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An alternative to the Pythagorean rule? Reevaluating Problem 1 of cuneiform tablet BM 34 568

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Abstract

The first problem of the Seleucid mathematical cuneiform tablet BM 34 568 calculates the diagonal of a rectangle from its sides without resorting to the Pythagorean rule. For this reason, it has been a source of discussion among specialists ever since its first publication, but so far no consensus in relation to its mathematical meaning has been attained. This paper presents two new interpretations of the scribe's procedure, based on the assumption that he was able to reduce the problem to a standard Mesopotamian question about reciprocal numbers. These new interpretations are then linked to interpretations of the Old Babylonian tablet Plimpton 322 and to the presence of Pythagorean triples in the contexts of Old Babylonian and Hellenistic mathematics.

Sumário

O primeiro problema do tablete cuneiforme selêucida BM 34 568 calcula a diagonal de um retângulo a partir de seus lados sem recorrer à regra pitagórica. Por essa razão, ele tem sido uma fonte de discussão entre especialistas desde sua primeira publicação, mas até o momento não se atingiu um consenso com relação a seu conteúdo matemático. O presente artigo apresenta duas novas interpretações do procedimento do escriba, assumindo que ele seria capaz de reduzir o problema a uma questão mesopotâmia padrão sobre números recíprocos. As novas interpretações são, em seguida, ligadas a interpretações do tablete Plimpton 322 da Antiga Babilônia e à presença de ternas pitagóricas nos contextos da matemática da Antiga Babilônia e da matemática helenística.

Keywords: Babylonia; Mesopotamia; Seleucid period; Pythagorean triples; BM 34 568; Plimpton 322; Historiography of mathematics

MSC: 01A17

1. Introduction

Thousands of mathematical cuneiform tablets from ancient Mesopotamia have survived to our days. Among those that have already been edited and translated into modern languages, there are many that continue to deserve specific studies, in order to clarify words and expressions that occur in them, mathematical devices that they employ, their location and date of composition, and, a point of growing importance, their context of production. The object of study of this paper is the first problem of the mathematical cuneiform tablet BM 34 568, which dates from the Seleucid

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period (311 B.C.E.–64 C.E.) and is now housed in the British Museum in London.[1] This first problem (hereafter referred to as BM 34 568:1) gives us a rectangle with length 4 and breadth 3 and asks us to obtain its diagonal. The two solutions presented find the diagonal respectively according to

$$3+\frac{1}{2}\cdot 4=5 \tag{1}$$

and

$$4+\frac{1}{3}\cdot 3=5. \tag{2}$$

These linear relations for the computation of the diagonal of a rectangle without mention of anything similar to the Pythagorean rule make this problem unique among the extant mathematical texts in cuneiform. All except one of the explanations that have been proposed for the problem by modern historians of mathematics view its solutions either as flawed or as limited in generality. In this paper, I will present two new interpretations of BM 34 568:1 according to which it embodies correct general arguments. In order to do this, I will assume, and make plausible, that whoever devised these solutions was able to reduce the problem of finding the diagonal of a rectangle to a problem on reciprocal numbers.

The remainder of this Introduction offers a global view of the nature of the available mathematical evidence in cuneiform and sketches the position of the tablet BM 34 568 within it. Section 2 describes BM 34 568:1 in detail and states my research questions. Earlier explanations for the mathematical contents of BM 34 568:1 are presented and briefly commented upon in Section 3. In Sections 4 and 5, I present my new interpretations of the solutions to the problem by translating it, first algebraically and then geometrically, into a standard Mesopotamian question on reciprocal numbers. In doing this, I make constant reference to interpretations of the much discussed Old Babylonian mathematical tablet Plimpton 322. Section 6 offers some further support for my new interpretations of BM 34 568:1 based on a consideration of the methods of generating Pythagorean triples in Old Babylonian and Hellenistic mathematics. In the final section, my main goal is to explore two consequences of this study, namely the assumed meaning of "generality" in Mesopotamian mathematics and the existence of a strong parallel between the explanations by historians of mathematics for the generality of Plimpton 322 and those for BM 34 568:1.

1.1. The nature of the evidence

The written evidence from ancient Mesopotamia has come down to us mainly through clay tablets, although inscriptions on bricks, buildings, stelae and statues have also survived and play an important role in informing us about the history of the region.[2] Most of this evidence was written using the cuneiform system. Even though there is no complete agreement on whether cuneiform was invented to write the Sumerian language, there is great probability that it was, for an early form of the script is present in a large group of clay tablets from around 3200 B.C.E. These are the oldest surviving texts in Sumerian, and perhaps the oldest written texts that are known [Michalowski, 2004, 19]. One important characteristic of the history of the cuneiform script is its complex and not yet well understood process of adaptation to serve as a writing system for languages other than Sumerian, such as Eblaite, Elamite, Hurrian, Hittite, Urartaean and Akkadian [Michalowski, 1997]. In the last centuries B.C.E., the Seleucid Empire, which came into being as an indirect consequence of the empire of Alexander the Great, became the setting where the cuneiform tradition was to produce some of its latest texts.[3]

Of all the languages to which cuneiform was adapted, Akkadian is the most relevant one. While, for the period from around 2500 to 2000 B.C.E., it is possible to speak of a relatively unitary Old Akkadian, in the following

[1] According to Jonathan Taylor, curator of Cuneiform Collections at the British Museum, the tablet was bought from dealers in 1879 (personal communication).

[2] Leick [2001] fully attests to the abundance of tablets as well as the importance of the other objects to the historian.

[3] To be exact, at the time of publication of Michalowski [1997] the latest cuneiform tablet known to us was a post-Seleucid astronomical almanac, dated 74 C.E.

millennia Akkadian is better considered as a set of dialects, of which the most important are Babylonian and Assyrian [Huehnergard and Woods, 2004].

The mathematical cuneiform evidence that has survived was almost exclusively written in the Babylonian dialect, and most of it originated in the period from around 2000 to 1600 B.C.E., known as the Old Babylonian Period. There are, however, some later mathematical tablets that are extant, namely two from the Kassite period (second half of the second millennium), a small group from the Late Babylonian Epoch (6th to 4th century B.C.E.), and another small group from the Seleucid Period [Høyrup, 2002]. There are also mathematical clay tablets from before the Old Babylonian period, written in Old Akkadian,[4] and a set of 20 tablets from early Old Babylonian Ur in which Sumerian is used instead of Akkadian [Muroi, 1998; Høyrup, 2002, 352]. One important characteristic of the texts from Old Babylonia is that they can be grouped into families according to linguistic and mathematical similarities. As for the more recent texts, although bearing resemblances to those of the Old Babylonian corpus, they also present their own distinguishing features, both in language and script and in content [Goetze, 1945; Høyrup, 2002, 319ff].

In spite of the huge quantity of clay tablets that have survived to our days, only three Seleucid mathematical procedure texts (that is, texts presenting problems and their solutions) have been discovered and published so far. The object of this paper, BM 34 568, is one of these. The other two are AO 6484, held by the department of Antiquités Orientales of the Louvre Museum in Paris, and VAT 7848, which is in the Vorderasiatisches Museum in Berlin. Together with only very few numerical tables, they constitute the whole of the mathematical cuneiform evidence for the period [Høyrup, 2002, 316].

