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Maps on matrix spaces ☆

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Dedicated to Richard Brualdi on the occasion of his 65th birthday

Abstract

It is well known that every automorphism of the full matrix algebra is inner. We give a short proof of this statement and discuss several extensions of this theorem including structural results for multiplicative maps on matrix algebras, characterizations of monotone and orthogonality preserving maps on idempotent matrices, some nonlinear preserver results, and some recent theorems concerning geometry of matrices. We show that all these topics are closely related and point out the connections with physics and geometry. Several open problems are posed.
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1. Introduction

Let $\mathbb{F}$ be an arbitrary field. We denote by $M_n(\mathbb{F})$ the algebra of all $n \times n$ matrices over $\mathbb{F}$. It is well known that every automorphism of this algebra is inner. More precisely, we have the following result.

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Theorem 1.1. *Let $\mathbb{F}$ be an arbitrary field and $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ a bijective linear map satisfying $\phi(AB) = \phi(A)\phi(B)$, $A, B \in M_n(\mathbb{F})$. Then there exists an invertible matrix $T \in M_n(\mathbb{F})$ such that*

$$\phi(A) = TAT^{-1}$$

for every $A \in M_n(\mathbb{F})$.

Every map $A \mapsto TAT^{-1}$, where T is any invertible matrix, will be called a similarity transformation.

The above theorem can be easily improved. Namely, every nonzero endomorphism of the algebra $M_n(\mathbb{F})$ is inner. Indeed, the kernel of an endomorphism is an ideal in $M_n(\mathbb{F})$. The algebra $M_n(\mathbb{F})$ is simple, that is, there are no nontrivial two-sided ideals in $M_n(\mathbb{F})$. So, if $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a nonzero endomorphism, it must be injective and thus, automatically bijective.

Theorem 1.1 is usually derived as a straightforward consequence of Noether-Skolem theorem [12, p. 93, Theorem 3.14] considering homomorphisms from a simple algebra into a finite-dimensional central simple algebra. We were able to find also several direct proofs in the literature. Here we will present the simplest of all proofs that we know. Although this proof is known to several mathematicians working on this kind of problems we were unable to find it in the literature. The idea comes from the paper of Chernoff [8] who studied representations of some operator algebras.

Let us first describe the idea. We identify $n \times n$ matrices with linear operators acting on the n-dimensional space $\mathbb{F}^n$ of all $n \times 1$ matrices over $\mathbb{F}$. If $x, y \in \mathbb{F}^n$ are nonzero column matrices then

$$xy^t = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} [\, y_1 \quad \cdots \quad y_n \,]$$

is a rank one $n \times n$ matrix and every $n \times n$ matrix of rank one can be represented in the above form. By linearity, any automorphism $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is uniquely determined by its behaviour on the set of all rank one matrices. If ϕ is an inner automorphism induced by an invertible matrix T, then

$$\phi(xy^t) = Txy^tT^{-1}$$

for every rank one matrix xy^t. Multiply this equation on the right hand side by a vector z with the property that $y^tT^{-1}z = \lambda$ is a nonzero scalar. We get

$$\phi(xy^t)z = \lambda Tx,$$

where λ is as above. Note also that $TAT^{-1} = (\lambda T)A(\lambda T)^{-1}$ for every nonzero scalar λ and every $A \in M_n(\mathbb{F})$. In other words, if an inner automorphism ϕ of $M_n(\mathbb{F})$ is induced by an invertible matrix T, then it is induced by any nonzero scalar multiple of T. Hence, the above equation gives the idea how to find T appearing in the conclusion of our theorem for a given automorphism ϕ. Based on this simple observation we get the following short proof.

Proof of Theorem 1.1. Choose and fix a pair of nonzero vectors $u, y \in \mathbb{F}^n$. Since ϕ is injective we can find $z \in \mathbb{F}^n$ such that $\phi(uy^t)z \neq 0$. Define $T : \mathbb{F}^n \to \mathbb{F}^n$ by $Tx = \phi(xy^t)z$, $x \in \mathbb{F}^n$. The linearity of T follows from the linearity of ϕ. Moreover, T is nonzero since $Tu \neq 0$. For arbitrary $A \in M_n(\mathbb{F})$ and $x \in X$ we have

$$TAx = \phi((Ax)y^t)z = \phi(A \cdot xy^t)z = \phi(A)\phi(xy^t)z = \phi(A)Tx,$$

and consequently,

$$TA = \phi(A)T.$$

Let w be any vector in $\mathbb{F}^n$. Since $Tu \neq 0$ and because ϕ is surjective we can find $B \in M_n(\mathbb{F})$ such that $\phi(B)Tu = w = TBu$. Thus, T is surjective, and therefore invertible. It follows that $\phi(A) = TAT^{-1}$, $A \in M_n(\mathbb{F})$, as desired. □

A more general approach is to consider $M_n(\mathbb{F})$ only as a ring. Then we are interested in ring automorphisms of $M_n(\mathbb{F})$, that is, bijective maps $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ satisfying $\phi(A + B) = \phi(A) + \phi(B)$ and $\phi(AB) = \phi(A)\phi(B)$, $A, B \in M_n(\mathbb{F})$. The easiest way to treat such maps is to note that the center $Z(M_n(\mathbb{F})) = \{A \in M_n(\mathbb{F}) : AB = BA$ for every $B \in M_n(\mathbb{F})\}$ of the ring $M_n(\mathbb{F})$ is the set of all scalar matrices λI, $\lambda \in \mathbb{F}$. Clearly, ϕ maps the center of $M_n(\mathbb{F})$ onto itself. Thus, $\phi(\lambda I) = f(\lambda)I$ for some function $f : \mathbb{F} \to \mathbb{F}$. Obviously, f is an automorphism of the field $\mathbb{F}$. For any matrix A we denote by $A_{f^{-1}}$ the matrix obtained from A by applying f^{-1} entry-wise, $A_{f^{-1}} = [a_{ij}]_{f^{-1}} = [f^{-1}(a_{ij})]$. The map $A \mapsto A_{f^{-1}}$ is a ring automorphism of $M_n(\mathbb{F})$. It follows that $\varphi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ defined by $\varphi(A) = \phi(A_{f^{-1}})$ is again a ring automorphism of $M_n(\mathbb{F})$. Moreover, it is linear. Indeed,

$$\varphi(\lambda A) = \phi(f^{-1}(\lambda)IA_{f^{-1}}) = \phi(f^{-1}(\lambda)I)\phi(A_{f^{-1}}) = \lambda\phi(A_{f^{-1}}) = \lambda\varphi(A).$$

The structural result for ring automorphisms follows now immediately from Theorem 1.1.

Corollary 1.2. *Let $\mathbb{F}$ be an arbitrary field and $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ a bijective additive map satisfying $\phi(AB) = \phi(A)\phi(B)$, $A, B \in M_n(\mathbb{F})$. Then there exist an automorphism f of the field $\mathbb{F}$ and an invertible matrix $T \in M_n(\mathbb{F})$ such that*

$$\phi(A) = TA_fT^{-1}$$

for every $A \in M_n(\mathbb{F})$.

Let $f : \mathbb{F} \to \mathbb{F}$ be an automorphism of the field $\mathbb{F}$. Then the map $A \mapsto A_f$ will be called a ring automorphism of $M_n(\mathbb{F})$ induced by f.

Recall that a map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is called an anti-automorphism of the algebra $M_n(\mathbb{F})$ if it is bijective, linear, and satisfies $\phi(AB) = \phi(B)\phi(A)$, $A, B \in M_n(\mathbb{F})$. The transposition map $A \mapsto A^t$ is an example of such maps. Moreover, if we compose any anti-automorphism of $M_n(\mathbb{F})$ with the transposition we get an automorphism of $M_n(\mathbb{F})$. Thus, we have another straightforward consequence of Theorem 1.1.

Corollary 1.3. *Let $\mathbb{F}$ be an arbitrary field and $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ a bijective linear map satisfying $\phi(AB) = \phi(B)\phi(A)$, $A, B \in M_n(\mathbb{F})$. Then there exists an invertible matrix $T \in M_n(\mathbb{F})$ such that*

$$\phi(A) = TA^tT^{-1}$$

for every $A \in M_n(\mathbb{F})$.

Let T be an arbitrary invertible matrix. Then the map $A \mapsto TA^tT^{-1}$ will be called an anti-similarity transformation.

A map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is called a Jordan automorphism of the algebra $M_n(\mathbb{F})$ if it is bijective, linear, and satisfies $\phi(A^2) = \phi(A)^2$ for every $A \in M_n(\mathbb{F})$. Obviously, every automorphism as well as every anti-automorphism is a Jordan automorphism of $M_n(\mathbb{F})$.

It follows from [23,78] that every Jordan automorphism ϕ of $M_n(\mathbb{F})$, char $\mathbb{F} \neq 2$, is either an automorphism, or an anti-automorphism. This together with the above structural results for automorphisms and anti-automorphisms yield the following result.

Corollary 1.4. *Let $\mathbb{F}$ be an arbitrary field*, char $\mathbb{F} \neq 2$, *and $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ a Jordan automorphism. Then there exists an invertible matrix $T \in M_n(\mathbb{F})$ such that either*

$$\phi(A) = TAT^{-1}$$

for every $A \in M_n(\mathbb{F})$, or

$$\phi(A) = TA^tT^{-1}$$

for every $A \in M_n(\mathbb{F})$.

