role of $PGL(2, F)$ in the preceding discussion. We may then define $\mathcal{P}(F, \ast)$ to be the quotient group of $\mathcal{P}(F)$ by adding the further relations

$$[x] = [x^\ast], \quad x \in \mathbb{P}^1(F) - \{\infty, 0, 1\}. \quad (1.21)$$

We may view $\mathcal{P}(F)$ as a module for $\langle \sigma \rangle$ or for $\langle \ast \rangle$. Evidently,

$$\mathcal{P}(F, \ast) \cong H_0(\langle \ast \rangle, \mathcal{P}(F)) \cong H_3(C_\ast \otimes_G \mathbb{Z}), \quad \text{where } G \text{ is any group that maps onto } GL(2, F) \cdot \langle \sigma \rangle / \text{center}. \quad (1.22)$$

The maximal subfields of $\mathbb{H}$ are all isomorphic to $\mathbb{C}$. By the theorem of Noether and Skolem, any isomorphism $\rho$ between two such subfields of $\mathbb{H}$ can be extended to an inner automorphism of $\mathbb{H}$ if and only if $\rho$ is the identity on the center $\mathbb{R}$ of $\mathbb{H}$. As a consequence of this observation, (1.21) follows from (1.4) and we have

$\mathcal{P}(\mathbb{H}) = \mathcal{P}(\mathbb{H}, \ast)$ is a quotient group of the divisible group $\mathcal{P}(\mathbb{C})$ and $\mathcal{P}(\mathbb{C}, \ast)$, where $\ast$ induces the complex conjugation map on $\mathbb{C}$. \quad (1.23)

The group $\mathcal{P}(\mathbb{H})$ should not be confused with $\mathcal{P}(\mathbb{C}, \ast)$ because, as we have already noted, the variables appearing in (1.5) do not have to commute with each other.
Let $\mathcal{H}^n$, $n > 0$, denote the real hyperbolic $n$-space. Various models of $\mathcal{H}^n$ will be used to carry out our discussions.

Let $\mathbb{R}^{1,n}$ denote the $\mathbb{R}$-vector space of all column vectors with $n + 1$ entries from $\mathbb{R}$ and define the inner product $\langle \cdot, \cdot \rangle_{1,n}$ by the rule
\[
\left\langle \sum_j e_j \alpha_j, \sum_j e_j \beta_j \right\rangle_{1,n} = -\alpha_0 \beta_0 + \sum_{j > 0} \alpha_j \beta_j.
\]

In Sah [18], $\mathcal{H}^n$ is modelled after the set $S_+(1,n,\mathbb{R}) = \{e_0 \alpha_0 + v \mid \alpha_0 = (1 + |v|^2)^{1/2}\}$, where $v \in \mathbb{R}^n = \sum_{j > 0} e_j \mathbb{R}$. This may be called the ray model with the identification of $e_0 \alpha_0 + v$ and the ray $(e_0 \alpha_0 + v)\mathbb{R}^+$. In terms of the ray model, each point $(e_0 \alpha_0 + v)\mathbb{R}^+$ has the unique representative $u = \frac{u}{v} \cdot \alpha_0^{-1}$ in the open unit ball of $\mathcal{H}^n$. This ball model shows that $\mathcal{H}^n$ is homeomorphic to the open unit ball of $\mathbb{R}^n$. The ideal boundary $\partial \mathcal{H}^n$ can now be identified with the unit sphere of $\mathcal{H}^n$. In other words, $\mathcal{H}^n = \mathcal{H}^n \cup \partial \mathcal{H}^n$ can be identified with the set $\{e_0 + u \mid u \in \mathbb{R}^n, |u| \leq 1\}$. In this model, geodesics of $\mathcal{H}^n$ appear as chords so that we can visualize hyperbolic simplices (including those with vertices on $\partial \mathcal{H}^n$ that are not compact but have finite volume) as Euclidean simplices. The angles are correctly measured at the "center" $e_0$ corresponding to $0 \in \mathbb{R}^n$. In particular, $\partial \mathcal{H}^n$ can be identified with the unit tangent sphere at $e_0$ of $\mathcal{H}^n$. However, this model is neither isometric nor conformal (at points other than $e_0$).

In terms of the group of all isometries $O'(1,n)$, the action is the obvious one on $S_+(1,n,\mathbb{R})$. The stability subgroup of $e_0$ is $O(n)$ and acts on $\partial \mathcal{H}^n$ in the obvious manner through $\mathbb{R}^n$. The full action of $O'(1,n)$ on $\partial \mathcal{H}^n$ leads to the conformal geometry of the $(n-1)$-sphere. This action can be better described in terms of the upper half space model. We denote the point $e_0 + e_1$ of $\partial \mathcal{H}^n$ by $\infty$ and place it at "+$\infty" above $\mathbb{R}^{n-1} = \sum_{2 \leq j \leq n} e_j \mathbb{R}$. By stereographic projection, the rest of $\partial \mathcal{H}^n$ is identified with the set $\mathbb{R}^{n-1}$ while $\infty$ is identified with the open half space above $\mathbb{R}^{n-1}$. The geodesics through $\infty$ are the "vertical" half lines orthogonal to $\mathbb{R}^{n-1}$ in the upper half space. Hyperbolic hyperplanes orthogonal to these geodesics are the hemispheres with centers at the feet of these half lines in $\mathbb{R}^{n-1}$. We note that $\partial \mathcal{H}^1$ consists of a pair of points and an ordering of this pair amounts to an orientation of $\mathcal{H}^1$. The stability subgroup of $\infty$ can be identified with the group $\text{Sim}(n-1)$ of all similarity transformations of the Euclidean space $\mathbb{R}^{n-1}$. $\text{Sim}(n-1)$ is the semidirect product of the group $\mathbb{R}^{n-1}$ of all translations by the stability subgroup of $(\infty,0)$: $O(n-1) \times \mathbb{R}^+$, where $0 = e_0 - e_1 \in \partial \mathcal{H}^n$. Thus the action of $O'(1,n)$ on $\partial \mathcal{H}^n$ is 2-transitive for $n > 0$ and 3-transitive for $n > 1$. When $n > 1$, the stability subgroup of three or more distinct points of $\partial \mathcal{H}^n$ is $O'(1,n)$-conjugate to $O(t)$ for some
$r \leq n - 2$. The subgroup $\mathbb{R}^+$ fixing $(\infty, 0)$ consists of hyperbolic translations along the geodesic $\mathbb{H}$ determined by $(\infty, 0)$.

If we combine the Hochschild–Serre spectral sequence with the lemma on "center kills," see Dupont [5], we have

$$H_\ast(\text{Sim}(n-1)) \cong H_\ast(O(n-1) \times \mathbb{R}^+) \cong H_\ast(O(n-1)) \otimes A_\ast(\mathbb{R}^+).$$

(2.1)

We note that $O^1(1, 1) \cong \mathbb{R}^+ \cdot O(1)$ with $O(1)$ acting by inversion on $\mathbb{R}^+$ so that

$$H_{\text{even}}(O^1(1, 1)) \cong A_\ast(\mathbb{R}^+) \quad \text{and} \quad H_{\text{odd}}(O^1(1, 1)) \cong H_{\text{odd}}(O(1)) \cong \mathbb{Z}/2\mathbb{Z}.$$  

(2.2)

For $2 \leq n \leq 5$, $O^1(1, n)$ can be described in other ways through "exotic" isomorphisms (see Helgason [10, p. 519] and [26, p. 1412]):

$$O^1(1, 2) \cong \text{PSU}(1, 1) \cdot \langle \rho \rangle \cong \text{PGL}(2, \mathbb{R}); \quad \text{SO}^1(1, 2) \cong \text{PSU}(1, 1).$$

$$O^1(1, 3) \cong \text{PSL}(2, \mathbb{C}) \cdot \langle \rho \rangle; \quad \text{SO}^1(1, 3) \cong \text{PSL}(2, \mathbb{C}).$$

$$O^1(1, 4) \cong \text{PSp}(1, 1) \cdot \langle \rho \rangle; \quad \text{SO}^1(1, 4) \cong \text{PSp}(1, 1).$$

$$O^1(1, 5) \cong \text{PSL}(2, \mathbb{H}) \cdot \langle \rho \rangle; \quad \text{SO}^1(1, 5) \cong \text{PSL}(2, \mathbb{H}).$$

(2.3)

In all cases, $\rho$ is a suitable element of order 2 and represents an orientation reversing isometry of $\mathbb{H}^n$. It may be assumed to be a hyperplane reflection in $\mathbb{H}^n$ with an appropriate choice of embedding. For $n = 2, 3, 5$, $\text{SO}^1(1, n)$ is doubly covered by $\text{SL}(2, \mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. This is most easily seen through the action of $\text{SL}(2, \mathbb{F})$ on the space of all $*$-hermitian symmetric sesquilinear forms on $\mathbb{F}^2$. Each such form $\langle \cdot, \cdot \rangle_M$ can be identified with a $*$-hermitian symmetric matrix $M$ of size $2 \times 2$:

$$M = \begin{pmatrix} x_0 - x_1 & q \\ q^* & x_0 + x_1 \end{pmatrix},$$

where $x_0, x_1 \in \mathbb{R}$, $q \in \mathbb{F}$, $\langle u, v \rangle_M = 'u^* \cdot M \cdot v$, $u, v \in \mathbb{F}^2$, $A[M] = (A^*)^{-1} \cdot M \cdot A^{-1}$, $A \in \text{GL}(2, \mathbb{F})$.

For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the $\text{SL}(2, \mathbb{F})$-orbit of $I_2$ then provides us with another model of $\mathbb{H}^n$ with $n = 2, 3, 5$, respectively. The effective group in each of these three cases is $\text{PSL}(2, \mathbb{F})$. In this setting, we have the obvious embeddings

$$\text{SL}(2, \mathbb{R}) \subset \text{SL}(2, \mathbb{C}) \subset \text{SL}(2, \mathbb{H}).$$
The automorphism \( \rho \) in (2.3) requires some care. We note that \( i \cdot w = (0, -i) \) is a \(*\)-hermitian symmetric and it is equivalent to \((1, 0)\) when viewed as forms on \( E^2 \). It is then easy to conjugate \( SL(2, \mathbb{R}) \) onto \( SU(1, 1) \) inside \( SL(2, \mathbb{C}) \). The automorphism \( \sigma \) defined in (1.3) stabilizes \( SU(1, 1) \) and \( SU(1, 1) \cdot \langle \sigma \rangle \) is then isomorphic to \( SL^\perp(2, \mathbb{R}) \). A similar description works for \( Sp(1, 1) \). As a result, \( \rho \) can be chosen uniformly as the automorphism induced by \( \sigma \) and we have the embeddings

\[
SU(1, 1) \subset Sp(1, 1)
\]

\[
\cap \quad \cap
\]

\[
SL(2, \mathbb{C}) \subset SL(2, \mathbb{H}).
\]

To see the corresponding isomorphisms with the groups \( SO^1(1, n) \), we just observe that the quadratic form \( x_0^2 - x_1^2 - |q|^2 \) is invariant by the action of \( SL(2, \mathbb{H}) \). Next we identify \( \partial H^5 \) with \( \mathbb{P}^1(\mathbb{H}) = SL(2, \mathbb{H})/B \). Here we use the unit ball model and identify the unit quaternions 1, \( i, j, k \) with \( e_2, e_3, e_4, e_5 \), respectively. Each point of \( \partial H^5 \) is represented as \( e_0 + e_1 x_1 + q, x_1 \in \mathbb{R}, q \in \mathbb{H}, x_1^2 + |q|^2 = 1 \). Such a point is then identified with the following point of \( \mathbb{P}^1(\mathbb{H}) \):

\[
\begin{pmatrix}
-q & * \\
1 - x_1 & *
\end{pmatrix}
B \in SL(2, \mathbb{H})/B = \mathbb{P}^1(\mathbb{H}), \quad \text{where } x_1 \neq 1.
\]

The isomorphism from \( SO^1(1, 5) \) to \( PSL(2, \mathbb{H}) \) then carries \( SO^1(1, 2) \) and \( SO^1(1, 3) \) onto \( PSL(2, \mathbb{R}) \) and \( PSL(2, \mathbb{C}) \), respectively. With a different choice of base point, \( SO^1(1, 2) \) and \( SO^1(1, 4) \) can be conjugated to \( PSU(1, 1) \) and \( PSU(1, 1) \), respectively. With a different choice of base point, \( SO^1(1, 2) \) and \( SO^1(1, 4) \) can be conjugated to \( PSU(1, 1) \) and \( PSU(1, 1) \), respectively.

In analogy with Section 1, let \( C_* = C_*(\partial H^n) \) be the normalized Eilenberg–MacLane chain complex based on the nonempty set \( \partial H^n, n \geq 1 \). We define \( C_*^{\text{gen.1}} \) to be the subcomplex of \( C_* \) spanned by the cells such that any 1-face of each of the cells spans a geodesic subspace of dimension 1 in \( H^n \). Since the vertices of our cells lie on \( \partial H^n \), this simply means that the cells must have distinct vertices. When \( n = 2, 3, \) and 5, \( \partial H^n \) can be identified with \( \mathbb{P}^1(\mathbb{F}) \) with \( \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H} \). In these three cases, \( C_*^{\text{gen.1}} \) is just \( C_*^{\text{gen.1}}(\mathbb{P}^1(\mathbb{F})) \) while \( C_* = C_*(\partial H^n) \) is just \( C_*(\mathbb{P}^1(\mathbb{F})) \) of Section 1. The reader should note that \( C_*^{\text{gen.1}}(\partial H^n) \) can be filtered by a decreasing sequence of subcomplexes \( C_*^{\text{gen.1}, i}, 0 \leq i \leq n \), through the dimension function on hyperbolic spaces. If \( \partial H^n \) is identified with the \( (n - 1) \)-sphere and \( O^1(1, n) \) is replaced by the subgroup \( O(n) \), then a different filtration may be defined by means of the dimension function on spherical spaces. By the same token, \( C_*^{\text{gen.1}}(\mathbb{P}^n(\mathbb{F})) \) can be filtered by means of the dimension function on the underlying projective space. In the present work, only \( C_*^{\text{gen.1}} \) and \( C_* \) are used in dealing with \( H^n, n \leq 5 \), so that confusion should not arise.
We now display some of the "E"-terms of the transposed spectral sequence associated to $C^{\text{gen.1}}_*$ with $G = O^1(1, n), n \geq 2$:

$C^{\text{gen.1}}_4 \otimes_G \mathbb{Z}$
$C^{\text{gen.1}}_5 \otimes_G \mathbb{Z}$

$Z \cdot (\infty, 0, 1) \quad H_1(O(n - 2)) \otimes (\infty, 0, 1) \quad H_2(O(n - 2)) \otimes (\infty, 0, 1)$
$Z \cdot (\infty, 0) \quad H_1(O(n - 1) \times \mathbb{R}^+) \otimes (\infty, 0) \quad H_2(O(n - 1) \times \mathbb{R}^+) \otimes (\infty, 0)$
$Z \cdot (\infty) \quad H_1(O(n - 1) \times \mathbb{R}^+) \otimes (\infty) \quad H_2(O(n - 1) \times \mathbb{R}^+) \otimes (\infty)$.

(2.4)

Here $\infty$, 0, 1 denote three distinct points of $\partial \mathbb{R}^n$, $n \geq 2$, and we have adopted the notation of $P^1(\mathbb{F})$. We recall that $O^1(1, n)$ is 3-transitive on $\partial \mathbb{R}^n$, $n \geq 2$. We also note that the stability subgroup $G_{(\infty, 0)}$ is a semidirect product of $G_{(\infty, 0)}$ and a unipotent subgroup. However, the lemma on "center kills" permits us to use the same coefficient groups for $(\infty, 0)$ and $(\infty)$ on the level of "E".

The map "d\textsubscript{1,1}" can be described as follows.

Let $c$ be an $i$-cycle representing an element of $H_i(O(n - 1) \times \mathbb{R}^+)$. The element $\text{diag}(1, -1, 1, \ldots, 1)$ exchanges $\infty$ and 0, inverts $\mathbb{R}^+$, and centralizes $O(n - 1)$ in $O^1(1, n)$. Up to a sign, "d\textsubscript{1,1}" carries the class of $c \otimes (\infty, 0)$ onto the class of $(c - \tau(c)) \otimes (\infty)$. In a similar way, the "E"-terms of the transposed spectral sequence associated to $C_*$ and $G = O^1(1, n)$ can be displayed. The rows are the same as in (2.4) for indices $j = 0, 1$. In the row indexed by $j = 2$, there is one more 2-cell $(\infty, 0, \infty)$. We note that any $j + 1$ points of $\partial \mathbb{R}^n$ for $j < n$ can be moved by a suitable element of $G = O^1(1, n)$ to $\partial \mathbb{R}^1$. This immediately leads to:

"E"_{0,j} stabilizes for $j \leq n$ and "E"_{0,j} stabilizes for $j < n$. These hold for the transposed spectral sequences associated to either $C^{\text{gen.1}}_*(\partial \mathbb{R}^n)$ or $C_*(\partial \mathbb{R}^n)$ with $G = O^1(1, n)$.

