Modeling Bounded Rationality in Congestion Games with the Quantal Response Equilibrium

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Abstract

This paper investigates the boundedly rational route choice problem with the framework of quantal response equilibrium in which users are noisy optimizers to make route choice decisions. In the congestion game, we establish the boundedly rational route choice model together with a numerical example, and then extend the model with heterogeneous types of users.

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1. Introduction

Consider a network where a set of users travel along directed paths, called routes, which connect origins to destinations. Users are at origins and wish to reach specific destination. In order to do so, they make travel decisions on the network, i.e., choose routes. Yet, users' decisions depend on route travel time over the network, itself depending on the flow of users taking each route and thus on the decisions of the other users. Hence, each user faces the following problem: Given a pattern of travel times for the different possible routes that reach the destination, find a feasible path. This kind of game belongs to the class of congestion games.

By assuming all road users behave in a completely rational way and seek to minimize their own disutility, Wardrop (1952) defined a state of route choice, so-called user equilibrium (UE). At the UE state, no user can further improve her or his utility by unilaterally changing routes. By relaxing some of the behavioral restrictions implied in a strict deterministic disutility minimization rule, Daganzo and Sheffi (1977) developed a stochastic user
equilibrium (SUE) model that considers the travellers’ imperfect perceptions of travel times. In this model, the travel time of a link is treated as a random variable which follows some known probability distribution. Gumbel (Dial, 1971) and normal (Daganzo and Sheffi, 1977) distributions are two commonly used ones, which result in the well-known logit-based and probit-based route choice models, respectively. The SUE is achieved when users can no longer change their perceived utility. Existence and uniqueness of UE or SUE in general networks have been well investigated in the literature, including the solution methods for obtaining these two states (Sheffi, 1985; Yang and Huang, 2005).

The third equilibrium type is boundedly rational user equilibrium (BRUE). As a relaxation of perfect rational and optimal assumption, the notion of bounded rationality was proposed by Simon (1955) and introduced to traffic modeling by Mahmassani and Chang (1987). It has been shown that bounded rationality is important in many contexts (see, e.g., Conlisk, 1996, and references cited therein). In the transportation field, Mahmassani and Chang (1987) studied the existence, uniqueness, and stability properties of BRUE in the standard single-link bottleneck network. Many simulation and experimental studies have incorporated travelers’ boundedly rational behaviors (e.g., Hu and Mahmassani, 1997; Mahmassani and Liu, 1997; Mahmassani, 2000). Lou et al. (2010) is the first to systematically examine the mathematical properties of BRUE in a network traffic assignment context. More specifically, as Mahmassani and Chang (1987) point out and discussed by Lou et al. (2010) and Di et al. (2013), BRUE flow distributions in a static network may not be unique and the set of all possible BRUE flow distributions is a non-convex and non-empty set. In these studies, travelers with bounded rationality still follow the behavior that exhibits a tendency toward utility maximization, but not necessarily to the absolute maximum level, and travelers are boundedly rational in the sense that they may choose non-shortest paths if the travel time saving offered by switching to the shortest path is not big enough (or no larger than a threshold value). Different from these studies, McKelvey and Palfrey (1995) proposed the idea of quantal response equilibrium (QRE), which provides a way to incorporate bounded rationality into game theoretic reasoning. In this study, we focus on the travelers’ route choice behavior with bounded rationality in the framework of QRE.

The remainder of this paper is organized as follows. In section 2, we establish the boundedly rational route choice model in the framework of quantal response equilibrium together with a numerical example. Section 3 extends the model with heterogeneous users. Section 4 concludes the study.

## 2. The boundedly rational route choice model with QRE framework

Congestion games model the interaction between users who select routes to go from the origin to the destination. The most widely studied cases in transportation networks are (a) user equilibrium (UE) and (b) stochastic user equilibrium (SUE). Existence and uniqueness of these two patterns have been established under a variety of conditions (Sheffi, 1985). Specifically, these two patterns entail two assumptions. First, every traveler is a perfect optimizer. Second, every traveler can perfectly predict other travelers’ choices. These assumptions give rise to the Nash equilibrium concept, which often yields sharp theoretical predictions. However, in practice, common knowledge of perfect rationality seldom holds.

