A two-variable generalization of the Stieltjes-type continued fraction

J. A. Murphy (*) and M. R. O'Donohoe (**)

ABSTRACT

A continued fraction expansion in two variables is described and shown to correspond to a double power series. This continued fraction is a natural generalization of the S-fraction and can be truncated to form a sequence of rational approximants.
A simple algorithm is derived for computing the coefficients of the two-variable fraction and the theorem of Van Vleck is adapted to establish the convergence of some simple examples. Numerical results are given comparing the method with Chisholm approximants.

1. INTRODUCTION

Chisholm [1] has defined a class of rational approximants in two variables. Such approximants correspond to double power series and are chosen so that they have five properties which are natural generalizations of properties of Padé approximants. The possible applications of bivariate rational approximation in physics and numerical analysis may be far-reaching and it would be convenient if the approximants could be directly related to continued fraction theory, as is the case in one variable. Although there are many feasible ways of defining rational approximants in two variables (cf. Lutterodt [2], Alabiso and Butera [3], Levin [4]) it appears there is no clear link with continued fractions of simple form. However, in this paper we consider a more general structure for continued fractions and define a new class of rational approximants which provide a means for analytic continuation of double power series. Further, these approximants have certain advantages over Chisholm approximants in suitable problems and can be related to well-studied aspects of continued fraction theory.

2. A CORRESPONDING CONTINUED FRACTION IN TWO VARIABLES

We shall examine the structure of a continued fraction which corresponds, in a certain sense, to the formal double power series

$$f(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}\, x^i y^j , \qquad (1)$$

where $a_{00} \neq 0$ and x and y are independent complex variables. The familiar type of continued fraction may be written

$$F = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \ldots + \frac{p_n}{q_n} + \ldots \qquad (2)$$

where p_n/q_n is called the nth partial quotient. If a fraction F of the form (2) is used as an expansion of an analytic function of a single variable then it is usual for p_n to be a monomial and for q_n to consist of a finite number of terms. In the case of a two-variable expansion it is desirable that the fraction be symmetrical in form between the two variables and that p_n still be a monomial. For this to be accomplished it is necessary for q_n to be a non-terminating expression. The continued fraction which we shall discuss has the form

$$f(x,y) = \frac{c_{00}}{1+g_0(x)+h_0(y)} + \frac{c_{11}\,xy}{1+g_1(x)+h_1(y)} + \frac{c_{22}\,xy}{1+g_2(x)+h_2(y)} + \ldots + \frac{c_{nn}\,xy}{1+g_n(x)+h_n(y)} + \ldots \qquad (3)$$

in which $g_n(x)$ and $h_n(y)$ are themselves continued fractions.
We write

$$\left[\begin{aligned} g_n(x) &= \frac{c_{n+1,n}x}{1} + \frac{c_{n+2,n}x}{1} + \ldots + \frac{c_{n+r,n}x}{1} + \ldots, \\ h_n(y) &= \frac{c_{n,n+1}y}{1} + \frac{c_{n,n+2}y}{1} + \ldots + \frac{c_{n,n+r}y}{1} + \ldots, \end{aligned}\right. \qquad (4)$$

so that (3) contains a double array of coefficients $\{c_{ij}\}$, where $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$ Continued fractions of the form (4) are called Stieltjes-type fractions or S-fractions (see [5]). We shall call (3) an S_2-fraction because it is a generalization of the S-fraction

(*) J. A. Murphy, Brunel University, Uxbridge, Middlesex, England.
(**) M. R. O'Donohoe, University of Cambridge, Computer Laboratory, Corn Exchange Street, Cambridge CB 2 3 QG, England.

to two variables, and we will refer to $g_n(x)$ and $h_n(y)$ as sub-fractions of (3). There is a correspondence between the coefficients $\{c_{ij}\}$ of the S_2-fraction (3) and the coefficients $\{a_{ij}\}$ of the double series (1) in the following sense. The value of c_{ij} depends on a_{ij} (and other coefficients of the series) but does not depend on the value of coefficients $a_{i+r,j+s}$, where r and s are non-negative integers and not both zero. We also note that if we set $a_{ij} = 0$ for all $i \neq j$ then $c_{ij} = 0$ for all $i \neq j$ and (3) reduces to an S-fraction in the product xy.

So far we have merely explained our choice of the structure (3) and we now prove its existence, a correspondence property and the uniqueness of the expansion. In order to establish existence we note that any formal double power series of the form (1), with $a_{00} \neq 0$, has a reciprocal series of similar form, and also that each of the sub-fractions (4), if it exists, corresponds to a single power series. We write

$$\begin{cases} g_n(x) = \sum_{i=1}^{\infty} u_i^{(n)} x^i , \\ h_n(y) = \sum_{j=1}^{\infty} v_j^{(n)} y^j , \end{cases} \tag{5}$$

for n = 0, 1, 2, ... A necessary and sufficient condition for the existence of an expansion (3) of the function f(x, y), defined by (1), is the existence of a sequence $\{T_n(x, y)\}$ of functions, each having a formal expansion

$$T_n(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{ij}^{(n)} x^i y^j \tag{6}$$

with reciprocal series

$$\frac{1}{T_n(x, y)} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \beta_{ij}^{(n)} x^i y^j , \tag{7}$$

so that $\beta_{00}^{(n)} = 1/\alpha_{00}^{(n)} \neq 0$, and satisfying the system of formal identities

$$T_n(x, y) = \frac{1}{1 + g_n(x) + h_n(y) + c_{n+1,n+1} xy\, T_{n+1}(x, y)} , \tag{8}$$

for n = 0, 1, 2, ... where $c_{00} = \alpha_{00}$ and $f(x, y) = c_{00} T_0(x, y)$. Assuming $T_n(x, y)$, $g_n(x)$ and $h_n(y)$ exist and rearranging (8), we obtain

$$T_{n+1}(x, y) = \frac{1}{c_{n+1,n+1} xy} \left\{ \frac{1}{T_n(x, y)} - 1 - g_n(x) - h_n(y) \right\}.$$

