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Matrix Near-Rings Contained in 2-Primitive Near-Rings with Minimal Subgroups

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1. INTRODUCTION

Matrix near-rings over general near-rings were introduced in [5]; more material appeared in [6–9]. Most of these results are concerned with the transfer of properties from the near-ring R to the matrix near-ring $\mathscr{M}_n(R)$ or vice versa. Notable exceptions are the construction of certain examples in [6, 7], as well as the proof in [8, 3.12] that a certain bicentralizer near-ring is actually a matrix near-ring.

In this paper we consider the question as to whether matrix near-rings have some significance in the structure theory of near-rings, and we find enough evidence of this to suggest that there is quite a bit more to be uncovered. Analogous to the ring case, we find that there is a *near-vector space* [1] lurking behind every 2-primitive near-ring with suitable finiteness conditions, and that such a near-vector space has a certain type of matrix near-ring, which is a generalization of that introduced in [5], associated with it. Moreover, it turns out that this setting forces the near-ring in question to be abelian, which is rather interesting in the sense that up to now very little in general near-ring theory has indicated that abelianness of the additive group has any special significance at all.

Our starting point is the fundamental work of Betsch on 2-primitive (left) near-rings with minimal right ideals. The results which we shall be referring to appear in his paper [2]. However, for the convenience of the reader we shall cite them, as well as some other results we need, from Meldrum [4]; thus "M4.5" will refer to the result numbered 4.5 in [4]. Of particular relevance are the following two theorems which we quote in full from M4.5 and M4.12 (with the obvious left–right switch):

THEOREM A. *Let R be a 2-primitive zerosymmetric right near-ring which has a minimal left ideal K. Suppose G is an R-module of type 2. Then we have:*

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(1) *G is R-isomorphic to* ${}_RK$.

(2) *There is an idempotent* $e \in K \setminus \{0\}$ *such that* (a) $K = Re = Ke$ *and* (b) $\mathrm{End}_R\, G$ *and* eRe *are ismorphic semigroups.*

(3) *Every nonzero element of K has rank* 1.

THEOREM B. *Suppose R is 2-primitive on the R-module G. If R contains an idempotent e of rank* 1 *then R has a minimal left ideal or else we have* $e \neq 1$ *and* $\mathrm{Ann}_R(G \setminus eG) = \{0\}$.

It turns out that if one focuses on minimal left R-subgroups instead of on minimal left ideals, then both these theorems remain valid, except that one can dispense with the exceptional case in Theorem B (which, as the example in [7] shows, is not so exceptional after all).

Terminology and notation not explained here may be looked up in [4].

2. PRIMITIVE NEAR-RINGS WITH MINIMAL SUBGROUPS

Throughout this paper R will denote a zerosymmetric right near-ring with identity 1; all R-modules will be required to be unitary.

An examination of the proof of Theorem A reveals that the only property of the left ideal K which is used is that it is also a left R-subgroup. The result we obtain by replacing "left ideal" by "left R-subgroup" in Theorem A will be referred to as Theorem A'; it gives us one part of the next result, which should be compared to Theorems A and B.

THEOREM 2.1. *Suppose R is 2-primitive on the R-module G. Then R has a minimal left R-subgroup if and only if R contains an idempotent e of rank* 1.

Proof. For the "if" part, suppose R contains an idempotent e of rank 1 and let $K := Re$. We wish to show (a) that there is an element $h \in G$ such that $\mathrm{Ann}_R\, h \cap K = \{0\}$. This will enable us to (b) construct an R-isomorphism τ: $K \to G$ which will imply that K is minimal.

(a) Since G is faithful and type 2, there exists $g \in G$ such that $Kg = G$. Now $h := eg \neq 0$ because $Kg = Reg$. So $eh = h$; also $Kh = G$. If $D := \mathrm{Aut}_R\, G$, then $eG = hD \cup \{0\}$ because e is of rank 1. We assert that $Mh \neq 0$ for any nonzero left R-subgroup M of K. To see this, note that, since $M \subseteq K = Re$, we have $M = Xe = (Xe)e$ for some $X \subseteq R$. Again, since G is faithful and type 2, there exists $g' \in G$ such that $Mg' = G$, so $mg' = h$ for some $m \in M$. But $mg' = (me)\, g' = m(eg')$, so $m(eg') \neq 0$, i.e., $m \notin \mathrm{Ann}_R(eg')$. However, eg' is in the same orbit as h, which implies that $\mathrm{Ann}_R(eg') = \mathrm{Ann}_R\, h$ by M3.18, so $m \notin \mathrm{Ann}_R\, h$. We have shown that no nonzero left R-subgroup M of K is contained in $\operatorname{Ann}_R h$, and therefore $\operatorname{Ann}_R h \cap K$, being a left R-subgroup of K, is zero.