McKelvey and Palfrey (1995) incorporated decision error into an analysis of non-cooperative games. They proposed a more general quantal response equilibrium model using an exponential function, which corresponds to a logistic distribution, yields the familiar logit form that is widely used in empirical work. The quantal response equilibrium notion can be viewed as an extension of standard random utility models of discrete choice, or as a generalization of Nash equilibrium that allows noisy optimizing behavior while maintaining the internal consistency of rational expectations (Haile et al., 2008). Individuals’ choices are assumed to be positively, but not perfectly, related to expected payoffs, in that decisions with higher expected payoffs are more likely to be selected. Next we will apply the quantal response equilibrium (QRE) framework to model the boundedly rational route choice problem.

Assume the following route choice situation: every decision maker faces a set of $I$ alternatives from origin to destination. Let $\Gamma = [N, I, \{u_i\}]$ be a congestion game in strategic form, where $N = \{1, \ldots, n\}$ is the set of players, $I = \{1, \ldots, i\}$ is the action set, $u_i$ is the users’ utility function when choosing route $i$. The user’s possible strategies are in $I$. Each route is endowed with a route travel time $t_i$, which is a function of the number of users having already used this route. We assume each player knows $N$ and $I$, that is, each player knows who is in the game and the strategy sets available to each. The focus is on predicting the choice probability for an alternative $i$ from the
alternatives set. Based on the idea of QRE, at equilibrium the route choice probability for the travelers is then given by the following logit formula,

\[ p_i = \frac{\exp(\eta u_i)}{\sum_{k \in I} \exp(\eta u_k)} , \quad i \in I . \] (1)

These choice probabilities are obtained by assuming that each player chooses a “noisy” best response by maximizing the travel time \( u_i + \epsilon_i \) instead of maximizing the travel time \( u_i \). We obtain the logit specification above by assuming that the noise terms \( \epsilon_i \) are independent and identically distributed with an extreme-value distribution. For more properties of Eq. 1, see McKelvey and Palfrey (1995).

The QRE model has a single parameter \( \eta \) \( (\eta \geq 0) \) which is called the bounded rationality parameter (Chen et al., 2012). The degree of bounded rationality is described by an error parameter, and the equilibrium probabilities converge to Nash equilibrium as this parameter goes to zero. Under perfect rationality, the decision with the higher utility is always chosen. Bounded rationality can be modeled by adding a random element in the utility function. In this section, we use a common parameter \( \eta \) for every traveler. As interpreted in Huang (1995), Chen et al. (2012) and Golman (2012), the parameter \( \eta \) represents the level of rationality in each traveler’s behavior. Specifically, it reflects the degree of cognitive of the traveler. When \( \eta \to 0 \), the traveler lacks the ability to make any rational judgment and thus randomizes over all alternatives with equal probabilities. When \( \eta \to \infty \), the traveler chooses the utility-maximizing alternative with certainty, i.e., the QRE in this limiting case is consistent with the Nash equilibrium.

Note that in the logit quantal response equilibrium model, travelers’ choice behavior follows logit quantal response functions. Logit responders can thus be seen as boundedly rational travelers, making errors while trying to choose optimal utility. This QRE model satisfies two important properties. First, travelers are noisy optimizers and choose stochastic best responses. Second, travelers face uncertainty over other travelers’ choices because they recognize that other travelers are also playing stochastic best responses.

The term \( u_i \) in Eq. 1 is given by:

\[ u_i = U - \theta t_i , \quad i \in I , \] (2)

where \( U \) is a constant term representing the utility received through a travel, \( t_i \) is the travel time function of alternative \( i \) which is assumed to be monotone increasing with the flow on it, \( \theta \) \( (\theta \geq 0) \) is the unit utility coefficient of travel time.

Then we can get the numbers choosing the alternative \( i \),

\[ x_i = d \times p_i , \quad i \in I , \] (3)

where \( d \) is the fixed total demand.