Or, using (5) and (7),

$$T_{n+1}(x,y) = \frac{1}{c_{n+1,n+1} xy} \left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \beta_{ij}^{(n)} x^i y^j - 1 - \sum_{i=1}^{\infty} u_i^{(n)} x^i - \sum_{j=1}^{\infty} v_j^{(n)} y^j \right\}. \tag{9}$$

Now, choosing

$$\beta_{00}^{(n)} = 1, \ \beta_{i0}^{(n)} = u_i^{(n)} , \ \beta_{0j}^{(n)} = v_j^{(n)}$$

for i = 1, 2, 3, ... and j = 1, 2, 3, ..., we see that $T_{n+1}(x, y)$ can be expressed in the form (6). Hence, a sufficient condition for the existence of the S_2-fraction expansion is that none of the coefficients $\{c_{ij}\}$ is zero. We shall deal with a practical test of this condition in Section 4. We also note that the S_2-fraction or any of its sub-fractions will terminate if and only if they represent rational functions.

Denoting by P_n/Q_n the nth convergent of (2), we have the results

$$\begin{cases} P_{n+1} = p_{n+1} P_{n-1} + q_{n+1} P_n , \\ Q_{n+1} = p_{n+1} Q_{n-1} + q_{n+1} Q_n , \end{cases} \tag{10}$$

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} \prod_{r=1}^{n} p_r , \tag{11}$$

for n = 0, 1, 2, ..., where $P_0 = 0$ and $Q_0 = 1$. We now write

$$f(x, y) = \frac{c_{00}}{1 + g_0 + h_0} + \frac{c_{11} xy}{1 + g_1 + h_1} + \dots + \frac{c_{nn}\ xy}{1 + g_n + h_n + c_{n+1,n+1} xy\, T_{n+1}(x, y)} . \tag{12}$$

The properties (10) and (11) (proved, for example, in [5]) are formal and can be applied when $\{q_n\}$ are non-terminating expressions. Applying (10) to (12) we have

$$f(x, y) = \frac{c_{nn} xy\, P_{n-1} + (1 + g_n + h_n + c_{n+1,n+1} xy\, T_{n+1}) P_n}{c_{nn} xy\, Q_{n-1} + (1 + g_n + h_n + c_{n+1,n+1} xy\, T_{n+1}) Q_n}$$

so that

$$f(x, y) - \frac{P_n}{Q_n} = \frac{(-1)^n c_{00} c_{11} c_{22} \dots c_{nn} (xy)^n}{Q_n (Q_{n+1} + c_{n+1,n+1} xy\, T_{n+1} Q_n)}$$

using (11). We can write

$$f(x, y) - \frac{P_n}{Q_n} = 0\, \{(xy)^n\} \tag{13}$$

where $0\, \{(xy)^n\}$ denotes error terms of order $x^i y^j$ for $i \geqslant n$ and $j \geqslant n$. Result (13) indicates the manner in which the convergents of the S_2-fraction (3) correspond to the double series (1).

To establish the uniqueness of the S_2-fraction expansion we suppose that there exist two distinct expansions, f (x, y) and $f^*(x, y)$, of a given double power series where

$$\begin{cases} f = \frac{c_{00}}{1 + g_0 + h_0} + \frac{c_{11}\, xy}{1 + g_1 + h_1} + \dots + \frac{c_{nn}\, xy}{1 + g_n + h_n} + \dots , \\ f^* = \frac{c_{00}^*}{1 + g_0^* + h_0^*} + \frac{c_{11}^* xy}{1 + g_1^* + h_1^*} + \dots + \frac{c_{nn}^*\, xy}{1 + g_n^* + h_n^*} + \dots . \end{cases} \tag{14}$$

As a starting point we assume the uniqueness of S-fraction expansions. Then by setting $y = 0$ in (14) we see that $c_{00} = c^*_{00}$ and $g_0 = g^*_0$.
Similarly, $h_0 = h^*_0$ from which we have $P_1 = P^*_1$ and $Q_1 = Q^*_1$, where P^*_n / Q^*_n is the nth convergent of f^*.
We also have $P_0 = P^*_0 = 0$ and $Q_0 = Q^*_0 = 1$ and, for proof by induction, we need to show that

$$\{c_{rr}, g_r, h_r, P_{r+1}, Q_{r+1}\} \equiv \{c^*_{rr}, g^*_r, h^*_r, P^*_{r+1}, Q^*_{r+1}\} \tag{15}$$

for $r = 0, 1, 2, \ldots. n-1$ implies

$$\{c_{nn}, g_n, h_n\} \equiv \{c^*_{nn}, g^*_n, h^*_n\}. \tag{16}$$

We consider the difference between the $(n+1)$th convergents of f and f^* and write

$$\frac{A_{n+1}}{B_{n+1}} = \frac{P_{n+1}}{Q_{n+1}} - \frac{P^*_{n+1}}{Q^*_{n+1}} = \frac{P_{n+1}\, Q^*_{n+1} - P^*_{n+1}\, Q_{n+1}}{Q_{n+1}\, Q^*_{n+1}}. \tag{17}$$

Using (10) and (15), we get

$$A_{n+1} = \{(1+g_n+h_n)c^*_{nn} - (1+g^*_n+h^*_n)c_{nn}\}(P_nQ_{n-1} - P_{n-1}Q_n)\, xy$$

or, using (11),

$$A_{n+1} = \{(1+g_n+h_n)\,c^*_{nn} - (1+g^*_n+h^*_n)c_{nn}\}\, 0\{(xy)^n\}$$

But, from (13) and (17), we know that

$$A_{n+1} = 0\,\{(xy)^{n+1}\}$$

so we must have

$$(1+g_n+h_n)\,c^*_{nn} - (1+g^*_n+h^*_n)\,c_{nn} \equiv 0$$

from which (16) follows.

