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The Drazin inverse of the linear combinations of two idempotents in the Banach algebra ☆

Shifang Zhang [a,*], Junde Wu [b]

[a] Department of Mathematics, Fujian Normal University, Fuzhou 350007, PR China
[b] Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China

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ABSTRACT

In this paper, some Drazin inverse representations of the linear combinations of two idempotents in a Banach algebra are obtained. Moreover, we present counter-examples to and establish the corrected versions of two theorems by Cvetković-Ilić and Deng.

1. Introduction

Let $\mathscr{A}$ be a Banach-$*$ algebra. An element $P \in \mathscr{A}$ is said to be an idempotent if $P^2 = P$ and a projection if $P^2 = P = P^*$. The set $\mathscr{P}(\mathscr{A})$ of all idempotents in $\mathscr{A}$ is invariant under similarity, that is, if $P \in \mathscr{P}(\mathscr{A})$ and $S \in \mathscr{A}$ is an invertible element, then $S^{-1}PS$ is still an idempotent.

Let us recall that the Drazin inverse of $A \in \mathscr{A}$ is the element $B \in \mathscr{A}$ (denoted by A^D) which satisfies

$$BAB = B, \quad AB = BA, \quad A^{k+1}B = A^k \tag{1}$$

for some nonnegative integer k. The least such k is the index of A, denoted by $\mathrm{ind}(A)$. The Drazin inverse for bounded linear operators on complex Banach spaces was investigated by Caradus [11]. Therein it

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* Corresponding author.
E-mail address: shifangzhangfj@163.com (S. Zhang).

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was established that the Drazin inverse of operator A exits if and only if 0 is at most a pole of the resolvent $R(\lambda, A)$, which is also equivalent to the descent and ascent of A to be both finite. Some more results about Drazin inverse can be found in [24] and references cited therein. It is well-known that if A is Drazin invertible, then the Drazin inverse is unique and $(aA)^D = \frac{1}{a}A^D$ for each nonzero scalar a. In particular, for an invertible operator A, the Drazin inverse A^D coincides with the usual inverse A^{-1} and $\mathrm{ind}(A) = 0$. The conditions (1) are also equivalent to

$$BAB = B, \quad AB = BA, \quad A - A^2B \text{ is nilpotent.}$$

The Drazin inverse of an operator in $\mathscr{A}$ is similarity invariant, that is, if T is Drazin invertible and $S \in \mathscr{A}$ is an invertible element, then $S^{-1}TS$ is still Drazin invertible and $(S^{-1}TS)^D = S^{-1}T^DS$. If $P \in \mathscr{P}(\mathscr{A})$, it is easy to verify that $P^D = P$.

This paper is concerned with the Drazin inverses $(aP + bQ)^D$ of the linear combinations of two idempotents in $\mathscr{A}$ for nonzero scalars a and b. In recent years, many authors paid much attention to properties of linear combinations of idempotents or projections (see [1–8,12–23]). In [14], Deng has discussed the Drazin inverses of the products and differences of two projections. Motivated by this paper, Böttcher and Spitkovsky wrote [1] and in that paper they proved that the Drazin invertibility of the sum $P + Q$ of two projections P and Q is equivalent to the Drazin invertibility of any linear combination $aP + bQ$ where $ab \neq 0$, $a + b \neq 0$. However, without some additional conditions, it is difficult to discuss the Drazin invertibility of linear combinations of two idempotents. More recently, under some conditions, Deng in [15] gave the Drazin inverses of sums and differences of idempotents on the Hilbert space. The methods used in [15] are space decompositions and operator matrix representations which are not available for general Banach-$*$ algebras, or general Banach algebras.

In this paper, by using direct calculation methods, we obtain some formulae for the Drazin inverse $(aP + bQ)^D$ of the linear combinations of idempotents P and Q in Banach algebra $\mathscr{A}$ under some conditions, and we also study the index $\mathrm{ind}(aP + bQ)$.

2. Main results and proofs

In this section, we always suppose that $\mathscr{A}$ is a Banach algebra with the unit I and $aP + bQ$ is a linear combination of two idempotents P and Q in $\mathscr{A}$ with nonzero scalars a and b. In order to prove that $aP + bQ$ is Drazin invertible, we only need to find out some $M \in \mathscr{A}$ which satisfies that

$$(aP + bQ)M = M(aP + bQ), \quad M(aP + bQ)M = M, \quad (aP + bQ)^{k+1}M = (aP + bQ)^k \tag{2}$$

for some nonnegative integer k.

The following result is essentially already in [2,20]; note $PQP = 0$ implies that $(PQ)^2 = (QP)^2$. We present a self-contained proof for the reader's convenience.