In this paper we will discuss several generalizations of the above theorems. Endomorphisms of matrix algebras are linear multiplicative maps. In the next section we will omit the linearity assumption and consider multiplicative maps on matrix algebras. Besides multiplicative maps on full matrix algebras we will be interested also in maps defined on some (Jordan, Lie) subalgebras that are multiplicative with respect to the usual product or Jordan product or Lie product. Multiplicative maps on $M_n(\mathbb{F})$ map every idempotent matrix into an idempotent. Moreover, such maps preserve orthogonality and the usual order on the set of idempotent matrices. We will survey some recent results on monotone and orthogonality preserving maps on idempotent matrices and point out the connection with physics. Observe that every Jordan automorphism of $M_n(\mathbb{F})$ has many nice preserving properties: it preserves invertibility, rank, commutativity, etc. The natural question is whether every linear map on $M_n(\mathbb{F})$ having a certain preserving property is a Jordan automorphism (or a map of a similar form). There is vast literature on these so called linear preserver problems. Besides linear also additive, multiplicative, and quadratic preservers were studied by many authors. Here we will be interested in general preservers, that is, maps on $M_n(\mathbb{F})$ having a certain preserving property that are not assumed to satisfy any additional algebraic assumption. In particular, we will discuss adjacency preserving maps on matrices. We will pay a special attention to connections between the above mentioned problems and show that they are also closely related to some problems in geometry. Several open problems will be posed. At the end we will give a long list of references. Although some of them will not be cited in the paper we decided to include them because of being so closely related to the research topics treated in this survey.

2. Multiplicative maps on matrix algebras

We started with the description of all bijective linear multiplicative maps on $M_n(\mathbb{F})$ and then presented a more general structural result for bijective additive multiplicative maps. Now we will go even one step further by omitting all but the multiplicativity assumption. So we will be interested in the description of all maps $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ satisfying $\phi(AB) = \phi(A)\phi(B)$, $A, B \in M_n(\mathbb{F})$.

Let us start with some examples. Assume that $k \leqslant n$. Following Jodeit and Lam [34] we will call a multiplicative map $\phi : M_n(\mathbb{F}) \to M_k(\mathbb{F})$ degenerate if $\phi(A) = 0$ for every singular matrix A. The structure of such maps is quite well understood. Namely, if $\phi(I) = 0$, then $\phi = 0$. Otherwise, $\phi(I)$ is a nonzero idempotent and the image of ϕ is contained in $PM_k(\mathbb{F})P$, where $P = \phi(I)$. Now, $PM_k(\mathbb{F})P$ is in a natural way isomorphic to $M_r(\mathbb{F})$, where $r = \text{rank } P$. This natural isomorphism maps P into the identity $r \times r$ matrix. Thus, if we want to understand the structure of degenerate multiplicative maps from $M_n(\mathbb{F})$ into $M_k(\mathbb{F})$ we have to understand the structure of unital degenerate multiplicative maps from $M_n(\mathbb{F})$ into $M_r(\mathbb{F})$ where r is any positive integer $\leqslant n$. Every such map sends invertible matrices into invertible matrices. And because the restriction of such a map to the set of all singular matrices is the zero map, we only need to understand the structure of homomorphisms between the general linear groups $\mathrm{GL}(n, \mathbb{F})$ and $\mathrm{GL}(r, \mathbb{F})$. Clearly, $\mathrm{GL}(r, \mathbb{F})$ can be embedded into $\mathrm{GL}(n, \mathbb{F})$. Thus, understanding the structure of degenerate multiplicative maps is the same as understanding the structure of endomorphisms of the general linear group. The structural theory for endomorphisms of the general linear group based on classical Borel-Tits results is highly nontrivial but well developed.

Let us now turn to nondegenerate multiplicative maps $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$. Every similarity transformation and every ring endomorphism of $M_n(\mathbb{F})$ induced by an endomorphism of the underlying field $\mathbb{F}$ is a nondegenerate multiplicative map on

$M_n(\mathbb{F})$. The map $A \mapsto (\operatorname{adj} A)^t$ which sends every matrix to its matrix of cofactors is also a multiplicative nondegenerate map. And finally, let r be an integer, $0 \leqslant r < n$, and $\varphi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ any degenerate multiplicative map. Then the formula

$$A \mapsto \begin{bmatrix} \varphi(A) & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

defines another nondegenerate multiplicative map on $M_n(\mathbb{F})$.

Clearly, any product of multiplicative maps on $M_n(\mathbb{F})$ is again a multiplicative map. In 1969 Jodeit and Lam [34] proved that every multiplicative map on $M_n(\mathbb{F})$ is a product of maps described above. More precisely, we have

Theorem 2.1. *Let n be a positive integer and $\mathbb{F}$ any field. Suppose that $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a multiplicative map. Then either ϕ is degenerate, or there exist an endomorphism $f : \mathbb{F} \to \mathbb{F}$ and an invertible matrix T such that*

$$\phi(A) = TA_fT^{-1}, \quad A \in M_n(\mathbb{F}),$$

or there exist an endomorphism $f : \mathbb{F} \to \mathbb{F}$ and an invertible matrix T such that

$$\phi(A) = T(\operatorname{adj} A_f)^tT^{-1}, \quad A \in M_n(\mathbb{F}),$$

or there exist a degenerate multiplicative map $\varphi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ and a nonzero idempotent $P \in M_n(\mathbb{F})$ such that

$$\phi(A) = \varphi(A) + P, \quad A \in M_n(\mathbb{F}).$$

It should be mentioned that Jodeit and Lam proved the above theorem under the weaker assumption that $\mathbb{F}$ is a principal ideal domain.

There are plenty of open problems here. Namely, $M_n(\mathbb{F})$ can be equipped with other products like Lie product $[A, B] = AB - BA$ or Jordan product $A \circ B = AB + BA$ (if the underlying field is not of characteristic 2 then Jordan product is usually defined by $A \circ B = \frac{1}{2}(AB + BA)$). Thus, instead of studying maps that are multiplicative with respect to the usual product one can study maps that are multiplicative with respect to Lie or Jordan product, that is, maps satisfying one of the following equations:

$$\phi(AB - BA) = \phi(A)\phi(B) - \phi(B)\phi(A), \tag{1}$$

$$\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A),$$

$$\phi\left(\frac{1}{2}(AB + BA)\right) = \frac{1}{2}(\phi(A)\phi(B) + \phi(B)\phi(A))$$

for all $A, B \in M_n(\mathbb{F})$. The last two equations look very similar. So it is interesting to observe that Molnár [53] had to use completely different approaches when characterizing their solutions. A related problem is to characterize maps that are multiplicative with respect to Jordan triple product, that is, maps ϕ on $M_n(\mathbb{F})$ satisfying

$$\phi(ABA) = \phi(A)\phi(B)\phi(A)$$

for all $A, B \in M_n(\mathbb{F})$.

Next, instead of considering maps that are multiplicative with respect to one of the above products on the full matrix algebra we can consider such maps on any subset that is closed under this product. For example, we can ask what is the general form of maps ϕ acting on upper triangular matrices that are multiplicative with respect to one of the above products. The set of all symmetric matrices and the set of all complex hermitian matrices are closed under any of the above mentioned Jordan products, while the set of skewsymmetric matrices and the set of skewhermitian complex matrices are closed under Lie product. So, we can study Jordan multiplicative or Lie multiplicative maps on these sets. Further, we can try to solve this kind of problems on matrices over certain rings or general division rings. And finally, instead of multiplicative maps on $n \times n$ matrices we can study such maps between multiplicative semigroups of matrices of different sizes. For example, we started this section with the result of Jodeit and Lam on multiplicative maps from $M_n(\mathbb{F})$ into $M_n(\mathbb{F})$. The special case of multiplicative maps f from $M_n(\mathbb{F}) \to \mathbb{F}$ (this is indeed a special case since every multiplicative map $f : M_n(\mathbb{F}) \to \mathbb{F}$ gives rise to a multiplicative map from $M_n(\mathbb{F})$ into $M_n(\mathbb{F})$ defined by $A \mapsto f(A)I$) has been treated much earlier. It is well known that every multiplicative map $f : M_n(\mathbb{F}) \to \mathbb{F}$ is of the form $f(A) = g(\det A)$, where $g : \mathbb{F} \to \mathbb{F}$ is a multiplicative function. It is much more difficult to treat multiplicative maps from $M_n(\mathbb{F}) \to M_m(\mathbb{F})$ with $n < m$ (see [36,37]).

Let us discuss here as an example the case of Lie multiplicative maps. If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a bijective map satisfying (1) then there exist an invertible matrix $T \in M_n(\mathbb{C})$, a function $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ satisfying $\varphi(C) = 0$ for every trace zero matrix C and an automorphism f of the complex field such that either $\phi(A) = TA_fT^{-1} + \varphi(A)I$ for every $A \in M_n(\mathbb{C})$, or $\phi(A) = -TA_f^tT^{-1} + \varphi(A)I$ for every $A \in M_n(\mathbb{C})$. If we do not assume that ϕ is bijective, then we conclude that ϕ must be either of one of the above forms (with f being a not necessarily bijective endomorphism), or the image of ϕ is contained in some subset of $M_n(\mathbb{C})$ consisting of pairwise commuting matrices (in this case we have $\phi(AB - BA) = 0$ for every pair $A, B \in M_n(\mathbb{C})$, or equivalently, $\phi(C) = 0$ for every trace zero matrix C). This has been recently proved by Dolinar [15]. Now we have here a whole set of open questions as described above. Can we extend this result to matrices over general fields or even general division rings? What happens on matrices over rings? It would be also natural to study Lie multiplicative maps on upper triangular or even more general block upper triangular matrices, on skewsymmetric matrices, on skewhermitian matrices, etc. Can we say something about Lie multiplicative maps $\phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ when $m > n$?