(b) Define $\tau: K \to G$ by $k \mapsto kh$. Clearly, this is an epimorphism with $\operatorname{Ker} \tau = \operatorname{Ann}_R h \cap K = 0$, i.e., an isomorphism.

From Theorem A′ we know that $\operatorname{End}_R G$ and eRe are isomorphic semigroups; in fact we have the following result:

COROLLARY 2.2. *Let R be 2-primitive on G and suppose K is a minimal left R-subgroup and that e is an idempotent such that $K = Re$. Then $(eRe, \cdot)$ is a group with zero, and the group $eRe\backslash\{0\}$ acts on Re by right multiplication as a fix point free group of automorphisms.*

Proof. Apply Theorem A′ and M3.35.

COROLLARY 2.3. *If R is 2-primitive then all minimal left R-subgroups and all minimal left ideals of R, when considered as R-modules, are R-isomorphic.*

Proof. This follows from Theorems A and A′.

COROLLARY 2.4. *Assume that R is 2-primitive and contains a distributive idempotent e of rank 1. Then eRe is a near-field.*

Proof. By Theorem 2.1, Re is a minimal left R-subgroup of R, and by Corollary 2.2 $(eRe, \cdot)$ is a group with zero. However, if e is distributive then $(eRe, +)$ is a group. This means that $(eRe, +, \cdot)$ is a near-field.

Since the additive group of a near-field is well known to be abelian, we have the following consequence:

COROLLARY 2.5. *Under the hypothesis of Corollary 2.4 $(eRe, +)$ is an abelian group.*

We can apply the above to derive a result of Maxson and Meldrum [3]:

COROLLARY 2.6. *A near-ring R is a near-field if $R = M_A(G)$, where A acts fix point free on G and $G\backslash\{0\}$ is an orbit.*

Proof. Since $M_A(G)$ is 2-primitive on G, and $1 \in M_A(G)$ is a distributive idempotent of rank 1, by Corollary 2.4, $M_A(G)$ is a near-field.

3. NEAR-RINGS WITH A COMPLETE SET OF DISTRIBUTIVE IDEMPOTENTS

Suppose R contains a *complete set of distributive idempotents*, i.e., it contains a finite set $\{e_1, ..., e_n\}$ of idempotents such that $e_1 + \cdots + e_n = 1$,

$e_i e_j = 0$ if $i \neq j$, each e_i is of rank 1, and each e_i is a distributive element of R. Let us agree to call such a near-ring a CDI-*near-ring*. If R is also 2-primitive on the R-module G, and $A := \mathrm{End}_R\, G$, then, by the results of Section 2, we have

(a) n *R-isomorphisms* φ_i: $G \to Re_i$, as well as

(b) n semigroup isomorphisms ψ_i: $A \to e_i Re_i$ such that

$$\varphi_i(rg\alpha) = r\varphi_i(g)\, \psi_i(\alpha)$$

for all $r \in R$, $g \in G$, $\alpha \in A$.

From φ_i: $G \to Re_i$ it follows that $\varphi_i(e_i G) = e_i Re_i$, i.e., φ_i is an isomorphism of the groups $(e_i G, +)$ and $(e_i Re_i, +)$. However, the latter is the additive structure of a near-field and therefore abelian; hence $(e_i G, +)$ is abelian. Now it is easy to see that

$$G = e_1 G + \cdots + e_n G$$

is a direct decomposition, and so G itself is abelian. Moreover, if Φ is the sum of the isomorphisms φ_i: $e_i G \to e_i Re_i$, then

$$\Phi: G \to e_1 Re_1 \oplus \cdots \oplus e_n Re_n =: H$$

is an isomorphism of groups, where

$$\Phi(g) = \varphi_1(e_1 g) + \cdots + \varphi_n(e_n g).$$

This fact enables us to turn H into an R-module by defining

$$r\Phi(g) := \Phi(rg),$$

which ensures that Φ is an R-isomorphism. In this situation (see M4.2) there is a semigroup isomorphism Ψ: $A \to \mathrm{End}_R\, H$, where

$$\Phi(g)\, \Psi(\alpha) := \varphi_1(e_1 g)\, \psi_1(\alpha) + \cdots + \varphi_n(e_n g)\, \psi_n(\alpha),$$

such that

$$\Phi(rg\alpha) = r\Phi(g)\, \Psi(\alpha).$$

So, provided it is not a ring, R is a dense subnear-ring of $M_{\Psi(A)}(H)$ (see M3.35).

In order to formulate the main theroems of this section we need a concept introduced by André [1]. A *near-vector space* is a pair (G, A), where G is an abelian group and A is a group with zero of endomorphisms of G with the following properties:

(a) If $\alpha \in A$ is nonzero it is a fix point free automorphism of $(G, +)$.

(b) The automorphism -1 is in A.