An analysis of these three texts reveals some points of contact with older mathematics. Yet they also seem to bring fresher developments. Specifically, VAT 7848 presents only four problems, all of them dealing with practical geometry, and is thought to be related to Late Babylonian tablets [Høyrup, 2002, 390]. AO 6484 contains new material dealing with the calculation of $\sum_0^9 2^i$ and $\sum_1^{10} i^2$. It also presents four problems on finding a number from the sum of the number and its reciprocal, known in the historiography of Mesopotamian mathematics "as *igûm* and *igibûm* problems."[5] These are very common problems in cuneiform, being attested in the Seleucid period, as well as in Old Babylonian mathematics. The recurrence of *igûm* and *igibûm* problems, together with the many tables of reciprocals that have survived to our days, reflect the strong use of and interest in pairs of reciprocal numbers in Mesopotamian mathematics. *Igûm* and *igibûm* problems will play an important role in the new interpretations that I present in Sections 4 and 5.

BM 34 568, the third surviving Seleucid problem text, is a collection of problems on the measures of the sides and diagonals of rectangles, all except one solved through the Pythagorean rule[6] or some consequence of it, plus an alloy problem. In comparison with Old Babylonian mathematical material, some of the problems of BM 34 568 are of entirely new types. Others are known from the more ancient sources, but are now solved through different methods. According to Høyrup [2002, 316], this text "seems to be a list of *new* problem types, either borrowed from elsewhere or fresh inventions of the area."

The Pythagorean rule plays a central role in this article. Among the Seleucid texts, it is used in the majority of the problems of BM 34 568, and in two problems of VAT 7848, once in the computation of the hypotenuse of a right triangle and once for obtaining the height of a trapezium whose area needs to be found. In Old Babylonian mathematics, the rule is used in nine surviving texts, disregarding the special and simple case of the diagonal of the square in YBC 7289 [Høyrup, 2002, 385–387].

One of these nine Old Babylonian texts is Plimpton 322, a tablet that has a special import to the discussions presented in this paper. Plimpton 322 is in fact the only known surviving piece of a broken tablet, and its interest for the history of mathematics lies mainly in the 4-column numeric table it contains, which is more easily described if we start with the last column and proceed backwards until the first one. The fourth column contains a numbering of the rows from 1 to 15. The other three numbers in each of the rows are associated with the sides and diagonal of a particular rectangle, in the following way. The third column displays the diagonals of the rectangles, and the second

4 A brief description of the Old Akkadian mathematical corpus can be found in Friberg [2005].

5 The terms *igûm* and *igibûm* are Sumerian loanwords in Akkadian, meaning "the igi and its igi." The term "igi" also appears in the expression "igi *n* g̃ál" and similar ones and may be translated as *the reciprocal of n* although the literal meaning of "igi *n* g̃ál" is still uncertain. Thus, an appropriate translation for *igûm* and *igibûm* would be *the reciprocal* and *its reciprocal* [Høyrup, 2002, 27–30].

6 Strictly speaking, the "Pythagorean theorem," in Mesopotamian cuneiform tablets, is neither "Pythagorean" nor a "theorem." Therefore, to mark the theorem-free context of Mesopotamian mathematics, I follow Høyrup's usage in [2002, 197, 385] and call it "Pythagorean rule".

column their short side. Due to the fact that the original tablet has been broken, the interpretation of the first column is uncertain: either it gives the square of the ratio between the short and the long sides of the rectangles, or the square of the ratio between the diagonal and the long side. Thus, Plimpton 322 strongly indicates the availability in Old Babylonia of some technique for generating Pythagorean triples, and it has been studied under this perspective by various historians of ancient mathematics. The main historiographical problems in these studies were to explain how the numeric table was constructed and what purpose it might have had.

2. The problem

BM 34 568 was first published in Neugebauer [1935–1937, III], receiving a hand copy [Appendix], a transliteration, a translation to German and a commentary [pp. 14–22]. In this paper I will follow the translation of the original of Problem 1 to English which has been published recently by Høyrup [2002, 392].[7] The text uses a sexagesimal representation for numbers. In the translation, integer digits are indicated by ..., `, ``, °, fractional digits by ′, ″, ... Høyrup's translation runs as follows:

1. [4 the length, 3 the wid]th, ⌞what the diagonal?⌟ Si[nc]e you do not know, 1/2 of your length
   [4 uš 3 sa]g̃.ki ⌞en bar.NUN⌟ m[u] nu.zu*ú* 1/2 *šá* uš*ka*
2. [to] your [w]idth you join: That is it. 4, the length, steps of 30′ go: 2.
   [*a-na* s]ag̃.ki*ka* tab-*ma šu-ú* 4 uš GAM 30 RÁ-*ma* 2
3. [2 t⌟o 3 you join: 5. 5 the diagonal. The th[i]rd of your width
   [2*a*⌟-*na* 3 tab-*ma* 5 5 bar.NUN *š*[*a*]*l-šú šá* sag̃.ki*ka*
4. [t]o your length you join: that is the diagonal. 3, the width, steps of 20′ you go, 1.
   [*a*-]*na* uš*ka* tab-*ma šu-ú* bar.NUN 3 sag̃ a.rá 20 RÁ 1
5. [1] to 4 you join: 5. 5 the diagonal.
   [1] *a-na* 4 tab-*ma* 5 5 bar.NUN

Lines 1–3 of the text are represented by relation (1) in the Introduction and lines 3–5 by relation (2). The validity of the final results is undeniable: the diagonal of a rectangle with sides 3 and 4 is in fact 5. However, we should expect to find the Pythagorean rule being used for finding the diagonal of the rectangle, as in all other extant cuneiform sources. Even if we conceded that it was not absolutely necessary to use the Pythagorean rule, we should at least expect to see why the scribe adopted this procedure and why the correct answers at which he arrived are not a coincidence.

It was in this frame of mind that I started my research. That there should be an explanation showing the scribe's procedure to be a correct one was hinted to me by the observation that Mesopotamian solutions of problems, although always specifically stated with particular numbers, usually represent general situations. Thus, in this particular case, the explanation sought should enable us to describe a general result on sides and diagonals of rectangles, of which BM 34 568:1 would be a particular instance. In fact, it is possible to interpret the scribe's solutions as general results in more than one way. Until now, Gillings [1966] has been the only scholar to do this, but with a somewhat anachronistic approach, as we will see in Section 3. My own explanations of BM 34 568:1 in Sections 4 and 5 are intended to be more historically adequate.

Secondarily, it would be quite informative to know where the scribe acquired this knowledge. However, as we know nothing about the scribe himself, it is virtually impossible to answer this question in a definitive way.[8] Thus, I will limit myself to studying whether my proposed interpretation is compatible with the Seleucid setting and with two enlarged contexts that may have influenced the scribe's knowledge as regards BM 34 568:1 (see Section 6).

