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Elliptic equations with singular BMO coefficients in Reifenberg domains

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ABSTRACT

$W^{1,p}$ estimate for the solutions of elliptic equations whose coefficient matrix can have large jump along the boundary of subdomains is obtained. The principal coefficients are supposed to be in the John–Nirenberg space with small BMO seminorms, and the domain and subdomains are Reifenberg flat domains. Moreover, it has been shown that the estimates are uniform with respect to the distance between the subdomains.

1. Introduction

We consider the following Dirichlet problem for the divergence form elliptic equation

$$\begin{cases} -(a_{ij}u_{x_j})_{x_i} = -\operatorname{div}\big(A(x)\nabla u(x)\big) = \operatorname{div} f = \big(f^i\big)_{x_i} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is an open and bounded subset of $\mathbb{R}^n$. Throughout this paper we assume that the $n \times n$ matrix A is defined on $\mathbb{R}^n$ as follows:

$$A = \sum_{i=0}^{K} A^i \chi_{\Omega^i}$$

where $\Omega^1, \ldots, \Omega^K$ are disjoint open subsets of Ω with flat boundary (see Definition 1.2), $\Omega^0 := \Omega \setminus \bigcup_{i=1}^{i=K} \Omega^i$, and A^i's for $i = 0, \ldots, K$ are uniformly bounded and uniformly elliptic with ellipticity

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constant Λ and also they are in the John–Nirenberg space BMO of the functions of bounded mean oscillation with small BMO seminorms, see [6].

This problem arises from the underground water flow through composite media with closely spaced interfacial boundaries. In particular, the coefficient matrix A has discontinuity across the boundaries of subdomains. There have been many results proving $C^{1,\alpha}$ regularity for a weak solution, see [8,7,1]. In this paper, we prove $W^{1,p}$ regularity for Elliptic Dirichlet problem with singular coefficient matrix A, under some necessary conditions. Our approach is influenced by [4] and [10]. However, additional difficulties are present due to the fact that we allow discontinuous coefficients, these are handled in the following sections.

We assume A^i's are (δ, R)-*vanishing* in Ω^i for $i = 0, \dots, K$. Let us also recall the following, see Section 2.2 for undefined notation.

Definition 1.1 *(Small BMO seminorm assumption).* We say that the matrix A of coefficients is $(\delta, R)-$ *vanishing* in Ω if

$$\sup_{0<r\leqslant R}\sup_{x\in\mathbb{R}^n}\sqrt{\frac{1}{|B_r|}\int_{B_r(x)}\left|A(y)-\bar{A}_{B_r(x)}\right|^2 dy}\leqslant\delta.$$

In our setting Ω_i, $i = 1, \dots, K$, are (δ, R)-Reifenberg Flat Domain defined as following:

Definition 1.2 *(Reifenberg Flat Domain assumption).* For $R > 0$, $\delta > 0$, we say that a domain Ω is (δ, R)-*Reifenberg flat* if for every $x \in \partial\Omega$ and every $r \in (0, R]$, there exists orthonormal coordinate system $(y_1, \dots, y_n)$ with origin at x so that in that coordinate system

$$B_r(0)\cap\{y_n > r\delta\}\subset\Omega,$$
$$B_r(0)\cap\{y_n < -r\delta\}\subset\Omega^c.$$

To deal with *flat* domains, this definition becomes meaningful when $\delta > 0$ is small and one can see δ depends on R. From this definition, we can see that if a domain Ω is (δ, R)-*Reifenberg flat*, then for any $x \in \partial\Omega$ and every $r \in (0, R]$, there exists an $(n-1)$ dimensional plane $\mathcal{P}(x, r)$ such that

$$\frac{1}{r}D\big[\partial\Omega\cap B_r(x),\mathcal{P}(x,r)\cap B_r(x)\big]\leqslant\delta,$$

where D denotes the Hausdorff distance.

We will get $W^{1,p}$ estimate for the classical weak solution of (1). As usual, the following is the definition for a weak solution.

Definition 1.3. We say that $u \in H_0^1(\Omega)$ is a weak solution of (1) if

$$\int_\Omega A\nabla u\nabla\varphi\,dx=-\int_\Omega f\nabla\varphi\,dx\quad\forall\varphi\in H_0^1(\Omega).$$

We are now ready to state the main result of this paper.

Theorem 1.4. *Let $1 < p < \infty$ be a real number. Then there is a small $\delta = \delta(\Lambda, p, n, R) > 0$ so that for all $f \in L^p(\Omega, \mathbb{R}^n)$ the Dirichlet problem (1), with the above notation and conditions, has a unique weak solution which satisfies the estimate*

$$\int_{\Omega} |\nabla u|^p \, dx \leqslant C \int_{\Omega} |f|^p \, dx. \tag{2}$$

The constant C is independent of u and f.

Let us just mention here that the constant C above does not depend on the distance between the subdomains, which allows the domains to touch each other.

Before our work, in the parabolic case, Fred Almgren and Lihe Wang proved the $C^{1,\alpha}$ estimates for heat flows across an interface in [1] where the coefficient matrix has singularity along the Hölder continuous boundaries of subdomains.

In the elliptic case, in [8], Y. Li and M. Vogelius considered an elliptic equation

$$\operatorname{div}(A\nabla u) = h + \operatorname{div}(g) \tag{3}$$

on a bounded domain D which has a finite number of disjoint subdomains with $C^{1,\alpha}$ boundary. They also allow the matrix A to have discontinuity across the boundaries. They proved a $C^{1,\alpha}$ regularity for the solution under reasonable Hölder continuity assumptions on A, h and g_i. Later in [7], Y. Li and L. Nirenberg extended the result in [8] to general second order elliptic systems with piecewise smooth coefficients. It is worth mentioning that this extension has applications in problems arising in elasticity.