We conclude this section by listing some papers treating this kind of problems that we are aware of. These are [1,7,15,20,22,34,36,37,40–43,49,51,53,54,62,67,82].

3. Maps on idempotent matrices

In this section we will consider matrices over any division ring $\mathbb{D}$. A matrix $P \in M_n(\mathbb{D})$ is called an idempotent if $P^2 = P$. The set of all $n \times n$ idempotent matrices with entries in $\mathbb{D}$ will be denoted by $P_n(\mathbb{D})$. For any integer k, $1 \leqslant k \leqslant n$, we denote by $P_n^k(\mathbb{D})$ and $P_n^{\leqslant k}(\mathbb{D})$ the set of all $n \times n$ idempotent matrices of rank k, and the set of all $n \times n$ idempotents of rank at most k, respectively.

In the previous section we have considered multiplicative maps on the set of all $n \times n$ matrices. Clearly, if $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ is a multiplicative map (with respect to the usual matrix product), then $\phi(P_n(\mathbb{D})) \subset P_n(\mathbb{D})$.

There are two natural relations on the set $P_n(\mathbb{D})$. First, it is well known that $P_n(\mathbb{D})$ is a poset (partially ordered set) with the partial order defined by

$$P \leqslant Q \iff PQ = QP = P, \quad P, Q \in P_n(\mathbb{D}).$$

Clearly, if $P = 0$ or $Q = I$ or $P = Q$, then $P \leqslant Q$. Otherwise, $P \leqslant Q$ implies that P and Q are simultaneously similar to

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

respectively.

The second natural relation on $P_n(\mathbb{D})$ is the orthogonality relation defined by

$$P \perp Q \iff PQ = QP = 0, \quad P, Q \in P_n(\mathbb{D}).$$

Obviously, $P \perp Q$ if $P = 0$ or $Q = 0$ or $P = I - Q$. If $P \perp Q$ and we do not have one of the trivial possibilities mentioned in the previous sentence, then P and Q are simultaneously similar to

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

respectively.

We already know that the set $P_n(\mathbb{D})$ is invariant under every multiplicative map acting on the whole matrix space $M_n(\mathbb{D})$. Moreover, it is obvious that for every multiplicative map $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ satisfying $\phi(0) = 0$ the restriction $\phi_{|P_n(\mathbb{D})} : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ preserves order and orthogonality, that is, for every $P, Q \in P_n(\mathbb{D})$ we have

$$P \leqslant Q \Rightarrow \phi(P) \leqslant \phi(Q)$$

and

$$P \perp Q \Rightarrow \phi(P) \perp \phi(Q).$$

The natural question here is, of course, what is the general form of maps on $P_n(\mathbb{D})$ satisfying the first or the second condition above.

We started with the structural result for automorphisms of the algebra of all square matrices over a given field, in the next step we considered maps on matrix algebras that are merely multiplicative, and now we have arrived to an even more general problem of characterizing maps on idempotent matrices preserving order or orthogonality. The study of this kind of problems has been motivated also by some problems in mathematical physics. In particular, the first result on automorphisms of the poset of idempotent matrices was obtained by Ovchinnikov [57]. The motivation for his work came from quantum mechanics (see the review MR 95a:46093). His result recently proved to be useful also in the study of quantum mechanical invariance transformations. Molnár [52] used it to considerably improve the classical Wigner's unitary–antiunitary theorem. Later a shorter proof of Molnár's theorem was found based on some structural results for maps on rank one idempotents preserving zero products, or more generally, preserving orthogonality [73,75]. Other applications of structural results for order preserving and orthogonality preserving maps on idempotents include theorems on automorphisms of operator and matrix semigroups [73,77], general preserver problems (see the next section) and geometry of matrices and Grassmanians (see the last section).

Thus, we are interested in the structure of maps on $P_n(\mathbb{D})$ that preserve either order, or orthogonality. The relations $\leqslant$ and $\perp$ are closely connected. Indeed, for every pair of idempotents $P, Q \in P_n(\mathbb{D})$ we have $P \leqslant Q$ if and only if $Q^\perp \subset P^\perp$. Here, $P^\perp$ denotes the set of all idempotents $R \in P_n(\mathbb{D})$ that are orthogonal to P. Moreover, if $P \leqslant Q$, then P and Q commute. Also, if P and Q are orthogonal then they commute. So, the problem of characterizing order preserving maps on idempotent matrices is closely related to the problem of characterizing orthogonality preserving maps on idempotent matrices and both problems are related to the structural problem for commutativity preserving maps on idempotent matrices. Of course, a map $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ is called a commutativity preserving map if $\phi(P)\phi(Q) = \phi(Q)\phi(P)$ for every pair of commuting idempotents $P, Q \in P_n(\mathbb{D})$.

First observe that if we want to get reasonable structural results for maps $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ that preserve one of our relations (order, orthogonality, commutativity) then we have to restrict ourselves to the case when $n \geqslant 3$. Indeed, the set of all rank one idempotents in $P_2(\mathbb{D})$ is a disjoint union of pairs $\{P, I - P\}$ of orthogonal idempotents of rank one. Clearly, two distinct rank one idempotents in $P_2(\mathbb{D})$ commute if and only if they are orthogonal. Every bijective map $\phi : P_2(\mathbb{D}) \to P_2(\mathbb{D})$ sending every pair of orthogonal rank one idempotents into a pair of orthogonal rank one idempotents and satisfying $\phi(0) = 0$ (note that then automatically $\phi(I) = I$) preserves order, orthogonality, and commutativity in both directions. Recall that a map $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ preserves order in both directions if for every pair $P, Q \in P_n(\mathbb{D})$ we have $P \leqslant Q$ if and only if $\phi(P) \leqslant \phi(Q)$. In the same way we define maps preserving orthogonality or commutativity in both directions.

So, in this section we will always assume that $n \geqslant 3$. For every invertible matrix $T \in M_n(\mathbb{D})$ and every automorphism σ of $\mathbb{D}$ the map $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ defined by

$$\phi(P) = TP_\sigma T^{-1}, \quad P \in P_n(\mathbb{D}), \tag{2}$$

is a bijective map preserving order, orthogonality, and commutativity in both directions.

Assume that $A, B \in M_n(\mathbb{D})$. Since the multiplication in $\mathbb{D}$ is not necessarily commutative we do not have $(AB)^t = B^t A^t$ in general. But if τ is an anti-endomorphism of $\mathbb{D}$ then $[(AB)_\tau]^t = B_\tau^t A_\tau^t$. It follows that for every invertible matrix $T \in M_n(\mathbb{D})$ and every anti-automorphism τ of $\mathbb{D}$ the map $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ defined by

$$\phi(P) = T(P_\tau)^t T^{-1}, \quad P \in P_n(\mathbb{D}), \tag{3}$$

is a bijective map preserving order, orthogonality, and commutativity in both directions. Every map of the form (2) or (3) will be called a standard map on $P_n(\mathbb{D})$. In other words, we get a standard map in two ways. We can either start with a similarity transformation on the whole space $M_n(\mathbb{D})$, compose it with a ring automorphism of $M_n(\mathbb{D})$ induced by an automorphism of the underlying division ring $\mathbb{D}$, and then restrict the obtained map to the set of all idempotents, or we do the same with a similarity transformation composed with the transposition and a map on $M_n(\mathbb{D})$ induced by an anti-automorphism of the underlying division ring $\mathbb{D}$.

If σ and τ in (2) and (3) are assumed to be a nonzero (not necessarily bijective) endomomorphism and anti-endomomorphism of $\mathbb{D}$, respectively, then the map ϕ is an injective map preserving order, orthogonality, and commutativity in both directions. We will call such maps almost standard maps on $P_n(\mathbb{D})$.

Choose any positive integer k, $1 \leqslant k \leqslant n-1$. A map $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$, which maps every idempotent of rank at most k into the zero idempotent and every idempotent of rank larger than k into itself preserves commutativity, order, and orthogonality. As most of us are used to work with matrices over fields it should be mentioned here that the definition of a rank of a matrix is slightly more complicated in the noncommutative case. The details will be given in the last section. Any map $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ whose image is contained in a simultaneously diagonalizable subset of $P_n(\mathbb{D})$ preserves commutativity. Any map $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ with the property $\phi(P) \leqslant P$, $P \in P_n(\mathbb{D})$, preserves orthogonality. Let $\phi : P_n^1(\mathbb{D}) \to P_n(\mathbb{D})$ be an arbitrary map. We will extend it inductively to an order preserving map $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$. As the starting map was chosen in an arbitrary way such maps are in general far from being of a standard or an almost standard form. We first define $\phi(0) = 0$. Assume that we have already extended ϕ to a map $\phi : P_n^{\leqslant k}(\mathbb{D}) \to P_n(\mathbb{D})$, where k is a positive integer $1 \leqslant k \leqslant n-1$, and that for every $P, Q \in P_n^{\leqslant k}(\mathbb{D})$ the relation $P \leqslant Q$ yields that $\phi(P) \leqslant \phi(Q)$. For every $P \in P_n^{k+1}(\mathbb{D})$ we can find $Q \in P_n(\mathbb{D})$ such that $\phi(R) \leqslant Q$ for every $R \in P_n(\mathbb{D})$ satisfying $R \leqslant P$, $R \neq P$. Indeed, the choice $Q = I$ works always but in general we have more freedom. We complete the inductive step by defining $\phi(P) = Q$.

These examples show that the maps $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ preserving one of our relations (commutativity, order, orthogonality) may be very far from being standard or almost standard. For more examples of such preservers with wild behaviour we refer to [77]. So, if we want to have reasonable structural results we have to impose additional conditions. The natural choices are the injectivity or the surjectivity assumption. We can also study such maps under the stronger assumption that the relation under the consideration is preserved in both directions.