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7 Høyrup's conformal translation always renders "a given expression by the same English expression, rendering different expressions differently." Also conserving word order, the conformal translation is intended to make it easier for a reader to follow the original even if he or she only knows the rudiments of the language [Høyrup, 2002, 41].

8 Concerning the identities of scribes in the Seleucid period, the colophon of AO 6484 informs us that it was written by an astrologer-priest, Anu-aba-uter; the colophons of VAT 7848 and BM 34 568 are destroyed [Høyrup, 2002, 389–390].

3. Earlier explanations

Neugebauer [1935–1937, III, 20], who first published the tablet BM 34 568, gives only a literal reading of Problem 1. He observes that, before each of the two calculations of the diagonal, the text gives a general formula, which Neugebauer himself presents in the modern forms[9]

$$\frac{l}{2}+b=d \tag{3}$$

and

$$\frac{b}{3}+l=d. \tag{4}$$

After both general formulae, the scribe calculates the result using the numerical data of the problem, that is, $b=3$ and $l=4$, whence in both cases $d=5$.[10]

One of the consequences of this reading is that the Seleucid scribe would have believed that relations (3) and (4) hold for all numerical values of b and l, or equivalently, for every given rectangle. On this interpretation, a misconception about the relation between the sides and the diagonal of a rectangle, or alternatively the sides of a right triangle, is to be assigned to the scribe.

Gandz [1937, 456–457] gives a different explanation for BM 34 568:1. He is very emphatic in stating that the attribution of a wrong result to the scribe conflicts with other beliefs about him: "To ascribe to a mathematician of rank, as the author of our document certainly is, such a nonsense is absurd indeed. It is my experience, that in such cases, i.e. wherever a modern commentator ascribes crass ignorance or obvious contradictions to an ancient author, the error is to be sought and found on the part of the modern interpreter." To Gandz, relations (3) and (4) do hold for the specific case of BM 34 568:1 and its multiples, but are not intended as general theorems. He argues for his position by noticing that, for every other triangle or rectangle treated by the whole set of problems in BM 34 568, there is a corresponding relation of the type given in Problem 1 that applies to that triangle or rectangle. Table 1 summarizes Gandz's evidence, gathering together the numerical values of (b,l,d) and possible expressions of d as a linear combination of b and l for each of the 19 problems—except Problem 16, the alloy problem.

On the basis of this evidence, Gandz considers it safe to assume that in choosing the values for the sides and diagonals of the rectangle in the problems of BM 34 568, the scribe purposely wanted to show the above "fluctuations" of the linear relations (1) and (2) [1937, 457].

Gillings [1966] presents a mathematically consistent context for the procedure used in BM 34 568:1. As we have seen, relations similar to (3) and (4) also hold for triples other than (3, 4, 5). Even for triples that are not among those of the tablet, we have similar identities. For example, for the triple (5, 12, 13) it is possible to write

$$5+\frac{2}{3}\cdot 12=13 \tag{5}$$

[9] In what follows, I will denote the breadth or front (Breite), length or flank (Länge), and hypotenuse or diagonal (Diagonale) by the triple (b,l,d). It is usual to have $b<l<d$. Sometimes the smallest two sides will be represented by x and y, without imposing any restriction on them.

[10] "Es soll d aus b und l berechnet werden. Der Text gibt zweierlei Vorschriften, einmal

$$d=\frac{l}{2}+b$$

das zweite Mal

$$d=\frac{b}{3}+l,$$

beidemale zunächst allgemein formuliert, dann erst in spezielle Zahlen umgesetzt" [Neugebauer, 1935–1937, III, 20]. (d should be calculated from b and l. The text gives two types of prescriptions, once $d=\frac{l}{2}+b$, the second time $d=\frac{b}{3}+l$, both first formulated in a general way, and only then turned into special numbers.)

Table 1
Relations between sides and diagonal for the rectangles in BM 34 568

Problems	b	l	d	Linear combinations
1, 2, 3, 4, 13, 17	3	4	5	$\frac{l}{2}+b=d$ and $\frac{b}{3}+l=d$
9	6	8	10	$\frac{l}{2}+b=d$ and $\frac{b}{3}+l=d$
12	9	12	15	$\frac{l}{2}+b=d$ and $\frac{b}{3}+l=d$
18, 19	15	20	25	$\frac{l}{2}+b=d$ and $\frac{b}{3}+l=d$
5, 6	32	60	68	$l+\frac{b}{4}=d$ and $b+\frac{36}{60}\cdot l=d$
10, 15	8	15	17	$l+\frac{b}{4}=d$ and $b+\frac{36}{60}\cdot l=d$
7, 8	25	60	65	$l+\frac{b}{5}=d$ and $b+\frac{40}{60}\cdot l=d$
11, 14	20	21	29	$l+\frac{2}{5}\cdot b=d$ and $b+\frac{3}{7}\cdot l=d$

and

$$12+\frac{1}{5}\cdot 5=13. \tag{6}$$

In all of the cases, the diagonal of the rectangle is expressed as the sum of one side with a fraction of the other. In symbols, it is possible to write

$$b+k_1\cdot l=d \tag{7}$$

and

$$l+k_2\cdot b=d. \tag{8}$$

Of course, any three numbers b, l, and d, be they a Pythagorean triple or not, satisfy relations such as (7) and (8). However, Gillings shows that if the three numbers form a Pythagorean triple, then the coefficients k_1 and k_2 are related to a possible way of generating the triple, as follows.

He first finds a general relation by generating Pythagorean triples in the following way,

$$b=m^2-n^2, \tag{9}$$

$$l=2mn, \tag{10}$$

$$d=m^2+n^2, \tag{11}$$

where m and n are positive integers with $m>n$.[11] In order to express the diagonal of the rectangle as the sum of one of the sides plus a fraction of the other, we solve (7) for k_1 and (8) for k_2. The results are, respectively:

$$k_1=\frac{n}{m} \tag{12}$$

and

$$k_2=\frac{m-n}{m+n}. \tag{13}$$

[11] It may occur that $b>l$ but this will not affect the points put forward here.

Table 2
Examples of the computation of k_1 and k_2

m	n	$m-n$	$m+n$	b	l	k_1	k_2	$d=\sqrt{b^2+l^2}=b+k_1\cdot l=l+k_2\cdot b$
2	1	1	3	3	4	$\frac{1}{2}$	$\frac{1}{3}$	5
3	1	2	4	8	6	$\frac{1}{3}$	$\frac{2}{4}$	10
3	2	1	5	5	12	$\frac{2}{3}$	$\frac{1}{5}$	13
4	1	3	5	15	8	$\frac{1}{4}$	$\frac{3}{5}$	17

A number of examples for various values of m and n are given in Table 2, which includes the data from Problem 1 in the first line.

