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An improvement of an inequality of Koksma

by Oto Strauch

Mathematical Institute of the Slovak Academy of Sciences, Stefanikova ul. 49, 81473 Bratislava, Czecho-Slovakia

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ABSTRACT

An expression for the Koksma integral [Indag. Math. **13** (1951), 285-287; MR **13**, 539] is given. The result is applied to show that the constant 1/3 in Koksma's inequality [ibidem] can be replaced by 1/4 and this constant is the best possible. Also the mean value of the integral is determined.

1. INTRODUCTION

The purpose of this note is to present an improvement, stated as Corollary 1, of Koksma's inequality [2]. In Theorem 1 we state the main result of this note, which gives an explicit formula of the Koksma integral [2] and the note concludes with a computation of the mean value of the Koksma integral for the set of intervals $[x,y)\subset[0,1]$. An analogous result is established for the set of intervals $[0,y)\subset[0,1]$. The proofs use only elementary properties of fractions.

We begin with definitions which we shall need later for the formulation of results. Our notations are standard and can be found in [1].

For a real number x, let $[x]$ denote the *integral part* of x and let $\{x\}=x-[x]$ be the *fractional part* of x.

Let $\omega_N=(x_n)_{n=1}^N$ be a given finite sequence of real numbers from the unit interval [0, 1] and let the *counting function* $A([0,x),\omega_N)$ be defined as

$$A([0,x),\omega_N)=\#\{n\le N; x_n\in[0,x)\},$$

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and the *remainder function* $R_N(x)$ by

$$R_N(x) = A([0,x), \omega_N) - Nx$$

for all $x \in (0,1)$ and $R_N(0) = R_N(1) = 0$.

Suppose we are given a finite sequence $q_1, q_2, \ldots, q_N$ of positive integers. Consider the sequence

$$\omega_N(\alpha) = (\{q_1\alpha\}, \{q_2\alpha\}, \ldots, \{q_N\alpha\}).$$

Set

$$R_N(x, y, \alpha) = A([x, y), \omega_N(\alpha)) - N(y - x).$$

Koksma [2] introduced and studied the integral $\int_0^1 R_N^2(x, y, \alpha)\,d\alpha$. He proved in a very simple way that

$$\int_0^1 R_N^2(x, y, \alpha)\,d\alpha \le \frac{1}{3} \sum_{m,n=1}^{N} \frac{(q_m, q_n)^2}{q_m q_n},$$

where (q_m, q_n) denotes the greatest common divisor. This is the inequality mentioned in the title. For the Koksma integral $\int_0^1 R_N^2(x, y, \alpha)\,d\alpha$ we first give

THEOREM 1

$$\int_0^1 R_N^2(x, y, \alpha)\,d\alpha = \sum_{m,n=1}^{N} \frac{(q_m, q_n)^2}{q_m q_n}$$
$$\times ((\{yc_{m,n}\} - \{xc_{m,n}\})(\{xc_{n,m}\} - \{yc_{n,m}\}) - \{yc_{m,n}\} + \{xc_{m,n}\}$$
$$+ \max(\{yc_{m,n}\}, \{xc_{n,m}\}) - \max(\{xc_{m,n}\}, \{xc_{n,m}\})$$
$$+ \min(\{yc_{m,n}\}, \{yc_{n,m}\}) - \min(\{xc_{m,n}\}, \{yc_{n,m}\})),$$

where

$$c_{m,n} = \frac{q_m}{(q_m, q_n)}.$$

PROOF. The remainder function $R_N(x, y, \alpha)$ can be expanded into

$$R_N(x, y, \alpha) = \sum_{n=1}^{N} c_{[x,y)}(\{q_n\alpha\}) - N(y - x).$$

Here $c_{[x,y)}$ denotes the characteristic function of the interval $[x, y)$ that

$$c_{[x,y)}(t) = \begin{cases} 1, & \text{for } t \in [x, y), \\ 0, & \text{for other cases.} \end{cases}$$

Let a and b be positive integers. For every pair of real numbers x and y, $0 \le x \le y \le 1$, we have

$$\int_0^1 c_{[x,y)}(\{a\alpha\})\,d\alpha = y - x.$$

Thus, we find that

$$\int_0^1 R_N^2(x, y, \alpha)\,d\alpha = \int_0^1 \Big(\sum_{n=1}^{N} c_{[x,y)}(\{q_n\alpha\})\Big)^2\,d\alpha - N^2(y - x)^2.$$

Now we shall compute

$$\int_0^1 c_{[x,y)}(\{a\alpha\})c_{[x,y)}(\{b\alpha\})d\alpha$$

$$=\sum_{k=0}^{a-1}\int_{k/a}^{k+1/a} c_{[x,y)}(\{a\alpha\})c_{[x,y)}(\{b\alpha\})d\alpha.$$