By substituting Eqs. 1 and 2 into the Eq. 3, we then have

\[ x_i = d \times \frac{\exp(\eta (U - \theta t_i))}{\sum_{k \in I} \exp(\eta (U - \theta t_k))} , \quad i \in I . \] (4)

This logit equilibrium is a fixed point: the choice probabilities that determine the utilities correspond to the probabilities determined by the utilities via a probabilistic choice rule (Rosenthal, 1989; Huang, 1995). Then the logit equilibrium solution route flow \( x_i \) \( (i \in I) \) can be uniquely obtained from solving Eq. 4. For more details, see Fisk (1980) and Sheffi (1985).
In addition, we enrich the standard QRE model by allowing the bounded rationality parameter $\eta$ to change over time. We assume players become “more rational” through repeated congestion game play. This indicates that the bounded rationality parameter $\eta$ may in fact increase over time due to learning. Specifically, we assume that individuals follow an exponential learning curve. To incorporate such an effect, we allow the bounded rationality parameter in round $t$, denoted by $\eta(t)$, to increase exponentially over time:

$$\eta(t) = \eta e^{\lambda(t-1)}.$$  \hspace{1cm} (5)

where $\lambda$ is the rate of learning.

In particular, we have $\eta(0) = \eta$ and $\eta(\infty) = \infty$. Similar approaches have been adopted in the literature (e.g., McKelvey and Palfrey 1992).

**Example 1.** We employ a simple two-route congestion game to illustrate the above model. The total demand is 100. The BPR (Bureau of Public Roads) travel time function is used as

$$t_i(x_i) = t_i^0 \left(1 + 0.15 \left(\frac{x_i}{C_i}\right)^4\right), \quad i = 1, 2,$$

where $t_i^0$ is the free flow travel time on route $i$, $C_i$ is the capacity of route $i$.

The values of parameters are given in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$t_1^0$</th>
<th>$t_2^0$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>15</td>
<td>20</td>
<td>60</td>
<td>50</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Fig. 1 depicts the relationship between $x_1$ and $\eta$. When $\eta = 0$, the $x_1^* = 50$. That means the travelers lack the ability to make any rational judgment and tend to be uniformly assigned on the two possible paths. As $\eta$ increases, the number choosing route 1 increases. When $\eta \to \infty$, the $x_1^* = 74$. By solving the Nash equilibrium route choice problem, we also get the result, $x_1^* = 74$. When $\eta \to \infty$, the QRE in this limiting case is consistent with the Nash equilibrium. The similar property can also be found in Fig. 2.
Fig. 1 and 2 also show that the number choosing route 1 is larger than that choosing route 2. The reason is that the free flow travel time of route 1 is lower than that of route 2, and the capacity of route 1 is higher than that of route 2.

3. Model extension with heterogeneous types of users

In this section, we explore a logit quantal response equilibrium model where players’ choice behavior follows logit quantal response functions but there is heterogeneity with respect to the responsiveness parameter. Let $\eta_j \in [0, \infty)$ denote the type of player $j$. According to the above model, the equilibrium strategies, which map types to choice probabilities, of all players are common knowledge in equilibrium, but players may have different beliefs about the type distributions. Denote the conditional subjective beliefs of player $m$ about the type of player $j$ by $F^m_j(\eta_j | \eta_m)$, where we assume that each $F^m_j$ has support contained in $[0, \infty)$, a smooth density function $f^m_j$.