In his paper, Gillings states that tables with the values of coefficients k_1 and k_2 for different values of the initial parameters m and n could have been constructed and that it was from one such table that the scribe of BM 34 568 solved Problem 1. In his opinion, "that the scribe used this technique in his first problem on Pythagorean triads in BM 34 568 is thus well attested" [1966, 226].[12]

I personally see no other reason to ascribe the knowledge of such relations to the scribe than the ease of handling and memorizing relations (7), (8), (12), and (13). Yet these are often anachronistic criteria, not enough to guarantee that *this* was in fact what the scribe had in mind.[13] But there is another difficulty with Gillings's explanation that needs to be pointed out. According to his interpretation, the scribe needed to know m and n explicitly in order to obtain the coefficients k_1 or k_2. However, from the practical point of view, it would be much easier to obtain d as m^2+n^2 than through relations (7) or (8).

Consequently, for the sake both of historical adequacy and practical convenience, Gillings's interpretation must be viewed with suspicion. Nevertheless, his paper has the great merit of proving that there is a possible general explanation for BM 34 568:1.

In spite of this possibility of a general interpretation, the tendency to ascribing an error to the scribe persisted. In a 1980 paper, van der Waerden compares the problems in BM 34 568 with those of the Chinese collection "Nine Chapters on the Mathematical Art" to argue that both come "most probably, from one common source" [van der Waerden, 1980, 2]. In trying to explain BM 34 568:1, he supposes "that the scribe who wrote BM 34 568 copied the problems and solutions from an earlier text, and that the correct solution of problem 1 was broken off" [van der Waerden, 1980, 9]. Even though he mentions Gandz's contrary views, he considers only $d=\sqrt{b^2+l^2}$ as the correct solution:

> Both solutions [in BM 34 568:1] are valid only in the case of the $(3,4,5)$-triangle. In this respect, our text is worse than the Chinese text, which gives the correct general solution
>
> $$z=\sqrt{x^2+y^2}.$$ [van der Waerden, 1980, 9; my addition between square brackets]

Friberg, in an article dedicated to reviewing interpretations of the Old Babylonian tablet Plimpton 322, states that the "author [of BM 34 568] wished to remind his readers that in any triangle which is a multiple of the basic $(1, 45, 1\,15)$ triangle,[14] the diagonal is related to the front and the flank of the triangle through one of the equations $c=b+\frac{a}{2}$ or $c=a+\frac{b}{3}$" [1981, 314, my addition between square brackets].

So BM 34 568:1 has a history of its own. Being a piece of evidence that deals with a well-known problem in a really nonstandard way, it has aroused dissonant opinions among historians of Mesopotamian mathematics. Although the field is consensual as regards the mere reading of the problem, it is not possible to say the same for its interpretation.

[12] A peculiar additional characteristic of the coefficients k_1 and k_2 that might help us in a search for a table of values for them is that they are a conjugate pair, in the sense that $k_1=\frac{1-k_2}{1+k_2}$ and $k_2=\frac{1-k_1}{1+k_1}$.

[13] As I will mention again in Section 6, the procedure for generating Pythagorean triples used by Gillings is very similar to the one we find in Euclid's *Elements*. So, strictly speaking, the weakness of Gillings's interpretation is not an *anachronism*, for if we take into account only the chronology, the procedure might have been available to Seleucid scribes. However, there is not the faintest evidence for this.

[14] In fractions, $(1, \frac{3}{4}, \frac{5}{4})$.

In the following two sections, I will present a pair of new explanations and will show that they are consistent with other remaining evidence.

4. A new algebraic explanation

In this and the following section I present a pair of new interpretations of BM 34 568:1, one algebraic and one geometric. Although I will constantly refer to interpretations of another Mesopotamian tablet, Plimpton 322, I will postpone a detailed discussion of the relationship between BM 34 568:1 and Plimpton 322, specifically between their interpretations, to Section 7.

Similar to Gillings's, my interpretations maintain that (1) and (2) are particular numerical instances of a general result on the sides and the diagonal of a rectangle. However, in my interpretations, one obtains the coefficients k_1 and k_2 in Eqs. (7) and (8) directly from the sides of the rectangle, that is, without resorting to auxiliary values such as m and n, as in Gillings's explanation. Most importantly, my interpretations are historically adequate, because they reduce the problem of finding the diagonal to the standard Mesopotamian problem of *igûm* and *igibûm*, i.e., the determination of a pair of reciprocal numbers $\frac{1}{t}$ and t from the knowledge of their difference (cf. p. 175 and footnote 5). This is a problem type found in Old Babylonian tablets and, with minor changes, also in the Seleucid evidence.

In the present section, I will use modern algebraic symbolism in order to easily treat *igûm* and *igibûm* problems as quadratic equations. This will allow us to find directly linear expressions for the sides of the rectangle of the forms (7) and (8). I will then continue to show in Section 5 that this algebraic apparatus can be replaced by a geometric procedure, which is compatible with the present general opinion on the strong geometric character of Mesopotamian mathematics.

Pairs of reciprocal numbers have already been used to explain the mathematics behind the Old Babylonian tablet Plimpton 322, in an interpretation first proposed by Bruins [apud Friberg, 1981] and then improved by Friberg [1981]. In the following, the strategy for determining the pair of reciprocals t and t' from the sides and diagonal of a rectangle is the one we find in Friberg [1981]. However, my use of this strategy to explain BM 34 568:1 is new.

Let a rectangle with sides x and y be given. Our aim is to show how the diagonal d can be obtained through identities of the form (7) and (8).

We first consider the similar rectangle with sides $\bar{x} = \frac{x}{y}$, $1 = \frac{y}{y}$, and $\bar{d} = \frac{d}{y}$ (for the required scaling, cf. p. 182 and footnote 15). From the Pythagorean rule we find that $\bar{d}^2 - \bar{x}^2 = 1$, which is algebraically equivalent to $(\bar{d} + \bar{x}) \times (\bar{d} - \bar{x}) = 1$. Since the two factors on the left side of this expression are each other's inverse, we can write

$$\bar{d} + \bar{x} = t' \quad \text{and} \quad \bar{d} - \bar{x} = t$$

for certain $t' > 1 > t > 0$ such that $t' \cdot t = 1$. From this, we obtain

$$\bar{x} = \frac{t' - t}{2} = \frac{\frac{1}{t} - t}{2},$$

which reduces to the quadratic equation

$$t^2 + 2\bar{x}t - 1 = 0.$$

Note that the coefficients of this quadratic equation depend only on $\bar{x} = \frac{x}{y}$, that is, on the sides of the rectangle. Furthermore, by definition the positive root of the equation satisfies $t = \bar{d} - \bar{x}$. Once this root has been found, d can be obtained from $d = \bar{d} \cdot y = (\bar{x} + t) \cdot y = \bar{x} \cdot y + t \cdot y = x + t \cdot y$. Thus, we have expressed the diagonal d of the rectangle as the sum of one of the sides (x) plus a fraction of the other ($t \cdot y$), i.e., in the form of the identities (7) and (8).