(2.5)

With knowledge of the appropriate stability results involving the coefficient groups, we can extend (2.5) to arbitrary terms in the transposed spectral sequence "E"\textsubscript{ij} under suitable restrictions on $r$, $i$, and $j$. For (2.5), $G$ can be replaced by any group that is mapped onto $O^1(1, n)$.

We will now examine the special cases with $2 \leq n \leq 5$. In particular, we indicate the relations with earlier works when $n = 2$ and 3. The most important case for the present work corresponds to $n = 5$.

Case 1. $n = 2$. $G \cong PGL(2, \mathbb{R}) = PSL^\pm(2, \mathbb{R}) = PSL(2, \mathbb{R}) \cdot \langle \tau \rangle$ with $\tau = (1 0)$. If we use the notation of Section 1, then the relevant "E"-terms associated to (2.4) are
The groups \( \mathcal{E}^2_{\text{odd},0} \cong \mathbb{Z}/2\mathbb{Z} \) in (2.6) survive to \( \mathcal{E}^\infty \) because they represent the nonzero homology groups of \( \langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z} \) complementary to \( \text{PSL}(2, \mathbb{R}) \) in \( \text{PGL}(2, \mathbb{R}) \). The groups \( \mathcal{A}^2_{\text{even}}(\mathbb{R}^+) \cong \mathcal{E}^2_{\text{even},1} \) are killed by \( \partial \) for suitable \( r \geq 2 \) starting from the column indexed by 0. This is seen by using the spectral sequence associated to \( C_\tau \) as in the discussion following (1.9). The relevant \( \mathcal{E}^2 \)-terms are

\[
\begin{array}{cccc}
\mathcal{P}(\mathbb{R}) & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} \\
0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} \\
0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} \\
\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathcal{A}^2_2(\mathbb{R}^+) & \mathbb{Z}/2\mathbb{Z}
\end{array}
\]

In (2.7), \( \mathcal{E}^2_{\tau,0} \) arise from the subcomplex formed by the cells with at most two distinct vertices. In particular, the death of \( \mathcal{E}^2_{\text{even},1} \) in (2.6) by means of suitable higher differentials is seen in (2.7) by means of \( \partial^1_{\text{even},2} \) and the cell \((\infty, 0, \infty)\). It is also clear how (1.15) fits into the comparison between the spectral sequences. The 2-divisibility of \( \mathcal{P}_\tau \) and its quotient \( \mathcal{P}(\mathbb{R}) \) evidently imply that \( \mathcal{E}^1_{1,1} \cong \mathbb{Z}/2\mathbb{Z} \) in (2.6) and (2.7) must survive to \( \mathcal{E}^\infty \). The groups \( \text{H}_s(\text{PGL}(2, \mathbb{R})) \) and \( \text{H}_s(\text{PGL}(2, \mathbb{R})) \) therefore fit into an exact sequence that can be read off from (2.6). Before exhibiting this exact sequence, we note that \( \text{H}_s(\text{PSL}(2, \mathbb{R}) \cdot \langle \tau \rangle) \) can also be described by the Hochschild–Serre spectral sequence associated to the semidirect product splitting of \( \text{PSL}(2, \mathbb{R}) \cdot \langle \tau \rangle \). We note that \( \text{H}_s(\text{PSL}(2, \mathbb{R}) \cdot \langle \tau \rangle) = 0 \) while \( \text{H}_s(\text{PSL}(2, \mathbb{R})) \cong K_2(\mathbb{C})^+ \oplus \mathbb{Z} \). Under the action of \( \tau \), \( K_2(\mathbb{C})^+ \) is a \( \mathbb{Q} \)-vector space that is pointwise fixed by \( \tau \) while \( \mathbb{Z} \cong \pi_1(\text{PSL}(2, \mathbb{R})) \) is negated by \( \tau \). It follows that \( \text{H}_s(\text{PSL}(2, \mathbb{R}) \cdot \langle \tau \rangle) \cong K_2(\mathbb{C})^+ \oplus \mathbb{Z}/2\mathbb{Z} \). We note that the "symbolic part" \( K_2(\mathbb{C})^+ \) is covered by \( \mathcal{A}^2_2(\mathbb{R}^+) \) so that the triviality of the action of \( \tau \) on \( K_2(\mathbb{C})^+ \) is justified. From the Hochschild–Serre spectral sequence \( \mathcal{E} \), we can obtain the exact sequence

\[
\mathbb{Z}/2\mathbb{Z} \rightarrow \text{H}_0(\langle \tau \rangle, \text{H}_s(\text{PSL}(2, \mathbb{R}))) \\
\quad \rightarrow \text{H}_s(\text{PSL}(2, \mathbb{R}) \cdot \langle \tau \rangle) \xrightarrow{\partial} \mathbb{Z}/2\mathbb{Z} \rightarrow 0. \quad (2.8)
\]

Here, the first \( \mathbb{Z}/2\mathbb{Z} \cong \mathcal{E}^2_{2,2} = \mathcal{H}_s(\langle \tau \rangle, \text{H}_s(\text{PSL}(2, \mathbb{R})) \) survives to \( \mathcal{E}^\infty \). This can be seen by looking at the surjective map from \( \text{H}_s(B(\text{PGL}(2, \mathbb{R})))^0 \).
to $H_*(B(PGL(2, \mathbb{R})), \mathbb{F}_2)$ as shown in Milnor [14]. The initial term of (2.8) can therefore be replaced by 0. We can now exhibit the exact sequence arising from (2.6):

$$0 \to H_0(\langle \tau \rangle, H_3(PSL(2, \mathbb{R})))$$

$$\to \mathcal{P}_3 \xrightarrow{\lambda_{\mathbb{R}}} A^2_2(\mathbb{R}^+) \xrightarrow{\text{sym}} K_2(\mathbb{C})^+ \to 0. \quad (2.9)$$

Equation (2.9) is identical to Parry and Sah [15, (C.13)] and it was shown in [15, Theorem C.14] that $H_0(\langle \tau \rangle, H_3(PSL(2, \mathbb{R}))) = H_3(PSL(2, \mathbb{R}))$. As in DuPont and Sah [7], we have the exact sequence

$$0 \to \mathbb{Z}/4\mathbb{Z} \to H_3(SL(2, \mathbb{R})) \to H_3(PSL(2, \mathbb{R})) \to 0.$$ 

The outer terms are trivial under the action of $\langle \tau \rangle$ and $H_3(PSL(2, \mathbb{R}))$ is 2-divisible by using (2.9). The long homology sequence of $\langle \tau \rangle$ with coefficients in the preceding exact sequence therefore yields

$$H_3(SL(2, \mathbb{R})) = H_0(\langle \tau \rangle, H_3(SL(2, \mathbb{R}))). \quad (2.10)$$

The maps $\lambda_{\mathbb{R}}$ and $\text{sym}$ in (2.9) will be discussed in Remark 2.14.

**Case 2.** $|\mathbb{Z}| = 3$. $G \cong PSL(2, \mathbb{C}) \cdot \langle \sigma \rangle$, $\sigma$ is complex conjugation. We go directly to the $E'$-terms of the transposed spectral sequence associated to $C_*$. The relevant terms are

$$\mathcal{P}(\mathbb{C}, \ast)$$

$$\begin{array}{cccc}
0 & 0 & & \\
0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & A^2_2(\mathbb{R}^+) & \bigoplus H_2(O(2)) & H_2(O(2)) \\
\end{array}$$

By the Hochschild–Serre spectral sequence, $H_2(O(2)) \cong H_2(SO(2)) \bigoplus \mathbb{Z}/2\mathbb{Z}$, where $H_2(SO(2)) \cong A^2_2(\mathbb{R}/\mathbb{Z})$ is a $\mathbb{Q}$-vector space which is pointwise fixed by $\sigma$. Similarly, $H_3(O(2)) \cong \mathbb{Q}/\mathbb{Z} \bigoplus \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Q}/\mathbb{Z} \cong \text{Tor}(SO(2), SO(2))$ is also pointwise fixed by $\sigma$. We note that $\mathbb{Z}/2\mathbb{Z} \cong E^{2,1}_2$ arises from $H_2(O(2))$ in $H_2(G_{(\infty,0)})$. It is not killed by $H_2(G_{(\infty,0)}) \otimes (\infty, 0, \infty)$.

As in Case 1, we may examine $H_*(PSL(2, \mathbb{C}) \cdot \langle \tau \rangle)$ by using the Hochschild–Serre spectral sequence associated to the semidirect product structure. $H_1(PSL(2, \mathbb{C})) = 0$ and $H_2(PSL(2, \mathbb{C})) \cong K_2(\mathbb{C}) \bigoplus \mathbb{Z}/2\mathbb{Z}$, where $K_2(\mathbb{C})$ is a $\mathbb{Q}$-vector space. The relevant terms of the spectral sequence on the level of $E'$ are
Here \( E_{0,3}^2 \cong H_0(\langle \sigma \rangle, H_3(PSL(2, \mathbb{C}))) \) is a divisible group and \( E_{0,2}^2 \cong K_2(\mathbb{C})^+ \amalg \mathbb{Z}/2\mathbb{Z} \). Moreover, \( E_{2,0}^2 - E_{2,0}^2 \) splits off as a direct summand of \( H_*(PSL(2, \mathbb{C}) \cdot \langle \tau \rangle) \) as the homology of \( \langle \sigma \rangle \). A comparison with (2.11) shows that \( \mathbb{Z}/2\mathbb{Z} \cong E_{2,1}^2 \) must survive to \( E_{2,2}^2 \) and leads to a direct summand that corresponds to \( \mathbb{Z}/2\mathbb{Z} \cong E_{2,2}^2 \) in the Hochschild–Serre spectral sequence.

If we use the fact that \( SL(2, \mathbb{C}) \) maps onto \( PGL(2, \mathbb{C}) \), the preceding spectral sequence argument may be carried out in analogy with Section 1. This would recover the exact sequence derived in Dupont and Sah [7] through the use of \( C^{\text{gen.1}}_* \):

\[
0 \to \mathbb{Q}/\mathbb{Z} \to H_3(SL(2, \mathbb{C}))
\]

\[
\to \mathcal{P}(\mathbb{C})\cdot \overset{\lambda_*}{\longrightarrow} A_2^2(\mathbb{C}^\times)\overset{\text{sym}}{\longrightarrow} K_2(\mathbb{C}) \to 0. \tag{2.12}
\]

Here, the group \( \mathbb{Q}/\mathbb{Z} \) is mapped onto the \( \mathbb{Q}/\mathbb{Z} \) summand in \( E_{2,2}^2 \) of (2.11) with kernel \( \mathbb{Z}/4\mathbb{Z} \). The injectivity of this term into \( H_3(SL(2, \mathbb{C})) \) is verified by the Cheeger–Chern–Simons invariant; see Dupont [5, 6]. If we combine the above with the exact sequence that arises from (2.11) as in Case 1, we have the exact sequence

\[
0 \to \mathbb{Q}/\mathbb{Z} \to H_0(\langle \sigma \rangle, H_3(SL(2, \mathbb{C})))
\]

\[
\to \mathcal{P}(\mathbb{C}, *)\cdot \overset{\lambda_*(\cdot)}{\longrightarrow} A_2^2(\mathbb{R}^+)\overset{\text{sym}}{\longrightarrow} K_2(\mathbb{C}) \to 0. \tag{2.13}
\]

Remark 2.14. We shall soon show that \( \mathcal{P}(\mathbb{C}) \) is uniquely 2-divisible. It then follows that (2.13) is the fixed part of (2.12) under the action of complex conjugation. The maps \( \lambda_* \) and \( \lambda_* \) in (2.12) and (2.9) are really \( d_{1,3}^0 \).

They are determined by brute force computation in Dupont and Sah [7] and Parry and Sah [15]. Both of them are forms of the Dehn invariant maps. They are the algebraic analogues of the dilogarithm functions and are found in the setting of the action of \( PGL(2, F) \) on \( C^{\text{gen.}}_*(\mathbb{P}^1(F)) \), \( F = \mathbb{C} \) and \( \mathbb{R} \). In each case, the cross-ratio symbol \( \{x\} \) is mapped onto 2. (\( \alpha - (\alpha - 1) \)), where \( \alpha \) is restricted to satisfy \( \alpha > 1 \), \( F = \mathbb{R} \). In these two cases, the target groups happen to be \( \mathbb{Q} \)-vector spaces so that the factor of 2 at the front is not very important. It does allow us to change \( \alpha - 1 \) to 1 – \( \alpha \). The explicit knowledge of the form of \( K_2(F) \) in these two cases allows
us to define $\text{sym}$ by sending $\alpha \wedge \beta$ onto the $K_2$-symbol $\{\alpha, \beta\}$ and questions of the correct 2-power factors in the front are again not important. For a general infinite field $F$, our special arguments can be modified by an examination of the action of $GL(2, F)$ on $C^*_{\text{gen}}(P^1(\mathbb{F}))$. The group $A_2(F^\times)$ must be replaced by $H_2(F^\times \times F^\times)/\text{im} \ "d_{2,1}"$. This latter group has a direct summand $H_2(F^\times)$ that survives to $\pi E\pi$ through the identification $F^\times \cong GL(1, F)$. The complementary direct summand is $F^\times \otimes F^\times/\langle \alpha \otimes \beta + \beta \otimes \alpha \rangle$. When $F^\times$ is not 2-divisible, this summand is not the same as $A_2(F^\times)$. In order to end up in $K_2(F) \cong H_2(SL(3, F))$, we can follow Suslin [23, p. 365], where it is shown that $K_2(F)$ is isomorphic to $H_0(F^\times, H_2(SL(2, F)))$. This involves the use of the Hochschild–Serre spectral sequence together with the map that sends $A \in GL(2, F)$ onto $\text{diag} (\det A^{-1}, A) \in SL(3, F)$. If we unravel all these maps, then sym $\lambda_{\pi}$ gives the known result, $K_2(F) \cong F^\times \otimes F^\times/\langle \alpha \otimes (1-\alpha) \rangle, \alpha \in F - \{0, 1\}$. The skew-symmetry relation is known to follow from the relation $\alpha \otimes (1-\alpha) = 0$. However, the definition of $\lambda_{\pi}$ as "$d_{0,3}^3$ requires us to have the skew-symmetry relation first. The factor of 2 does not appear in this alternate approach. Since the unravelling process is quite tedious, we will not carry it out. When $F = \mathbb{H}$, $\mathcal{P}(\mathbb{H})$ is a quotient of $\mathcal{P}(\mathbb{C})$ and each element of $\mathbb{H}$ is conjugate to an element of $\mathbb{C}$. Functoriality allows us to define $\lambda_{\mathbb{C}}$ in terms of $\lambda_{\mathbb{C}}$. This avoids the task of repeating the brute force computation of "$d_{0,3}^3"$.