Theorem 2.1. *Let P and Q be the idempotents in Banach algebra $\mathscr{A}$ and $PQP = 0$. Then $aP + bQ$ is Drazin invertible for any nonzero scalars a and b, $\mathrm{ind}(aP + bQ) \leqslant 1$ and*

$$(aP + bQ)^D = \frac{1}{a}P + \frac{1}{b}Q - \left(\frac{1}{a} + \frac{1}{b}\right)PQ - \left(\frac{1}{a} + \frac{1}{b}\right)QP + \left(\frac{1}{a} + \frac{2}{b}\right)QPQ.$$

Moreover, $\mathrm{ind}(aP + bQ) = 0$ if and only if $P + Q + QPQ = I + PQ + QP$.

Proof. Let

$$M = \frac{1}{a}P + Q - \left(\frac{1}{a} + 1\right)PQ - \left(\frac{1}{a} + 1\right)QP + \left(\frac{1}{a} + 2\right)QPQ.$$

We claim that

$$(aP + Q)^D = M.$$

In fact, by the assumption that $PQP = 0$, we have that

$$M(aP + Q) = (aP + Q)M = P + Q - PQ - QP + QPQ.$$

Also a direct calculation shows that

$$\begin{aligned} M(aP + Q)M &= (P + Q - PQ - QP + QPQ)M \\ &= \left[\frac{1}{a}P + PQ - \left(\frac{1}{a} + 1\right)PQ\right] \\ &\quad + \left[\frac{1}{a}QP + Q - \left(\frac{1}{a} + 1\right)QPQ - \left(\frac{1}{a} + 1\right)QP + \left(\frac{1}{a} + 2\right)QPQ\right] \\ &\quad - PQ - \frac{1}{a}QP - QPQ + \left(\frac{1}{a} + 1\right)QPQ + QPQ = M \end{aligned}$$

and that

$$M(aP + Q)^2 = (P + Q - PQ - QP + QPQ)(aP + Q) = aP + Q. \tag{3}$$

Thus, from (2) we get that $(aP + Q)^D = M$. So we have

$$(aP + bQ)^D = \frac{1}{b}\left(\frac{a}{b}P + Q\right)^D = \frac{1}{a}P + \frac{1}{b}Q - \left(\frac{1}{a} + \frac{1}{b}\right)PQ - \left(\frac{1}{a} + \frac{1}{b}\right)QP + \left(\frac{1}{a} + \frac{2}{b}\right)QPQ.$$

Moreover, it follows from (3) and the definition of Drazin index that $\text{ind}(aP + bQ) = \text{ind}\left(\frac{b}{a}P + Q\right) \leqslant 1$.In addition, a direct calculation shows that

$$(aP + bQ)^D(aP + bQ) = P + Q - PQ - QP + QPQ.$$

Note that $\text{ind}(aP + bQ) = 0$ if and only if $(aP + bQ)^D(aP + bQ) = I$, so $\text{ind}(aP + bQ) = 0$ if and only if $I = P + Q - PQ - QP + QPQ$. This completes the proof. □

Theorem 2.2. *Let P and Q be the idempotents in Banach algebra $\mathscr{A}$ and $PQP = P$. Then $aP + bQ$ is Drazin invertible for any nonzero scalars a and b, and*

$$(aP + bQ)^D = \begin{cases} \frac{a^2}{(a+b)^3}P + \frac{1}{b}Q + \frac{ab}{(a+b)^3}(PQ + QP) + \left(\frac{b^2}{(a+b)^3} - \frac{1}{b}\right)QPQ, & \text{if } a + b \neq 0; \\ \frac{1}{a}Q(P - I)Q, & \text{if } a + b = 0. \end{cases}$$

Moreover, $\text{ind}(aP - aQ) \leqslant 3$ and $\text{ind}(aP + bQ) \leqslant 2$ when $a + b \neq 0$.

Proof. Case (1) Let $M = \frac{a^2}{(a+1)^3}P + Q + \frac{a}{(a+1)^3}(PQ + QP) + \left(\frac{1}{(a+1)^3} - 1\right)QPQ$. We claim that if $a \neq -1$, then $(aP + Q)^D = M$. In fact, by the assumption that $PQP = P$, we have

$$(aP + Q)M = M(aP + Q) = \frac{a^2}{(a + 1)^2}P + Q + \frac{a}{(a + 1)^2}(PQ + QP) + \left(\frac{1}{(a + 1)^2} - 1\right)QPQ.$$