To see how the above procedure works in a numerical case, suppose, as in BM 34 568:1, we are given $x = 3$ and $y = 4$. Then t is the positive solution of the equation

$$\frac{3}{4} = \frac{\frac{1}{t} - t}{2} \quad \text{or} \quad t^2 + 2 \cdot \frac{3}{4} \cdot t - 1 = 0.$$

From this we find $t = \frac{1}{2}$, which implies that d can be expressed as

$$d = x + \frac{1}{2} \cdot y = 3 + \frac{1}{2} \cdot 4 = 5.$$

Suppose, on the other hand, that we are given $x = 4$ and $y = 3$. Then we arrive at the equation

$$\frac{4}{3} = \frac{\frac{1}{t} - t}{2} \quad \text{or} \quad t^2 + 2 \cdot \frac{4}{3} \cdot t - 1 = 0.$$

From this it follows that $t = \frac{1}{3}$, and hence that we can write

$$d = x + \frac{1}{3} \cdot y = 4 + \frac{1}{3} \cdot 3 = 5.$$

This algebraic argument confirms Gillings's result that relations like (1) and (2) can be regarded as instances of a general property of the sides and diagonal of a rectangle. Gillings's explanation required the use of the auxiliary variables (9)–(11), which is not attested in Mesopotamian mathematics. On the other hand, my own interpretation assumes knowledge of the Pythagorean rule, techniques for generating reciprocal numbers, and a procedure to solve a quadratic equation, all of which are well attested in the surviving Mesopotamian evidence.

In the absence of algebraic symbolism in Babylonian mathematics, it might be argued that the above algebraic procedure for finding the linear expressions for the sides and diagonal of a rectangle is historically less plausible. In the following section, I will show that a procedure to the same effect can be carried out geometrically, thus being more in line with the current general opinion on the strong geometric character of cuneiform mathematics.

5. An improved, geometric interpretation

In a reassessment of the Old Babylonian tablet Plimpton 322 and its interpretations, Robson [2001, 2002] reintroduces the interpretation originally presented algebraically by Bruins [Robson, 2001, 185], but now derived from geometric procedures as evidenced by the Old Babylonian sources. In what follows, I will show that the same can be done for the algebraic explanation of BM 34 568:1 that I have presented in the previous section, thus producing a geometric one. The algebraic and geometric interpretations will turn out to possess important common features: pairs of reciprocals as a starting point, the resolution of a quadratic equation, and the scaling of a rectangle. However, the two explanations cannot be considered equivalent, for they are based on fundamentally different sets of tools—algebraic symbolism for the former and geometric operations for the latter—which cannot be put in a one-to-one correspondence.

The key point of the geometric explanation can be obtained from Figs. 1 and 2. In Fig. 1, we see a rectangle of area 1, that is, one whose sides are a pair of reciprocals, in Mesopotamian terminology referred to as *igûm* and *igibûm* (cf. p. 175 and footnote 5). The rectangle is divided into three parts: a square (on the right end) and two congruent smaller rectangles. In Fig. 2, the leftmost rectangle has been cut and pasted to another position, forming an L-shaped figure. The larger, external sides of this figure have a length equal to half the sum of the *igûm* and *igibûm*; the internal ones have a length equal to half their difference.

As is clear from Fig. 2, the L-shaped figure can be supplemented with the internal square with sides $\frac{1}{2} \times (igib\hat{u}m - ig\hat{u}m)$, indicated by the dotted line, in order to build a larger square with sides $\frac{1}{2} \times (igib\hat{u}m + ig\hat{u}m)$ which encloses the L-shape. Thus the area of the internal square plus that of the L-shape is equal to the area of the external square.

In YBC 6967, an Old Babylonian tablet that plays a major role in Robson's explanation of Plimpton 322, the scribe uses this property to find the *igûm* and *igibûm* from the difference of the pair of reciprocals. Squaring half this difference gives him the area of the internal square; adding 1 then produces the area of the external square, whose side is half the sum of the reciprocals. Finally, half the sum plus half the difference gives the *igibûm*; half the sum minus half the difference gives the *igûm*.

In AO 6484, one of the three known Seleucid procedure texts, the scribe starts with the sum of the *igûm* and *igibûm*. By exactly reversing the steps found in YBC 6967, he obtains the difference of the *igûm* and *igibûm*, after which the reciprocals themselves are obtained in the same way as above.

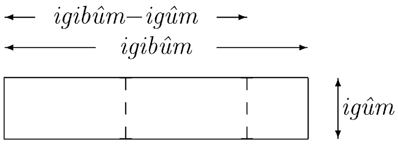

Fig. 1. Rectangle of area 1.

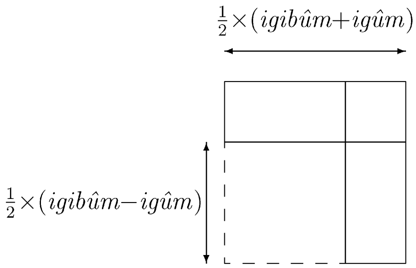

Fig. 2. Configuration resulting from "cut-and-paste."

Since the L-shaped figure in Fig. 2 (or, equivalently, the rectangle in Fig. 1) has the same area as a square of side 1, the geometrical procedure outlined above can in fact be used to generate Pythagorean triples, for

$$\underbrace{\left[\frac{1}{2}\times(\textit{igibûm}-\textit{igûm})\right]^2}_{\text{internal square}}+\underbrace{[1]^2}_{\text{L-shape}}=\underbrace{\left[\frac{1}{2}\times(\textit{igibûm}+\textit{igûm})\right]^2}_{\text{external square}}.$$

Furthermore, the sides and diagonal of the rectangles defined by these triples satisfy the straightforward linear relation

$$\underbrace{\left[\frac{1}{2}\times(\textit{igibûm}-\textit{igûm})\right]}_{\text{one side}}+\underbrace{\textit{igûm}\times[1]}_{\textit{igûm}\times\text{other side}}=\underbrace{\left[\frac{1}{2}\times(\textit{igibûm}+\textit{igûm})\right]}_{\text{diagonal}}, \tag{14}$$

which is exactly the type of relationship for the sides and diagonal of a rectangle on which the solutions of BM 34 568:1 (identities (1) and (2)) are based. Thus, starting with an *igûm* and *igibûm* pair, it is possible to obtain geometrically a Pythagorean triple in such a way that the linear relation above is an immediate consequence of its definition.

I will now show how the scribe could have used this result in order to find the diagonal of a specific rectangle. Starting with the sides x and y, he would search for the number t such that $x + t \cdot y$ is equal to the unknown diagonal. In order to be able to consider the sides and the diagonal of the rectangle as a Pythagorean triple generated by the procedure in Figs. 1 and 2, one of the sides needs to have length 1. The scribe would therefore rescale the original rectangle, as done arithmetically in Section 4, and obtain a new rectangle with sides $\frac{x}{y}$ and 1.[15] He could then treat the side $\frac{x}{y}$ as half the difference of an *igûm* and *igibûm* pair, from which he would find the value of the *igûm* as explained above.[16] As we have seen in identity (14), the *igûm* is then exactly the needed coefficient t, i.e., the coefficient k_1 or k_2 in Eqs. (7) and (8), valid both for the rescaled and the original rectangle.