The remaining cases when $n > 3$ are new. We quickly summarize our goals as well as the various technical arguments. The principal goal is to obtain exact sequences involving $H_3(SL(2, F))$ and $H_3(SL(2, \mathbb{F}) \cdot \langle \sigma \rangle)$ that are similar to those obtained in Cases 1 and 2 and to determine their relations. For this purpose, we will examine the transposed spectral sequence associated to $C^*_{\text{gen}}(\partial \mathcal{X}^5) \cong C^*_{\text{gen}}(P^1(\mathbb{F}))$ and the double covering groups $SL(2, \mathbb{F}) \cdot \langle \sigma \rangle$ and $SL(2, \mathbb{F})$ of $O^1(1, 5)$ and $SO^1(1, 5)$, respectively. The chain complex $C_*$ can be used to get rid of the term "$E_{2,1}^2$ in the manner described in earlier cases and in Section 1. The case $n = 5$ with $G = O^1(1, 5)$ actually takes care of the cases of $n > 5$ with $G = O^1(1, n)$ or $SO^1(1, n)$ in the sense that all the relevant terms stabilize. We will therefore restrict our attention to four cases corresponding to $n = 4$ and 5 and $G = O^1(1, n)$ or $SO^1(1, n)$. This means that we can use the exotic isomorphisms of (2.3). In general, if $n \geq 3$, then $SO^1(1, n)$ and $SO(n)$ have $\pi_1 \cong \mathbb{Z}/2\mathbb{Z}$. If $S$ denotes one of these groups and $\tilde{S}$ denotes its universal covering group, then $H_2(S) \cong H_2(\tilde{S}) \Pi \pi_1$; see Milnor [14]. In all cases, $H_2(\tilde{S})$ is divisible and uniquely 2-divisible since it is made up from $K_2(\mathbb{C}), K_2(\mathbb{C})^+, \text{and } H_2(SU(2))$. The direct sum decomposition of $H_2(S)$ is compatible with the obvious embeddings. Since $H_1(S) = H_1(\tilde{S}) = 0$, the Hochschild–Serre spectral sequence associated to the universal covering
sequence yields the result that $H_3(\mathcal{N})$ maps surjectively onto $H_3(S)$ with a kernel of order dividing 4. If we use the universal complexification homomorphisms and repeat the argument used in Dupont and Sah [7] (cf. the discussion after (2.9)), it follows that the kernel is $\mathbb{Z}/4\mathbb{Z}$. This corresponds to the multiplication by 4 isogeny on $\mathbb{Q}/\mathbb{Z} \cong H_3(\mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}/\mathbb{Z}$ is the torsion subgroup of $SO(2)$. Under the obvious embeddings involving $SO(n)$, $SO(n + 1)$, $SO^1(1, n)$, $SO^1(n + 1)$, we have isomorphisms on $\pi_1$ as well as on this cyclic group of order 4. These facts allow us to transfer the exact sequences back and forth between $S$ and $\mathcal{N}$. Moreover, these assertions can be extended to $O(n)$ and $O^1(1, n)$. Some care has to be exercised because covering groups of disconnected topological groups are not always classified by fundamental groups. For an algebraic version of this problem, see MacLane [12, pp. 124–131]. We also note that (2.5) can be modified to accommodate $SO^1(1, n)$. The main point is that any isometry of hyperbolic $n$-space can be extended to an orientation-preserving isometry of hyperbolic $(n + 1)$-space. When $n > 3$ and $G$ is either $SO^1(1, n)$ or $O^1(1, n)$, then $E_0^j$ is bijectively stable for $j \leq 3$ while $E_0^{n, 4}$ stabilizes bijectively for $n > 4$ and surjectively as we go from $SO^1(1, 4)$ to $O^1(1, 4)$ to $SO^1(1, 5)$. In all these cases, $G$ may be replaced by a group that is mapped surjectively onto $SO^1(1, n)$ or $O^1(1, n)$. In particular, for $n = 4$ and 5, we can use appropriate subgroups of $SL(2, \mathbb{H})$. Finally, we recall from Sah [18] that $H_3(O(n)) \cong H_3(SU(2)) \amalg \mathbb{Z}/2\mathbb{Z}$, $n \geq 3$, and that $H_3(SO(n)) \cong H_3(O(n))$ for $n = 3$ or $n \geq 5$, while $H_3(SO(4)) \cong H_3(SU(2)) \amalg H_2(SU(2)) \amalg \mathbb{Z}_2$. When $n \geq 3$, we also have $H_3(O(n)) \cong H_3(O(3))$. This slight improvement on the bijective stability requires the special structure of $O(4)$; see the discussion in Sah [18, Remark 2.11] and in the proof of Theorem 4.4.

Case 3. $n = 5$.

We begin with $G = O^1(1, 5)$ and $C_{\text{gen.}}^1$. There is no difficulty exhibiting the relevant terms of $E^2$ (this takes care of all cases with $n \geq 5$):

$$
\begin{align*}
E^2_{0,4}(\text{stable}) \\
\begin{array}{ccc}
\mathcal{F}(\mathbb{H}) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & A_2^2(\mathbb{R}^+) & 0 \\
\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & A_2^2(\mathbb{R}^+) & H_2(O(4)) & H_3(O(4))
\end{array}
\end{align*}
$$

(2.15)

The $\mathbb{Z}/2\mathbb{Z}$ direct summands in $E^2_{0,0}$ survive to $E^\infty$ and represent the nonzero positive-dimensional homology of $\langle \sigma \rangle$. As discussed already, we may lift to $SL(2, \mathbb{H}) \cdot \langle \sigma \rangle$ and extract the exact sequence

$$
\text{(2.15)}
$$
The map $\text{sym}$ in (2.16) requires a little bit of explanation. The point is that $K_2(\mathbb{C})^+ \cong H_2(SL(2, \mathbb{H}))$ under the map induced by the obvious embedding of $SL(2, \mathbb{C})$ into $SL(2, \mathbb{H})$. From a theorem of Mathe, see [1] or [18], $H_2(SO(2))$ maps surjectively to $H_2(SU(2))$. The map $\text{sym}$ in (2.16) is induced by the corresponding map in (2.13).

By using the functorial nature of our spectral sequences, we can combine (2.9), (2.13), and (2.16) into the following commutative row-exact diagram (2.17). As usual, $\hookrightarrow$ denotes an injective map while $\twoheadrightarrow$ denotes a surjective map. (See diagram 2.17).

**Remark 2.18.** The group $\mathbb{Q}/\mathbb{Z}$ in fact injects into $H_0(\langle \sigma \rangle, H_4(SL(2, \mathbb{C})))$. It is stably detected by the Cheeger-Chern-Simons class as described in Dupont [5, 6]. One of the consequences of the results to be proved is the surjectivity of the maps denoted by $j_R$, $j_C$, and $A$ in (2.17). The map $A_C$ is induced by the inclusion of $SU(2)$ into $SL(2, \mathbb{C})$. This requires a little bit of care since $SU(2)$ denotes the following subgroup of $SL(2, \mathbb{H})$: \[
\{\text{diag}(q, q) \mid q \text{ is a unit quaternion}\} = SL(2, \mathbb{H})_{(\infty, 0, 1)}.
\]

As it stands, it is not a subgroup of the obvious $SL(2, \mathbb{C})$ subgroup of $SL(2, \mathbb{H})$. On the other hand, $SL(2, \mathbb{C})$ has just one conjugacy class of maximal compact subgroups and all of them are conjugate to the subgroup $SU(2)$ of $SL(2, \mathbb{C})$. A direct calculation shows that the standard $SU(2)$ in $SL(2, \mathbb{C})$ becomes the preceding subgroup with respect to the basis $(e_1)$ and $(e_2)$ of the right $\mathbb{H}$-vector space $\mathbb{H}^2$. Since $GL(2, \mathbb{H}) = SL(2, \mathbb{H}) \times \mathbb{R}^+ \cdot I_2$ and conjugation in $SL(2, \mathbb{H})$ induces the identity map on its homology, we have found the lift $A_C$ of $A$. It will be proved later that $A_C$ is surjective.

We next consider the action of $G = SL(2, \mathbb{H})$ on $C^\text{gen.1}$ rather than the effective group $SO^1(1, 5)$. There is no problem exhibiting the relevant $E^2$-terms:

\[
\begin{array}{cccc}
\mathcal{P}(\mathbb{H}) & \mathbb{R}/\mathbb{Z} \otimes \mathcal{P}(\mathbb{C})^- & \mathcal{P}(\mathbb{H}) \\
0 & 0 & 0 \\
0 & 0 & A_2^+(\mathbb{R}) \\
\mathbb{Z} & 0 & A_2^+(\mathbb{R}) \cup H_2(SU(2)) \cup H_3(SU(2)) \cup H_2(SU(2)) \otimes \mathbb{R}
\end{array}
\]
\[
\begin{array}{cccccccccc}
0 & \rightarrow & \mathbb{H}^3(\mathbb{SL}_2, \mathbb{R}) & \rightarrow & \mathbb{H}^3(\mathbb{SL}_2, \mathbb{C}) & \rightarrow & \mathbb{H}^3(\mathbb{SU}_2, \mathbb{C}) & \rightarrow & 0 \\
& & & & \downarrow \tilde{\lambda}_R & & & \downarrow \tilde{\lambda}_R & & \\
& & & & \mathbb{H}^3(\mathbb{SL}_2, \mathbb{R}) & \rightarrow & \mathbb{H}^3(\mathbb{SL}_2, \mathbb{C}) & \rightarrow & \mathbb{H}^3(\mathbb{SU}_2, \mathbb{C}) & \rightarrow & 0 \\
& & & & & & \mathbb{H}^3(\mathbb{SL}_2, \mathbb{R}) & \rightarrow & \mathbb{H}^3(\mathbb{SL}_2, \mathbb{C}) & \rightarrow & \mathbb{H}^3(\mathbb{SU}_2, \mathbb{C}) & \rightarrow & 0 \\
& & & & & & & & \mathbb{H}^3(\mathbb{SL}_2, \mathbb{R}) & \rightarrow & \mathbb{H}^3(\mathbb{SL}_2, \mathbb{C}) & \rightarrow & \mathbb{H}^3(\mathbb{SU}_2, \mathbb{C}) & \rightarrow & 0 \\
\end{array}
\]
Some explanations of the terms are in order. The column "$E_{i,*}^2$" is handled by the stability result (2.5). The lemma on "center kills" allows us to replace $H_{i,*}(G_{(x)})$ by $H_{i,*}(G_{(x,0)})$ and $G_{(x,0)}$ is the diagonal subgroup of $SL(2, \mathbb{H})$ so that it is isomorphic to $\mathbb{R}^+ \times Sp(1) \times Sp(1)$. The element $(0, 1, 0)$ exchanges the two $Sp(1)$ factors and inverts $\mathbb{R}^+$ so that it fixes the subgroup $G_{(x,0,1)}$ of $SL(2, \mathbb{H})$ and exchanges $\infty$ and 0. In terms of $SO^1(1, 5)$, $Sp(1) \times Sp(1)$ is the universal covering group of $SO(4)$. These are used in the determination of "$E_{i,*}^2$" exhibited in (2.19) for $i > 0$. When $i = 1$, a 3-cell has the nonzero coefficient group $H_1(SO(2)) \cong \mathbb{R}/\mathbb{Z}$ if and only if it has the cross-ratio symbol $\{z\}$ with $z$ not real. Since $SL(2, \mathbb{H})$ covers $SO^1(1, 5)$, elements that carry $(x, 0, 1, z)$ onto $(\infty, 0, 1, z)$ must induce negation on the coefficient group $\mathbb{R}/\mathbb{Z}$. It is then easy to determine "$E_{i,*}^2$" in (2.19). We note that $\mathbb{R}/\mathbb{Z} \otimes P(\mathbb{C})^-$ and $H_2(SU(2)) \otimes \mathbb{R}^+$ are both $\mathbb{Q}$-vector spaces because $P(\mathbb{C})^-$ is divisible while $\mathbb{R}^+$ is a $\mathbb{Q}$-vector space. These two terms are both negated by the automorphism $\sigma$ of $SL(2, \mathbb{H})$ defined in (1.3) and $\sigma$ covers $\rho$ in terms of the action on $H_{i,*}(SL(2, \mathbb{H}))$. The remaining terms in (2.19) are pointwise fixed by the action of $\sigma$. Because the differentials of the spectral sequence are $\langle \sigma \rangle$-equivariant and $\mathbb{R}/\mathbb{Z} \otimes P(\mathbb{C})^-$, $H_2(SU(2)) \otimes \mathbb{R}^+$ are $\mathbb{Q}$-vector spaces, it follows that $\text{im}(\"d_3^3\)) \subseteq H_3(SU(2)) \otimes \mathbb{R}^+$ and $\text{im}(\"d_3^0\)) \subseteq H_3(SU(2)).$ We recall that the term $A^{'2}(\mathbb{R}^+)$ is actually killed by $\"d_3^3\)$ while $\"d_3^0\)$ is 0. In Theorem 5.10, $H_3(SU(2))$ will be shown to be 2-divisible. $P(\mathbb{H})$ is 2-divisible by (1.23). $H_3(SU(2))$ is uniquely 2-divisible; see Dupont [5] or Sah [18]. These imply that $H_3(SL(2, \mathbb{H}))$ is also 2-divisible. It is therefore possible to speak of the $\pm$-eigenspace decomposition of $H_3(SL(2, \mathbb{H}))$ under the action of $\sigma$, although a priori this does not give a direct sum decomposition because it involves a little bit of ambiguity concerning the elements of order 2 fixed by $\sigma$. However, a comparison of the exact sequence arising from (2.19) with that of (2.16) gives the following.

**Theorem 2.20.** $H_3(SL(2, \mathbb{H})) \cong H_3(SL(2, \mathbb{H}))^{\sigma(-)} \amalg H_3(SL(2, \mathbb{H}))^{\sigma(+)}$ is the \pm-eigenspace decomposition of $H_3(SL(2, \mathbb{H}))$ as a 2-divisible group. The negative eigenspace $H_3(SL(2, \mathbb{H}))^{\sigma(-)}$ is isomorphic to the $\mathbb{Q}$-vector space $H_2(SU(2)) \otimes \mathbb{R}^+ / \text{im}(\"d_3^3\)).$

**Remark 2.21.** It should be noted that Theorem 2.20 can be stated with the phrase "as a 2-divisible group" deleted. $H_3(SL(2, \mathbb{H}))^{\sigma(-)}$ is then defined as the kernel of the surjective map from $H_3(SL(2, \mathbb{H}))$ to $H_3(SL(2, \mathbb{H}))$. The identification of this kernel with a $\mathbb{Q}$-vector space then yields the desired direct sum decomposition of $H_3(SL(2, \mathbb{H}))$ without invoking Theorem 5.10. In any case, Theorem 2.20 is not used in the proof of Theorem 5.10. The map $\"d_3^3\)$ in (2.19) can be described in terms of $\lambda_\mathbb{C}$. Namely, $\lambda_\mathbb{C}$ maps $P(\mathbb{C})^-$ into $(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{R}^+$ of $A^{'2}(\mathbb{C}^+)$ by
using the polar decomposition of $\mathbb{C}^\times$, and \(1 \otimes \lambda_\mathbb{C}\) then maps \((\mathbb{R}/\mathbb{Z}) \otimes \mathcal{P}(\mathbb{C})\) into \((\mathbb{R}/\mathbb{Z}) \otimes (\mathbb{R}/\mathbb{Z}) \otimes \mathbb{R}^+\). \((\mathbb{R}/\mathbb{Z}) \otimes (\mathbb{R}/\mathbb{Z})\) maps onto \(H_2(SO(2)) \cong \Lambda^2_\mathbb{Z}(\mathbb{R}/\mathbb{Z}) \cong H_2(U(1))\) and this maps onto \(H_2(SU(2))\). By tensoring this with the identity map of \(\mathbb{R}^+\) and composing the result with \(1 \otimes \lambda_\mathbb{C}\) we get a description of "\(d_{1,3}\) up to some nonzero integer factor. Since we are dealing with \(\mathbb{Q}\)-vector spaces, the nonzero integer factor may be ignored. Eventually, \(H_3(SL(2, \mathbb{H}))^{\sigma(-)}\) turns out to be isomorphic to the divisible part \(K^M_2(\mathbb{R})\) of the Milnor \(K\)-group \(K^M_2(\mathbb{R})\); see Sah [19]. In order to obtain this identification, Theorem 2.20 is needed. The summand \(H_0(\langle \sigma \rangle, H_3(SL(2, \mathbb{H}))\) turns out to be isomorphic to a direct sum of \(\mathbb{Q}/\mathbb{Z}\) and a suitable \(\mathbb{Q}\)-vector space. It is in fact isomorphic to \(H_3(SL(2, \mathbb{R}))/\text{(cyclic group of order 2)}\) in terms of the obvious inclusion of \(SL(2, \mathbb{R})\) into \(SL(2, \mathbb{H})\). All of these assertions require the deep results of Suslin [23–25] that confirm the conjecture of Lichtenbaum and Quillen.

We are left with the two cases where \(n = 4\). In one of these, we see a modified form of the Dehn invariant map in hyperbolic 4-space which is also a form of the Gauss–Bonnet map.

**Case 4. \(n = 4\).**

We can exhibit the relevant terms of "\(E^2\) for \(G = O^1(1, 4)\) acting on \(C^\text{gen.1}\):

\[
\begin{array}{cccc}
\text{"}E^2_{0,4} & \mathcal{P}(\mathbb{H}) & 0 & \downarrow \text{"}d^2 \\
0 & 0 & \text{"}E^2_{2,2} & \\
0 & 0 & \Lambda^2_\mathbb{Z}(\mathbb{R}^+) & \\
\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \Lambda^2_\mathbb{Z}(\mathbb{R}^+) \sqcup H_2(O(3)) \sqcup H_3(O(3))
\end{array}
\]

To get "\(E^2_{*,}\), the 2-divisibility of \(\mathcal{P}(\mathbb{C})\) is used. "\(E^2_{0,4}\) maps surjectively on the stable group described in (2.15). The "\(E^2_{2,1}\)-term is \(\Lambda^2_\mathbb{Z}(\mathbb{R}^+)\) because the map \(H_2(O(2)) \to H_2(O(3))\) is surjective by Sah [18], and it is killed as usual by "\(d^3_{0,4}\). As indicated before Case 3, \(H_3(O(3)) \cong H_3(O(4))\) so that "\(d^2_{2,5}\) = 0 by comparison with (2.15).