Our conjecture when starting our work in this direction was that surjective maps on $P_n(\mathbb{D})$ preserving one of our relations must be of a standard form. Surprisingly, it turned out that this is not true. Namely, there exist nonstandard surjective preservers of order or orthogonality or commutativity on complex idempotent matrices. It is not easy to describe an example due to its complexity. We refer the interested readers to [77]. The construction of these counterexamples is based on the fact that there are many "wild" endomorphisms of the complex field [35]. On the other hand, the only nonzero endomorphism of the real field is the identity map. Our conjecture is that every surjective order preserving map on $P_n(\mathbb{R})$ is standard. In fact, it is tempting to conjecture that even more is true. First we need one more definition. A division ring $\mathbb{D}$ is an EAS-division ring if every nonzero endomorphism of $\mathbb{D}$ is automatically surjective (note that in the EAS-case every almost standard map is automatically standard). Let us mention here that besides the field of real numbers also the field of rational numbers and the division ring of quaternions have this property. Assume that $\mathbb{D}$ is an EAS-division ring and $n \geqslant 3$. Is it then true that every surjective order preserving map on $P_n(\mathbb{D})$ is of a standard form? We conjecture that the answer to this question as well as to the analogous question for orthogonality preserving maps are in the affirmative. At the end of this section we will consider commutativity preserving maps. It will be then easy to guess what is our conjecture on the structure of surjective commutativity preserving maps on idempotent matrices over EAS-division rings.

Let $\mathbb{D}$ be an infinite division ring. Then we can find injective maps $\varphi_1 : P_3^1(\mathbb{D}) \to \mathbb{D}$ and $\varphi_2 : P_3^2(\mathbb{D}) \to \mathbb{D}$. It is easy to verify that a map $\phi : P_3(\mathbb{D}) \to P_3(\mathbb{D})$ defined by

$$\phi(0) = 0, \quad \phi(I) = I,$$

$$\phi(P) = \begin{bmatrix} 1 & \varphi_1(P) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P \in P_3^1(\mathbb{D}),$$

and

$$\phi(P) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varphi_2(P) & 0 \end{bmatrix}, \quad P \in P_3^2(\mathbb{D}),$$

is an injective order preserving map. In fact, all we have to do is to verify that

$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leqslant \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 0 \end{bmatrix}$$

holds true for any choices of the entries denoted by $*$. If we compose such a map with the transposition then the obtained map is again an injective order preserving map on $P_3(\mathbb{D})$. We have obtained two types of injective order preserving maps and if we compose any of them with a similarity transformation we again arrive at an injective order preserving map. Any such map will be called a degenerate injective order preserving map on $P_3(\mathbb{D})$. It is quite obvious how to extend the notion of a degenerate injective order preserving map to higher dimensions (for the details see [77]). One of the main results in [77] is the following.

Theorem 3.1. *Let $\mathbb{D}$ be any EAS-division ring. Assume that $n \geqslant 3$ and let $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ be an injective order preserving map. Then either ϕ is a degenerate injective order preserving map, or it is of a standard form.*

It is rather easy to prove that every map $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ preserving order in both directions must be injective. Obviously, a degenerate injective order preserving map cannot preserve order in both directions. Consequently, if $\mathbb{D}$ is any EAS-division ring, $n \geqslant 3$, and $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ a map preserving order in both directions, then ϕ is standard. In [77] one can find a counterexample showing that the EAS-assumption is indispensable in the above theorem as well as in its corollary. We have already mentioned that surjective order preserving maps may have a wild behaviour if the underlying division ring is not EAS. However, the bijectivity assumption is strong enough to give the expected nice structural result for general division rings. Namely, in [77] it was proved that for any division ring $\mathbb{D}$ every bijective order preserving map on $P_n(\mathbb{D})$, $n \geqslant 3$, is standard.

We continue with maps $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ preserving orthogonality. We already know that surjective orthogonality preserving maps are not necessarily standard. For rather "wild" examples of such maps we refer to [77]. As in the case of order preserving maps it is a rather simple observation that maps preserving orthogonality in both directions are automatically injective. Thus, the main problem here is to characterize injective orthogonality preserving maps.

Because of some already mentioned applications in physics and some applications in the theory of general preservers (see the next section) we are interested also in orthogonality preserving maps acting on the subset of rank one idempotents. Let $\mathbb{D}$ be any division ring and n an integer $\geqslant 3$. In [77] it was proved that every injective orthogonality preserving map $\phi : P_n^1(\mathbb{D}) \to P_n^1(\mathbb{D})$ is a restriction of an almost standard map.

If ϕ is an injective orthogonality preserving map defined on the whole set $P_n(\mathbb{D})$ then a rather simple argument shows that it maps rank one idempotents into rank one idempotents, and therefore, by the previous statement, the restriction of ϕ to the set of rank one idempotents is of an almost standard form. At this point it would be tempting to conjecture that it is of an almost standard form on the whole set $P_n(\mathbb{D})$. However, this is not true in general. The counterexample can be found in [77]. Once again, the conjecture holds true for idempotent matrices over EAS-division rings.

Theorem 3.2. *Let $\mathbb{D}$ be any EAS-division ring. Assume that $n \geqslant 3$ and let $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ be an injective orthogonality preserving map. Then ϕ is of a standard form.*

As in the case of order preserving maps we can replace in the above theorem the two assumptions, that is the injectivity assumption and the orthogonality preserving assumption, by a single assumption of preserving orthogonality in both directions and get the same conclusion. To get the same conclusion for idempotent matrices over an arbitrary division ring we need stronger assumptions. In order to get a standard form in the general case we have to assume that either ϕ is a bijective map preserving orthogonality, or a surjective map preserving orthogonality in both directions.

It would be interesting to prove analogous results for orthogonality preserving maps on $P_n^k(\mathbb{D})$, $1 < k < n/2$. For the results on orthogonality preserving maps defined on idempotents or projections of a fixed finite rank in the infinite-dimensional case we refer to [21,50,74,76]. In these four papers and references therein one can find further results on order preserving, orthogonality preserving, and commutativity preserving maps on idempotent operators on infinite-dimensional spaces. It should be mentioned here that for all known results in the infinite-dimensional case we need much stronger assumptions than in the finite-dimensional case. In contrast to the finite-dimensional case we have no counterexamples showing the optimality of the theorems in the infinite-dimensional case and in fact we conjecture that all known infinite-dimensional results are far from being optimal.

Let $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ be any map which sends every idempotent either into itself, or into its orthocomplement, that is, $\phi(P) \in \{P, I - P\}$, $P \in P_n(\mathbb{D})$. Then ϕ preserves commutativity in both directions. Such maps will be called orthopermutations. So, on one hand, the study of commutativity preservers on $P_n(\mathbb{D})$ is slightly more complicated than the study of order or orthogonality preservers because we have to take into consideration besides almost standard and standard maps also orthopermutations. But on the other hand, it turns out that we do not need to deal with the EAS-assumption as in the case of order and orthogonality preservers. Namely, the following was proved in [77].

Theorem 3.3. *Let $\mathbb{D}$ be any division ring. Assume that $n \geqslant 3$ and let $\phi : P_n(\mathbb{D}) \to P_n(\mathbb{D})$ be an injective commutativity preserving map. Then ϕ is an almost standard map composed with a bijective orthopermutation.*

If we replace the injectivity assumption and the preserving property in the above theorem by a single assumption of preserving commutativity in both directions then we get the same conclusion with the only difference that the orthopermutation appearing in the assertion need not be bijective.

Let us conclude this section with a rather general open problem. Throughout this section we have considered maps acting on the set of all $n \times n$ idempotents or on the set of all $n \times n$ idempotents of a fixed rank. We can restrict our attention to the set of all idempotents belonging to some multiplicative subsemigroup of $M_n(\mathbb{D})$ and then try to characterize order and orthogonality preserving maps on this set. For example, we can consider such maps on upper triangular idempotent matrices or more generally, on block upper triangular idempotents. We are aware of only one paper [17] treating this kind of problems. The results and examples in this paper show that the structure of order preserving maps on upper triangular idempotents is essentially more complicated than in the case of the set of all idempotents.

4. General preservers

In the last few decades a lot of results on linear preservers on matrix algebras have been obtained. Also, a more general problem of characterizing additive preservers and a related problem of characterizing multiplicative preservers on matrix algebras were studied a lot. It is surprising that in some special cases we can get nice structural results for preservers without any algebraic assumption like linearity, additivity or multiplicativity. In this section we will briefly survey some recent results on general preservers and explain the main ideas in their proofs.

We start with spectrum preserving maps. Given a complex matrix A we will denote its spectrum by $\sigma(A)$ with the convention that eigenvalues are counted according to multiplicity. A map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called spectrum preserving if $\sigma(\phi(A)) = \sigma(A)$ for every $A \in M_n(\mathbb{C})$. The problem of characterizing linear maps preserving spectrum on matrix and more general Banach algebras has a long history (see [2]). In particular, Marcus and Moyls [44] showed that every linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserving spectrum is either a similarity transformation, or an anti-similarity transformation.

Let us now give an example of a nonlinear spectrum preserving map on $M_n(\mathbb{C})$. For every $A \in M_n(\mathbb{C})$ we choose an invertible matrix $T_A \in M_n(\mathbb{C})$. Obviously, the map $A \mapsto T_A A T_A^{-1}$ preserves spectrum. Every such map will be called a local similarity.

The zero 3×3 matrix and

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

have the same spectrum, but they are not similar. Having in mind this and similar examples it is easy to construct spectrum preserving bijective maps on $M_n(\mathbb{C})$ that are not local similarities. Baribeau and Ransford [3] proved a surprising result that under the additional differentiability assumption spectrum preserving bijective maps must be local similarities.