To revisit the numeric examples of Section 4, suppose the scribe is given the breadth 3 and length 4 of a rectangle and is required to find its diagonal. He would ask "what multiplied by 4 and then added to 3 will give the diagonal?" The equivalent rescaled question would be "what multiplied by 1 and then added to one fourth of 3 (that is, three fourths) will give one fourth of the diagonal?" Following the procedure associated with Figs. 1 and 2, he asks for an *igûm* and *igibûm* pair under the condition that half their difference is equal to $\frac{3}{4}$. By solving the associated *igûm* and *igibûm* problem (as explained with the help of Figs. 1 and 2), he obtains $\frac{1}{2}$ for the value of the *igûm*, and hence for k, exactly as in (1).

Suppose, further, that the scribe asks "what multiplied by 3 and then added to 4 will give the diagonal?" or equivalently "what multiplied by 1 and then added to one third of 4 will give one third of the diagonal?" He would now look

[15] There is some indirect evidence that Old Babylonian scribes were able to deal with the similarity of rectangles [Høyrup, 2002, 228] and it seems possible that, like some other mathematical techniques, this ability was transmitted to the Seleucid setting. More importantly, the rescaling of numbers while preserving their linear relations is a very well attested technique in Old Babylonian tablets through the so-called "false position" technique [Høyrup, 2002, 101–103].

[16] Note that we are dealing with three different rectangles: the original one with sides x and y; the rescaled one with sides $\frac{x}{y}$ and 1; and finally a rectangle with area 1 whose sides are the *igûm* and *igibûm* pair that generate the Pythagorean triple defining the second rectangle.

for another pair of reciprocals, half the difference of which is equal to $\frac{4}{3}$. He would obtain $\frac{1}{3}$ as the *igûm*, and hence for k, exactly as in (2).

In this way, the procedure used in BM 34 568:1 can be understood as an application of the geometric techniques associated with *igûm* and *igibûm* problems. Also in Section 4, *igûm* and *igibûm* pairs were used, but with an algebraic symbolism, which is historically less plausible. The purely geometric interpretation is thus much more consistent towards the extant cuneiform evidence.

6. Evidence from other contexts

In Sections 4 and 5, I have presented a pair of new interpretations of the mathematical contents of the problem BM 34 568:1. I have also argued that my interpretations are historically more adequate than the explanation by Gillings, who resources to relations (9)–(11) for generating Pythagorean triples, although these are nowhere attested in the cuneiform evidence. Of my two interpretations, the geometric one is historically more plausible, since the required mathematics—for the solution of an *igûm* and *igibûm* problem and for the geometric generation of Pythagorean triples from reciprocal pairs—is attested in Seleucid sources.

However, in order to confirm the scribe's awareness of linear relations between the sides and the diagonal of a rectangle as expressed by (7) and (8), explicit evidence is needed. At the moment, no such evidence is available, but it is nevertheless possible to increase the plausibility of my new interpretations by examining how some of their characteristics fit into two contexts that might have borne influence on the practices of Seleucid scribes.

The first of these contexts is Old Babylonian mathematics, whose import on the practices of Seleucid scribes is shown mainly by the continuity of relevant problem types and the persistence of the use of the Akkadian language. Let us then consider how BM 34 568:1, specifically, might be linked to Old Babylonian mathematics.

The first key point here is that we know that Seleucid scribes had an interest in pairs of reciprocals as they appear in *igûm* and *igibûm* problems. This is explicitly attested by the already mentioned tablet AO 6484, and is reinforced by a multiplace table of reciprocals, AO 6456, both dating from the Seleucid period. A second point is that this problem type is also found in the Old Babylonian tablet YBC 6967, as we have seen in Section 5. As a consequence of these points, we can assume with a strong degree of reliability that the context of production of BM 34 568:1 possessed an interest in pairs of reciprocals that is already found in Old Babylonian mathematics.

Further, given the evidence from Plimpton 322, we can assume that interest in the generation of Pythagorean triples was available in Old Babylonian times. Since various other Old Babylonian techniques are known to have been used by Seleucid scribes a millennium and a half later, the same might be the case with the interest in Pythagorean triples and techniques for dealing with them.

However, the very indirect and delicate character of this evidence reminds us that this is not a closed issue and we should not take the "compatibility" of the new interpretation with Old Babylonian mathematics for a "confirmation" of any kind.

The second context that deserves consideration is Hellenistic mathematics, for there is a well-attested interaction between Eastern and Western cultural and political elements after the conquests of Alexander the Great. But before we enter into details, a note of caution is necessary. The question of the mutual influence between the Near East and Greece, or between East and West, as it is commonly referred to, has been intensively discussed. The specialized studies focus mainly on two distinct periods, Archaic and Ancient Greece on the one side, and the Seleucid period on the other. Some awareness of the difficulty in dealing historically with matters of cultural interaction between East and West is necessary before we go on to discuss the specific issue of the possible contacts between Hellenistic mathematics and Seleucid scribal practices.

I will first consider Ancient Greece, because it reinforces the point to be stated below for the Seleucid period. Until the 1950s, the dominant view, especially among Hellenists, was that Greece had blossomed without any influence from its eastern neighbors. The decipherment, in 1952–1953, of Linear B, the writing of Mycenaean Greece, revealed a culture that can be explained only if we assume that a strong influence from the East had taken place in its formation. This understanding was one of the factors that provided a clearer understanding of the Eastern influence on Greek culture.[17] In the history of science, a corresponding opinion implying that only the achievements of the Greeks could

[17] The introductory chapter of Burkert [2004] offers a brief but informative history of how historians and classicists constructed the myth of an isolated Greece and how this has been abandoned by more recent studies.

be considered as science persisted until the last decades of the 20th century. As more insight into cuneiform and other non-European evidence was gained, this view was also emphatically criticized, for example in Pingree [1992]. Nevertheless, the question of the mutual influence of East and West is still a very subtle one. To use the words of Walter Burkert, when searching for explanations of cultural phenomena, "the mere statement of influence is unsatisfactory. One has to seek out the kinds of response to cultural influence, the modifications that occurred, including possible progress by misunderstanding" [2004, 5].

Regarding the Seleucid period, until very recently the common approach was to study the Hellenistic East after Alexander the Great by giving prime importance to the establishment and spread of Greek culture. However, as interest in the history of late Babylonia grew and more cuneiform texts from the Seleucid period were published, specialists intensified studies on Seleucid history from a point of view more in accordance with Seleucid sources [Kuhrt and Sherwin-White, 1987, ix–x]. As a result, the view that the Seleucid empire had undergone a process of Hellenization has weakened. To some, it is still possible to speak of a slight degree of Hellenization [Leick, 2001, 273]. To others, the very concept of " 'Hellenization' is in practice too simplistic a concept for the study of that interaction" [Austin, 1989]. So, when dealing with the possible Hellenistic influence on or interaction with Seleucid mathematical practices, we should keep in mind this is indeed an exacting question.