We can also exhibit the relevant terms for \(G = SO^1(1, 4)\) acting on \(C^\text{gen.1}\):

\[
\begin{array}{cccc}
\text{"}E^2_{0,4} & \mathcal{P}(\mathbb{H}) & 0 & \downarrow \text{"}d^2 \\
0 & 0 & \text{"}E^2_{2,2} & \\
0 & 0 & \Lambda^2_\mathbb{Z}(\mathbb{R}^+) \sqcup \mathbb{Z}/2\mathbb{Z} & \\
\mathbb{Z} & 0 & \Lambda^2_\mathbb{Z}(\mathbb{R}^+) \sqcup H_2(SO(3)) \sqcup H_3(SO(3))
\end{array}
\]
The vanishing of the term $E_{1,3}^2$ arises from the fact that $\mathcal{P}_2$ is 2-divisible and is fixed by an orientation-reversing isometry of hyperbolic 2-space and the extension of this to an orientation-preserving isometry of hyperbolic 4-space must induce the negation map on the coefficient group $\mathbb{R}/\mathbb{Z}$. The group $E_{0,4}^d$ maps surjectively onto $E_{0,4}^d$ of (2.22). The map $d_{0,4}^2$ is a form of the Dehn invariant map in hyperbolic 4-space. Namely, a totally asymptotic hyperbolic 4-simplex is such that all the codimension 2 faces are isometric and can be identified with $(\infty, 0, 1)$. Each of them has an interior dihedral angle corresponding to an element of $\mathbb{R}/\mathbb{Z}$. The sum of all these angles is the codimension 2 Dehn invariant. It is also related to the volume of the totally asymptotic 4-simplex; see Sah [16, 17]. To make the relation with the scissors congruence group more precise, it would be better to use $C_*$ rather than $C_*^\text{gen.}$; compare Dupont and Sah [7]. In any case, $\mathbb{R}/\mathbb{Z} \otimes (\infty, 0, 1)$ is negated by the action of an orientation-reversing isometry $\rho$ of hyperbolic 4-space. This term indicates that $E_{0,4}^d$ is definitely larger than $E_{0,4}^d$ of (2.22). In contrast, $E_{2,2}^d$ of (2.23) and (2.22) are the same and equal the kernel of the homomorphism from $H_2(SO(2))$ to $H_3(SO(3))$. This kernel can be identified with the kernel of the surjective map from $H_2(U(1))$ to $H_2(SU(2))$ by lifting the groups to their double covering groups. Both summands of $E_{2,1}^d$ are killed by $d_{0,4}^2$ after $E_{1,2}^d$ is killed by $d_{0,4}^2$, $A_2^d(\mathbb{R}^+)$ is killed as usual. The summand $\mathbb{Z}/2\mathbb{Z}$ is related to the double covering group $Sp(1, 1)$ of $SO^1(1, 4)$. Namely, if we replace $SO^1(1, 4)$ by $Sp(1, 1)$, then (2.23) becomes

\[
E_{0,4}^d \quad 0 \quad "d^2"
\]

\[
\mathcal{P}(\mathbb{H}) \quad 0 \quad U(1) \otimes (\infty, 0, 1) \quad E_{2,2}^d \quad (2.24)
\]

\[
0 \quad 0 \quad A_2^d(\mathbb{R}^+) \quad Z \quad 0 \quad A_2^d(\mathbb{R}^+) \sqcup H_2(SU(2)) \quad H_3(SU(2))
\]

$U(1)$ now denotes the double covering group of $SO(2) \cong \mathbb{R}/\mathbb{Z}$. As explained after Remark 2.14, $H_3(SU(2)) \rightarrow H_3(SO(3))$ and $H_3(Sp(1, 1)) \rightarrow H_3(SO^1(1, 4))$ are both surjective with compatible kernel $\mathbb{Z}/4\mathbb{Z}$. The terms $\mathbb{R}/\mathbb{Z} \otimes (\infty, 0, 1)$ and $\mathbb{Z}/2\mathbb{Z}$ in (2.23) make up $U(1) \otimes (\infty, 0, 1)$ in terms of the double covering map. $U(1) \otimes (\infty, 0, 1)$ in (2.24) is killed by $d_{0,4}^2$ as before by the “Dehn invariant.”

3. Structure of $\mathcal{P}(\mathbb{C})/\text{im} \mathcal{P}_\mathbb{R}, \mathcal{P}(\mathbb{H})$, and $H_2(SU(2))$

The principal result in the present section is:

**Theorem 3.1.** Let $\lambda(\mathbb{R}^+) : \mathcal{P}(\mathbb{H}) \rightarrow A_2^d(\mathbb{R}^+)$ be the composition of $\lambda_{3d}$ in
followed by projection onto $A^2_2(\mathbb{R}^+)$. Then $\lambda(\mathbb{R}^+)$ is an isomorphism.

As a consequence, $H_2(SU(2)) \cong K_2(\mathbb{C})^+$ under the inclusion of $SU(2)$ into $SL(2, \mathbb{C})$ and both $i_*$ and $\Delta$ in (2.17) are surjective maps.

Along the way, various other results are proved.

We begin by recalling some results from Dupont and Sah [7, (5.12) through Theorem 5.14]. Let $\mathbb{C}(t)$ denote the field of rational functions in the variable $t$ over $\mathbb{C}$. Let $f(t) = a \prod (\alpha_i - t)^{d(i)}$, $g(t) = b \prod (\beta_j - t)^{e(j)}$, $\alpha_i$ distinct in $\mathbb{C}$, $\beta_j$ distinct in $\mathbb{C}$, $a, b \in \mathbb{C}$, and $d(i), e(j) \in \mathbb{Z}$. Define $f^* g$ in $P(\mathbb{C})$ as follows:

$$f^* g = \sum_i \alpha_i \beta_j [\alpha_i^{-1} \beta_j]$$

with the understanding that all meaningless symbols are set equal to 0 and the sum is 0 if either $f$ or $g$ lies in $\mathbb{C}$. In particular, (1.5), (1.7), (1.8), and (1.16) hold in the target group $P(\mathbb{C})$.

The main properties of this $P(\mathbb{C})$-valued product are:

$$f^* g$$ is bimultiplicative in $f, g$; $f^* f = 0$; and

$$f^* (1 - f) = [f(0)] - [f(\infty)].$$

We note that $f(0)$ and $f(\infty)$ have the usual meaning in terms of evaluation of rational functions and (1.16) is used whenever needed. As an application, we show:

**Proposition 3.4.** $P(\mathbb{C})$ is uniquely 2-divisible. (In fact, the same holds for $P(\mathbb{F})$, $\mathbb{F}$ is any algebraically closed field.)

**Proof.** As mentioned in (1.19), $P(\mathbb{C})$ is divisible through the distribution formula. For $z \in \mathbb{C}$, define $[z]/2 = [z^{1/2}] + [-z^{1/2}]$. Since $P(\mathbb{C}) \cong P_{\mathbb{C}}$, we only have to show that the map sending $[z]$ to $[z]/2$ respects the defining relation (1.5) for $P_{\mathbb{C}}$, $z \in \mathbb{C} - \{0, 1\}$. By (1.7), (1.8), and (1.16), this simply means that we have to show

$$[z_1]/2 - [z_2]/2 + [z_2/z_1]/2 - [(z_2 - 1)/(z_1 - 1)]/2$$

$$+ [(z_2^{-1} - 1)/(z_1^{-1} - 1)]/2 = 0$$

holds in $P(\mathbb{C})$ for $z_1, z_2 \in \mathbb{C} - \{0, 1\}$, $z_1 \neq z_2$. (3.5)

We begin by simplifying (3.5) a bit. We set $z_i = w_i^2$, $w_i \in \mathbb{C}$. Then definition of $[z]/2$ makes it clear that the choice of $w_i$ or $-w_i$ makes no difference. We then consider the following element $f_2(t)$ of $\mathbb{C}(t)$:

$$f_2(t) = -w_1(t - \alpha_1)(t - \alpha_2)/(t - 1)(t + \alpha_2),$$

$$\alpha_1 = w_2/w_1, \alpha_2 = (1 + w_2)/(1 - w_1).$$

(3.6)
Direct computation shows that

\[ 1 - f_2(t) \text{ has simple zeroes at } \pm (w_2^2 - 1)^{1/2}/(w_1^2 - 1)^{1/2} \]

and simple poles at 1 and \(-\alpha_2\).

By (3.3) and \([-1] = 2 \cdot ([i] + [-i]) = 0\) in \(\mathcal{P}(\mathbb{C})\), we can compute \(f_2^{-1} (1 - f_2)\) through (3.2) to obtain

\[
\begin{align*}
[w_2] - [-w_1] &= \left[\frac{(z^{-1} - 1)/(z_1^{-1} - 1)}{2} - \frac{(z_2 - 1)/(z_1 - 1)}{2}\right] \\
&\quad - \left[w_1/w_2\right] - \left[-\frac{(w_1 - 1)/(w_2 + 1)}{2}\right] \\
&\quad - \left[-\frac{(w_2^{-1} + 1)/(w_1^{-1} - 1)}{2}\right].
\end{align*}
\]

(3.8)

Setting \(\alpha = -w_2, \beta = w_1\) in the defining relation (1.5) and adding the result to (3.8), we obtain (3.5). \(\blacksquare\)

**Remark 3.9.** A more complicated, but similar, argument can be carried out to show the unique 3-divisibility of \(\mathcal{P}(\mathbb{C})\). The unique 3-divisibility of \(\mathcal{P}(\mathbb{C})\) will not be needed for the proof of Theorem 3.1. (\(\mathcal{P}(\mathbb{C})\) is in fact uniquely divisible; see Sah [19].)

With Proposition 3.4 at hand, \(\mathcal{P}(\mathbb{C})\) is the direct sum of the \(\pm\)-eigenspaces under the action of complex conjugation. In particular, \(\mathcal{P}(\mathbb{C}, \ast)\) is just the \(+\)-eigenspace and contains the image of \(\mathcal{P}(\mathbb{R})\). As mentioned in Remark 2.14, \(\lambda_\mathbb{C}\) sends the generator \([z]\) of \(\mathcal{P}_\mathbb{C}\) onto \(2 \cdot (z \wedge (1 - z))\) of the \(\mathbb{Q}\)-vector space \(\Lambda_2^\mathbb{C}(\mathbb{C}^\times)\). Under the polar decomposition of \(\mathbb{C}^\times, \Lambda_2^\mathbb{C}(\mathbb{C}^\times)\) becomes the direct sum of \(\Lambda_2^\mathbb{R}(\mathbb{R}^+) \sqcup \Lambda_2^\mathbb{R}(U(1))\) and \(\mathbb{R}^+ \otimes \mathbb{Z} U(1)\). They are respectively the \(+\) and \(-\) eigenspaces under complex conjugation. \(\lambda_\mathbb{C}(\ast)\) is just \(\lambda_\mathbb{C}\) followed by the projection onto the \(+\)-eigenspace.

By the commutativity in (2.17), we define a map as follows:

\[ \lambda(\mathbb{R}/\mathbb{Z}): \mathcal{P}(\mathbb{C}, \ast)/\text{im } \mathcal{P}_\mathbb{R} \to \Lambda_2^\mathbb{R}(\mathbb{R}/\mathbb{Z}) \text{ is the composition of } \lambda_\mathbb{C}(\ast) \text{ followed by the projection into the group } \Lambda_2^\mathbb{R}(\mathbb{R}/\mathbb{Z}), \]

where \(U(1)\) is identified with \(\mathbb{R}/\mathbb{Z}\) and we have ignored a power of 2. More precisely, let \(z \in \mathbb{C} - \{0, 1\}\) and write \(z = r \cdot \exp(i\pi \alpha)\) and \(1 - z = s \cdot \exp(i\pi \beta)\), \(r, s \in \mathbb{R}^+, \alpha, \beta \in \mathbb{R}\) mod \(2\mathbb{Z}\). Then \(\lambda(\mathbb{R}/\mathbb{Z})[z] = \alpha \wedge \beta \in \Lambda_2^\mathbb{R}(\mathbb{R}/\mathbb{Z}).\) (3.10)

**Proposition 3.11.** The map \(\lambda(\mathbb{R}/\mathbb{Z})\) defined in (3.10) is bijective.

**Proof.** We will construct the inverse map in the following manner: Define \(v: \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathcal{P}(\mathbb{C}, \ast)/\text{im } \mathcal{P}_\mathbb{R}\) according to the following rule:
If \( \alpha, \beta \in \mathbb{R} \) are not congruent mod \( \mathbb{Z} \), then \( v(\alpha, \beta) \) is the coset of \([z]\), where \( z \in \mathbb{C} \) is the unique point of intersection of the lines in the complex plane through 0 and 1 making the respective angles \( \pi \alpha \) and \( \pi \beta \) relative to the positive x-axis. Otherwise, \( v(\alpha, \beta) = 0 \). In all cases, (1.7), (1.8), and (1.16) are in force. (3.12)

It is now straightforward to check that

\[
v(\beta, \alpha) = [1 - z] \text{ or } 0, \text{ respectively, in (3.12) so that } v \text{ is skew-symmetric.}
\]

(3.13)

We recall that \( \mathcal{P}(\mathbb{C}, \ast) \) is generated by the elements \([z]\) and \([z] = [\bar{z}]\) holds in \( \mathcal{P}(\mathbb{C}, \ast) \), \( z \in \mathbb{C} \). Once we show that \( v \) is biadditive, the map it induces from \( M_{2}(\mathbb{R}/\mathbb{Z}) \) to \( \mathcal{P}(\mathbb{C}, \ast)/\text{im } \mathcal{P}_{\mathbb{R}} \) will evidently be the inverse to \( \lambda(\mathbb{R}/\mathbb{Z}) \). In view of (3.13), it is enough to show additivity of \( v \) in \( \beta \).

Suppose we are given \( \alpha, \beta, \beta_1, \beta_2 \) with \( \beta = \beta_1 + \beta_2 \). Set \( z = v(\alpha, \beta), \ z_i = v(\alpha, \beta_i), \ i = 1, 2, \) as in (3.12) with the obvious abuse of notation. The cases where \( z_1z_2 = 0 \) and \( z = 1 \) are trivial. The case of \( z = 0 \) is similar and simpler. We therefore assume \( z_1z_2 \neq 0 \) and \( z \neq 0 \) or 1. It follows that \( z = r_1z_i, \ i = 1, 2, \) and \( r_i \in \mathbb{R}^\ast \) as well as

\[
(z - r_1) \cdot (z - r_2) \cdot (z - 1)^{-1} = r \in \mathbb{R}^\ast. 
\]

(3.14)

The required additivity condition is

\[
[z] \equiv [z_1] + [z_2] \mod \mathcal{P}_{\mathbb{R}}. 
\]

(3.15)

As in the proof of Proposition 3.4, consider the following element of \( \mathcal{C}(t) \):

\[
f(t) = (t - r_1)(t - r_2) \cdot (r(t - 1))^{-1}. 
\]

(3.16)

\( f(t) \) has real coefficients and \( 1 - f(t) \) has simple zeroes at \( z, \bar{z}, \) and a simple pole at 1. By (3.2), (3.3), we compute \( f \ast (1 - f) \) and obtain

\[
[z/r_1] + [\bar{z}/r_1] + [z/r_2] + [\bar{z}/r_2] - [z] - [\bar{z}] \in \text{im } \mathcal{P}_{\mathbb{R}}. 
\]

(3.17)

If we combine Proposition 3.4 and (1.20), then \( \mathcal{P}(\mathbb{C}, \ast)/\text{im } \mathcal{P}_{\mathbb{R}} \) is uniquely 2-divisible. Condition (3.15) follows from (3.17) by reading it in \( \mathcal{P}(\mathbb{C}, \ast)/\text{im } \mathcal{P}_{\mathbb{R}} \) and dividing the image by 2. \( \blacksquare \)

**Corollary 3.18.** The map \( j_\alpha : H_3(SL(2, \mathbb{R})) \to H_0(\langle \sigma \rangle, H_3(SL(2, \mathbb{C}))) \) defined in (2.17) is surjective.

**Proof.** This is a “diagram chase.” The term \( \mathbb{Q}/\mathbb{Z} \) in (2.17) arises from the third homology of the finite cyclic subgroups of \( SL(2, \mathbb{C}) \). These can be
conjugated uniformly into $SO(2) \subset SL(2, \mathbb{R})$. According to Parry and Sah [15], these are just the "rational multiples" of $[2]$ in $\mathcal{P}_r$. Since $[2] + [-1] = 0 = [-1]$ holds in $\mathcal{P}_c$ and the rational multiples correspond to $[z]$ with $z$ ranging over the roots of 1 in $\mathbb{C}$, all these rational multiples lie in the kernel of the map from $\mathcal{P}_r$ to $\mathcal{P}_c = \mathcal{P}(\mathbb{C})$, hence to $\mathcal{P}(\mathbb{C}, *)$, through the distribution formula (1.19). If we terminate the top two rows of (2.17) with $\text{im } \lambda_{r}$ and $\text{im } \lambda_{c}(\ast)$ and replace $H_0(\langle\sigma\rangle, H_2(SL(2, \mathbb{C})))$ by its quotient modulo the trivial torsion $\mathbb{Q}/\mathbb{Z}$, the desired surjectivity of $j_R$ then follows from Proposition 3.11, diagram (2.17), and the snake lemma.