The integral $\int_{k/a}^{k+1/a} c_{[x,y)}(\{a\alpha\})c_{[x,y)}(\{b\alpha\})d\alpha$ is expressed by the substitution $\alpha=(1/a)\beta+k/a$ as

$$\frac{1}{a}\int_0^1 c_{[x,y)}(\beta)c_{[x,y)}\left(\left\{\frac{b}{a}\beta+\frac{b}{a}k\right\}\right)d\beta.$$

Now assume that $(a,b)=1$. It is then evident that the set of fractional parts

$$\left\{\frac{b}{a}\beta+\frac{b}{a}k\right\},\ k=0,1,\dots,a-1$$

is the same as

$$\frac{\{b\beta\}+k}{a},\quad k=0,1,\dots,a-1.$$

Moreover

$$\sum_{k=0}^{a-1} c_{(x,y]}\left(\frac{\{b\beta\}+k}{a}\right)=[ay-\{b\beta\}]-[ax-\{b\beta\}]$$
$$=a(y-x)+\{ax-\{b\beta\}\}-\{ay-\{b\beta\}\}.$$

It is easy to see that

$$\{ax-\{b\beta\}\}=\begin{cases}\{ax\}-\{b\beta\}, & \text{if } \{ax\}\ge\{b\beta\},\\ \{ax\}-\{b\beta\}+1, & \text{if } \{ax\}<\{b\beta\},\end{cases}$$

and consequently, we obtain

$$\int_0^1 c_{[x,y)}(\{a\alpha\})c_{[x,y)}(\{b\alpha\})d\alpha=(y-x)^2+\frac{1}{a}(y-x)(\{ax\}-\{ay\})$$
$$+\frac{1}{a}\cdot\int_{\substack{x\le\beta\le y\\ \{ax\}<\{b\beta\}}} 1d\beta-\frac{1}{a}\cdot\int_{\substack{x\le\beta\le y\\ \{ay\}<\{b\beta\}}} 1d\beta.$$

We shall distinguish two following cases:

I. Consider first the case where $s/b\in(x,y]$ for some integer s, $0\le s\le b$. Then the number of intervals $[s/b,(s+1)/b]\subset(x,y]$ is equal to $[by]-[bx]-1$ and for all $[s/b,(s+1)/b]\subset(x,y]$ we will have

$$\frac{1}{a}\cdot\int_{\substack{s/b\le\beta\le s+1/b\\ \{ax\}<\{b\beta\}}} 1d\beta-\frac{1}{a}\cdot\int_{\substack{s/b\le\beta\le s+1/b\\ \{ay\}<\{b\beta\}}} 1d\beta=\frac{\{ay\}-\{ax\}}{ab}.$$

Integrating over the intervals $[x, ([bx]+1)/b]$ and $[[by]/b, y]$ we have

$$\frac{1}{a}\cdot\int\limits_{\substack{x\leq\beta\leq[bx]+1/b\\ \{ax\}<\{b\beta\}}} 1d\beta-\frac{1}{a}\cdot\int\limits_{\substack{x\leq\beta\leq[bx]+1/b\\ \{ay\}<\{b\beta\}}} 1d\beta=\frac{\max(\{ay\},\{bx\})-\max(\{ax\},\{bx\})}{ab}$$

and

$$\frac{1}{a}\cdot\int\limits_{\substack{[by]/b\leq\beta\leq y\\ \{ax\}<\{b\beta\}}} 1d\beta-\frac{1}{a}\cdot\int\limits_{\substack{[by]/b\leq\beta\leq y\\ \{ay\}<\{b\beta\}}} 1d\beta=\frac{\min(\{ay\},\{by\})-\min(\{ax\},\{by\})}{ab},$$

respectively. Collecting all these results, we obtain

$$(1)\qquad \begin{cases} \int_0^1 c_{[x,y)}(\{a\alpha\})c_{[x,y)}(\{b\alpha\})d\alpha \\ \quad =(y-x)^2+\dfrac{1}{ab}((\{ay\}-\{ax\})(\{bx\}-\{by\})-\{ay\}+\{ax\} \\ \qquad +\max(\{ay\},\{bx\})-\max(\{ax\},\{bx\}) \\ \qquad +\min(\{ay\},\{by\})-\min(\{ax\},\{by\})). \end{cases}$$

II. Now consider the interval $(x, y]\subset[0, 1]$ satisfying the condition $[bx]=[by]$. By using the same line as in the above proof, it can be shown that

$$(2)\qquad \begin{cases} \int_0^1 c_{[x,y)}(\{a\alpha\})c_{[x,y)}(\{b\alpha\})d\alpha \\ \quad =(y-x)^2+\dfrac{(\{ax\}-\{ay\})(\{by\}-\{bx\})}{ab} \\ \qquad +\begin{cases} \dfrac{\{by\}-\{bx\}}{ab}, & \text{for } \{ax\}<\{bx\} \\ \dfrac{\{by\}-\{ax\}}{ab}, & \text{for } \{bx\}\leq\{ax\}<\{by\} \\ 0, & \text{for } \{by\}\leq\{ax\} \end{cases} \\ \qquad -\begin{cases} \dfrac{\{by\}-\{bx\}}{ab}, & \text{for } \{ay\}<\{bx\} \\ \dfrac{\{by\}-\{ay\}}{ab}, & \text{for } \{bx\}\leq\{ay\}<\{by\} \\ 0, & \text{for } \{by\}\leq\{ay\}. \end{cases} \end{cases}$$