Theorem 4.1. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a spectrum preserving C^1-diffeomorphism. Then ϕ is a local similarity.*

So, if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a spectrum preserving C^1-diffeomorphism then we can find for every $A \in M_n(\mathbb{C})$ an invertible $T_A \in M_n(\mathbb{C})$ such that $\phi(A) = T_A A T_A^{-1}$. Of course, there is a certain freedom of choice of T_A and it is natural to ask whether T_A can be chosen to depend nicely on A. Baribeau and Ransford observed that there are holomorphic spectrum-preserving bijective maps for which it is impossible to choose T_A to be a continuous function of A. The transposition map is an example of such a map. Note that the transposition map is a local similarity. Namely, every matrix A is similar to its transpose A^t.

The proof of the above theorem is not easy to understand. But one can prove the following much simpler theorem which illustrates the main idea of the proof. We say that $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a determinant preserving map if $\det \phi(A) = \det A$ for every $A \in M_n(\mathbb{C})$. If we associate to every $A \in M_n(\mathbb{C})$ matrices P_A and Q_A with $\det P_A = \det Q_A = 1$ then the map $A \mapsto P_A A Q_A$ preserves determinant. However, if $\varphi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ is any bijective map (here, $S_n(\mathbb{C})$ denotes the set of all singular complex $n \times n$ matrices) and if we define $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\phi(A) = \varphi(A)$ if $A \in S_n(\mathbb{C})$ and $\phi(A) = A$ if A is invertible, then ϕ is a bijective determinant preserving map which is in general not of the form $A \mapsto P_A A Q_A$. Similarly as above we have the following statement.

Proposition 4.2. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a determinant preserving C^1-diffeomorphism. Then for every $A \in M_n(\mathbb{C})$ there exist $P_A, Q_A \in M_n(\mathbb{C})$ such that* $\det P_A = \det Q_A = 1$ *and* $\phi(A) = P_A A Q_A$.

Note that in order to prove this proposition it is enough to show that $\operatorname{rank} \phi(A) = \operatorname{rank} A$ for every $A \in M_n(\mathbb{C})$. Indeed, assume that this is true. Then for every $A \in M_n(\mathbb{C})$ there exist invertible matrices P_A and Q_A such that $\phi(A) = P_A A Q_A$. From $\det \phi(A) = \det A$ we conclude that $\det P_A = (\det Q_A)^{-1}$. After replacing P_A and Q_A by μP_A and $\mu^{-1} Q_A$, respectively, where $\mu^n = \det Q_A$, we get $\det P_A = \det Q_A = 1$. In fact, as ϕ^{-1} has the same properties as ϕ, it is enough to show only that $\operatorname{rank} \phi(A) \geqslant \operatorname{rank} A$ for every $A \in M_n(\mathbb{C})$.

It is now clear that Baribeau and Ransford had to prove that under the assumptions of Theorem 4.1 we have

$$\operatorname{rank}((\phi(A) - \lambda I)^k) = \operatorname{rank} (A - \lambda I)^k \tag{4}$$

for every $\lambda \in \mathbb{C}$ and every $k = 1, \ldots, n$. Once we have this we can use the Jordan canonical form to see that because A and $\phi(A)$ have the same spectrum, they must be similar. Of course, it is much more difficult to get (4) from the assumptions of Theorem 4.1 than to prove that every determinant preserving C^1-diffeomorphism preserves rank. Still, the proof of this easier fact gives some insight into the main ideas needed in the proof of (4).

Before proving our proposition we make one more remark. The assumption that $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a C^1-diffeomorphism yields that for every positive real number p and every $A \in M_n(\mathbb{C})$ we can find $\delta, M > 0$ such that

$$\|\phi(A + H) - \phi(A)\| < M\|H\|^{1-p} \tag{5}$$

for every $H \in M_n(\mathbb{C})$ with $\|H\| < \delta$.

Proof of Proposition 4.2. As already mentioned we have to show that rank $\phi(A) \geqslant$ rank A, $A \in M_n(\mathbb{C})$. If A is invertible, then $0 \neq \det A = \det \phi(A)$, and therefore, $\phi(A)$ is invertible as well. So, assume that rank $A = r < n$ and rank $\phi(A) = k < r$. In order to get a contradiction we first recall that singular values of a matrix B are defined as eigenvalues of $(B^*B)^{1/2}$. We usually order them in decreasing order $s_1(B) \geqslant s_2(B) \geqslant \cdots \geqslant s_n(B)$. Note that $s_1(B) = \|B\|$. Fix a real number $p < \frac{1}{n}$. Then

$$\begin{aligned}(n-k)(1-p) - (n-r) &> (n-k)(1-(1/n)) - (n-r) \\ &= r - k - 1 + \frac{k}{n} \geqslant \frac{k}{n} \geqslant 0.\end{aligned} \tag{6}$$

There exist invertible matrices P and Q such that

$$A = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q.$$

For any positive real ε define

$$H_\varepsilon = P\begin{bmatrix} 0 & 0 \\ 0 & \varepsilon I_{n-r} \end{bmatrix} Q.$$

Clearly,

$$\|H_\varepsilon\| \leqslant \|P\|\|Q\|\varepsilon \tag{7}$$

and

$$\det(\phi(A + H_\varepsilon)) = \det(A + H_\varepsilon) = \det P \det Q\varepsilon^{n-r}. \tag{8}$$

On the other hand

$$\det(\phi(A + H_\varepsilon)) = \det(\phi(A) + T_\varepsilon),$$

where $T_\varepsilon = \phi(A + H_\varepsilon) - \phi(A)$. It follows from (5) and $\|H_\varepsilon\| \leqslant \|P\|\|Q\|\varepsilon$ that there exist $\delta, M > 0$ such that

$$\|T_\varepsilon\| < M\varepsilon^{1-p} \tag{9}$$

for all positive $\varepsilon < \delta$. Assume from now on that $\varepsilon < \delta$. It is well known that the kth singular value of a matrix B can be characterized as

$$s_k(B) = \min\{\|B - C\| : \text{rank } C < k\}$$

and that the absolute value of the determinant is majorized by the product of singular values (see for example [18]). Thus,

$$|\det(\phi(A+H_\varepsilon))| \leqslant \prod_{j=1}^{n} s_j(\phi(A+H_\varepsilon)).$$

Now, by (9)

$$\begin{aligned} s_k(\phi(A+H_\varepsilon)) &\leqslant s_{k-1}(\phi(A+H_\varepsilon)) \leqslant \cdots \leqslant s_1(\phi(A+H_\varepsilon)) \\ &= \|\phi(A+H_\varepsilon)\| \leqslant \|\phi(A)\| + \|T_\varepsilon\| \leqslant \|\phi(A)\| + M\varepsilon^{1-p}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} s_n(\phi(A+H_\varepsilon)) &\leqslant s_{n-1}(\phi(A+H_\varepsilon)) \leqslant \cdots \leqslant s_{k+1}(\phi(A+H_\varepsilon)) \\ &= \min\{\|\phi(A) + T_\varepsilon - C\| : \operatorname{rank} C \leqslant k\} \\ &\leqslant \|\phi(A) + T_\varepsilon - \phi(A)\| < M\varepsilon^{1-p}. \end{aligned}$$

Thus,

$$\begin{aligned} |\det(\phi(A+H_\varepsilon))| &\leqslant (\|\phi(A)\| + M\varepsilon^{1-p})^k (M\varepsilon^{1-p})^{n-k} \\ &= \|\phi(A)\|^k M^{(n-k)} \varepsilon^{(1-p)(n-k)} \\ &\quad \times (1 + c_1\varepsilon^{(1-p)} + \cdots + c_k\varepsilon^{(1-p)k}) \end{aligned}$$

for some constants $c_1, \ldots, c_k$. Putting $\|\phi(A)\|^k M^{(n-k)} |\det P \det Q|^{-1} = L$ and comparing the obtained inequality with (8) we get

$$\varepsilon^{n-r} \leqslant L\varepsilon^{(1-p)(n-k)}(1 + c_1\varepsilon^{(1-p)} + \cdots + c_k\varepsilon^{(1-p)k}),$$

or equivalently,

$$1 \leqslant L\varepsilon^{(1-p)(n-k)-(n-r)}(1 + c_1\varepsilon^{(1-p)} + \cdots + c_k\varepsilon^{(1-p)k}).$$

According to (6), the right hand side tends to 0 when $\varepsilon \to 0$. This contradiction completes the proof. □

A careful reader has observed that we did not need (5) in full generality. In fact, all we need is that (5) is satisfied for some fixed $p < \frac{1}{n}$. Thus, the assumption that ϕ is a C^1-diffeomorphism can be replaced by a milder form of differentiability. Moreover, we do not need to assume that ϕ is defined on the whole matrix algebra. It is enough to assume that it is defined on some open subset of $M_n(\mathbb{C})$. For the details we refer to [3].

Baribeau and Ransford asked whether Theorem 4.1 holds true under the weaker assumption that ϕ is a homeomorphism. A possible strategy to attack this problem would be to study neighbourhoods of matrices in certain subsets of $M_n(\mathbb{C})$ satisfying some spectral conditions. Let us explain this in the 2×2 case. So, let $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a spectrum preserving homeomorphism. For every $A \in M_2(\mathbb{C})$ with two different eigenvalues $\phi(A)$ has the same eigenvalues, and so they are both diagonalizable, and thus, similar. So, all we have to do is to show that every matrix similar to $\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$ is mapped into a similar matrix and every scalar matrix λI is mapped into itself. Fix $\lambda \in \mathbb{C}$. We know that the restriction of ϕ to the subset $\mathscr{T} = \{A \in M_2(\mathbb{C}) : \sigma(A) = (\lambda, \lambda)\}$ is a homeomorphism of $\mathscr{T}$ onto itself. We do not know whether there exists a neighbourhood of $\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$ in $\mathscr{T}$ that is homeomorphic to $\{A \in \mathscr{T} : \|A - \lambda I\| < 1\}$. If the answer to this question is negative, then obviously ϕ maps scalar matrices into themselves and the set of matrices similar to $\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$ onto itself, and must be therefore a local similarity. If this approach works then we believe that especially in higher dimensions it requires nontrivial topological tools.