We are now ready to tackle Theorem 3.1. We begin by deriving the last two assertions of Theorem 3.1 from the first assertion of Theorem 3.1. The surjectivity of $\lambda$ and the isomorphism $H_3(SU(2)) \cong K_2(\mathbb{C})^+$ are immediate corollaries of the exactness of the last row of (2.17) and the bijectivity of $\lambda(\mathbb{R}^+)$. As discussed in Remark 2.18, $\lambda$ can be lifted to $\lambda_2$. This gives the surjectivity of $j_{\mathbb{C}}$.

As in (3.10), $\lambda(\mathbb{R}^+)$ is defined by sending $[z] = [\bar{z}]$ of $\mathcal{P}(\mathbb{H})$ onto $r \wedge s$. If we ignore 2-power factors, then $\lambda(\mathbb{R}^+)$ extends $\lambda_{r}$ in an obvious way. To construct the inverse map to $\lambda(\mathbb{R}^+)$, we imitate the proof of Proposition 3.11 and define a map $\mu: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathcal{P}(\mathbb{H})$. Unlike the proof of Proposition 3.11, $\mu$ is defined only "locally" as follows:

Let $r, s \in \mathbb{R}^+$ so that $r + s > 1 > |r - s|$. Let $z$ be the unique point in the upper half complex plane with $|z| = r$ and $|1 - z| = s$. Set $\mu(r, s) = [z] = [\bar{z}]$ in $\mathcal{P}(\mathbb{H})$. (3.19)

We note that the restrictions on $r$ and $s$ in (3.19) simply mean that there is a Euclidean triangle with sides of length 1, $r$, and $s$. A similar construction was used in Sah and Wagoner [20]. It is now immediate that

$$\mu(s, r) = [1 - \bar{z}] = [1 - z] \text{ in } \mathcal{P}(\mathbb{H}) \text{ so that } \mu \text{ is skew-symmetric}$$

in accordance with (1.7), (1.8), and (1.16). (3.20)

The remainder of the proof is divided into two steps.

**Step 1.** We first show that $\mu$ induces a "local" homomorphism from $\lambda^2(\mathbb{R}^+) \rightarrow \mathcal{P}(\mathbb{H})$. In view of (3.19) and (3.20), this means showing additivity in $s$ in a suitable neighborhood of $(1, 1)$ in $\mathbb{R}^+ \times \mathbb{R}^+$. Unique divisibility in $\mathbb{R}^+$ and the Archimedean property of $\mathbb{R}$ then imply that $\mu$ induces a homomorphism from $\lambda^2(\mathbb{R}^+) \rightarrow \mathcal{P}(\mathbb{H})$.

**Step 2.** We then show that $\mathcal{P}(\mathbb{H})$ can be generated by $[z]$ with $z$ ranging over a corresponding neighborhood of $\exp(2\pi i/6)$. Once this is accomplished, $\lambda(\mathbb{R}^+)$ and the homomorphism induced by $\mu$ are mutually
inverses of each other for elements \([z]\) with \(z\) ranging over such a neighborhood of \(\exp(2\pi i/6)\) and \((r, s)\) over a corresponding neighborhood of \((1, 1)\) in \(\mathbb{R}^+ \times \mathbb{R}^+\), where \(r, s,\) and \(z\) are related by (3.19). This then implies that \(\lambda(\mathbb{R}^+)\) is bijective with inverse induced by \(\mu\).

Step 1 is accomplished by the following result. For later purposes we work with a neighborhood not only of \(\exp(2\pi i/6)\) but of the intersection of the unit circle and the open first quadrant of the complex plane.

**Lemma 3.21.** There is an open subset \(U\) of \(\mathbb{C}\) containing \(\{\exp(i\theta)|0 < \theta < \pi/2\}\) and satisfying the following statement:

Given \(z_1, z_2 \in U\) and \(|z_1| = |z_2| = r\), then there exists \(z_3 \in \mathbb{C}\)

with the properties

\[
\begin{align*}
(a) & \quad |z_3| = r, |1 - z_3| = |1 - z_1| \cdot |1 - z_2|; \\
(b) & \quad [z_1] + [z_2] = [z_3] \text{ holds in } \mathcal{P}(\mathbb{H}).
\end{align*}
\]

We note that once \(z_3\) exists, then it is unique up to complex conjugation because by (a) it lies on the intersection of the circles centered at 0 and 1 with respective radii \(r\) and \(1 - z_1 \cdot |1 - z_2|\).

**Proof:** It suffices to prove for every \(\delta \in \mathbb{R}\) with \(0 < \delta < \frac{1}{2}\) that there exists an \(\varepsilon \in \mathbb{R}^+\) depending on \(\delta\) such that (3.22) holds with \(U\) replaced by \(U_\delta = \{z \in \mathbb{C}|\delta < \Re(z) < 1 - \delta\ \text{ and } (1 + \varepsilon)^{-1} < |z| < 1 + \varepsilon\}\). \(U\) can then be taken to be \(\bigcup U_\delta\). We prepare for this by making two observations.

First, we consider (1.5) with \(F = \mathbb{H}, a=(1-w_1)^{-1}, \beta=1-w_2\) for elements \(w_1, w_2 \in \mathbb{H} - \{0, 1\}\). By (1.7), (1.8), and (1.16), we rewrite (1.5) as

\[
[w_1] + [w_2] + [(1 - w_1)(1 - w_2)] + [w_1^{-1}(1 - (1 - w_1)(1 - w_2))] \\
+ [w_2^{-1}(1 - (1 - w_2)(1 - w_1))] = 0. \quad (3.23)
\]

Second,

\[
\text{If } w \in \mathbb{H} \text{ with } |w| = 1, \text{ then } [w] = 0 \text{ in } \mathcal{P}(\mathbb{H}). \quad (3.24)
\]

To see this, note that \(|w| = 1\) implies that \(w^* = w^{-1}\). By (1.4), (1.7), (1.8), (1.16), (1.21), and (1.23), we may assume \(w \in \mathbb{C}\) and conclude that \(2[w] = 0\) holds in the uniquely 2-divisible group \(\mathcal{P}(\mathbb{C}, \ast)\). Since \(\mathcal{P}(\mathbb{H})\) is a quotient group of \(\mathcal{P}(\mathbb{C}, \ast)\), we obtain (3.24).

Now suppose we are given \(\delta \in \mathbb{R}\) with \(0 < \delta < \frac{1}{2}\). We wish to prove the following.

There exists \(\varepsilon \in \mathbb{R}^+\) such that given \(z_1, z_2 \in U_\delta\) with \(|z_1| = |z_2| = r\), then there exists \(w_i \in \mathbb{H}\) conjugate in \(\mathbb{H}\) to \(z_i, i = 1, 2\), for which \(|1 - (1 - w_1)(1 - w_2)| = r\). \quad (3.25)
Once (3.25) is proven, we let \( z_1, z_2 \in \mathbb{C} \) be conjugate to \( 1 - (1 - w_1)(1 - w_2) \) in \( \mathbb{H} \). Then (a) follows easily. Furthermore, the last two terms in (3.23) are 0 by (3.24), and so (b) follows easily from (1.4) and (1.8).

To prove (3.25), suppose we are given \( z_1, z_2 \in \mathbb{C} \) with \( |z_1| = |z_2| = r \) and \( \delta < \text{Re}(z_i) < 1 - \delta \) for \( i = 1, 2 \). Let 1, i, j, k denote the usual quaternion units. Let \( w_1 = z_1 = a + bi \) with \( a, b \in \mathbb{R} \), and let \( c = \text{Re}(z_2) \). We seek \( d, e \in \mathbb{R} \) so that \( w_2 = c + di + ej \) is conjugate in \( \mathbb{H} \) to \( z_2 \) and \( |1 - (1 - w_1)(1 - w_2)| = r \). Computation shows that

\[
|1 - (1 - w_1)(1 - w_2)|^2 = 2(ac + bd) + 2(1 - a - c)r^2 + r^4.
\]

Equating this with \( r^2 \) and solving for \( d \) gives

\[ d = -(r^4 + (1 - 2a - 2c)r^2 + 2ac)/2(r^2 - a^2)^{1/2}. \]  

(3.26)

It remains for us to show that we can choose \( e \) so that \( c^2 + d^2 + e^2 = r^2 \). Equivalently, we need \( |d| \leq (r^2 - c^2)^{1/2} \). Combining this with (3.26) shows that it suffices for us to choose \( e \) so that \((1 + \varepsilon)^{-1} < r < (1 + \varepsilon)\) implies

\[
2(r^2 - a^2)^{1/2}(r^2 - c^2)^{1/2} \geq |r^4 + (1 - 2a - 2c)r^2 + 2ac|.
\]

When \( r = 1 \), the left-hand side is strictly greater than the right-hand side. The result therefore follows from uniform continuity for \( (r, a, c) \) lying in a neighborhood of the compact set \( \{1\} \times [\delta, 1 - \delta] \times [\delta, 1 - \delta] \subset \mathbb{R}^3 \).

Step 2 is accomplished by the following result:

**Lemma 3.27.** Let \( U \) be as in Lemma 3.21. Then \( \mathcal{P}(\mathbb{H}) \) is generated by \( \{[z] \mid z \in U\} \).

**Proof.** Let \( A \) be the subgroup of \( \mathcal{P}(\mathbb{H}) \) generated by \( \{[z] \mid z \in U\} \). Evidently, there are intervals \( I_1 \) and \( I_2 \) about \( \pi/3 \) and \( - \pi/3 \), respectively, in \( \mathbb{R} \) such that for every \( (\alpha, \beta) \in I_1 \times I_2 \) there exists \( z \) in \( U \) with

\[ \text{arg}(z) = \alpha \quad \text{and} \quad \text{arg}(1 - z) = \beta. \]

Recalling the isomorphism \( \lambda(\mathbb{R}/\mathbb{Z}) : \mathcal{P}(\mathbb{C}, \ast)/\text{im} \mathcal{R}_\mathbb{R} \cong A_2^2(\mathbb{R}/\mathbb{Z}) \) of Proposition 3.11 and neglecting 2-power factors, we see that

\[ \lambda(\mathbb{R}/\mathbb{Z})([z]) = (\text{arg}(z)/2\pi) \land (\text{arg}(1 - z)/2\pi) \]

and

\[ A_2^2(\mathbb{R}/\mathbb{Z}) \] is generated by \( \{\lambda(\mathbb{R}/\mathbb{Z})([z]) \mid z \in U\} \).

In particular, \( \mathcal{P}(\mathbb{H}) = A + \text{im} \mathcal{R}_\mathbb{R} \). It is therefore sufficient to show that
\{[r] | r \in \mathbb{R}\} \subset A. Since \([1 + r^2] = [-r^2]\) holds in \(P(\mathbb{C})\) through (1.8) and (1.16), hence holds in the quotient \(P(\mathbb{H})\), it is enough to show that
\[
\{[-a^2] | a \in \mathbb{R}\} \subset A. \quad (3.28)
\]

As the elements \(z_1, z_2\) of Lemma 3.21 range over a neighborhood of \(\exp(2\pi i/8)\), \(z_3\) ranges over neighborhood of \(i = \exp(2\pi i/4)\). This and (1.21) imply that there is an open subset \(V\) of \(\mathbb{C}\) with
\[
\{\exp(i\theta) | 0 < |\theta| \leq \pi/2\} \subset V \quad \text{and} \quad \{[z] | z \in V\} \subset A.
\]

We now consider the relation (1.5) for pure quaternions \(\alpha = ai, \beta = bj\) in \(\mathbb{H}, a, b \in \mathbb{R}^+\). By (1.4) and (1.21), we obtain
\[
[ai] - [bi] + [(b/a)i] - [z_1] + [z_2] = 0 \quad \text{in} \quad P(\mathbb{H}),
\]
where \(z_1, z_2 \in \mathbb{C}\) are determined up to complex conjugation by the conditions
\[
|z_1| = (1 + b^2)^{1/2}/(1 + a^2)^{1/2}, \quad \text{Re}(z_1) = (1 + a^2)^{-1};
\]
\[
|z_2| = a(1 + b^2)^{1/2}/(b(1 + a^2)^{1/2}), \quad \text{Re}(z_2) = a^2/(1 + a^2).
\]

For a fixed \(a \in \mathbb{R}^+\), there is an open interval \(I_a \subset \mathbb{R}^+\) about \(a\) such that \(b \in I_a\) implies that \(b/a, |z_1|, |z_2|\) are so close to 1 that \((b/a)i, z_1, z_2 \in V\).

It follows that
\[
[ai] \equiv [bi] \mod A \text{ holds for } b \in I_a.
\]

Since \(i \in V\), a straightforward connectedness argument then implies that
\[
[ai] \in A \text{ holds for every } a \in \mathbb{R}^+.
\]

From (1.19) with \(n = 2\), we have \([-a^2] = 2[ai] + 2[-ai] = 4[ai]\) in \(P(\mathbb{H})\). This gives the desired (3.28).

This completes the proof of Theorem 3.1. It should be noted that the argument used in proving the preceding results is similar in spirit to the argument used by Jessen [11] in simplifying the proof of the theorem of Dehn and Sydler. Parts of our arguments can be presented geometrically in terms of scissors congruences of totally asymptotic 3-dimensional hyperbolic polyhedra. The main point is that the relation (1.5) is much stronger than a scissors congruence because the five distinct points on \(\partial \mathcal{H}^5\) leading to (1.5) usually span a totally asymptotic hyperbolic 4 simplex, while a scissor congruence arises when these five points lie on \(\partial \mathcal{H}^3\).
4. RELATIONS WITH SCHUR MULTIPLIERS

We can now improve the result in Sah [18] on Schur multipliers.

**Theorem 4.1.** Let $G$ be a simple, connected, simply-connected nonabelian Lie group such that its Lie algebra is absolutely simple and not among 10 exceptional ones of type $E$ and $F$ (3 compact and 7 noncompact and all are not $\mathbb{R}$-split). Let $\rho: G \to SL(n, \mathbb{C})$ denote any nontrivial Lie group homomorphism. Then $\rho_*: H_2(G) \to H_2(G) \to H_2(SL(n, \mathbb{C}))$ is injective and the image is $K_2(\mathbb{C})^+$, where $H_2(SL(n, \mathbb{C})) \cong K_2(\mathbb{C})$ for $n \geq 2$. In particular, $H_2(SL(n, \mathbb{H})) \to H_2(SL(n + 1, \mathbb{H}))$ is bijective for $n \geq 1$ and $K_2(\mathbb{H}) \cong K_2(\mathbb{C})^+$.

*Proof.* In the case of complex Lie groups and complex Lie group homomorphisms, $\rho_*$ is known to be bijective and we can include all the exceptional groups. The point is that all the groups are then algebraic and $\rho$ can be described by character theory. The induced map $\rho_*$ can then be described on the level of $K_2(\mathbb{C})$ as multiplication by a suitable positive integer in terms of $K_2$-symbols; see Milnor [13] or Sah and Wagoner [20]. The bijectivity of $\rho_*$ follows from the theorem of Bass and Tate [3], which asserts that $K_2(\mathbb{C})$ is a $\mathbb{Q}$-vector space. See also [22].

In the present case, $\rho_*$ factors through the universal complexification homomorphism. As a consequence, we can replace $\rho$ by the universal complexification homomorphism from $G$ to $G_{\mathbb{C}}$. We note that the complexification homomorphism may have a discrete nontrivial kernel. When $G$ is noncompact, it was shown in Sah [18] that $H_2(G)$ is isomorphic to $K_2(\mathbb{C})^+$ under the induced map. When $G$ is compact, the result in Sah [18] showed that $G$ can be taken to be $SU(2)$. The desired assertion then follows from Theorem 3.1. \[\square\]

**Remark 4.2.** Outside of the 10 exceptional cases, Theorem 4.1 was first posed by Milnor as a question in an oral discussion with one of us. The last assertion of Theorem 4.1 improves the result of Alperin and Dennis [1] in the case of a quaternion division algebra over any real closed field. This amounts to a “proof analysis.” In any case, there is no problem in replacing $\mathbb{R}$ by any of its real closed subfields. The most interesting case is that of the algebraic closure $\mathbb{R}^{alg} = \mathbb{R} \cap \mathbb{Q}$ of $\mathbb{Q}$ in $\mathbb{R}$. In this case, $K_2(\mathbb{Q}) = 0$ follows from the deep results of Garland [8], which imply that it is torsion, and the theorem of Bass and Tate [3], which implies that it is a $\mathbb{Q}$-vector space. As noted by Cheeger [4], the nonvanishing of $H_3(SU(2))$ depends on the algebraic points of $SU(2)$. Similar examples can be found in Parry and Sah [15] for $H_3(SL(2, \mathbb{R}))$ and $H_3(SL(2, \mathbb{C}))$. Cheeger’s observation is based on a rigidity argument. As a result, if we replace $\mathbb{C}$ by $\mathbb{Q}$, then $H_3(SU(2, \mathbb{Q}))$ is in fact the first nonzero positive dimension Eilenberg-
MacLane integral homology group of \( SU(2, \mathbb{Q}) \). This gives a rough answer to a question of Harris [9] that formalizes Cheeger's observation. In this connection, we note that \( H_3(SL(2, \mathbb{Q})) \) maps injectively into \( H_3(SL(2, \mathbb{C})) \) by Hilbert's Nullstellensatz. The surjectivity of this map is sometimes called the rigidity conjecture for \( H_3(SL(2, \mathbb{C})) \). This question is still open.