(c) If

$$Q(G, A) := \{ g \in G \mid (\forall \alpha, \beta \in A)(\exists \gamma \in A) \text{ with } g\alpha + g\beta = g\gamma \}$$

is the *quasi-kernel* of G, then Q generates G as a group.

If (G, A) is a near-vector space, it seems natural to call the elements of $M_A(G)$ *near-linear transformations.*

We first prove

LEMMA 3.1. *$(H, \Psi(A))$ is a near-vector space.*

Proof. We begin by proving that -1 is in $\Psi(A)$. Recall that the multiplicative group of a near-field has just one element of order 2, namely -1, (unless $1 = -1$, in which case there is nothing to prove). From the isomorphisms $\psi_i \colon A \to e_i r e_i$ it therefore follows that A has just one element of order 2, say α_0, and consequently

$$\Phi(g)\, \Psi(\alpha_0) = \varphi_1(e_1 g)(-e_1) + \cdots + \varphi_n(e_n g)(-e_n) = -\Phi(g).$$

To end the proof, we show that

$$\bigcup_{i=1}^{n} e_i R e_i \subseteq Q(H, \Psi(A)) =: Q$$

and since the set on the left is a set of generators for H, so is Q. If $\Psi(\alpha)$, $\Psi(\beta) \in \Psi(A)$ and $e_i r e_i \neq 0$ then

$$\begin{aligned}(e_i r e_i)\, \Psi(\alpha) + (e_i r e_i)\, \Psi(\beta) &= (e_i r e_i)\, \psi_i(\alpha) + (e_i r e_i)\, \psi_i(\beta) \\ &= (e_i r e_i)\, \psi_i(\gamma) \\ &= (e_i r e_i)\, \Psi(\gamma),\end{aligned}$$

where γ is determined by

$$\psi_i(\gamma) = (e_i r e_i)^{-1}\, (e_i r e_i \psi_i(\alpha) + e_i r e_i \psi_i(\beta)),$$

because $e_i R e_i \backslash \{0\}$ is a multiplicative group.

What we have proved up to this point amounts to the following theorem:

THEOREM 3.2. *A 2-primitive CDI-near-ring which is not a ring is abelian, symmetric (i.e., $a(-b) = -ab$), and isomorphic to a dense subnear-ring of a* *near-ring of near-linear transformations of a finite dimensional near-vector space.*

Going the other way, we next establish

THEOREM 3.3. *Suppose (G, A) is a finite dimensional near-vector space. Then $M_A(G)$ is a CDI-near-ring which is 2-primitive on G.*

Proof. Let $\{u_1, ..., u_n\}$ be a basis for $Q(G, A)$. Then every element $g \in G$ is representable in a unique way as $g = u_1\alpha_1 + \cdots + u_n\alpha_n$, where $\alpha_i \in A$. Now define e_i in $M_A(G)$ by $e_i(u_1\alpha_1 + \cdots + u_n\alpha_n) := u_i\alpha_i$, for $i = 1, ..., n$. Then the set $\{e_1, ..., e_n\}$ has the necessary properties to ensure that $M_A(G)$ is CDI. That $M_A(G)$ is also 2-primitive on G follows from a well-known theorem (see M3.34).

This theorem, together with the material preceeding Theorem 3.2, now gives the following two characterizations of finite dimensional near-vector spaces:

THEOREM 3.4. *Let G be a group and let $A := D \cup \{0\}$, where D is a fix point free group of automorphisms of G.*

(1) *(G, A) is a finite dimensional near-vector space if and only if $M_A(G)$ is a CDI-near-ring.*

(2) *(G, A) is a finite dimensional near-vector space if and only if there exist a finite number of near-fields, $F_1, ..., F_n$, semigroup isomorphisms $\psi_i\colon A \to F_i$, and a group isomorphism $\Phi\colon G \to F_1 \oplus \cdots \oplus F_n$ such that if*

$$\Phi(g) = x_1 + \cdots + x_n,$$

then

$$\Phi(g\alpha) = x_1\psi_1(\alpha) + \cdots + x_n\psi_n(\alpha),$$

for all $g \in G$ and $\alpha \in A$.

The picture of a finite dimensional near-vector space that emerges contrasts quite interestingly with that of a finite dimensional vector space. In the former case we have, in general, not one but a finite set of associated near-fields, all having isomorphic multiplicative semigroup structures, but not necessarily being isomorphic as near-fields. The second part of Theorem 3.4 should also be compared to what [1, Theorem 4.6] asserts for the finite dimensional case.