Let us now turn to commutativity preserving maps. A map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserves commutativity if for every pair $A, B \in M_n(\mathbb{C})$ we have $\phi(A)\phi(B) = \phi(B)\phi(A)$ whenever $AB = BA$. It preserves commutativity in both directions if for every pair $A, B \in M_n(\mathbb{C})$ we have $\phi(A)\phi(B) = \phi(B)\phi(A)$ if and only if $AB = BA$. The study of bijective commutativity preserving linear maps on matrix algebras started in [81]. After this first paper there have been many others treating bijective linear maps preserving commutativity. One motivation to study such maps comes from the theory of Lie algebras. Namely, the assumption of preserving commutativity can be reformulated as preserving zero Lie products. The most general result on bijective linear commutativity preserving maps can be found in [5] where such maps were treated on prime algebras. Only very recently the first results on nonbijective linear commutativity preserving maps were obtained first for matrix algebras over algebraically closed fields $\mathbb{F}$ with char $\mathbb{F} = 0$ [56] and then for arbitrary finite-dimensional central simple algebras over such fields [6].

Can we describe the general form of not necessarily linear commutativity preserving maps on $M_n(\mathbb{C})$? The difficulties that we entered in our attempts to solve the problem of characterizing linear maps preserving commutativity without assuming bijectivity or the stronger both directions preserving property suggest that it is reasonable to start with these stronger assumptions when treating the more difficult nonlinear case. So, we will be interested in bijective maps $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserving commutativity in both directions. What are examples of such maps? Of course, every similarity transformation is a bijective map preserving commutativity in both directions. The same holds true for anti-similarity transformations. To find nonlinear examples observe that if A and B commute and if p and q are arbitrary complex polynomials then $p(A)$ and $q(B)$ commute as well. So, if we associate to each $A \in M_n(\mathbb{C})$ a polynomial p_A, then the map $A \mapsto p_A(A)$ preserves commutativity. Every such map will be called a locally polynomial map. This kind of maps are in general neither bijective nor they preserve commutativity in both directions. However, if such a map ϕ is bijective and if the polynomials p_A, $A \in M_n(\mathbb{C})$, are chosen in such a way that for every $A \in M_n(\mathbb{C})$ we can find a polynomial q_A such that $A = q_A(p_A(A))$ (in other words, if ϕ is bijective and its inverse is again a locally polynomial map), then it preserves commutativity in both directions. Such maps will be called regular locally polynomial maps. Another example of a bijective map preserving commutativity in both directions is the entrywise complex conjugation. The main result in [75] states that every continuous bijective map on $M_n(\mathbb{C})$ preserving commutativity in both directions is a composition of the maps described above.

Theorem 4.3. *Let $n \geqslant 3$ and let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a continuous bijective map preserving commutativity in both directions. Then there exist an invertible matrix $T \in M_n$ and a regular locally polynomial map $A \mapsto p_A(A)$ such that either $\phi(A) = Tp_A(A)T^{-1}$ for all $A \in M_n$, or $\phi(A) = Tp_A(A^t)T^{-1}$ for all $A \in M_n$, or $\phi(A) = Tp_A(\overline{A})T^{-1}$ for all $A \in M_n$, or $\phi(A) = Tp_A(A^*)T^{-1}$ for all $A \in M_n$. Here, $\overline{A} = \overline{[a_{ij}]} = [\overline{a_{ij}}]$, and $A^* = \overline{A}^t$.*

The most interesting open problem here is whether an analogue holds true for real matrices.

The continuity assumption and the assumption that $n \geqslant 3$ are indispensable in this theorem (see [75] for counterexamples). Let us just mention that we have also a rather good understanding of the structure of noncontinuous bijective maps preserving commutativity in both directions (see [75, Theorem 2.1]). The above result is closely connected with the structural result for maps on matrix algebras that are multiplicative with respect to Lie product [15]. Here we have the weaker assumption that only the zero Lie product is preserved. The cost for obtaining a reasonable structural result under this much weaker assumption are additional bijectivity and continuity assumptions.

It is much easier to study bijective maps preserving commutativity in both directions on the real subspace of all self-adjoint matrices. Namely, it is easy to characterize commuting pairs of self-adjoint matrices. Two such matrices commute if and only if they are simultaneously unitarily similar to diagonal matrices. On the other hand, in the self-adjoint case we can obtain a complete description of such maps also in the infinite-dimensional case [55].

The proof (see [75]) of Theorem 4.3 is rather long. We will present here the main ideas in order to point out the connection with some results from the third section concerning maps on idempotents. Let $\mathscr{S} \subset M_n(\mathbb{C})$ be any subset. We denote by $\mathscr{S}'$ the commutant of $\mathscr{S}$, $\mathscr{S}' = \{A \in M_n(\mathbb{C}) : AB = BA \text{ for every } B \in \mathscr{S}\}$. When $\mathscr{S}$ is a singleton we write shortly $\{A\}' = A'$. Clearly, under assumptions of Theorem 4.3 we have $\phi(\mathscr{S}') = \phi(\mathscr{S})'$ for every subset $\mathscr{S} \subset M_n(\mathbb{C})$. The scalar matrices λI can be characterized as matrices A with the property that $A' = M_n(\mathbb{C})$. Therefore, ϕ maps the set of scalar matrices onto itself. Next we consider nonscalar matrices having maximal or minimal commutants with respect to inclusion. More precisely, we call a nonscalar matrix $A \in M_n(\mathbb{C})$ maximal if every $B \in M_n(\mathbb{C})$ satisfying $A' \subset B'$ and $A' \neq B'$ has to be a scalar matrix. Obviously, ϕ maps the set of maximal matrices onto itself. It is not very difficult to see that a nonscalar matrix is maximal if and only if it is of the form $\lambda P + \mu I$ for some scalars λ, μ, $\lambda \neq 0$, and some nontrivial idempotent P or of the form $\lambda I + N$ for some scalar λ and some square-zero matrix $N \neq 0$. Recall that a matrix is nonderogatory if its Jordan form has exactly one Jordan block corresponding to each eigenvalue. One can prove that these are the matrices with minimal commutants, and therefore, ϕ preserves nonderogatory matrices. In the next step we want to prove that ϕ preserves matrices with n distinct eigenvalues. Such matrices are nonderogatory and to prove our assertion we have to characterize such matrices among all nonderogatory matrices using the commutativity relation. This requires quite some work but the basic idea can be explained with the following 4×4 example. The matrices

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $\lambda_i \neq \lambda_j$ whenever $i \neq j$, are both nonderogatory. We want to show that A cannot be mapped into B by a bijective map preserving commutativity in both directions. To show this we observe that any matrix $C \in A'$ is of one of the following forms:

$$\begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix}, \quad \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix},$$

$$\begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix},$$

$$\begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix}, \quad \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix},$$

$$\begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix}, \quad \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix},$$

$$\begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{bmatrix},$$

where $\alpha, \beta, \gamma, \delta$ are any pairwise distinct scalars. Thus, all the matrices belonging to A' were divided into 15 classes. Any two matrices belonging to the same class have the same commutant. For example the commutant of

$$\begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix},$$

where α, β, and γ are pairwise distinct, is the space of all matrices of the form

$$\begin{bmatrix} * & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix},$$

no matter what are the values of α, β, and γ. It is also clear that matrices from A' belonging to different classes have different commutants. We have shown that the set $\{C' : C \in A'\}$ has cardinality 15. If A was mapped by a bijective map preserving commutativity in both directions into B, then B would have the same property, that is, we would have $\#\{C' : C \in B'\} = 15$. But clearly,

$$C_\omega = \begin{bmatrix} 0 & 0 & 1 & \omega \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

belongs to B' for every $\omega \in \mathbb{C}$ and $C'_\omega \neq C'_\tau$ whenever $\omega \neq \tau$. Thus, the set $\{C' : C \in B'\}$ is not finite, and consequently, A cannot be mapped into B.

This kind of reasoning brings us to the conclusion that ϕ maps matrices with n distinct eigenvalues into matrices of the same kind. Now, it is easy to see that diagonalizable matrices are exactly those matrices that commute with some matrix having n distinct eigenvalues. So, ϕ preserves diagonalizable matrices. We denote by $\mathscr{D}_k$, $k = 1, \ldots, n$, the set of all diagonalizable matrices with exactly k eigenvalues. The set $\mathscr{D}_1$ is the set of all scalar matrices and we already know that $\phi(\mathscr{D}_1) = \mathscr{D}_1$. In order to see that $\phi(\mathscr{D}_2) = \mathscr{D}_2$ we only have to observe that $\mathscr{D}_2$ is the intersection of the set of all diagonalizable matrices and the set of maximal matrices. It is not difficult to verify that for a diagonalizable matrix A the following two statements are equivalent:

- $A \in \mathscr{D}_3$,
- $A \notin \mathscr{D}_1 \cup \mathscr{D}_2$ and every matrix $B \in \mathscr{D}$ satisfying $B \in A'$, $A' \subset B'$, and $A' \neq B'$ belongs to $\mathscr{D}_1 \cup \mathscr{D}_2$.