The establishment of Greek cities on non-Greek territory—e.g., Seleucia-Tigris and Seleucia-on-the-Red Sea—, a Greek ostracon informing us of the presence of an army under Greek offices at Babylon, the Greek theater at Babylon—which presupposes a Greek or Hellenized audience—, the use made by the Seleucids of local languages for their administration, and the involvement of Seleucid rulers with local rituals and values are some of the most prominent examples of Greek influence on the Seleucid Empire [Kuhrt and Sherwin-White, 1987]. Having to do with the state administration, this interaction occurred specifically at court settings, thus having a direct impact on the activities of scribes.

What then can be said about mathematical activity? First, although the literature commonly refers to the authors of Seleucid mathematical and astronomical tablets as scribes, it has been suggested that in the Seleucid Kingdom there might have been a shift of the locus of mathematical practice from scribes, in the sense of administrative employees whose duties included the production of written material, to priests [Robson, 2005, 13], as is exemplified by the signature of the colophon of AO 6484 (cf. footnote 8). Thus the more elaborate mathematical activity in the Seleucid empire would have been produced at temples and not any more in the purely administrative setting to which scribes were restricted. Further, there is evidence that whatever had been the degree of interaction of local and Greek culture it did not affect the temples, which continued to conduct their activities according to longstanding local traditions. This would, in fact, be a reason to discard the possibility of a link from BM 34 568:1 to the Greek mathematical tradition. Nevertheless, late Babylonian astronomy was also a product of the activities of priests working in temples, and we have the evidence that Greek astronomy, for instance in the work of Hipparchus, was influenced by the Babylonian astronomy from Seleucid times [e.g., Neugebauer, 1975, 341–342]. Therefore we can affirm that the context of Greek mathematics was somehow in contact with the context that produced astronomical and mathematical tablets like ours.

As regards my interpretations of BM 34 568:1, it is important to notice that the Greek tradition of mathematics knew more than one procedure to generate Pythagorean triples. One of these is treated in Euclid's *Elements*, Book X, in a lemma between Propositions 28 and 29 [Stamatis, 1969, III, 44–45].[18] Other ways of generating Pythagorean triples are attributed by Proclus to Plato and to the Pythagoreans [Friedlein, 1873, 428.7–429.8]. Independent of the peculiarities of each of these Greek methods, they point to the direction that, in Euclid's time, around the third century B.C.E. (an approximate *terminus post quem* for the production of BM 34 568), the generation of Pythagorean triples was explicitly present in the Greek mathematical tradition.

Therefore, without trying to impose a definite influence of Greek mathematicians on Seleucid scribes, and taking into account that some interaction between these cultures took place, we are forced at least to admit the presence of techniques for the generation of Pythagorean triples in Greek mathematics as a point to be considered.

To sum up, we do not have any direct confirmation for my interpretations of the problem BM 34 568:1 as presented in Sections 4 and 5. Neither do we have any textual evidence for the origin of the type of knowledge on which the interpretations are based. However, the present section shows that the generation of Pythagorean triples was a concept present in Old Babylonian as well as Hellenistic mathematics, two contexts with which Seleucid scribes may be

[18] Incidentally, except of course for the notation, this procedure is the same as used by Gillings in his interpretation of BM 34 568:1 (see Section 3, formulae (9)–(11)).

assumed to have been familiar to some degree. Therefore, we can say that the concept was most likely accessible to the scribe of BM 34 568. This compatibility of my new interpretations with contexts familiar to the Seleucid scribes provides some more confidence in—although not necessarily a definite confirmation of—the possibility that a method of generating Pythagorean triples by means of reciprocals lay at the basis of the linear expressions for the diagonal of a rectangle in the problem BM 34 568:1.

7. Concluding remarks

7.1. Historiographic aspects

History is seen as a series of events that occur to human beings, as well as the narratives human beings construct about these events. In this way, the second meaning of history has a history of its own, sometimes referred to as historiography. It is in this sense that I want to make the following two historiographic remarks.

The first one is related to the manner the historiography of Mesopotamian mathematics has viewed the concept of "generality." We usually read (numerical) problems in cuneiform sources under the assumption that they represent general (numerical) situations. This may be exemplified by the following two quotations from recent publications of the field:

> "[...] solutions to specific problems serve as generic examples from which generalisations are inferred (not always correct)" [Robson, 2005, 10]

> "Since Mesopotamian problems always use specific numbers for various quantities, there are two topics of interest: the general algorithm or procedure used, and the specific numbers chosen. [...] We take for granted that the algorithms are 'correct'; a concerned reader may easily construct an algebraic proof from the algorithm as a check." [Melville, 2004, 151]

Is this a historically fair reading? Is the criterion of generality a Mesopotamian concern as much as ours? Indeed, there is vast evidence that Mesopotamian mathematics embodies general procedures and rules. Three examples are quadratic problems (side and surface, or surface and confrontation problems), diagonal problems (using the Pythagorean rule), and *igûm* and *igibûm* problems (on reciprocal numbers). In all these cases, the evidence on clay tablets shows us that the same group of procedures is applied again and again, subject only to minor modifications. Thus, the evidence tells us that Mesopotamian scribes knew how to solve these problems *independently* of their numerical values. It is precisely in this sense that we can say that Mesopotamian knowledge was about the general side and surface problem, the general diagonal problem and the general *igûm* and *igibûm* problem.

As we have seen, also in relation to BM 34 568:1 historians of mathematics have looked for a path leading to the corresponding general problem. It has been suggested that this general problem might deal with:

- All rectangles similar to the rectangle with sides 3 and 4. This is Friberg's view as explained in Section 3, and in a certain sense it is also van der Waerden's position.[19]
- All rectangles. In one sense this is the position of Gandz, who maintained that the scribe's intention was to show that for any rectangle a similar relation could be applied. In another sense, this is Gillings's and my own position, stating that well defined linear relations of the form (7) or (8) could be applied to each rectangle.

The search for generality is, thus, one of the reasons that still legitimate investigations as the present one.

The second historiographic point I want to make concerns the strong parallel between the proposed explanations for the generality of the tablet Plimpton 322 and those for BM 34 568:1.

[19] Høyrup's opinion is basically the same: "[BM 34 568:1] is not particularly interesting from our present point of view but only as an early instance of a mathematically trivial interest in the rectangle or right triangle where w, l and d form an arithmetical progression, an interest which surfaces again in various later practical geometries." [Høyrup, 2002, 396, my addition between square brackets]

First, the generation of Pythagorean triples by means of the procedure from Euclid's *Elements*, as expressed by relations (9)–(11), was used by Neugebauer and Sachs [1945, 38–41] to explain Plimpton 322, and later by Gillings [1966] to establish a rationale for BM 34 568:1.

A second explanation for Plimpton 322 by Bruins [apud Friberg, 1981] was based on pairs of reciprocals, used in an algebraic manner. Much in the same way, I have presented an algebraic strategy with pairs of reciprocals to explain BM 34 568:1 in Section 4.

A third, geometrical explanation for the general procedure behind Plimpton 322 uses the diagrams of Figs. 1 and 2 [Robson, 2001]. These diagrams seem to be in the background of the *igûm* and *igibûm* problems in YBC 6967 and AO 6484, and I have used them to give an historically more plausible rationale for the procedure behind BM 34 568:1.