Since $\{bx\}\leq\{by\}$, the right-hand sides of (1) and (2) coincide. Hence we readily find that (1) is true for each of these two possible cases. This completes the proof of (1) for any $[x, y)\subset[0, 1]$. To complete the proof of our theorem let us assume that $(a, b)=c>1$. To the integral $\int_0^1 c_{[x,y)}(\{a\alpha\})c_{[x,y)}(\{b\alpha\})d\alpha$ we apply the following formula of integration [3]

$$\int_0^1 f(\{c\alpha\})\,d\alpha=\int_0^1 f(\alpha)\,d\alpha$$

which holds for all Riemann's integrable functions $f:[0,1]\to\Re$. The theorem has thus been proved.

Since

$$(3)\qquad \begin{cases} \max\limits_{0\le x,y,u,v\le 1}\ ((y-x)(u-v)-y+x \\ \qquad +\max(y,u)-\max(x,u) \\ \qquad +\min(y,v)-\min(x,v))=\dfrac{1}{4} \end{cases}$$

we obtain the following corollary.

COROLLARY 1. *For any finite sequence* $q_1,\dots,q_N$ *of* N *positive integers, we have*

$$\int_0^1 R_N^2(x,y,\alpha)\,d\alpha\le\frac{1}{4}\sum_{m,n=1}^{N}\frac{(q_m,q_n)^2}{q_mq_n}$$

for every interval $[x,y)\subset[0,1]$.

The constant 1/4 is best possible, since we may have equality here, for instance when $q_1=\cdots=q_N=1$, $x=0$ and $y=1/2$.

We shall demonstrate now that from the viewpoint of measure theory, the previous inequality can be improved for a large set of intervals $[x,y)\subset[0,1]$. Namely, the mean value of the Koksma integral can be written in the following form.

THEOREM 2

$$2\cdot\iint_{0\le x\le y\le 1} dxdy\int_0^1 R_N^2(x,y,\alpha)\,d\alpha=\frac{1}{6}\sum_{\substack{m,n=1\\ q_m=q_n}}^{N}1.$$

PROOF. For the remainder function $R_N(x)$ of any ω_N we have [4]

$$\iint_{0\le x\le y\le 1}(R_N(y)-R_N(x))^2\,dxdy=\int_0^1 R_N^2(x)\,dx-\Big(\int_0^1 R_N(x)\,dx\Big)^2$$

$$=\frac{1}{2\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\Big|\sum_{n=1}^{N}e^{2\pi ihx_n}\Big|^2.$$

The rest follows from

$$\int_0^1 e^{2\pi ih(a-b)\alpha}\,d\alpha=\begin{cases}0, & \text{for } a\ne b,\\ 1, & \text{for } a=b,\end{cases}$$

and from

$$\sum_{h=1}^{\infty} \frac{1}{h^2} = \frac{\pi^2}{6}.$$

To obtain an information about the efficiency of (3) we look for the value $N=2$ in Theorem 2.

COROLLARY 2

$$2 \cdot \iint_{0 \le x \le y \le 1} ((\{ay\} - \{ax\})(\{bx\} - \{by\}) - \{ay\} + \{ax\} + \max(\{ay\}, \{bx\}) - \max(\{ax\}, \{bx\}) + \min(\{ay\}, \{by\}) - \min(\{ax\}, \{by\}))\, dxdy$$

$$= \begin{cases} 0, & \text{for } a \ne b, \\ \frac{1}{6}, & \text{for } a = b, \end{cases}$$

holds for all positive integers a and b.

Along the same lines, upon the restriction of the set of intervals $[x, y) \subset [0, 1]$ to $[0, y) \subset [0, 1]$, it can be shown that

$$\int_0^1 \int_0^1 R_N^2(0, y, \alpha)\, dy d\alpha = \frac{1}{12} \sum_{m,n=1}^{N} \frac{(q_m, q_n)^2}{q_m q_n} + \frac{1}{12} \sum_{\substack{m,n=1 \\ q_m = q_n}}^{N} 1,$$

and

$$\int_0^1 (\min(\{ay\}, \{by\}) - \{ay\}\{by\})\, dy = \begin{cases} \frac{1}{12}, & \text{for } a \ne b, \\ \frac{1}{6}, & \text{for } a = b. \end{cases}$$

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