It follows easily that $\phi(\mathscr{D}_3) = \mathscr{D}_3$. Repeating this procedure we get $\phi(\mathscr{D}_k) = \mathscr{D}_k$, $k = 1, \ldots, n$.

In the next step we show that the set of all matrices of the form $\lambda P + \mu I$, where $\lambda \neq 0$ and P is an idempotent of rank one, is mapped by ϕ onto itself. Thus, we have to characterize such matrices among all diagonalizable matrices with exactly two eigenvalues. If a diagonalizable matrix B has two eigenvalues none of them being of multiplicity one then we may assume that it is of the diagonal form

$$\begin{bmatrix} \alpha I_1 & 0 & 0 & 0 \\ 0 & \alpha I_2 & 0 & 0 \\ 0 & 0 & \beta I_3 & 0 \\ 0 & 0 & 0 & \beta I_4 \end{bmatrix},$$

where $\alpha \neq \beta$ and the I_j's are the identity matrices of appropriate sizes. The matrix

$$C = \begin{bmatrix} \alpha I_1 & 0 & 0 & 0 \\ 0 & \beta I_2 & 0 & 0 \\ 0 & 0 & \beta I_3 & 0 \\ 0 & 0 & 0 & \alpha I_4 \end{bmatrix}$$

commutes with B and also belongs to $\mathscr{D}_2$. The first commutant $\{B, C\}'$ is equal to the set of all matrices with the block diagonal form

$$\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix},$$

where the $*$'s represent arbitrary square matrices of the sizes corresponding to the above block representation of B. It is then easy to see that the second commutant $\{B, C\}'' = \{F \in M_n(\mathbb{C}) : FT = TF \text{ for every } T \in \{B, C\}'\}$ contains the diagonal matrix

$$\begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & 2I_2 & 0 & 0 \\ 0 & 0 & 3I_3 & 0 \\ 0 & 0 & 0 & 4I_4 \end{bmatrix}$$

having exactly 4 eigenvalues. It is rather easy to see that if a diagonalizable matrix A has two eigenvalues one of them being of multiplicity one and if $C \in \mathscr{D}_2$ commutes with A, then any diagonalizable matrix contained in $\{A, C\}''$ has at most 3 eigenvalues.

This shows that the set of all matrices of the form $\lambda P + \mu I$, where $\lambda \neq 0$ and P is an idempotent of rank one, is mapped by ϕ onto itself. If $A = \lambda P + \mu I$ and $B = \lambda_1 P + \mu_1 I$, where $\lambda \neq 0$, $\lambda_1 \neq 0$, and P is an idempotent of rank one, then we already know that $\phi(A) = \alpha Q + \beta I$ and $\phi(B) = \alpha_1 Q_1 + \beta_1 I$, where $\alpha \neq 0, \alpha_1 \neq 0$, and Q and Q_1 are idempotents of rank one. As $A' = B'$ we have $\phi(A)' = \phi(B)'$ which yields that $Q = Q_1$. Thus ϕ induces in a natural way a map $\varphi : P_n^1(\mathbb{C}) \to P_n^1(\mathbb{C})$ and this map preserves commutativity. But two rank one idempotents commute if and only if they are equal or orthogonal. Thus, φ preserves orthogonality and we can apply the results discussed in the third section to conclude that ϕ has a desired form on all diagonalizable matrices with exactly two eigenvalues one of them being of multiplicity one. From here it is rather easy to conclude that ϕ has the desired form on the set of all diagonalizable matrices. It is then possible to conclude the proof of Theorem 4.3 using the continuity assumption.

Another recent result on general preservers was motivated by the theory of Lie algebras. Let $\mathscr{L}$ be a Lie algebra. For some basic definitions and facts concerning

Lie algebras we refer to [66]. One of the fundamental concepts in this theory is that of a solvable Lie algebra. In [64] bijective maps $\phi : \mathscr{L} \to \mathscr{L}$ with the property that both ϕ and its inverse map every solvable Lie subalgebra into some solvable Lie subalgebra were characterized in the special case when $\mathscr{L}$ is the Lie algebra $M_n(\mathbb{C})$ equipped with the Lie product $[\cdot, \cdot]$, $[A, B] = AB - BA$. The famous Lie's theorem [66, pp. 21–23] states that every solvable Lie subalgebra of $M_n(\mathbb{C})$ is similar to a triangular one. In other words, a Lie subalgebra $\mathscr{L} \subset M_n(\mathbb{C})$ is solvable if and only if there exists a triangularizing chain of invariant subspaces for $\mathscr{L}$. Here, of course, by an invariant subspace of $\mathscr{L}$ we mean a subspace that is invariant under every member of $\mathscr{L}$. Using Lie's theorem it is possible to show that preserving solvability in both directions is equivalent to preserving simultaneous triangularizability of matrix pairs in both directions. And simultaneous triangularizability of matrices A and B can be considered as a generalization of commutativity of this matrix pair. Therefore, the study of solvability preserving maps was based on some ideas from [75]. The obvious examples of linear bijective maps preserving solvability in both directions are similarity transformations, anti-similarity transformations and ring automorphisms of $M_n(\mathbb{C})$ induced by automorphisms of the complex field. All these examples are semilinear. To get nonadditive examples we define two matrices A and B to be lattice-equal, denoted by $A \sim B$, if they have exactly the same lattice of invariant subspaces. The complete description of this equivalence relation can be found in [19, Theorem 10.2.1] and [79]. Lie's theorem yields that a bijective map $\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying $\tau(A) \sim A$, $A \in M_n(\mathbb{C})$, preserves solvability in both directions. Such a map is just an arbitrary permutation on each of the equivalence classes with respect to $\sim$. Every such map is called a bijective lattice preserving map. In [64] it was shown that every bijective map on $M_n(\mathbb{C})$ preserving solvability in both directions is a composition of the types of maps described in this paragraph.

We continue with an infinite-dimensional result on nonlinear preservers. Let X be an infinite-dimensional real or complex Banach space. A bijective map $\phi : B(X) \to B(X)$ is called biseparating if $AB = 0 \iff \phi(A)\phi(B) = 0$, $A, B \in B(X)$. Clearly, every bijective map $\eta : B(X) \to B(X)$ with the property that $\overline{\operatorname{Im} \eta(A)} = \overline{\operatorname{Im} A}$ and $\operatorname{Ker} \eta(A) = \operatorname{Ker} A$, $A \in B(X)$, is biseparating. In [76] we have proved that every bijective biseparating map on $B(X)$ is a composition of such a map and an inner linear or (in the complex case) conjugate-linear automorphism of $B(X)$. The main idea was to reduce this problem to the problem of characterizing bijective maps on rank one idempotents preserving orthogonality in both directions and then to apply results discussed in the third section. How far can we relax the assumptions in the finite-dimensional case and still get a reasonable result? More precisely, can we omit the bijectivity assumption or replace it by the weaker injectivity or surjectivity assumption, can we assume that zero products are preserved in one direction only, and finally, can we replace the real or the complex field by a more general field or even an arbitrary division ring?

Note that in contrast to linear preservers, general preservers have much richer structures. In a certain sense, it is natural (though sometimes surprising to see) that the linearity assumption together with some preserving property will give rise to a "standard map" acting on the matrix space. On the other hand, if one just imposes an analytic or a topological assumption on the maps together with some preserving property, then one get maps on matrices with local behaviours such as local similarity and equivalence. If one further relaxes the assumptions on the preservers, the mappings will only send matrices in the same equivalence classes such as matrices having the same eigenvalues, the same Jordan form, etc.

Certainly, these are only the first steps and a lot of work will have to be done to achieve an understanding of the structure of general preservers comparable to our understanding of linear preservers.

In [58–60] maps on the full matrix algebra as well as on some subalgebras were considered having more than just one preserving property. If we have enough preserving properties (for example, if we study continuous maps preserving spectrum and commutativity in both directions), then such maps must be necessarily linear Jordan automorphisms. It would be interesting to find other collections of sets, properties or relations whose preservation characterizes Jordan automorphisms.

When discussing possible applications of the structural results for maps on idempotents [72] we suggested that several preserver problems concerning partial orders on matrices can be solved without the linearity assumption and indicated how to reduce this kind of problems to the structural problems for order preserving maps on idempotents. Following this idea Legiša [38] recently characterized surjective maps on matrix algebras preserving minus partial ordering in both directions. It is interesting to note that such maps are automatically semilinear. It is remarkable that in some preserver problems the semilinear character of the maps under consideration is not an assumption but we get it as a conclusion. The most important examples of this phenomenon are so called fundamental theorems of geometry of matrices that will be considered in our last section.

5. Geometry of matrices

The study of geometry of matrices was initiated by Hua in [24–31]. Let $\mathbb{D}$ be any division ring and $M_{m\times n}(\mathbb{D})$ the set of all $m \times n$ matrices over $\mathbb{D}$. It is easy to verify that this is a metric space with the distance defined by $d(A, B) = \operatorname{rank}(A - B)$, $A, B \in M_{m\times n}(\mathbb{D})$. In fact, all we have to know to check this is that rank is subadditive, that is, $\operatorname{rank}(A + B) \leqslant \operatorname{rank} A + \operatorname{rank} B$, $A, B \in M_{m\times n}(\mathbb{D})$. Note that since $\mathbb{D}$ is not necessarily commutative we have to be careful when defining the rank of a matrix A. We denote by $\mathbb{D}^n$ the set of all $1 \times n$ matrices and consider it always as a left vector space over $\mathbb{D}$. Correspondingly, we have the right vector space of all $m \times 1$ matrices $(\mathbb{D}^m)^t$. We first take the left vector subspace of $\mathbb{D}^n$ generated by the rows of A (the row space of A) and define the row rank of A to be the dimension of this subspace. The column rank of A is the dimension of the right vector space generated by columns of A. This space is called the column space of A. These two ranks are equal for every matrix over $\mathbb{D}$ and this common value is called the rank of a matrix. If rank $A = r$ then there exist invertible matrices $T \in M_m(\mathbb{D})$ and $S \in M_n(\mathbb{D})$ such that

$$TAS = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Here, I_r is the $r \times r$ identity matrix and zeroes stand for zero matrices of the appropriate size. Note also that rank of a matrix and its transpose are not necessarily the same. However, if τ is a nonzero anti-endomorphism of $\mathbb{D}$ then rank $A =$ rank A_τ^t.