We note that Theorem 3.1 results in a statement about the Schur multiplier of the compact group \( SO(3) \) and the noncompact group \( SO^1(1, 3) \). From this point of view, \( SO(3) \) is the stability group of a point in the symmetric space \( SO^1(1, 3)/SO(3) \) which is just the real hyperbolic 3-space. The argument we used involves the boundary of the hyperbolic 5-space. We will now examine our result from a more direct viewpoint. Consider the following commutative diagram of maps arising from the obvious inclusions of groups:

\[
\begin{array}{ccc}
H_i(O(n)) & \xrightarrow{\gamma(i,n)} & H_i(O^1(1, n)) \\
\downarrow{\alpha(i,n)} & & \downarrow{\beta(i,n)} \\
H_i(O(n+1)) & \xrightarrow{\gamma(i,n+1)} & H_i(O^1(1, n+1))
\end{array}
\]

**Theorem 4.4.** In the diagram (4.3), we have:

(a) \( \alpha(i, n) \) is surjective for \( i \leq n \) and bijective for \( i < n \); however, both \( \alpha(1, 1) \) and \( \alpha(3, 3) \) are bijective.

(b) \( \beta(i, n) \) is surjective for \( i \leq n \) and bijective for \( i < n \); however, both \( \beta(1, 1) \) and \( \beta(2, 2) \) are bijective.

(c) \( \gamma(1, n) \) is bijective for \( 1 \leq n \); \( \gamma(2, n) \) is surjective for \( 2 \leq n \) and bijective for \( 2 < n \); \( \gamma(3, n) \) is surjective for \( 3 \leq n \).

(d) the surjectivity of \( \gamma(3, 3) \) in (c) is equivalent to the surjectivity of \( \Delta E : H_3(SU(2)) \rightarrow H_0(\langle \sigma \rangle, H_3(SL(2, \mathbb{C}))) \).

**Proof.** (a) The first assertion is known from Sah [18]. The bijectivity of \( \alpha(1, 1) \) is trivial. The bijectivity of \( \alpha(3, 3) \) was noted at the end of Remark 2.14. A careful argument uses (1) Künneth's formula for \( H_3(O(3)) \) and \( H_3(O(5)) \), (2) bijectivity of \( \alpha(3, 4) \), and (3) Hochschild–Serre spectral sequences applied to semidirect products \( SO(4) = SU(2) \cdot SO(3) \) and \( O(4) = SO(4) \cdot O(1) \). We omit further details.

(b) The first assertion can be derived from Sah [18]. The point is that \( O(1, n) = O^1(1, n) \times \langle \pm I_{n+1} \rangle \) so that stability results for \( O^1(1, n) \) can be derived from the stability statements for \( O(1, n) \) by means of Künneth's theorem. The bijectivity of \( \beta(1, 1) \) is trivial. The bijectivity of \( \beta(2, 2) \) is well known and is used in Sah [18].

(c) and (d) The bijectivity of \( \gamma(1, n) \) is trivial. The assertions about \( \gamma(2, n) \) follow from the stability results of Sah [18] and Theorem 3.1. In the
case of $\gamma(3, n)$, the commutativity of (4.3) and (b) immediately imply that the critical case is that of $\gamma(3, 3)$. The argument involving universal coverings described after Remark 2.14 shows that the surjectivity of $\Lambda_C$ is equivalent with the surjectivity of $H_3(SO(3)) \to H_0(\langle \sigma \rangle, H_3(SO^1(1, 3)))$. Another Hochschild-Serre spectral sequence argument shows that the latter surjectivity is equivalent with the surjectivity of $\gamma(3, 3)$. This reduces the proof of (c) and (d) to showing the surjectivity of $\gamma(3, 3)$.

As in Sah [18], we consider the normalized Eilenberg-MacLane chain complex $C_\ast(\mathcal{H}^n) = \ast C_\ast(1, n, \mathbb{R})$ based on the model $S^\ast_\ast(1, n, \mathbb{R})$ of real hyperbolic space $\mathcal{H}^n$. Let $G$ denote any group that maps onto $O^1(1, n)$. By using the existence of a midpoint of a geodesic line segment in $\mathcal{H}^n$, $n > 1$, it was noted in Sah [18] that $C_\ast(\mathcal{H}^n) \otimes G \mathbb{Z}$ is 1-acyclic with augmentation $\mathbb{Z}$. In the transposed spectral sequence associated to $C_\ast(\mathcal{H}^n)$ and $G = O^1(1, n)$, $\varepsilon^{1,0}_i = H_i(O(n))$ and $\gamma(i, n)$ is just the edge homomorphism starting from $\varepsilon^{1,0}_i$. For $i = 1 \leq n$, the maps of (4.3) consist of bijections of $\mathbb{Z}/2\mathbb{Z}$. This and the 1-acyclicity result give us information about $\varepsilon^{1,0}_i$. With this at our disposal, we immediately have:

Let $n \geq 2$. Then $C_\ast(\mathcal{H}^n) \otimes G \mathbb{Z}$ is 2-acyclic if and only if $\gamma(2, n)$ in (4.3) is surjective. \hfill (4.5)

As indicated in (c), $\gamma(2, n)$ is surjective for $n \geq 2$. It follows that:

Let $G = O^1(1, n)$ and let $n \geq i$. Suppose that $i \leq 2$. Then $C_\ast(\mathcal{H}^n) \otimes G \mathbb{Z}$ is $i$-acyclic with augmentation $\mathbb{Z}$. \hfill (4.6)

A direct geometric proof of the 2-acyclicity assertion of (4.5) can be given. It is somewhat similar to the arguments used in Sah [18]. The idea is to show that a 2-cycle is homologous to one that is made up from 2-cells that have circumcenters and then use the existence of reflections in $O^1(1, 2)$ to kill off these 2-cycles.

As noted already, $\gamma(2, n)$ is actually bijective when $n > 2$. With this at hand, we use (4.6) and let $i = 3$ in (4.3) and deduce the following analogue of (4.5):

Let $n \geq 3$. Then $C_\ast(\mathcal{H}^n) \otimes G \mathbb{Z}$ is 3-acyclic if and only if $\gamma(3, n)$ is surjective. \hfill (4.7)

The critical case of $\gamma(3, 3)$ can be seen by looking at (2.22). By Theorem 3.1 and the use of $C_\ast(\partial \mathcal{H}^4)$, we obtain $\varepsilon^{4,0}_{0,3} = 0$ and $\varepsilon^{4,1}_{2,1} = 0$ and conclude the surjectivity of the following map

$\gamma(3, 4) : \alpha(3, 3). H_3(O(3)) \to H_3(O^4(1, 4)) \cong H_3(O^1(1, 5))$. 


This gives us the surjectivity of $\gamma(3, 4)$ so that $C_\ast(\mathbb{H}^n) \otimes_G \mathbb{Z}$ is 3-acyclic for $n \geq 4$. At this point, the Boot–Strapping Lemma stated in Sah [18] (only sparingly mentioned because its use has been superceded by other arguments) now takes over in reverse to give the 3-acyclicity for $n \geq 3$. To be precise, $C_\ast(\mathbb{H}^3) \otimes_G \mathbb{Z}$ can be viewed as a subcomplex of $C_\ast(\mathbb{H}^4) \otimes_G \mathbb{Z}$ through Witt's theorem (the two appearances of $G$ are different; in each case, it denotes the corresponding group $O'(1, n)$). The quotient complex begins in degree 4 and is spanned by the independent 4-cells. These correspond to geodesic hyperbolic 4-simplices with an ordering of the vertices. Each of these 4-cells becomes a boundary in the quotient complex through the inscribed center construction so that the quotient complex is 4-acyclic. An examination of the long homology sequence (as in the Boot–Strapping Lemma) shows that the 3-acyclicity of $C_\ast(\mathbb{H}^3) \otimes_G \mathbb{Z}$ follows from the 3-acyclicity of $C_\ast(\mathbb{H}^4) \otimes_G \mathbb{Z}$. We have completed the proof of the surjectivity of $\gamma(3, 3)$, hence of (c) and (d).

Remark 4.8. It should be evident that the preceding argument is roundabout. In an earlier draft, we sketched a geometric proof of (4.6). Attempts by us to give a geometric argument to prove the 3-acyclicity assertion run into a messy bookkeeping problem. Such a direct proof would give a proof of the bijectivity between $H_3(SU(2))$ and $K_3(\mathbb{C})^+$. In the meantime we found the present argument which gave us more information. The present approach is really geometric. The bookkeeping problem is solved through the use of the identity (1.5). In a careful reading of a draft of the present work, the referee found a geometric argument to prove the 3-acyclicity assertion. It involves an analytic volume estimate as well as a number of clever combinatorial topology arguments in dimension 3. The combinatorial arguments can be systematically absorbed through the use of spectral sequences (and appropriate filtrations) so that the entire argument can be viewed as a “step-by-step skeletal modification.” Unfortunately, the argument does not yield any definite result beyond dimension 3. We therefore pose the following question:

**Problem 4.9.** Let $G = O'(1, n), n > 3$. Is $C_\ast(\mathbb{H}^n) \otimes_G \mathbb{Z}$ n-acyclic?

An affirmative solution for $n = 4$ would determine $\ker A$ in (2.17) as $\mathbb{Z}/2\mathbb{Z}$ and imply that the stabilization inclusion of $SL(1, \mathbb{H})$ into $SL(2, \mathbb{H})$ (distinct from $A$) induces $H_3(SL(1, \mathbb{H})) \cong H_3(SL(2, \mathbb{H}))$. In Sah [19], $j_R$ will be shown to be bijective while $j_C$ will be shown to have kernel $\mathbb{Z}/2\mathbb{Z}$. These depend on the deep results of Suslin [23–25]. For ease of future references, we collect the various assertions about the maps in (2.17).

**Theorem 4.10.** In diagram (2.17), the maps $j_R$, $j_C$, $A$, and $A_C$ are all surjective. Moreover, we have:
(a) $\ker \{ \mathcal{P}_n \to \mathcal{P}(C, \ast) \}$ contains the trivial torsion $\mathbb{Q}/\mathbb{Z}$ which consists of all the "rational multiples" of the element $[2]$ in $\mathcal{P}_n$.

(b) $\Delta$ is injective on the trivial torsion $\mathbb{Q}/\mathbb{Z}$ of $H_3(SU(2))$.

(c) $\ker \Delta$ contains all the elements of order 2 in $H_3(SU(2))$ so that its order is at least 2.

(d) $\ker j_C$ has order at least 2.

Proof. We prove (c) and (d) only. The other assertions are just repetitions. Here $SU(2) = \{ \text{diag}(q, q) \mid q \in \text{Sp}(1) \}$. Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{H})$ exchanges the two $\text{Sp}(1)$ factors in the diagonal subgroup $\text{Sp}(1) \times \text{Sp}(1)$ of $SL(2, \mathbb{H})$, it follows that if $c \in H_\bullet(\text{Sp}(1))$, then the elements $(c, 0)$ and $(0, c)$ in $H_\bullet(\text{Sp}(1)) \amalg H_\bullet(\text{Sp}(1)) \subset H_\bullet(\text{Sp}(1) \times \text{Sp}(1))$ have the same image in $H_\bullet(SL(2, \mathbb{H}))$. Thus the images of $H_\bullet(SL(1, \mathbb{H}))$ and $H_\bullet(SU(2))$ in $H_\bullet(SL(2, \mathbb{H}))$ are related by multiplication by 2. This yields (c) since $H_\bullet(SU(2))$ contains the trivial torsion subgroup $\mathbb{Q}/\mathbb{Z}$. Assertion (d) follows from (b) and (c).

The next result collects the known assertions about $H_\bullet(SL(n, \mathbb{H}))$. Strictly speaking, one of the assertions is proved in the following section. However, the results in the next section do not depend on the results from the present section so we do not have a circular argument.

**Theorem 4.11.** For $n \geq 1$, $SL(n, \mathbb{H})$ has a center $\pm I_n$ of order 2 and is an abstract simple group modulo its center. In particular, $H_1(SL(n, \mathbb{H})) = 0$.

(a) $H_2(SL(n, \mathbb{H})) \to H_2(SL(n + 1, \mathbb{H}))$ is bijective for $n \geq 1$ and the stable value is $K_2(\mathbb{H}) \cong K_2(C)^+$, a $\mathbb{Q}$-vector space.

(b) $H_3(SL(2, \mathbb{H})) \to H_3(SL(n + 1, \mathbb{H}))$ is surjective for $n \geq 2$ and bijective for $n \geq 3$. $H_3(SL(2, \mathbb{H})) \cong H_3(SL(2, \mathbb{H}))^{p^1} \amalg H_0(\langle \sigma \rangle, H_3(SL(2, \mathbb{H})))$. Here $H_3(SL(2, \mathbb{H}))^{p^1}$ is a $\mathbb{Q}$-vector space. The stabilization inclusion of $SL(1, \mathbb{H})$ into $SL(2, \mathbb{H})$ maps $H_3(SL(1, \mathbb{H}))$ surjectively onto $H_0(\langle \sigma \rangle, H_3(SL(2, \mathbb{H})))$. In particular, $H_3(SL(n, \mathbb{H}))$ is 2-divisible for $n \geq 1$ and the stabilization map is injective on the trivial torsion $\mathbb{Q}/\mathbb{Z}$ of $H_3(SL(1, \mathbb{H}))$.

**Proof.** By Theorem 4.1, we only have to justify (b). The first assertion in (b) is a special case of the general stability result proved in Sah [18, App. B]. The direct sum decomposition of $H_3(SL(2, \mathbb{H}))$ was shown in Theorem 2.20. The 2-divisibility of $H_3(SU(2))$ to be proven in Theorem 5.10 with the above gives the 2-divisibility of $H_3(SL(n, \mathbb{H}))$, $n \geq 1$. It also yields the surjectivity of the map from $H_3(SL(1, \mathbb{H}))$ to $H_0(\langle \sigma \rangle, H_3(SL(2, \mathbb{H})))$ because $\Delta$ is surjective and, as shown in the proof of Theorem 4.10, the images of $H_3(SL(1, \mathbb{H}))$ and $H_3(SU(2))$ in $H_3(SL(2, \mathbb{H}))$ are related by multiplication by 2. The last assertion of (b) is
proved by using the Cheeger–Chern–Simons class; see Dupont [5, 6]. On
the level of $H_3(SL(n, \mathbb{H}))$, we are dealing with $\tilde{P}_1$ rather than $\tilde{C}_2$, where $P_1$
is the first Pontrjagin class while $C_2$ is the second Chern class.

5. RELATIONS WITH SCISSORS CONGRUENCES

This section is mainly concerned with the connection between the results
obtained in the present work and the problem of scissors congruences in
3-dimensional hyperbolic as well as spherical spaces.