With logit response functions, choice probabilities are logit transformations of expected utilities, the probability of $j$ choosing action $i$ as a function of $\eta_j$ is:

$$p_{ji}(\eta_j, u_i) = \frac{\exp(\eta_j u_{ji})}{\sum_{i'} \exp(\eta_j u_{ji'})}.$$ (7)

We next turn to the beliefs that other players $m$ have about $j$’s choice probabilities without knowing $\eta_j$, but with their subjective beliefs $F^m_j$ about its distribution. We denote type $\eta_m$ of player $m$’s belief about player $j$’s choice probabilities by $\sigma^m_j(p_j)$. Therefore, given $j$’s strategy, $p_j(\bullet)$, the belief of type $\eta_m$ of player $m$ that player $j$ will choose action $i$ is

$$\sigma^m_j(p_j | \eta_m) = \int_0^{\infty} p_{ji}(\eta_j) f^m_j(\eta_j | \eta_m) d\eta_j.$$ (8)

Given $\sigma^m_j(p_j | \eta_j)$, the beliefs of type $\eta_j$ of player $j$ about the profile of choice probabilities of all players other than $j$, type $\eta_j$ of player $j$’s expected payoffs are

$$u^\eta_{ji}(\sigma^j) = \sum_{i_j \in Z_j} \left( \prod_{m \neq j} \sigma^m_{ji}(t_m | \eta_j) \right) u_i(i_j, i_{-j}).$$ (9)
With the logit response functions, Eqs. 7, 8, and 9 must all be satisfied simultaneously. This leads to the following:

**Definition 1.** \( p^* \) is a subjective heterogeneous quantal response equilibrium (SQRE) if:

\[
p^*_j(\eta_j) = \frac{\exp \left( \eta_j u^*_{jj} \left( \sigma^*_j \left( p^* | \eta_j \right) \right) \right)}{\sum_{k=1}^{n} \exp \left( \eta_k u^*_{jk} \left( \sigma^*_j \left( p^* | \eta_k \right) \right) \right)}, \text{ for all } i, j \text{ and } \eta_j. \tag{10}
\]

Next we consider the special case of SQRE corresponding to the assumption of rational expectations of players’ type distributions. That is, we eliminate the element of subjectivity from SQRE, leaving only heterogeneous QRE. In particular, we require that \( F_j^m = F_j \) for every \( j \), and that the distributions of each player’s type is common knowledge. Each player is independently assigned by nature a response sensitivity, \( \eta_j \), drawn from a commonly known distribution, \( F_j(\eta_j) \). Therefore, given the \( j \)’s choice probability functions, \( p_j(\bullet) \), the probability \( j \) chooses route \( i \) is

\[
\sigma_j(p) = \int_0^\infty p_j(\eta)f_j(\eta)d\eta. \tag{11}
\]

Both the strategies and distributions of types are common knowledge, given \( \sigma_{-j} \), the strategies of all players other than \( j \), \( j \)'s payoffs can be expressed as

\[
u_{ji}(\sigma) = \sum_{i_j \in I_j} \left( \prod_{m \neq j} \sigma_m(i_m) \right) u_j(i_j, i_{-j}). \tag{12}\]

In a heterogeneous quantal response equilibrium with logit response functions, Eqs. 7, 11, and 12 must all be satisfied simultaneously. This leads to the following:

**Definition 2.** \( \hat{p}^* \) is a heterogeneous quantal response equilibrium (HQRE) if:

\[
p^*_j(\eta_j) = \frac{\exp \left( \eta_j u^*_j \left( \sigma(\hat{p}^*) \right) \right)}{\sum_{k=1}^{n} \exp \left( \eta_k u^*_{jk} \left( \sigma(\hat{p}^*) \right) \right)}, \text{ for all } j, i \text{ and } \eta_j. \tag{13}
\]

**Theorem 1.** (Rogers et al., 2009) In finite congestion game, a heterogeneous quantal response equilibrium exists. Note that if \( \eta_j = \eta_m \) for all \( j \) and \( m \), HQRE collapses to standard (homogeneous) QRE.

### 4. Conclusions

In this paper, bounded rationality is considered to be incorporated into the route choice model. We applied the framework of quantal response equilibrium to model the kind of route choice behavior in a congestion game. The model extension with heterogeneous users is investigated. Actually, the modeling method used in this paper is similar with the well-known Logit model, and both models cannot be used in the general network. In the future research, we plan to carry out the route choice experiment to calibrate the model parameters, and study the learning adjustment mechanism to realize the Wardrop equilibrium. Other kinds of choice models such as departure time choice models or mode choice models with bounded rationality are also our future research topics.
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References