and

$$(aP + Q)^3M = (aP + Q)^2 = a^2P + Q + a(PQ + QP).$$

Moreover, by calculating, we get that

$$
\begin{aligned}
&M(aP+Q)M\\
&=\frac{a^4}{(a+1)^5}P+\frac{a^2}{(a+1)^3}QP+\frac{a^3}{(a+1)^5}QP+\frac{a^3}{(a+1)^5}P+\left(\frac{1}{(a+1)^2}-1\right)\frac{a^2}{(a+1)^3}QP\\
&\quad+\frac{a^2}{(a+1)^2}PQ+Q+\frac{a}{(a+1)^2}QPQ+\frac{a}{(a+1)^2}PQ+\left(\frac{1}{(a+1)^2}-1\right)QPQ\\
&\quad+\frac{a^3}{(a+1)^5}PQ+\frac{a}{(a+1)^3}QPQ+\frac{a^2}{(a+1)^5}QPQ\\
&\quad+\frac{a^2}{(a+1)^5}PQ+\left(\frac{1}{(a+1)^2}-1\right)\frac{a}{(a+1)^3}QPQ\\
&\quad+\frac{a^3}{(a+1)^5}P+\frac{a}{(a+1)^3}QP+\frac{a^2}{(a+1)^5}QP+\frac{a^2}{(a+1)^5}P+\left(\frac{1}{(a+1)^2}-1\right)\frac{a}{(a+1)^3}QP\\
&\quad+\frac{a^2}{(a+1)^2}\left(\frac{1}{(a+1)^3}-1\right)PQ+\frac{a}{(a+1)^2}\left(\frac{1}{(a+1)^3}-1\right)QPQ+\left(\frac{1}{(a+1)^3}-1\right)QPQ\\
&\quad+\frac{a}{(a+1)^2}\left(\frac{1}{(a+1)^3}-1\right)PQ+\left(\frac{1}{(a+1)^2}-1\right)\left(\frac{1}{(a+1)^3}-1\right)QPQ\\
&=\frac{a^2}{(a+1)^3}P+Q+\frac{a^3+2a^2+a}{(a+1)^5}PQ+\frac{a^3+2a^2+a}{(a+1)^5}QP\\
&\quad+\left\{\frac{a^2}{(a+1)^5}+\frac{1}{(a+1)^2}\frac{a}{(a+1)^3}+\frac{a}{(a+1)^2}\frac{1}{(a+1)^3}\right.\\
&\quad\left.+\left(\frac{1}{(a+1)^3}-1\right)+\left(\frac{1}{(a+1)^2}-1\right)\frac{1}{(a+1)^3}\right\}QPQ=M.
\end{aligned}
$$

Thus, it follows from (2) that $(aP+Q)^D=M$ and $\mathrm{ind}(aP+Q)\leqslant 2$ when $a\neq -1$. Similarly to the discussion in the proof of Theorem 2.1, when $a+b\neq 0$, we have

$$
(aP+bQ)^D=\frac{a^2}{(a+b)^3}P+\frac{1}{b}Q+\frac{ab}{(a+b)^3}(PQ+QP)+\left(\frac{b^2}{(a+b)^3}-\frac{1}{b}\right)QPQ
$$

and

$$
\mathrm{ind}(aP+bQ)=\mathrm{ind}\left(\frac{a}{b}P+Q\right)\leqslant 2.
$$

Case (2) Suppose that $a+b=0$. By calculating, we have

$$
\begin{aligned}
&(aP-aQ)\frac{1}{a}Q(P-I)Q=\frac{1}{a}Q(P-I)Q(aP-aQ)=Q-QPQ,\\
&(aP-aQ)\left(\frac{1}{a}Q(P-I)Q\right)^2=\frac{1}{a}Q(P-I)Q,\\
&(aP-aQ)^4\left(\frac{1}{a}Q(P-I)Q\right)=a^3(QPQ-Q)=(aP-aQ)^3.
\end{aligned}
$$

Therefore,$(aP-aQ)^D = \frac{1}{a}Q(P-I)Q$, $(aP-aQ)^4(aP-aQ)^D = (aP-aQ)^3$ and $\mathrm{ind}(aP-aQ) \leqslant 3$. This completes the proof. □

Remark 2.3. (1) Under the assumption of Theorem 2.2, we have $\mathrm{ind}(aP-aQ) = 3$ if and only if $P+QPQ \neq PQ+QP$. For this, we only need to note that $(aP-aQ)^3(aP-aQ)^D = a^2(Q-QPQ)$ and $(aP-aQ)^2 = a^2(P-PQ-QP+Q)$.

(2) Our results recovered most of the main conclusions in [15], but our methods are very different from the methods used in [15]. In particular, the methods used in [15] cannot yield any information about the Drazin index.