Two matrices $A, B \in M_{m\times n}(\mathbb{D})$ are said to be adjacent if $d(A, B) = 1$. Suppose that $T \in M_m(\mathbb{D})$ and $S \in M_n(\mathbb{D})$ are invertible matrices and $R \in M_{m\times n}(\mathbb{D})$ any matrix. Then, obviously, the map ϕ defined by $A \mapsto TAS + R$ is a bijective map on $M_{m\times n}(\mathbb{D})$ preserving adjacency in both directions, that is, for every pair $A, B \in M_{m\times n}(\mathbb{D})$ the matrices A and B are adjacent if and only if $\phi(A)$ and $\phi(B)$ are adjacent. If $A \in M_{m\times n}(\mathbb{D})$ is any matrix and $\sigma : \mathbb{D} \to \mathbb{D}$ an automorphism of division ring $\mathbb{D}$ then the matrix A_σ has the same rank as A. So, the map $A \mapsto A_\sigma$ is bijective and preserves adjacency in both directions. Similarly, if $m = n$ and τ is an anti-automorphism of $\mathbb{D}$ then $A \mapsto A_\tau^t$ is a bijective map preserving adjacency in both directions.

The fundamental theorem of the geometry of matrices states that every bijective map $\phi : M_{m\times n}(\mathbb{D}) \to M_{m\times n}(\mathbb{D})$ preserving adjacency in both directions is of the form $A \mapsto TA_\sigma S + R$, where T, S, R, and σ are as above. If $m = n$, then we have the additional possibility that $\phi(A) = TA_\tau^t S + R$ where $T, S, R \in M_n(\mathbb{D})$ with T and S invertible and τ is an anti-automorphism of $\mathbb{D}$. This theorem was proved by Hua [31] under the additional assumption that $\mathbb{D} \neq \mathbb{F}_2$. The special case $\mathbb{D} = \mathbb{F}_2$ was solved by his followers (see [80], where also similar results for symmetric matrices, skewsymmetric matrices, and hermitian matrices can be found). In the case of upper triangular matrices the structure of bijective maps preserving adjacency in both directions is more complicated [9,10].

It is remarkable that we get such a strong conclusion (the map ϕ has a very simple form and in particular, up to a translation it is additive and even semilinear when $\mathbb{D}$ is a field) under rather weak assumptions of bijectivity and preserving adjacency in both directions. It is therefore not surprising that this theorem has many applications. Let us mention just the most important ones. Hua obtained the structural results for Jordan and Lie automorphisms of matrix rings as easy consequences of the fundamental theorem of the geometry of matrices. The most frequently used method in the theory of linear preservers is the reduction of a problem of characterizing certain linear preservers to the problem of characterizing linear maps preserving rank one matrices. Obviously, linear maps preserving rank one matrices preserve the pairs of adjacent matrices. Therefore, the fundamental theorem of the geometry of matrices can be applied when studying linear preservers. And finally, some theorems considering the geometry of Grassman spaces can be deduced from the fundamental theorem of the geometry of matrices. Let us explain this very briefly. Let m, n be positive integers. We will consider the Grassman space of all vector subspaces of $\mathbb{D}^{m+n}$ of dimension m. Chow [11] and Dieudonné [13,14] studied bijective maps on the Grassman space preserving adjacent pairs of points in the Grassman space (vector subspaces of dimension m) in both directions. Recall that the m-dimensional subspaces U and V are adjacent if $\dim(U+V)=m+1$. Now, to each m-dimensional subspace U of $\mathbb{D}^{m+n}$ we can associate an $m\times(m+n)$ matrix whose rows are coordinates of the vectors that form a basis of U. Each $m\times(m+n)$ will be written in the block form $[X \ \ Y]$, where X is an $m\times n$ matrix and Y is an $m\times m$ matrix. Two matrices $[X \ \ Y]$ and $[X' \ \ Y']$ are associated to the same subspace U (their rows represent two bases of U) if and only if $[X \ \ Y]=P[X' \ \ Y']$ for some invertible $m\times m$ matrix P. If this is the case, then Y is invertible if and only if Y' is invertible. So, we have associated to each point in a Grassman space a (not uniquely determined) matrix $[X \ \ Y]$. If Y is singular, we call the corresponding point in the Grassman space point at infinity. Otherwise, we observe that this point can be represented also with the matrix $[Y^{-1}X \ \ I]$. The matrix $Y^{-1}X$ is uniquely determined by the point in the Grassman space. So, if U and V are two m-dimensional subspaces that are finite points in the Grassman space, then they can be represented with two uniquely determined $m\times n$ matrices T and S and it is easy to see that the subspaces U and V are adjacent if and only if the matrices T and S are adjacent. Using this connection it is possible to deduce the result of Chow on bijective maps on a Grassman space preserving adjacency in both directions from the fundamental theorem of geometry of matrices.

We hope that we have succeeded to convince the reader that the fundamental theorem of the geometry of matrices is a strong result with important consequences. When we say strong we mean that we get a strong conclusion under very weak assumptions. Still, one may ask if we can get the same conclusion under even weaker assumptions.

Can we get an almost the same conclusion without the bijectivity assumption (with an almost the same conclusion we mean the same assertion with the only difference that σ and τ are not necessarily bijective)? Surprisingly, the answer depends on the underlying division ring. In [68] it was proved that the answer is in the affirmative for real matrices but negative for complex matrices. Is this really surprising? It is perhaps less surprising if we recall structural results for maps on idempotents. There we get nice structural results for idempotents over EAS-division rings and many counterexamples in the complex case.

We have already mentioned that besides the fundamental theorem of the geometry of matrices we have also fundamental theorems of the geometry of symmetric matrices, skewsymmetric matrices, and hermitian matrices [80]. These theorems, of course, characterize bijective maps preserving adjacency in both directions on the set of symmetric matrices, skewsymmetric matrices, and hermitian matrices, respectively. As far as we know the problem of validity of these results without the bijectivity assumption is still open.

Can we get the same conclusion as in the fundamental theorem of the geometry of matrices under the weaker assumption that the adjacency is preserved in one direction only? This long standing open question has been recently answered in the affirmative by Huang and Wan [33]. The same authors together with Höfer solved positively also the analogous problem for symmetric and hermitian matrices [32]. Everybody working in linear preservers knows that characterizing linear maps preserving a certain property or relation in one direction only is usually more difficult than characterizing linear maps preserving this property or relation in both directions. This holds even more in the absence of the linearity assumption. So, the above results are substantial improvements of the fundamental theorem and we believe they will prove to be important in the theory of linear and general preservers.

Let us conclude with the connection between the geometry of matrices and the structural results for order preserving maps on idempotents discovered in [72]. To present the most important idea we consider just the special case that ϕ is a map on the set of all $n \times n$ matrices preserving adjacency. Moreover, we assume that $\phi(0) = 0$ and $\phi(I) = I$. We then claim that ϕ maps idempotents into idempotents and that the restriction of ϕ to the set of idempotents preserves order. Indeed, let P and Q be idempotents with $P \leqslant Q$. Then, up to a similarity, P and Q are diagonal idempotents, and therefore, we can find a chain of idempotents $0 = P_0 \leqslant P_1 \leqslant P_2 \leqslant \cdots \leqslant P_{n-1} \leqslant P_n = I$ such that rank $P_k = k$, $k = 0, 1, \ldots, n$, P_k and P_{k+1} are adjacent, $k = 0, 1, \ldots, n-1$, and P and Q are members of this chain. We denote $\phi(P_k) = Q_k$, $k = 0, 1, \ldots, n$. We know that $Q_0 = 0$ and $Q_n = I$. Now, Q_1 is adjacent to 0, and therefore, Q_1 is of rank one. Since Q_2 is adjacent to Q_1, it is of rank at most two. Proceeding in the same way we conclude that rank $Q_k \leqslant k$, $k = 0, 1, \ldots, n$. In particular, Q_{n-1} is a matrix of rank at most $n-1$ adjacent to I. One can prove that then it must be an idempotent of rank $n-1$. Further, Q_{n-2} is a matrix of rank at most $n-2$ that is adjacent to Q_{n-1} which is an idempotent of rank $n-1$. It follows rather easily that Q_{n-2} is an idempotent of rank $n-2$ satisfying $Q_{n-2} \leqslant Q_{n-1}$. Continuing in this way we conclude that all the Q_k's are idempotents with $Q_k \leqslant Q_{k+1}$, $k = 0, 1, \ldots, n-1$. In particular, $\phi(P)$ and $\phi(Q)$ are idempotents satisfying $\phi(P) \leqslant \phi(Q)$, as desired. Thus, the results from the third section can be applied to study adjacency preserving maps.

In our forthcoming paper we will use this approach to unify and extend some of the above results and to clarify the problem of characterizing (not necessarily bijective) maps preserving adjacency in both directions.

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