3. S_2-APPROXIMANTS

In the formal proofs in section 2 we have used "convergents" which are themselves non-terminating expressions. However, for practical purposes we now define a sequence of rational expressions which we call the S_2-approximants to the expansion (3). We adopt the notation $0(x,y)^n$ to denote error terms of order $x^r y^{n-r}$ for $r = 0, 1, 2, \ldots, n$ and define the sequence $\{K_n(x,y)\}$ of S_2-approximants by

$$f(x,y) - K_n(x,y) = 0(x,y)^n \tag{18}$$

for $n = 1, 2, 3, \ldots\ldots$ Using this definition the first five approximants may be written

$$K_1(x,y) = c_{00}, \quad K_2(x,y) = \frac{c_{00}}{1+c_{10}x+c_{01}y},$$

$$K_3(x,y) = \cfrac{c_{00}}{1+\cfrac{c_{10}x}{1+c_{20}x} + \cfrac{c_{01}y}{1+c_{02}y}} + \frac{c_{11}\,xy}{1},$$

$$K_4(x,y) = \cfrac{c_{00}}{1+\cfrac{c_{10}x}{1+\cfrac{c_{20}x}{1+c_{30}x}} + \cfrac{c_{01}y}{1+\cfrac{c_{02}y}{1+c_{03}y}}} + \frac{c_{11}\,xy}{1+c_{21}x+c_{12}y},$$

$$K_5(x,y) = \cfrac{c_{00}}{1+\cfrac{c_{10}x}{1+\cfrac{c_{20}x}{1+\cfrac{c_{30}x}{1+c_{40}x}}} + \cfrac{c_{01}y}{1+\cfrac{c_{02}y}{1+\cfrac{c_{03}y}{1+c_{04}y}}}} + \cfrac{c_{11}\,xy}{1+\cfrac{c_{21}x}{1+c_{31}x}+\cfrac{c_{12}y}{1+c_{13}y}} + \frac{c_{22}}{1}$$

In order to represent the structure as clearly as possible, we have combined two of the common notations for continued fractions.
(See, for example, [6].). Let $g^{(n)}_r(x)$ and $h^{(n)}_r(y)$ denote the nth convergents of the sub-fractions $g_r(x)$ and $h_r(y)$, respectively.
We can now summarise the form of the S_2-approximants by

$$\begin{cases} K_{2n-1} = \dfrac{c_{00}}{1+g^{(2n-2)}_0+h^{(2n-2)}_0} + \dfrac{c_{11}\,xy}{1+g^{(2n-4)}_1+h^{(2n-4)}_1} + \ldots + \dfrac{c_{n-1,\,n-1}\,xy}{1} \\ K_{2n} = \dfrac{c_{00}}{1+g^{(2n-1)}_0+h^{(2n-1)}_0} + \dfrac{c_{11}\,xy}{1+g^{(2n-3)}_1+h^{(2n-3)}_1} + \ldots + \dfrac{c_{n-1,\,n-1}\,xy}{1+g^{(1)}_{n-1}+h^{(1)}_{n-1}} \end{cases} \tag{19}$$

for $n = 1, 2, 3, \ldots\ldots$ We note that $K_n(x,y)$ is computed from the coefficients in the triangular array $\{c_{ij} : i+j < n\}$.
Also, when expressed as rational functions, it may be seen that S_2-approximants do not bear any relationship to other definitions ([1], [2], [3] and [4]) of rational approximants.
Although the recurrence formulae (10) do not appear to generalise to the two-variable case, each S_2-approximant can be evaluated directly by an algorithm such as algorithm 1, below. Mayers [7] has remarked that a direct evaluation algorithm may even be preferable to the use of recurrence formulae in the one-variable case. Algorithm 1 requires a minimum of storage space, the value of K_{2n-1} or K_{2n} being held by the variable F_1 on exit from the algorithm.

Algorithm 1

$i := n-1$

$$k := \begin{cases} 1, & \text{for } K_{2n-1} \\ 2, & \text{for } K_{2n} \end{cases}$$

$F_1 := 0$

$$F_2 := \begin{cases} 0 & , \text{ if } k = 1 \\ c_{i,\,i+1}y, & \text{if } k = 2 \end{cases}$$

$$F_3 := \begin{cases} 0 & , \text{ if } k = 1 \\ c_{i+1,i}x\,, & \text{if } k = 2 \end{cases}$$

for $r = 1, 2, \ldots. (n-1)$

$F_1 := c_{ii}xy/(1 + F_1 + F_2 + F_3)$

$j \;:= i + k$

$i \;:= i - 1$

$F_2 := c_{ij}y$

$F_3 := c_{ji}x$

for $s = 1, 2, \ldots.\ k$

$j \;:= j - 1$

$F_2 := c_{ij}y/(1 + F_2)$

$F_3 := c_{ji}x/(1 + F_3)$

$k := k + 2$

$F_1 := c_{00}/(1 + F_1 + F_2 + F_3)$

In a computer implementation of this algorithm it is necessary to test for division by zero as some approximants may not exist.