Thus, we have three types of explanations that can each be applied to Plimpton 322 as well as to BM 34 568:1. The similarity of the explanations is that they were all used to establish the generality of the studied texts. Their difference, however, is that each of them presupposes a different view on Mesopotamian mathematics and represents a different period of its historiography.

In relation to this difference, I will restrict myself to commenting on the third explanation, the geometrical one. The procedure behind Figs. 1 and 2, closely associated with tablets YBC 6967 (Old Babylonian period) and AO 6484 (Seleucid), has been called an example of "naive cut-and-paste geometry" [Høyrup, 2002, 96–99]. It is "cut-and-paste," because, in forming the larger square, one of the pieces of the rectangle whose sides are the reciprocal numbers was cut from its position and pasted into another. It is "naive" in the sense that no *critique* is made of the process: "We *see* immediately that the procedure is correct, and we have to make an effort to see what precisely we have presupposed [...]" [Høyrup, 2002, 98]. As such, "cut-and-paste geometry" would be one of a set of recurrent techniques used by Mesopotamian scribes, especially Old Babylonian. This set of techniques has been understood mostly because of Høyrup's investigations:

> Jens Høyrup's groundbreaking study of Akkadian 'algebraic' terminology has shown, however, that the scribes themselves conceptualised unknowns much more concretely as lines, areas, and volumes. These imaginary geometric figures could then be manipulated, as described in the model solutions, until the magnitude of the unknown was found using techniques such as completing the square. [Robson, 2000, 14]

As a consequence of these historiographic remarks, the following can be said. First, the search for a general explanation for the rationale of the problem BM 34 568:1 is a legitimate one. Second, among the three explanations produced so far by historians of mathematics, one in Gillings [1966] and two in the present paper, the geometric one I have presented in Section 5 represents best the current trends in the historiography of the subject.

7.2. *Further research questions*

One of the remarkable characteristics of BM 34 568:1 is that it computes the solution in two different ways. What was the intention of the scribe in giving these two different rules? Maybe from his point of view the calculations showed an essential difference. In relation to this possibility, it should be remembered that breadth and length do not commute. The word used in the tablet for breadth is "sag̃.ki," possibly the same as "sag̃,"[20] the latter translated as "front," "width," or "breadth," meaning the smaller side of the rectangle; the word used for length is "uš," "flank" or "length," the larger side. So the purpose of the double calculation might be to show that the procedure can be applied in relation both to the breadth and to the length, and that the two applications of the procedure were felt as different. But this evidently requires further investigation. I do not know of any other instance where we may assume this kind of noncommutability between "breadth" and "length," so that we cannot exclude that the repetition was simply carried out for didactical reasons.

Another question that requires further research is the following: should the scribe have told us how to obtain the coefficients $\frac{1}{2}$ and $\frac{1}{3}$ in his solutions (1) and (2)? If we want to interpret BM 34 568:1 as a numerical case of a general

[20] Neugebauer [1935–1937, III, 14], pointing to a possible deviation from common terminology in BM 34 568:1, adds the following note to the transcription: "Man beachte, daß dieses erste Beispiel eine etwas ausführlichere Terminologie zeigt, als die folgenden: sak-ki bezw. sak-ki-*ka* statt sag, uš-*ka* statt uš, *šu-ú* in der Resultatangabe." (Note that this first example shows a somewhat more elaborate terminology than the following ones: sak-ki, respectively sak-ki-*ka* instead of sag, uš-*ka* instead of uš, *šu-ú* in the presentation of the result.)

relation—in the sense of *generality* discussed above—we have to admit that the coefficients were calculated apart from the tablet and simply used in it. Then BM 34 568:1 would be an example of a problem in whose solution at least one step of the calculation (or the solution) is not explained in the tablet itself. Is this consistent with the remaining evidence? My opinion is that indeed it is, and that BM 34 568:1 is not an isolated case in this respect. Something similar happens with Problem 13 of the same tablet, in which an occurrence of the number 1 used in the solution of the problem is brought into play without any explanation of how it was obtained.[21]

Thus, Problems 1 and 13 might enable us to formulate an hypothesis about the textual habits of the scribe: sometimes nontrivial operations are left out from the text. It remains as an open problem to evaluate whether this was a systematic procedure of his, and whether he had some criteria to guide him in making the omissions. If it is possible to establish such criteria, we can see the absence of the calculations of the coefficients $\frac{1}{2}$ and $\frac{1}{3}$ at least as a lesser oddity.

7.3. Epilogue

The last decades have been rich in discussions about how the historical investigation of ancient mathematics should be carried on.[22] One of the central topics of debate has been the opposition between rational reconstructions and more historical approaches to the subject. Eventually, the feeling grew that the logical or mathematical character of purely rational reconstructions could not always be accepted as historical—more than once, they were vectors of *anachronism*, the capital sin of historians. Historians of mathematics started to feel that ancient sources should be more important than anything else in the justification of their works and have been trying more and more to include the cultural context and the perspective of the *longue durée* in their research [Netz, 2003, 276].

Still, the chronical lack of sources that affects the field turns this historical enterprise always into a risky one. As I mentioned in the Introduction, there are no more than three cuneiform mathematical problem texts that have survived—and are known to this date—from the Seleucid period. What should the historian do when trying to interpret aspects of this so rarefied an evidence? The difficulty of the decision involved is nicely put by Melville [2004, 153]:

> "All we can do when faced with a recurrence of similar problems in the Seleucid period is tentatively to decide whether to join the dots of these isolated pieces of evidence."

This is exactly the problem we have in relation with BM 34 568:1. In the present article, the dots have been joined: the obtaining of the coefficients $\frac{1}{2}$ and $\frac{1}{3}$—more generally, k_1 and k_2 in (7) and (8)—by solving an *igûm* and *igibûm* problem—especially in the geometric fashion—fits the Seleucid framework, in relation both to techniques and to concepts. Furthermore, the availability of knowledge on the generation of Pythagorean triples—or rather, of triples of breadths, lengths, and diagonals of rectangles—is consistent with (a) what we know from Hellenistic mathematics, which was part of a context with attested political and cultural contact with the Seleucid setting, and most of all (b) with the influences Seleucid scribes might have indirectly received from their Old Babylonian heritage.

For these reasons, with the knowledge we possess at the moment, the interpretations from Sections 4 and 5 may be successful in presenting a historically sensitive picture of BM 34 568:1. If new evidence arises, a new history might have to be written.

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[21] In trying to account for this number 1, we may note the following. Neugebauer's *algebraic* interpretation in this case is that it is the result of the expression $((d+l)-(d+b))^2$, where $d+l$ and $d+b$ were given [1935–1937, III, 21]. A more geometric view of how the number could have been derived is presented by Høyrup [2002, 398]. These examples illustrate again two different views on the nature of Mesopotamian mathematics, a point I have brought to attention in Section 6.

[22] See Høyrup [1996] for a survey of the historiography of Mesopotamian mathematics.

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