We now turn to the near-ring perspective again. According to the previous theorem we can specify a finite dimensional near-vector space by taking n near-fields $F_1, ..., F_n$ such that there are semigroup isomorphisms

ϑ_{ij}: $(F_j, \cdot) \to (F_i, \cdot)$ with $\vartheta_{ij}\vartheta_{jk} = \vartheta_{ik}$ for $1 \leqslant i, j, k \leqslant n$. We then take $G := F_1 \oplus \cdots \oplus F_n$ as the additive group of the near-vector space and any one of the semigroups $(F_{i_0}, \cdot)$ as the semigroup of endomorphisms by defining

$$(x_1 + \cdots + x_n)\alpha := x_1 \vartheta_{1i_0}(\alpha) + \cdots + x_n \vartheta_{ni_0}(\alpha)$$

for all $x_j \in F_j$ and all $\alpha \in F_{i_0}$. Our object is to study the near-ring

$$R := M_{F_{i_0}}(G).$$

More specifically, we would like to determine the smallest subnear-ring S of R which contains the complete set of distributive idempotents $\{e_1, ..., e_n\}$, where e_j is defined by $e_j(x_1 + \cdots + x_n) := x_j$, and which is 2-primitive on G. Since S is supposed to be 2-primitive, given any $(b_{1j}, ..., b_{nj})' \in G$, (we are now adopting vector notation, letting transposes be indicated by primes), S must contain an element s_j such that $s_j(0, ..., 0, 1, 0, ..., 0)' = (b_{1j}, ..., b_{nj})'$, where the 1 is in position j. Now if $(x_1, ..., x_n)' \in G$ then

$$\begin{aligned} s_j e_j(x_1, ..., x_n)' &= s_j(0, ..., 0, x_j, 0, ..., 0)' \\ &= s_j(0, ..., 0, 1, 0, ..., 0)' \, \vartheta_{ji_0}^{-1}(x_j) \\ &= (b_{1j}, ..., b_{nj})' \, \vartheta_{ji_0}^{-1}(x_j) \\ &= (b_{1j}\vartheta_{1j}(x_j), b_{2j}\vartheta_{2j}(x_j), ..., b_{nj}\vartheta_{nj}(x_j))'. \end{aligned}$$

It follows that $\sum_{j=1}^{n} s_j e_j$ is in S; and the above calculation shows that it can adequately be denoted by the matrix (b_{ij}). In fact, every square matrix $C := (c_{ij})$, where $c_{ij} \in F_i$, represents an element of S, with the action on G defined by

$$\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} c_{11}\vartheta_{11}(x_1) + \cdots + c_{1n}\vartheta_{1n}(x_n) \\ \vdots \\ c_{n1}\vartheta_{n1}(x_1) + \cdots + c_{nn}\vartheta_{nn}(x_n) \end{pmatrix}.$$

It is not difficult to see that the subring of $M_{F_{i_0}}(G)$ generated by the set of all these elements is indeed 2-primitive on G, so it equals S. Let us call it the *near-ring of matrices determined by the near-fields* $F_1, ..., F_n$ *and the matrix of isomorphisms* (ϑ_{ij}), and denote it by $\mathscr{M}_n(\{F_i\}, (\vartheta_{ij}))$. Note that the choice of i_0 does not figure in the definition of this near-ring. We formulate the preceeding observations in

THEOREM 3.5. *Suppose* (G, A) *is an* n*-dimensional near-vector space. Then* $M_A(G)$ *contains a subnear-ring* S *isomorphic to a near-ring of matrices* *determined by n near-fields with isomorphic multiplicative semigroups. If S is not a ring, then S is dense in* $M_A(G)$.

The ring case does not occur when (G, A) is not a vector space, and so we have the next corollary.

COROLLARY 3.6. *Suppose* (G, A) *is a finite near-vector space which is not a vector space. Then* $M_A(G)$ *is isomorphic to a near-ring of matrices determined by a finite number of finite near-fields with isomorphic multiplicative semigroups.*

Let us close by considering the following simple

EXAMPLE 3.7. Let $F_1 = F_2 := \mathbf{R}$, the field of real numbers, let ϑ_{11} be the identity function, and let ϑ_{21} be defined by $\vartheta_{21}(x) := x^3$. Then the action of a matrix (c_{ij}) in $\mathcal{M}_2(\{F_i\}, (\vartheta_{ij}))$ on $F_1 \oplus F_2 = \mathbf{R}^2$ is given by

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_{11}x_1 + c_{12}x_2^{1/3} \\ c_{21}x_1^3 + c_{22}x_2 \end{pmatrix},$$

while the action of the semigroup of endomorphisms, which is of course isomorphic to the multiplicative semigroup of the reals, is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \alpha_a = \begin{pmatrix} x_1 a \\ x_2 a^3 \end{pmatrix},$$

where α_a is the endomorphism corresponding to the real number a. Clearly, the near-ring $\mathcal{M}_2(\{F_i\}, (\vartheta_{ij}))$ is not a ring, and it can be shown that it does not contain a single minimal left ideal, although it does, of course, have at least two minimal left R-subgroups corresponding to the complete set of idempotents

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

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