4. CORRESPONDING SEQUENCE ALGORITHMS

In a previous paper [8] we showed how any continued fraction of the form (2) can be derived from a system of recurrence relations. In particular, if $f_0(x)$ has both a formal series expansion

$$f_0(x) = x \sum_{r=0}^{\infty} w_r^{(0)} x^r \tag{20}$$

and an S-fraction expansion

$$f_0(x) = \frac{d_1x}{1} + \frac{d_2x}{1} + \ldots. + \frac{d_nx}{1} + \ldots. \tag{21}$$

then we can write

$$\begin{cases} f_1 = d_1x - f_0 \\ f_n = d_nxf_{n-2} - f_{n-1} \end{cases} \tag{22}$$

for $n = 2, 3, 4, \ldots.$ where $f_n(x)$ has a series expansion of the form

$$f_n(x) = x^{n+1} \sum_{r=0}^{\infty} w_r^{(n)} x^r\,. \tag{23}$$

The coefficients $\{d_n\}$ may be obtained from $\{w_r^{(0)}\}$ by equating coefficients of powers of x in (22). This method, essentially that of Viskovatov [6], is algorithm 2., below. The idea can be generalized to all types of corresponding fractions in one variable [9] and also to the two-variable fraction (3). We refer to $\{f_n\}$ as the corresponding sequence of a continued fraction and describe the algorithm 2. as a corresponding sequence algorithm. To obtain the corresponding sequence algorithm for the S_2-fraction we first observe that an expansion of the form (3) may be derived from the recurrence relations

$$\begin{cases} f_1(x,y) \;=\; c_{00} - f_0(x,y)\,\{1+g_0(x)+h_0(y)\}\,, \\ f_{n+1}(x,y) = c_{nn}xyf_{n-1}(x,y) - f_n(x,y)\,\{1+g_n(x)+h_n(y)\} \end{cases} \tag{24}$$

for $n = 1, 2, 3, \ldots.$ where $f_n(x, y)$ has a double series expansion of the form

$$f_n(x,y) = (xy)^n \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}^{(n)}\, x^i y^j \tag{25}$$

for $n = 0, 1, 2, \ldots.,$ and each sub-fraction also has a formal series expansion. We write

$$\begin{cases} g_n(x) = x \sum_{k=0}^{\infty} u_k^{(n)}\, x^k \\ h_n(y) = y \sum_{k=0}^{\infty} v_k^{(n)}\, y^k \end{cases} \tag{26}$$

for $n = 0, 1, 2, \ldots.$. By equating coefficients of x^iy^j in (24) we obtain algorithm 3., below. The coefficients $\{c_{ij} : i \neq j\}$ are computed by applying algorithm 2 to the series (26).

Algorithm 2

$$d_1 \;=\; w_0^{(0)}\,,$$

$$w_r^{(1)} \;= -w_{r+1}^{(0)}\,,\quad r = 0, 1, 2, \ldots.,$$

$$d_n \;=\; \frac{w_0^{(n-2)}}{w_0^{(n-2)}}\,,\quad n = 2, 3, 4, \ldots.,$$

$$w_r^{(n)} \;=\; d_n\, w_{r+1}^{(n-2)} - w_{r+1}^{(n-1)}\,,\quad r = 0, 1, 2, \ldots.,\; n = 2, 3, 4, \ldots..$$

Algorithm 3

$$c_{00} \;=\; a_{00}^{(0)}\,,\quad c_{nn} = \frac{a_{00}^{(n)}}{a_{00}^{(n-1)}}\,,\quad n = 1, 2, 3, \ldots.\,,$$

$$u_i^{(n)} \;=\; \frac{1}{a_{00}^{(n)}} \left[c_{nn}\, a_{i+1,0}^{(n-1)} - a_{i+1,0}^{(n)} - \sum_{k=0}^{i-1} a_{i-k,0}^{(n)} u_k^{(n)} \right]$$

$$i = 0, 1, 2, \ldots.,\quad n = 0, 1, 2, \ldots.\,,$$

$$v_j^{(n)} = \frac{1}{a_{00}^{(n)}} \left[c_{nn}\, a_{0,j+1}^{(n-1)} - a_{0,j+1}^{(n)} - \sum_{k=0}^{j-1} a_{0,j-k}^{(n)} v_k^{(n)} \right]$$

$$j = 0, 1, 2, \ldots, \quad n = 0, 1, 2, \ldots,$$

$$a_{ij}^{(n+1)} = c_{nn}\, a_{i+1,j+1}^{(n-1)} - a_{i+1,j+1}^{(n)} - \sum_{k=0}^{i} a_{i-k,j+1}^{(n)} u_k^{(n)} - \sum_{k=0}^{j} a_{i+1,j-k}^{(n)} v_k^{(n)},$$

$$i = 0, 1, 2, \ldots, \quad j = 0, 1, 2, \ldots, \quad n = 0, 1, 2, \ldots .$$

In algorithm 3 we set $a_{ij}^{(-1)} = 0$ for all i and j.

The algorithms 2 and 3 are useful for establishing the existence of the S_2-fraction expansion in particular cases. Specifically, algorithm 3 breaks down only if one of the coefficients $\{a_{00}^{(n)}\}$ is zero, in which case one of the coefficients $\{c_{nn}\}$ is zero and the S_2-fraction does not exist. Similarly, if algorithm 2 breaks down this implies that one of the coefficients $\{c_{ij} : i \neq j\}$ is zero and the appropriate sub-fraction does not exist.