The following exact sequence arises by applying Proposition 3.4 and
taking the negative eigenspaces of the terms in (2.12) under the action of
complex conjugation:

$$0 \to H_3(SL(2, \mathbb{C})) \to \mathcal{P}(\mathbb{C})^{-} \xrightarrow{D} \mathbb{R} \otimes (\mathbb{R}/\mathbb{Z}) \to K_2(\mathbb{C}) \to 0. \quad (5.1)$$

Strictly speaking, the unique element of order 2 in $H_3(SL(2, \mathbb{C}))$ also
belongs to $H_3(SL(2, \mathbb{C}))^{-}$. Since $H_3(SL(2, \mathbb{C}))$ is the direct sum of $\mathbb{Q}/\mathbb{Z}$
and a suitable uniquely 2-divisible group and $\mathbb{Q}/\mathbb{Z}$ is pointwise fixed by the
complex conjugation map, we may ignore the element of order 2 in forming
(5.1). The map $D$ in (5.1) is the “negative” component of $\tilde{\iota}_C$. An exact
sequence of the form (5.1) was first found in Dupont [5] with $\mathcal{P}(\mathbb{C})$
replaced by the scissors congruence group $\mathcal{P}H^3$ in hyperbolic 3-space and
with 0 at the beginning of (5.1) replaced by an unknown group $A$ satisfying
$2^NA = 0$ for some $N$. The map $D$ is then the Dehn invariant map in hyper-
bolic 3-space; see Dupont and Sah [7]. We quickly recall the natural map
relating $\mathcal{P}(\mathbb{C})^{-}$ and $\mathcal{P}H^3$. For $n > 1$, $\mathcal{P}H^n$ is first shown to be isomorphic
to $\mathcal{P}H^n$. For odd $n$, $\mathcal{P}H^n$ is next shown to be generated by the classes of
totally asymptotic hyperbolic $n$-simplices. When $n = 3$, each such simplex
has vertices $\infty, 0, 1, z$ with $z \in \mathbb{C} - \mathbb{R}$ after we apply an isometry. With a
fixed choice of an orientation of $\mathcal{H}^3$, the scissors congruence class of the
totally asymptotic 3-simplex is associated to $\pm [z]$, where $\pm$ denotes the
orientation of the ordered 3-simplex $(\infty, 0, 1, z)$. This forces us to impose
the condition $[z] = -[z]$. The defining relation (1.5) and the unique 2-
divisibility of $\mathcal{P}(\mathbb{C})$ therefore imply $\mathcal{P}(\mathbb{C})^{-} \cong \mathcal{P}H^3$. A comparison then
identifies (5.1) with the exact sequence of Dupont [5] with the unknown
group $A$ replaced by 0.

We record the improvement:

**Theorem 5.2.** $\mathcal{P}(\mathbb{C})^{-} \cong \mathcal{P}H^3$ and $K_2(\mathbb{C})^{-}$ is the cokernel of the
hyperbolic Dehn invariant map in dimension 3. $\mathcal{P}H^3$ is divisible and has no
2-torsion.

The only new result in Theorem 5.2 is the assertion that $\mathcal{P}H^3$ has no
2-torsion. In Sah [19], this will be improved further by showing that $\mathcal{P}(\mathbb{C})$ is uniquely divisible so that $\mathcal{P}\mathcal{H}^3$ will turn out to be uniquely divisible. It appears that the absence of 2-torsion has to be proved first. It should be noted that the presence of torsion in $\mathcal{P}\mathcal{H}^3$ would simultaneously contradict the conjecture of Milnor [14] (also called the Friedlander–Milnor conjecture) as well as the proposed solution to Hilbert’s Third Problem in hyperbolic 3-space (namely, the scissors congruence class of a 3-dimensional geodesic hyperbolic polytope is determined by its volume and its Dehn invariant). In terms of (5.1), the latter statement is equivalent to the conjecture that $H_3(SL(2, \mathbb{C}))$ is detected by the volume invariant that corresponds to the Cheeger–Chern–Simons class; see Dupont [5].

We next turn our attention to the scissors congruence problem in spherical 3-space. In Dupont [5], there is a similar exact sequence:

$$0 \to H_3(SU(2)) \to \mathcal{P}S^3/\mathbb{Z} \xrightarrow{D} \mathbb{R} \otimes (\mathbb{R}/\mathbb{Z}) \to H_2(SU(2)) \to 0. \quad (5.3)$$

Again, there is a possible kernel $B$ at the beginning of (5.3) with $2^NB = 0$ for some $N$. As we shall see, $B$ will turn out to be 0 and $H_3(SU(2))$ will then be 2-divisible. In general, $\mathcal{P}S^n$ is the scissors congruence group of spherical polytopes in dimension $n$ and $\mathbb{Z}$ in $\mathcal{P}S^3$ denotes the infinite cyclic subgroup generated by the class of the 3-sphere. $D$ denote the spherical Dehn invariant map in dimension 3. We want to show that (5.3) is analogous to (5.1) in the sense that (5.3) is related to the positive eigenspaces of the terms in (2.12); cf. Dupont and Sah [7, Remark 2 following Theorem 4.10].

Our first task is to show the exactness at the beginning of (5.3). For this purpose, we compare the approaches used in Sah [18] and in DuPont [5]. This will account for the last remark in DuPont and Sah [7, App. A]. This will also take care of Theorem 2.20 and Theorem 4.11.

Let $C_\ast = C_\ast(S^n)$ denote the normalized Eilenberg–MacLane chain complex based on $S^n$ with $SO(n+1)$ acting on the complex through its action on $S^n = S(n+1, \mathbb{R})$. $C_\ast(S^n) \otimes_{SO(n+1)} \mathbb{Z}$ can be filtered by the subcomplexes $\mathcal{F}^i_\ast$, $0 \leq i \leq n$, through the $\mathbb{R}$-dimension filtration in the same way as $C_\ast(S^n) \otimes_{O(n+1)} \mathbb{Z}$ in Sah [18]. Denote the latter filtration by $\mathcal{F}^i_\ast$. Evidently $\mathcal{F}^i_\ast$ maps surjectively onto $\mathcal{F}^i = H_0(O(n+1)/SO(n+1), \mathcal{F}^i_\ast)$. We assert that:

For $0 \leq i < n$, $\mathcal{F}^i_\ast \to \mathcal{F}^i$ is an isomorphism of chain complexes. (5.4)

This follows from the observation that any isometry of $S^n$ can be extended to an orientation-preserving isometry of $S^{n+1}$. We have the following exact sequence by looking at the long homology sequence:

$$\to H_n(\mathcal{F}^i_{n-1}) \to H_n(\mathcal{F}^i_\ast) \to H_n(\mathcal{F}^i_\ast/\mathcal{F}^{i-1}_\ast) \to H_{n-1}(\mathcal{F}^{i-1}_\ast) \to . \quad (5.5)$$
We now take advantage of the existence of the orthogonal cone construction.

**Proposition 5.6.** With the preceding notation, $\mathcal{F}_+^n$ is $i$-acyclic with augmentation $\mathbb{Z}$ for $0 \leq i \leq n - 1$ while $\mathcal{F}_+^{n-1}$ is $(n-1)$-acyclic with augmentation $\mathbb{Z}$. Moreover,

(a) $H_n(\mathcal{F}_+^n/\mathcal{F}_+^{n-1}) \cong \mathcal{P}S^n/\Sigma(\mathcal{P}S^{n-1})$, where $\Sigma$ denotes suspension or orthogonal join with $S^0$ (same as $2 \cdot [\text{point}]$);

(b) the image of $H_n(\mathcal{F}_+^{n-1})$ in $H_n(\mathcal{F}_+^n)$ is annihilated by $2$;

(c) $H_n(\mathcal{F}_+^n/\mathcal{F}_+^{n-1}) = 0$ when $n$ is even and positive.

**Proof.** $\mathcal{F}_+^n/\mathcal{F}_+^{n-1}$ and $\mathcal{F}_+^n/\mathcal{F}_+^{n-1}$ both begin in degree $n$ with generators corresponding to spherical $n$-simplices. By means of either the circumscribed center or the inscribed center construction, each such generator of $\mathcal{F}_+^n/\mathcal{F}_+^{n-1}$ becomes a boundary. This was noted in Sah [18] and uses the fact that $O(n+1)$ contains all the hyperplane reflections. In view of (5.4) and in view of the orthogonal cone construction (see Sah [18]), we have the acyclicity assertions about $\mathcal{F}_+^n$. We note also that Witt's theorem has been used implicitly in some of our identifications.

Assertion (a) follows from Dupont [5]. Namely, $H_0(O(n+1), St(\mathbb{R}^{n+1}'))$, where $St(\mathbb{R}^{n+1}')$ is the Steinberg module with twisted action by $O(n+1)$ that involves orientation. This latter is identified with $\mathcal{P}S^n/\Sigma(\mathcal{P}S^{n-1})$ in Dupont [5]. This notation requires a little bit of care. When $n > 1$, $\mathcal{P}S^{n-1}$ is 2-divisible so that the suspension $\Sigma$ has the same image as the cone construction. When $n = 1$, $\mathcal{P}S^0 \cong \mathbb{Z} \cdot [\text{point}]$ and the quotient $\mathcal{P}S^1/\Sigma(\mathcal{P}S^0)$ is to be interpreted as $\mathcal{P}S^1/\mathbb{Z} \cdot 4(e_1) \neq (e_2) \cong \mathbb{R}/\mathbb{Z} \cdot 2\pi$. Assertion (c) follows from (a) and the surjectivity of $\Sigma$ when $n$ is even. In fact, for $n$ even and positive, $\Sigma$ is bijective; see Sah [17].

We are left with the proof of (b). Let $c$ be an $n$-cycle of $\mathcal{F}_+^{n-1}$. Since $SO(n+1)$ is transitive on the collection of all $n$-dimensional subspaces of $\mathbb{R}^{n+1}$, $c$ may be assumed to be made up of $n$-cells lying on $S(n, \mathbb{R})$ with poles $\pm e_{n+1}$. Consider $\partial((\pm e_{n+1}) \neq c) = c - (\pm e_{n+1}) \neq \partial c$. Since $c$ is an $n$-cycle of $\mathcal{F}_+^{n-1}$, the $(n-1)$-cells in $\partial c$ must cancel out in pairs under the action of $SO(n+1)$. If such paired $(n-1)$-cells have rank (= dimension of the $\mathbb{R}$-subspace spanned by the vertices of the individual cells) less than $n$ or have rank $n$ and are similarly oriented in $\mathbb{R}^n$, then the cancellation can be achieved by an element of $SO(n+1)$ that fixes both $e_{n+1}$ and $-e_{n+1}$. When such paired cells have rank $n-1$ and are oppositely oriented, then an examination of $(e_{n+1}) \neq \partial c + (-e_{n+1}) \neq \partial c$ shows that we can carry out the cancellation under $SO(n+1)$ of both pairs by a “crisscrossing” process. (This is similar to a double wedding where the two couples discovered during the ceremony that they were better off if they...
exchanged partners and proceeded to do so! It follows that $2c$ becomes a boundary in $F^n_+$ and (b) is proved.

**Proposition 5.7.** With the preceding notation, we have:

(a) $0 \to H_1(F^n_+) \to H_1(F^n_+/F^n_0) \to H_1(F^n_+/F^n_0) \to 0$ is exact and corresponds to $0 \to \mathbb{Z} \cdot \pi \cdot \mathbb{Z} \cdot 2\pi \to \mathbb{R} \cdot \mathbb{Z} \cdot 2\pi \to \mathbb{R} \cdot \pi \cdot \pi \to 0$ when $n > 0$;

(b) for $n > 1$, $H_n(F^n_+) \to H_n(F^n_+/F^n_0)$ is bijective. In particular, for $n > 0$, $H_n(F^n_+) \cong \mathcal{P}S^n/2 \cdot [\text{point}] \neq \mathcal{P}S^n$ is $2$-divisible and is $0$ when $n$ is even.

**Proof.** We begin with the observation that the notation in (b) of Proposition 5.7 follows that of Sah [16], while the notation of (a) of Proposition 5.6 follows that of Dupont [5]. They lead to the same thing when $n > 1$. They differ by a factor of $2$ when $n = 1$. The assertion (a) is straightforward and precisely accounts for this difference. We now proceed to the proof of (b) and let $S^j$ be the $j$-sphere of unit vectors in $\Sigma_{1 \leq s \leq j+1} e_s \mathbb{R}$. Let $c \in C_*^j(S^j) \otimes_{SO(j+1)} \mathbb{Z}$ with $c'$ denoting the image of $c$ under a reflection of $S^j$. $c - c'$ is $0$ in $C_*^j(S^j+1) \otimes_{SO(j+2)} \mathbb{Z}$. If $c$ is a $j$-cell, then $c - c'$ is a $j$-cycle of $C_*^j(S^j) \otimes_{SO(j+1)} \mathbb{Z}$. Suppose that $c$ is a $j$-cell with rank $j$ and orientation $e(c)$. Then $c$ is mapped onto the scissors congruence class $e(c)[c]$ in $\mathcal{P}S^j/\Sigma(\mathcal{P}S^j-1)$. In this definition, $j$-cells of rank less than $j$ and $\partial(C_{j+1}(S^j) \otimes_{SO(j+1)} \mathbb{Z})$ are mapped onto $0$. This definition gives the isomorphism in (a) of Proposition 5.6. We now consider the map that assigns to $c \in C_{n-1}^j(S^{n-1}) \otimes_{SO(n)} \mathbb{Z}$ the element $(e_{n+1}) \neq (c - c')$ of $C_n(S^n) \otimes_{SO(n+1)} \mathbb{Z}$. This evidently induces a homomorphism from $C_{n-1}^j(S^{n-1}) \otimes_{SO(n)} \mathbb{Z}$ into $H_n(S^n)$. If rank $c < n - 1$, then $(e_{n+1}) \neq (c - c')$ is $0$. If $c = \partial a$, then $\partial((e_{n+1}) \neq (a - a')) = a - a' - (e_{n+1}) \neq (\partial a - \partial a') = -e_{n+1} \neq (c - c')$ because $a - a' = 0$ holds in $C_n(S^n) \otimes_{SO(n+1)} \mathbb{Z}$. Since $\mathcal{P}S^n \otimes_{SO(n)} \mathbb{Z}$ is $2$-divisible, the homomorphic image is therefore $2$-divisible when $n > 1$. Since $(e_{n+1}) \neq (c - c')$ is the same as $((e_{n+1}) - (-e_{n+1})) \neq c$, the exactness of (5.5) and assertion (a) of Proposition 5.6 show that $(e_{n+1}) \neq (c - c')$ determines an element of the image of $H_n(S^n-1) \otimes_{SO(n+1)} \mathbb{Z}$ in $H_n(S^n)$. The proof of (b) in Proposition 5.6 shows that these generate the image of $H_n(S^n-1) \otimes_{SO(n+1)} \mathbb{Z}$ in $H_n(S^n)$. By (b) of Proposition 5.6, this image is $2$-divisible as well as of exponent $2$, hence it is $0$ for $n > 1$. This proves (b).

We concentrate our attention on $S^3$ and $SO(4)$. The other cases are also interesting, but they do not lead to anything substantially new. Since $SO(4)$ has universal covering group $SPin(4) = SL(1, \mathbb{H}) \times SL(1, \mathbb{H})$, the structure of $H_3(SPin(4))$ can be described in two different ways. Namely, we can use either Küneth's theorem associated to the direct product decomposition of $SPin(4)$ or the transposed spectral sequence associated to $C_*^j(S^3)$ and
SPin(4). The following exact sequence can then be extracted from the transposed spectral sequence \( E^2 \) by using the known results

\[
0 \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow H_3(SU(2)) \rightarrow H_3(C_*(S^3) \otimes_{SO(4)} \mathbb{Z}) \rightarrow 0.
\]

With \( n = 3 \) in Propositions 5.6 and 5.7, \( H_3(C_*(S^3) \otimes_{SO(4)} \mathbb{Z}) \cong \mathcal{S}^3/\Sigma(\mathcal{S}^2) \). By using 2-divisibility, \( \partial^2 \) can be determined explicitly on generating 3-cycles of the form \( (x_1, x_2, y_1, y_2) \), where \( \langle x_i, y_j \rangle = r \) is independent of \( i \) and \( j \) (these special 3-cells are called “box kites”). The determination of \( \partial^2 \) shows that the result is compatible with \( D \) in (5.3) so that (5.3) may be mapped into (5.8). The orthogonal join \( (x_1, x_2) \# (y_1, y_2) \) corresponds to the generators of the subgroup \( \mathcal{S}^3 \# \mathcal{S}^1 \) and contains \( \Sigma(\mathcal{S}^2) = \Sigma^2(\mathcal{S}^1) \). These latter are the “lunes” and \( \Sigma^2(\mathcal{S}^1) \cong \mathbb{R}/\mathbb{Z} \) under the reduced volume map. The Dehn invariant \( D \) is known to have kernel \( \mathbb{Q}/\mathbb{Z} \) on \( \mathcal{S}^3 \# \mathcal{S}^1 \) corresponding to the rational lunes and \( D(\mathcal{S}^1 \# \mathcal{S}^1) \) is equal to the symmetric tensors in \( \mathbb{R}/\mathbb{Z} \otimes \mathbb{R}/\mathbb{Z} \). Moreover, the lunes that correspond to \( \mathbb{Q}/\mathbb{Z} \) are scissors congruent to the fundamental domains of finite cyclic subgroups of \( SU(2) \) acting on \( SU(2) \cong S^3 \) through left multiplications. These correspond to the trivial torsion subgroup \( \mathbb{Q}/\mathbb{Z} \) of \( H_3(SU(2)) \). By combining all these observations, it is not very difficult to conclude that there is no problem with 2-torsion in (5.3). Namely, (5.3) is exact as given.