In the rest of this paper, we consider the group inverse. The group inverse of $A \in \mathscr{A}$ [9,10,21] is the element $B \in \mathscr{A}$ (denoted by A^g) which satisfies

$$BAB = B, \quad AB = BA, \quad ABA = A.$$

Obviously, A has group inverse if and only if$A$ has Drazin inverse with $\mathrm{ind}(A) \leqslant 1$.

Before giving the revised versions of Theorems 3.2 and 3.3 in [13], we present the following two counter-examples to these theorems.

Example 2.4. Let $A = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \in B(l_2 \oplus l_2)$ and $B = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \in B(l_2 \oplus l_2)$ with S and T in $B(l_2)$ such that $TS \neq 0$. Consider the operators

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in B(H_2 \oplus H_2), \quad Q = \begin{pmatrix} I & A \\ B & 0 \end{pmatrix} \in B(H_2 \oplus H_2),$$

where $H_2 = l_2 \oplus l_2$. Direct calculations show that $BA \neq 0$, $(BA)^2 = AB = 0$. Hence we have $P^2 = P$, $Q^2 = Q$, $PQP = P$. From Theorem 2.2, we know that $P+Q$ has Drazin inverse and $(P+Q)^D = \frac{1}{8}P + Q + \frac{1}{8}(PQ+QP) - \frac{7}{8}QPQ$. Hence $(P+Q) - (P+Q)^2(P+Q)^D = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & BA \end{pmatrix} \neq 0$, which implies that $\mathrm{ind}(P+Q) > 1$. From this and Theorem 2.2, it is clear that $\mathrm{ind}(P+Q) = 2$. So the group inverse $(P+Q)^g$ of $P+Q$ does not exist.

Example 2.5. Define operators P and Q in $B(\mathbb{C}^5)$ by

$$P = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&1 \end{pmatrix},$$

respectively. Obviously,

$$P^2 = P, \quad Q^2 = Q, \quad PQP = P = PQ.$$

This means that both P and Q are idempotents in $B(\mathbb{C}^5)$. Then it results from Theorem 2.2 that $(P-Q)^D = Q(P-1)Q$. But a direct calculation shows that

$$(P-Q)^2(P-Q)^D = QPQ - Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \neq P - Q.$$

This implies that $\mathrm{ind}(P-Q) > 1$, and so the group inverse $(P-Q)^g$ of $P-Q$ does not exist.

In spite of the above two counter-examples, we have the following theorems.

Theorem 2.6. *Let P and Q be the idempotents in Banach algebra $\mathscr{A}$ and $PQP = P$. Then*

(i) $(P+Q)^D = \frac{1}{8}P + Q + \frac{1}{8}(PQ+QP) - \left(\frac{7}{8}\right)QPQ$,
(ii) $(P-Q)^D = Q(P-I)Q$,
(iii) $P+Q$ *has a group inverse if and only if* $P + QPQ = PQ + QP$ *,*
(iv) $P-Q$ *has a group inverse if and only if* $P = QPQ$.

Proof. Since the results of parts (i) and (ii) are special cases of Theorem 2.2, it suffices to show part (iii) and part(iv). For this, we only need to note that $(P+Q)-(P+Q)^2(P+Q)^D = \frac{1}{2}(P+QPQ-PQ-QP)$ and that $(P-Q)-(P-Q)^2(P-Q)^D = P - QPQ$, which can be obtained by direct calculations. This completes the proof. □

Theorem 2.7. *Let P and Q be the idempotents in Banach algebra $\mathscr{A}$ and $PQP = PQ$. Then*

$$(P+Q)^g = P + Q - 2QP - \frac{3}{4}PQ + \frac{5}{4}QPQ,$$

$$(P-Q)^D = P - Q - PQ + QPQ.$$

Moreover, $\mathrm{ind}(P-Q) \leqslant 2$ and $P-Q$ has a group inverse if and only if $PQ = QPQ$.

Proof. Since the group inverse of $P+Q$ can by checked directly, its proof is omitted. It can be checked directly that the indicated $(P+Q)^g$ is the group inverse. Now let $M = P - Q - PQ + QPQ$. By direct calculations we have that

$$M(P-Q)M = M,\ (P-Q)^2M = M, \tag{4}$$

and that

$$(P-Q)^3M = (P-Q)^2 = (P-Q)M = M(P-Q) = P - PQ - QP + Q.$$

This implies that $(P-Q)^D = P - Q - PQ + QPQ$ and that $\mathrm{ind}(P-Q) \leqslant 2$. In this case, from equation (4) and the definition of a group inverse, we know that $P-Q$ has a group inverse if and only if $(P-Q)^2(P-Q)^D = (P-Q) = (P-Q)^D = P - Q - PQ + QPQ$. This completes the proof. □

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