The computation of the coefficients of the Chisholm rational approximants by the "prong" method of Hughes Jones and Makinson [10] requires the solution of sets of linear equations and uses more storage space and computing time.

5. CONVERGENCE OF AN S_2-FRACTION EXPANSION

We will now show how the well-known convergence theorem of Van Vleck [5] may be applied to S_2-fraction expansions whose coefficients have limits. If we denote the nth convergent of (2) by P_n/Q_n then we say that F converges if $\lim_{n\to\infty} P_n/Q_n$ exists. If, however, we permit $\{q_n\}$ to be a sequence of non-terminating expressions then it is necessary to be more precise.

Definition

If at all points (x, y) in some region R all the sub-fractions of an S_2-fraction converge to finite limits, and the S_2-fraction as a whole (with sub-fractions replaced by their limits) converges to a finite limit, then the S_2-fraction converges everywhere in R.

As explained above, we are interested in the sequence of S_2-approximants for practical applications so we now prove the following theorem.

Theorem 1

If an S_2-fraction converges to a finite limit at each point (x, y) of a region R then the sequence of its S_2-approximants converges to the same limit at each point of R.

To prove this theorem we require the following lemma.

Lemma

Define the transformation $t_n(w)$ by

$$t_n(w) = \frac{p_n}{q_n + w}$$

for n = 1, 2, 3, ... The value F of a convergent continued fraction of the form (2) may be expressed as

$$F = \lim_{n\to\infty} t_1 t_2 \ldots t_n(w) \tag{27}$$

and is independent of the value of w.

Proof : It may be shown by induction that

$$t_1 t_2 \ldots t_n(w) = \frac{P_{n-1} w + P_n}{Q_{n-1} w + Q_n} \tag{28}$$

for n = 1, 2, 3, ... and if the continued fraction is convergent then we may write

$$P_n = Q_n (F + \epsilon_n), \quad \lim_{n\to\infty} \epsilon_n = 0.$$

Substituting for P_{n-1}, P_n in (28) we get

$$t_1 t_2 \ldots t_n(w) = F + \frac{\epsilon_{n-1} w + \epsilon_n (Q_n/Q_{n-1})}{w + (Q_n/Q_{n-1})} .$$

As $n \to \infty$ we obtain result (27) even if the sequence $\{Q_n/Q_{n-1}\}$ is unbounded.

We now let (2) represent an S_2-fraction and consider the first n terms of the mth approximant. We write

$$\phi_{mn} = \frac{p_1}{q_1 + \eta_{m1}} + \frac{p_2}{q_2 + \eta_{m2}} + \ldots + \frac{p_n}{q_n + \eta_{mn}}$$

where η_{mr} represents the truncation error in the rth partial quotient.

If all the sub-fractions converge then $\lim_{m\to\infty} \eta_{mr} = 0$ for r = 1, 2, ... n and we have

$$\lim_{m\to\infty} \phi_{mn} = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \ldots + \frac{p_n}{q_n} .$$

To prove theorem 1 we let $n \to \infty$ and apply the lemma. We now quote Van Vleck's theorem. The proof will be found in [5].

Theorem 2

Let $k_1, k_2, k_3, \ldots$ be a sequence of numbers having a finite limit k. If $k \neq 0$, let L denote the rectilinear cut from $-(4k)^{-1}$ to ∞ in the direction of the vector from 0 to $-(4k)^{-1}$. Let G denote an arbitrary closed region whose distance from L is positive or, if k = 0, an entirely arbitrary finite closed region. There exists N, depending only on G, such that the S-fraction

$$\frac{1}{1+} \frac{k_n z}{1} + \frac{k_{n+1} z}{1} + \frac{k_{n+2} z}{1} + \ldots$$

converges uniformly over G for n > N.

In order to apply this theorem to the S_2-fraction (3) we suppose that all the sub-fractions converge uniformly over some closed region and we replace them by their limits. We can then rewrite (3) as

$$f(x,y)=\frac{\gamma_0(x,y)}{1}+\frac{\gamma_1(x,y)}{1}+\ldots+\frac{\gamma_n(x,y)}{1}+\ldots \tag{29}$$

where

$$\begin{cases} \gamma_0(x,y)=\dfrac{c_{00}}{1+g_0(x)+h_0(y)}, \\ \gamma_n(x,y)=\dfrac{c_{nn}\,xy}{\{1+g_{n-1}(x)+h_{n-1}(y)\}\{1+g_n(x)+h_n(y)\}} \end{cases} \tag{30}$$

for $n = 1, 2, 3, \ldots$. We are led to consider $\lim_{n\to\infty} \gamma_n(x,y)$ over a region in which all $\{\gamma_n(x,y)\}$ exist for n sufficiently large.

From theorem 2, we get the following two theorems for the S_2-fraction.

Theorem 3

Let R be a finite closed region in which all the sub-fractions of an S_2-fraction (3) converge uniformly and for which there exists N such that $1+g_{n-1}(x)+h_{n-1}(y)\neq 0$ for all $n > N$ and $(x,y)\in R$. A sufficient condition for the S_2-fraction to converge uniformly over R is that

$$\lim_{n\to\infty} c_{nn}=0. \tag{31}$$

Proof: If condition (31) holds then, from (30), $\lim_{n\to\infty}\gamma_n(x,y)=0$ for $(x,y)\in R$ and we can apply theorem 2 to the S_2-fraction in the form (29).