It is also possible to give a direct proof that \( \partial^2 \) in (5.8) establishes an isomorphism between the subgroup of \( H_3(C_*(S^3) \otimes_{SO(4)} \mathbb{Z}) \) generated by the orthogonal kites and the space of symmetric tensors in \( \mathbb{R}/\mathbb{Z} \otimes \mathbb{R}/\mathbb{Z} \). If we quotient out these corresponding subgroups, then we have the exact sequence

\[
0 \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow H_3(SU(2)) \rightarrow H_3(C_*(S^3) \otimes_{SO(4)} \mathbb{Z})^{\text{indec}} \rightarrow A^2_2(\mathbb{R}/\mathbb{Z}) \rightarrow H_2(SU(2)) \rightarrow 0,
\]

where \( H_3(C_*(S^3) \otimes_{SO(4)} \mathbb{Z})^{\text{indec}} \cong \mathcal{S}^3/(\mathcal{S}^3 \# \mathcal{S}^1) \). (5.9)

We can now state the following analogue of Theorem 5.2.

**Theorem 5.10.** \( K^+_2(\mathbb{C}) \cong H_2(SU(2)) \) is isomorphic to the cokernel of the spherical Dehn invariant map (reduced or not) in dimension 3. \( H_3(SU(2)) \) is a 2-divisible group, and \( D(\mathcal{S}^3) \) is a \( \mathbb{Q} \)-vector space.

The 2-divisibility of \( H_3(SU(2)) \) is an easy consequence of (5.9) since \( \mathcal{S}^3 \) is classically known to be 2-divisible. The identification of \( H_2(SU(2)) \) with the cokernel of the spherical Dehn invariant map was first obtained in
DuPont [5] through (5.3). The identification of $H_3(SU(2))$ with the $\mathbb{Q}$-vector space $K_2(\mathbb{C})^+$ now implies that the image of the Dehn invariant map $D$ (in its various forms) is a $\mathbb{Q}$-vector space. This resolves a question left open in DuPont [5]. The point is that the explicit formula of the Dehn invariant map on special generators of $\mathcal{P}S^3$ described in DuPont [5] does not give useful information on the divisibility of the image. Conjecturally, $\mathcal{P}S^3$ is a $\mathbb{Q}$-vector space. This conjecture is equivalent to the conjecture that $H_3(SU(2))$ is the direct sum of its trivial torsion with a $\mathbb{Q}$-vector space.

We will next describe a connection between (5.9) and the positive eigenspace exact sequence deduced from (2.12):

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(SL(2, \mathbb{C}))+ \to \mathcal{P}(\mathbb{C}, *) \to K_2(\mathbb{R}^+) \cup K_2(\mathbb{Z}) \to 0. \quad (5.11)$$

This connection arises through the Hopf map from $S^3$ to $S^2$. It is somewhat roundabout.

We begin by identifying $S^3 = S(4, \mathbb{R})$ with the spaces $S(2, \mathbb{C})$ or $S(1, \mathbb{H})$ of unit vectors and then replace the action of $SO(4)$ by the action of $U(2)$. We then lift $U(2)$ to the double covering group $\tilde{U}(2) = U(1) \times SL(1, \mathbb{H})$ and identify the latter with a subgroup of $SL(1, \mathbb{H}) \times SL(1, \mathbb{H}) \cong Spin(4)$. We recall that $(q_1, q_2) \in SL(1, \mathbb{H}) \times SL(1, \mathbb{H})$ acts on $S(1, \mathbb{H})$ by sending the unit quaternion $q$ onto $q_1qq_2^{-1} = q_1qq_2^*$. We analyze the transposed spectral sequence for the action of $\tilde{U}(2)$ on $C_*(S^3)$ and notice that the $E^{\infty}$ terms can be computed by Künneth’s theorem. By using this and by comparing with the corresponding transposed spectral sequence associated to $Spin(4)$, we find that $E^2_4$ is in fact 2-acyclic so that we may derive the exact sequence

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(SU(2)) \to H_3(C_*(S^3) \otimes U(2), \mathbb{Z}) \to 0. \quad (5.12)$$

The inclusion map of $\tilde{U}(2)$ into $Spin(4)$ together with the 5-lemma induces an isomorphism between (5.8) and (5.12).

We next identify $S^3 = S(3, \mathbb{R})$ with $\mathbb{P}^1(\mathbb{C})$ or $\partial \mathcal{H}^3$ and consider the action of $SL(2, \mathbb{C})$ on it. We map $\tilde{U}(2)$ to $SU(2) \subset SL(2, \mathbb{C})$ by projecting onto its second factor. This map is compatible with the Hopf map from $S^3$ to $S^2$. We now compare (2.12) and (5.12). Our discussion can be summarized by the diagram of maps

$$H_3(C_*(S^3) \otimes U(2), \mathbb{Z}) \xrightarrow{\text{ Hopfmap }} H_3(C_*(S^3) \otimes SO(4), \mathbb{Z}) \cong \mathcal{P}S^3/\Sigma(\mathcal{P}S^2)$$

$$\cong \mathcal{P}(\mathbb{C}). \quad (5.13)$$
We note that the effective groups $U(2)$ and $SO(4)$ have been used in place of $\overline{U}(2)$ and $SPin(4)$ in the top row in (5.13) since the homology groups depend only on the effective groups.

In order to obtain a map of (5.8) or (5.9) into (2.12), we need to find a map $\eta: \mathbb{R}/\mathbb{Z} \otimes \mathbb{R}/\mathbb{Z} \to A^2_x(\mathbb{C}^*)$ so that it is compatible with $d^2$ and $d^3$ in the appropriate manner. It should be noted that $A^2_x(\mathbb{C}^*)$ is really the space of skew-symmetric tensors in $\mathbb{C}^* \otimes \mathbb{C}^*$. The identification of this space with $A^2_x(\mathbb{C}^*)$ involves a factor of 2. The obvious candidate for $\eta$ requires some justification. Namely, the required compatibility condition involves $d^2$ and $d^3$ and we cannot invoke the naturality of spectral sequences. Generating 3-cycles for $H_3(C_*(S^3) \otimes SO(4), \mathbb{Z})$ can be described easily. Unfortunately, they are not easily described for $H_3(C_*(S^3) \otimes U(2), \mathbb{Z})$. Among the 3-cycles of the complex $C_*(S^3) \otimes SO(4), \mathbb{Z}$, the orthogonal joins $(u, v) \neq (x, y)$ also represent 3-cycles of $C_*(S^3) \otimes U(2), \mathbb{Z}$. Under the Hopf map, these project to 0. This shows that the obvious candidate for $\eta$ has the correct property on the subgroup generated by the orthogonal joins. The action of $U(2)$ on $S(2, \mathbb{C})$ is evidently the restriction of the action of $U(2)$ on $\mathbb{C}^2 - \{0\}$. The Hopf map is evidently the restriction of the projection map from $\mathbb{C}^2 - \{0\}$ to $\mathbb{P}^1(\mathbb{C})$. The action of $GL(2, \mathbb{C})$ on $C_*(\mathbb{C}^2 - \{0\})$ was analyzed in Sah [18, App. B]. With a little bit of care that amounts to evaluating $d^3$ on 4-cycles of the form

$$(e_1, e_\alpha, e_2, e_2\beta, e_2\beta\gamma) - (e_1, e_\alpha, e_2, e_2\gamma, e_2\beta\gamma), \quad \alpha, \beta, \gamma \in \mathbb{C}^*,$$

the argument leading to (5.12) can be adapted to yield the exact sequence

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow H_3(SL(2, \mathbb{C})) \longrightarrow H_3(C_*(\mathbb{C}^2 - \{0\}) \otimes GL(2, \mathbb{C}), \mathbb{Z}) \longrightarrow 0.$$  

The problem of "compatibility" between $d^2$ and $d^3$ can now be transferred to the problem involving (5.14) and (2.12). This time, the generators of $\mathcal{P}(\mathbb{C})$ are all 3-cells $(\infty, 0, 1, z), z \in \mathbb{P}^1(\mathbb{C}) - \{\infty, 0, 1\}$. The map $d^3$ was described explicitly in Dupont and Sah [7]. To obtain the compatibility of $d^2$ and $d^3$, it is enough to find a 3-cycle $c(z)$ in $C_*(\mathbb{C}^2 - \{0\}) \otimes GL(2, \mathbb{C}), \mathbb{Z}$ to cover $(\infty, 0, 1, z)$ and show the compatibility of $d^2$ and $d^3$ on $c(z)$ and $z$ as well as on those elements that generate $H_3(C_*(\mathbb{C}^2 - \{0\}) \otimes GL(2, \mathbb{C}), \mathbb{Z})$ in conjunction with $c(z)$. The checking process will involve the computation of $d^2$ and the precise definition of $\eta$. The computation of $d^2$ is similar to the computation of $d^3$ presented in Dupont and Sah [7]. Since it is simpler, we will leave the details to the reader. Instead, we will describe $c(z)$ and the other generators needed for doing the computation.

In $C_*(\mathbb{C}^2 - \{0\}) \otimes GL(2, \mathbb{C}), \mathbb{Z}$, $(e_1, e_1\alpha, e_2, e_2\beta)$ is a 3-cycle for any $\alpha, \beta \in \mathbb{C}^* - \{1\}$. These correspond to the orthogonal joins and project to 0.
in \( \mathcal{P}(\mathbb{C}) \) since the latter is based on the normalized Eilenberg-MacLane chain complex. The image of this element under \( "d^2 \) is easily shown to be equal to \( \alpha \otimes \beta + \beta \otimes \alpha \). We note that the computation of \( "d^2 \) is purely algebraic. When \( \alpha \) and \( \beta \) range over \( U(1) \), \( (e_1, e_2 \alpha, e_2, e_2 \beta) \) represents an orthogonal "kite" and \( "d^2 \) can be identified with the Dehn invariant map in spherical 3-space and the image is easily seen to be \( \alpha \otimes \beta + \beta \otimes \alpha \) as claimed.

It is straightforward to check that \( c(z) \) can be taken to be the 3-chain
\[
c(z) = (e_1, e_2, e_1 + e_2, e_1 z + e_2) - (e_1 z(1 - z)^{-1}, e_2, e_1 + e_2, (e_1 + e_2)z) + (e_1, e_2, 1 - e_2, e_1 z^{-1}(1 - z) - e_2, e_1 z^{-1}(1 - z))
- (e_1, -e_2, e_1(z - 1) - e_2, e_1(z - 1))
+ (e_1, -e_2, e_1 z - e_2, e_1 z^{-1} - e_2, e_1 z^{-1} - e_2, e_1 z^{-1})
- (e_1, -e_2, -e_1 - e_2, -e_1).
\]

In particular, the last five terms of \( c(z) \) project to 0 in \( \mathcal{P}(\mathbb{C}) \). With some patience, \( "d^2 \) carries \( c(z) \) onto \( -\{(1 - z)z^{-1} \otimes z\} \). We define the map \( \eta: \mathbb{C}^x \otimes \mathbb{C}^x \to A^2_2(\mathbb{C}^x) \) by the rule \( \eta(\alpha \otimes \beta) = 2(\alpha \wedge \beta) \). From Dupont and Sah [7], the map \( "d^3 \) carries \( (\infty, 0, 1, z) \) onto \( 2(z \wedge (1 - z)) \). Now \( H_3(SL(2, \mathbb{C})) \cong K_2(\mathbb{C}) \) is the quotient of \( A^2_2(\mathbb{C}^x) \) with respect to the subgroup generated by all \( 2(z \wedge (1 - z)) \). The extractness of (5.14) therefore shows that \( H_3(C_*(\mathbb{C}^2 - \{0\}) \otimes G_L(2, \mathbb{C}) \mathbb{Z}) \) is generated by the image of \( H_3(SL(2, \mathbb{C})) \) and the classes of the 3-cycles represented by \( c(z) \) and \( (e_1, e_1 \alpha, e_2, e_2 \beta) \). The Hopf map evidently induces a map of the first three terms of (5.14) into the corresponding first three terms in (2.12) such that it is the identity on \( H_3(SL(2, \mathbb{C})) \). We therefore have the desired compatibility condition involving \( "d^2 \), \( "d^3 \), \( \eta \), and the Hopf map. Compatibility of spectral sequences therefore pulls \( \eta \) back to give us the desired map from the exact sequence (5.8) into the exact sequence (5.11). In view of the discussion on orthogonal joins, this also defines a map of the exact sequence (5.9) into the exact sequence (5.11). By Theorem 3.1, the image of \( \mathcal{P}S^1/(\mathcal{P}S^1 \times \mathcal{P}S^1) \cong H_3(C_*(S^3) \otimes SO(4)) \mathbb{Z}_{\text{indee}} \) in \( \mathcal{P}(\mathbb{C}, \ast) \) must be contained in the kernel of the map from \( \mathcal{P}(\mathbb{C}, \ast) \) to \( \mathcal{P}(\mathbb{H}) \). A diagram chase shows that it is in fact equal to the kernel described. We can summarize our discussions:

**Theorem 5.15.** There is a commutative diagram
\[
\begin{array}{ccc}
\mathcal{P}S^1(\mathcal{P}S^2) & \xrightarrow{D} & \mathbb{R}/\mathbb{Z} \otimes \mathbb{R}/\mathbb{Z} \\
\downarrow{\xi} & & \downarrow{(0, \eta)} \\
\mathcal{P}(\mathbb{C}, \ast) & \xrightarrow{\mathcal{H}(\ast)} & A^2_2(\mathbb{R}^+) \amalg A^2_2(\mathbb{R}/\mathbb{Z}),
\end{array}
\]
where \( \xi \) is given by the Hopf map as in (5.13), and \( \eta(\alpha \otimes \beta) = 2(\alpha \wedge \beta) \). This
extends to a map of the exact sequence (5.9) into the exact sequence (5.11) and leads to the exact sequence

$$0 \to \ker \{ H_3(SU(2)) \to H_3(SL(2, \mathbb{C}))^+ \} \to \mathcal{P}S^3/(\mathcal{P}S^1 \times \mathcal{P}S^1)$$

$$\to \mathcal{P}(\mathbb{C}, *) \to \mathcal{P}(\mathbb{H}) \to 0.$$ (5.16)

Here $H_3(SL(2, \mathbb{C}))^+$ is the fixed points of complex conjugation on $H_3(SL(2, \mathbb{C}))$ and $2(\alpha \wedge \beta)$ is mapped onto the $K_7$-symbol $\{ \alpha, \beta \} \in K_7(\mathbb{C}) \cong H_2(SL(2, \mathbb{C}))$.

**Remark 5.17.** (a) The obvious identification of $S^3$ with $\partial \mathcal{H}^4$ together with the inclusion of $SO(4)$ into $SO^1(1, 4)$ does not work. This would lead to the zero map from $H_3(C_\ast(S^3) \otimes SO(4) \mathbb{Z})$ to $\mathcal{P}(\mathbb{H})$. Roughly speaking, the 3-cycles can all be coned to the fixed point $e_0$ in $\mathcal{H}^4$ for the action of $SO(4)$. Of course, $e_0$ is not in $C_\ast(S^4) = C_\ast(\partial \mathcal{H}^4)$ and the cone construction has to be explained in a bigger complex. However, the vanishing assertion is clear from (5.16).

(b) Conjecturally, $H_3(SU(2))$ is isomorphic to $H_3(SL(2, \mathbb{C}))^+$. This would follow from an affirmative solution to Problem 4.9 for $n = 4$. Also it would follow if $\mathcal{P}S^3$ is detected by volume and the Dehn invariant. In any case it would imply, by (5.16) and results of Sah [19], that $\mathcal{P}S^3$ is a $\mathbb{Q}$-vector space.

(c) It should be noted that the surjectivity of the maps from $H_3(SL(2, \mathbb{R}))$ and $H_3(SU(2))$ to $H_3(SL(2, \mathbb{C}))^+$ vaguely suggest something like the noncompact–compact duality principle in Lie theory. Namely, $SL(2, \mathbb{R})$ and $SU(2)$ are the noncompact and compact forms of $SL(2, \mathbb{C})$ while $\mathcal{H}^3$ and $S^3$ are symmetric spaces in noncompact–compact duality. The method described in Cheeger [4] and in Parry and Sah [15] is based on a “Galois transfer principle” in order to get a lower bound on the ranks of $H_3(SU(2))$ and $H_3(SL(2, \mathbb{R}))$ as abelian groups. As mentioned before, $H_3(SL(2, \mathbb{R}))$ turns out to be isomorphic to $H_3(SL(2, \mathbb{C}))^+$; see Sah [19]. However, there does not seem to be any direct way to relate $H_3(SU(2))$ and $H_3(SL(2, \mathbb{R}))$. In some sense, the surjectivity of the map from $H_3(SU(2))$ to $H_3(SL(2, \mathbb{C}))^+$ is analogous to the surjectivity of the map $H_2(SU(2))$ to $H_2(SL(2, \mathbb{C}))^+$ observed in Sah and Wagoner [20]. The determination of the kernels appears to be related to Problem 4.9 for $n = 4$ and 3, respectively.

**References**


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