Theorem 4

Let there exist c, g(x) and h(y) such that, in an S_2-fraction (3),

$$\lim_{n\to\infty} c_{nn}=c,\quad \lim_{n\to\infty} g_n(x)=g(x),\quad \lim_{n\to\infty} h_n(y)=h(y) \tag{32}$$

and let R be defined as in theorem 3. The S_2-fraction will converge uniformly over R except when

$$\frac{c\,xy}{\{1+g(x)+h(y)\}^2}=-\frac{1}{4}-\zeta \tag{33}$$

where ζ is any real non-negative number.

Proof: Under the conditions (32), $\lim_{n\to\infty}\gamma_n(x,y)$ exists and we apply theorem 2 to obtain the restriction (33).

In practice the information required to apply theorems 3 and 4 may not be available. However, these theorems may be used empirically to estimate regions of uniform convergence or to indicate the singularity structure of the expanded function. To provide examples to which these convergence theorems can be strictly applied we consider the following class of S_2-fraction expansions. Given the S-fraction expansions

$$\begin{cases} X(x)=\dfrac{\lambda_0}{1}+\dfrac{\lambda_1 x}{1}+\dfrac{\lambda_2 x}{1}+\ldots+\dfrac{\lambda_n x}{1}+\ldots \\ Y(y)=\dfrac{\mu_0}{1}+\dfrac{\mu_1 y}{1}+\dfrac{\mu_2 y}{1}+\ldots+\dfrac{\mu_n y}{1}+\ldots \end{cases} \tag{34}$$

it is easily verified by formal multiplication that $f(x,y)=X(x)\,Y(y)$ has the S_2-fraction expansion

$$f(x,y)=\frac{\lambda_0\mu_0}{1+g_0+h_0}+\frac{\lambda_1\mu_1 xy}{1+g_1+h_1}+\ldots+\frac{\lambda_n\mu_n xy}{1+g_n+h_n}+ \tag{35}$$

where

$$\begin{cases} g_n(x)=\dfrac{\lambda_{n+1}x}{1}+\dfrac{\lambda_{n+2}x}{1}+\dfrac{\lambda_{n+3}x}{1}+\ldots \\ h_n(y)=\dfrac{\mu_{n+1}y}{1}+\dfrac{\mu_{n+2}y}{1}+\dfrac{\mu_{n+3}y}{1}+\ldots \end{cases} \tag{36}$$

for $n = 0, 1, 2, \ldots$. If $\lambda_n\to\lambda$, $\mu_n\to\mu$ as $n\to\infty$ then it follows that $c_{nn}\to\lambda\mu$ and

$$g(x)=\frac{\lambda x}{1}+\frac{\lambda x}{1}+\ldots,\quad h(y)=\frac{\mu y}{1}+\frac{\mu y}{1}+\ldots \tag{37}$$

Writing $g(x)=\lambda x/\{1+g(x)\}$ we see that $g(x)=\frac{1}{2}(\sqrt{\{1+4\lambda x\}}-1)$ except when $\lambda x=-\frac{1}{4}-\xi_1$. Similarly, $h(y)=\frac{1}{2}(\sqrt{\{1+4\mu y\}}-1)$ except when $\mu y=-\frac{1}{4}-\xi_2$. Applying theorem 4, the S_2-fraction (35) converges uniformly except when

$$\lambda x=-\frac{1}{4}-\xi_1,\quad \mu y=-\frac{1}{4}-\xi_2 \tag{38}$$

or

$$\frac{4\lambda\mu xy}{(\sqrt{\{1+4\lambda x\}}+\sqrt{\{1+4\mu y\}})^2}=-\frac{1}{4}-\xi_3 \tag{39}$$

where ξ_1, ξ_2 and ξ_3 are real non-negative numbers.

In fact condition (39), which arises from theorem 4, is redundant. To prove this we let $\alpha=\sqrt{\{1+4\lambda x\}}$, $\beta=\sqrt{\{1+4\mu y\}}$ have their principal values and observe that (39) may be written

$$\frac{1}{4}\left[\frac{\alpha\beta+1}{\alpha+\beta}\right]^2=-\xi_3$$

which implies that

$$\mathrm{Re}\,\{(\alpha\beta+1)(\bar{\alpha}+\bar{\beta})\}=0. \tag{40}$$

Writing $Re(\alpha) = a$ and $Re(\beta) = b$, condition (40) reduces to

$$(|\alpha|^2 + 1)b + (|\beta|^2 + 1)a = 0. \qquad (41)$$

Since we are dealing with the principal branch of the square root, a and b are non-negative real numbers so that the only admissible solution of (41) occurs when $a = b = 0$, which implies that conditions (38) both hold. Hence the S_2-fraction expansion (35) converges uniformly over any finite region in which the sub-fractions converge uniformly.

In tables 1 and 2, below, we compare the rates of convergence of the sequences $\{K_n(x, y)\}$, of S_2-approximants, and $\{X_n(x) Y_n(y)\}$, where $X_n(x)$ and $Y_n(y)$ are the nth convergents of the S-fractions (34). We note that $K_n(x, y)$ matches $\frac{1}{2}n(n+1)$ terms of the double series expansion, whereas the product $X_n(x) Y_n(y)$ matches n^2 terms. In the example in table 1, the sequence $\{K_n(x,y)\}$ converges slightly faster that $\{X_n(x) Y_n(y)\}$ for the chosen values of x and y. However the reverse is true for the example in table 2.

Table 1

$1/\sqrt{(1+x)(1+y)}$	n	$K_n(x,y)$	$X_n(x)Y_n(y)$
$x = 0.1,\ y = 0.5$	4	0.778405	0.77833
	6	0.7784979	0.7784973
	8	0.778498934	0.778498927
	10	0.77849894406	0.77849894399
	12	0.77849894416	0.77849894416
$x = 1.0,\ y = 2.0$	6	0.408200	0.4079
	8	0.4082452	0.408226
	10	0.40824804	0.4082467
	12	0.408248274	0.40824818
	14	0.408248290	0.408248282
	16	0.408248290	0.408248290

Table 2

$e^{-(x+y)}$	n	$K_n(x, y)$	$X_n(x)Y_n(y)$
$x = 0.1,\ y = 0.1$	4	0.81870	0.818729
	6	0.818730772	0.81873075330
	8	0.81873075308	0.81873075308
$x = 0.1,\ y = 0.2$	4	0.74067	0.740802
	6	0.74081840	0.7408182272
	8	0.74081822058	0.74081822068
	10	0.74081822068	0.74081822068

6. COMPARISON OF S_2-APPROXIMANTS WITH CHISHOLM APPROXIMANTS

We will denote by $[n/n](x, y)$ the nth diagonal Chisholm approximant to the double series (1).

Writing

$$[n/n](x, y) = \frac{P_n(x, y)}{Q_n(x, y)} = \frac{\sum_{p=0}^{n} \sum_{q=0}^{n} b_{pq} x^p y^q}{\sum_{r=0}^{n} \sum_{s=0}^{n} d_{rs} x^r y^s} \qquad (42)$$

the coefficients $\{b_{pq}\}$ and $\{d_{rs}\}$ are defined by

$$Q_n(x, y) f(x, y) - P_n(x, y) = 0(x, y)^{2n+1}, \qquad (43)$$

with $d_{00} = 1$, together with n "symmetrisation" conditions.

The latter are formed by equating to zero the sums of coefficients of the pairs of terms

$$x^k y^{2n-k+1}, \quad x^{2n-k+1} y^k$$

for $k = 1, 2, \ldots, n$. These additional n conditions are required to uniquely define the $[n/n]$ approximant and their effect is to increase the order of approximation to $0(x, y)^{2n+2}$ when $x = y$. Consequently, exact comparison between Chisholm approximants and S_2-approximants is not possible.

Chisholm [1] defined his approximants so that they have five properties analogous to properties of Padé approximants. These are : (i) symmetry between x and y; (ii) uniqueness; (iii) if $x = 0$ or $y = 0$, Chisholm approximants reduce to Padé approximants; (iv) Chisholm approximants are invariant under the transformations

$$x = \frac{Au}{1 - Bu}, \quad y = \frac{Av}{1 - Cv}$$

for constants A, B and C such that $A \neq 0$; (v) the reciprocal of an approximant is an approximant of the reciprocal series.

Now, properties (i), (ii) and (iii) are shared by S_2-approximants but properties (iv) and (v) are not satisfied by any subsequence of S_2-approximants. However, S_2-approximants are invariant under the elementary transformations

$$x = Au, \quad y = Bv$$

for constants A and B, which is not true of Chisholm approximants.

A disadvantage of S_2-approximants is that the S_2-fraction expansion of a function $f(x, y)$ may not exist. One way to overcome this difficulty is to form the S_2-fraction expansion of $f(x, y) + g(x, y)$, where $g(x, y)$ is any suitable function such that the expansion exists. The drawback of this technique is that a poor choice of $g(x, y)$ may result in slow convergence, or if $|g(x, y)| \gg |f(x, y)|$ in some region then the value of $f(x, y)$ will be obscured in that region.

From limited computational experience it appears that both Chisholm approximants and S_2-approximants satisfactorily represent the zeros and singularities of a function, but the assessment of this property is a very difficult problem in two complex variables.

In tables 3-6, below, a numerical comparison is made between the two methods of approximation for the following functions :

(i) $1/\sqrt{\{1+x+y\}}$, (ii) $1/\sqrt{\{(1+x)(1+y)\}}$,

(iii) $e^{-x}/\sqrt{\{1+y\}}$, (iv) $e^{-(x+y)}$.

The Chisholm approximants were computed using an algorithm given by Graves-Morris, Hughes Jones and Makinson [11]. The results given are inconclusive, but it appears that the method to be preferred depends on the function chosen and the values of x and y.

Table 3

(i) $f(x,y) = 1/\sqrt{\{1+x+y\}}$					
n	[n/n]	K_{2n+1}	n	[n/n]	K_{2n+1}
	$x = 2, y = 2, f = 0.4472136$			$x = 1, y = 2, f = 0.5$	
2	0.4455	0.450	2	0.4986	0.5020
3	0.4483	0.452	3	0.500088	0.5016
4	0.44718	0.44706	4	0.5000055	0.499966
5	0.447200	0.447293	5	0.49999966	0.500014
6	0.447215	0.4472156	6	0.499999978	0.500000088
7	0.4472137	0.4472157	7	0.5000000013	0.50000021
8	0.4472136	0.4472153	8	0.5000000002	0.50000015
	$x = 3, y = 3, f = 0.377964$			$x = 5, y = 5, f = 0.301511$	
2	0.392	0.383	2	0.041	0.312
3	0.391	0.388	3	0.17	0.326
4	0.3740	0.3775	4	-0.27	0.3001
5	0.3778	0.3784	5	8.49	0.3035
6	0.3783	0.377974	6	0.287	0.30152
7	0.3790	0.377978	7	-0.36	0.301639
8	0.377955	0.377969	8	0.065	0.301695

Table 4

(ii) $f(x,y) = 1/\sqrt{\{(1+x)(1+y)\}}$					
n	[n/n]	K_{2n+1}	n	[n/n]	K_{2n+1}
	$x = 1, y = 1, f = 0.5^*$			$x = 1, y = 2, f = 0.4082482905$	
2	0.50030	0.50037	2	0.4095	0.4094
3	0.5000088	0.499989	3	0.40833	0.408215
4	0.50000026	0.50000032	4	0.408254	0.408251
5	0.5000000076	0.4999999906	5	0.40824871	0.408248258
6	0.5000000002	0.5000000003	6	0.40824832	0.4082482995
			7	0.4082482926	0.4082482906
			8	0.4082482906	0.4082482905
	$x = 1, y = 5, f = 0.2886751$			$x = 5, y = 5, f = 0.1666667^*$	
2	0.296	0.294	2	0.176	0.183
3	0.290	0.28887	3	0.1682	0.1640
4	0.28891	0.28876	4	0.16694	0.1672
5	0.28872	0.288686	5	0.16671	0.16658
6	0.288682	0.2886774	6	0.166675	0.166682
7	0.2886764	0.2886755	7	0.1666682	0.1666640
8	0.2886754	0.2886752	8	0.1666669	0.1666671

* For this example the even approximants $\{K_{2n}\}$ are exact when $x = y$.

Table 5

(iii) $f(x, y) = e^{-x}/\sqrt{\{1 + y\}}$					
n	[n/n]	K_{2n+1}	n	[n/n]	K_{2n+1}
	x = 1, y = 1, f = 0.2601300475			x = 1, y = 2, f = 0.2123952944	
2	0.26059	0.256	2	0.2133	0.201
3	0.2601296	0.260113	3	0.21244	0.21213
4	0.26013013	0.2601318	4	0.2123983	0.2123952
5	0.2601300495	0.260130059	5	0.21239551	0.2123947
6	0.2601300476	0.2601300472	6	0.21239531	0.212395236
7	0.2601300475	0.2601300475	7	0.2123952955	0.2123952906
			8	0.2123952945	0.2123952941
	x = 1, y = 5, f = 0.1501862			x = 5, y = 5, f = 0.002751	
2	0.1544	0.110	2	0.044	0.15
3	0.15088	0.148	3	-0.0024	-0.0073
4	0.15031	0.1499	4	0.0032	0.0069
5	0.15021	0.15012	5	0.002728	0.00280
6	0.1501900	0.150175	6	0.002752	0.00249
7	0.1501868	0.1501842	7	0.002751	0.002748
8	0.1501863	0.1501858	8	0.002751	0.002754

Table 6

(iv) $f(x, y) = e^{-(x + y)}$					
n	[n/n]	K_{2n+1}	n	[n/n]	K_{2n+1}
	x = 0.3, y = 0.3, f = 0.5488116361			x = 1, y = 1, f = 0.1353352832	
2	0.5488154	0.548889	2	0.13573	0.143
3	0.5488116337	0.54881175	3	0.1353325	0.13543
4	0.5488116361	0.5488116364	4	0.135335294	0.1353387
5	0.5488116361	0.5488116361	5	0.1353352832	0.13533530
			6	0.1353352832	0.1353352835
			7	0.1353352832	0.1353352832
	x = 1, y = 2, f = 0.04978706837			x = 5, y = 5, f = 0.00004540	
2	0.053	0.067	2	0.011	0.042
3	0.049713	0.04984	3	0.000035	-0.0022
4	0.0497882	0.04982	4	0.000060	0.0029
5	0.049787057	0.04978719	5	0.0000446	0.00032
6	0.04978706844	0.049787078	6	0.00004543	0.00037
7	0.04978706837	0.04978706839	7	0.00004540	0.000056
8	0.04978706837	0.04978706837			

7. CONCLUSION

In this paper we have shown how continued fractions in two variables may be constructed, restricting our investigation to a Stieltjes-type fraction. It is also possible to carry the generalisation to N variables and to show how other types of continued fraction may be generalised. In particular, a two-variable fraction can be used to interpolate to point values of a function given at the nodes of a rectangular mesh. The details of this work will be left to a separate paper.

REFERENCES

1. CHISHOLM, J. S. R. : "Rational approximants defined from double power series", Math. Comp., v.27, 1973, No. 124, pp. 841-848.

2. LUTTERODT, C. A. : "A two-dimensional analogue of Padé approximant theory", J. Phys. A : Math., Nucl. Gen., v.7, 1974, No. 9, pp. 1027-1037

3. ALABISO, C. & BUTERA, P. : "N-variable rational approximants and method of moments", J. Math. Phys., v.16, 1975, No. 4, pp. 840-845.

4. LEVIN, D. : "General order Padé-type rational approximants defined from double power series", ES-74-041, 1974, Dept. of Environmental Sciences, Tel-Aviv University.

5. WALL, H. S. : *Analytic theory of continued fractions,* Chelsea, 1948.

6. KHOVANSKII, A. N. : *The application of continued fractions and their generalisations to problems in approximation theory,* (trans. by P. Wynn), Noordhoff, 1963.

7. HANDSCOMB, D. C. (ed.) : *Methods of numerical approximation,* Pergamon, 1966.

8. MURPHY, J. A. & O'DONOHOE, M. R. : "Some properties of continued fractions with applications in Markov processes" J.I.M.A., v.16, 1975, pp. 57-71.

9. O'DONOHOE, M. R., Ph.D. thesis, 1974, Brunel University.

10. HUGHES JONES, R. & MAKINSON, G. J. : "The generation of Chisholm rational polynomial approximants to power series in two variables", J.I.M.A., v.13, 1974, pp. 299-310.

11. GRAVES-MORRIS, P. R.; HUGHES JONES, R. & MAKINSON, G. J. : "The calculation of some rational approximants in two variables", J.I.M.A., v.13, 1974, pp. 311-320.