Structure of the paper: in Section 2, we state preliminary notation, definitions and assumptions throughout this paper. Mathematical background and main tools are given in Section 3. In the first subsection of Section 4, we discuss the interior $W^{1,p}$ regularity for a weak solution of (1) and in the second subsection, a global $W^{1,p}$ regularity is derived.

2. Definitions and notation

2.1. Geometric notation

(1) A typical point in $\mathbb{R}^n$ is $x = (x', x_n)$. A typical point in $\mathbb{R}^n \times \mathbb{R}$ is $(x,t) = (x', x_n, t)$.
(2) $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n;\ x_n > 0\}$ and $\mathbb{R}^n_- = \{x \in \mathbb{R}^n;\ x_n < 0\}$.
(3) $B_r = \{x \in \mathbb{R}^n\colon\ |x| < r\}$ is an open ball in $\mathbb{R}^n$ centered at 0 and radius $r > 0$, $B_r(x) = B_r + x$, $B_r^+ = B_r \cap \{x_n > 0\}$, $B_r^+(x) = B_r^+ + x$, $T_r = B_r \cap \{x_n = 0\}$, and $T_r(x) = T_r + x$.
(4) $\Omega_r = \Omega \cap B_r$, $\Omega_r(x) = \Omega \cap B_r(x)$.
(5) $\partial\Omega_r$ is the boundary of Ω_r, $\partial_w \Omega_r = \partial\Omega \cap B_r$ is the wiggled part of $\partial\Omega_r$, and $\partial_c \Omega_r = \partial\Omega_r \backslash \partial_w \Omega_r$ is the curved part of $\partial\Omega_r$.
(6) $\mathcal{P}_i^\delta(y)$ is the $(n-1)$ dimensional plane which is translated hyperplane at $y \in \partial\Omega^i$ by δ along the normal direction toward Ω^i.
(7) Hausdorff distance D is defined as

$$D[A,B] = \sup\bigl\{\operatorname{dist}(a,B)\colon\ a \in A\bigr\} + \sup\bigl\{\operatorname{dist}(b,A)\colon\ b \in B\bigr\}.$$

2.2. Matrix of coefficients

A is supposed to be $A = \sum_{i=0}^K A^i \chi\Omega^i$ where A^i's for any $i = 0,\dots,K$ are $(\delta, R)-$*vanishing* on Ω^i, uniformly bounded and uniformly elliptic, i.e., there exists a positive constant Λ such that

$$\Lambda^{-1}|\xi|^2 \leqslant A^i(x)\xi \cdot \xi \leqslant \Lambda|\xi|^2 \quad \text{a.e. } x \in \mathbb{R}^n,\ \forall \xi \in \mathbb{R}^n$$

and

$$\|A\|_\infty = \sup_y \left|A(y)\right| < C.$$

We also fix the following notation.

(1) $|A| = \sqrt{(A:A)} = \sqrt{\sum_{i,j=1}^n a_{ij}^2}$.
(2) The average of A over Ω is $\bar{A}_\Omega = \frac{1}{|\Omega|}\int_\Omega |A(x)|\,dx$.
(3) $\tilde{A}_{B_r} := \sum_{i=0}^K \overline{A^i}_{\Omega_r^i}\chi_{\Omega_r^i}$.

2.3. Notation for estimates

We employ the letter C to denote a universal constant usually depending on the dimension, ellipticity and the geometric quantities of Ω.

3. Preliminary tools and mathematical background

In this section we recall standard facts from measure theory and functional analysis which will be needed in the sequel.

One of our main tools will be the Hardy–Littlewood maximal function since a function value at a point in L^p does not make a good sense. The maximal function controls the local behavior of a function in an analytical way.

Definition 3.1. For a locally integrable function f on $\mathbb{R}^n$. Let

$$(\mathcal{M}f)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \left|f(y)\right| dy$$

be the Hardy–Littlewood maximal function of f. We also define

$$\mathcal{M}_\Omega f = \mathcal{M}(\chi_\Omega f)$$

if f is not defined outside Ω.

The basic theorem for the Hardy–Littlewood maximal function is the following:

Theorem 3.2. *(Cf. [9].) We have*

(a) *If $f \in L^p(\mathbb{R}^n)$ with $p > 1$, then $\mathcal{M}f \in L^p(\mathbb{R}^n)$. Moreover,*

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}.$$

(b) *If $f \in L^1(\mathbb{R}^n)$, then*

$$\left|\left\{x \in \mathbb{R}^n \colon (\mathcal{M}f)(x) > \lambda\right\}\right| \leqslant \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

To show $\nabla u \in L^p$, we will use the following lemma:

Lemma 3.3. *(Cf. [3].) Suppose that f is a nonnegative measurable function in a bounded domain Ω. Let $\theta > 0$ and $m > 1$ be constants. Then for $0 < p < \infty$,*

$$f \in L^p(\Omega) \quad \textit{iff} \quad S = \sum_{k \geqslant 1} m^{kp} \left| \left\{ x \in \Omega \colon f(x) > \theta m^k \right\} \right| < \infty$$

and

$$\frac{1}{C} S \leqslant \|f\|^p_{L^p(\Omega)} \leqslant C\left(|\Omega| + S\right),$$

where $C > 0$ is a constant depending only on θ, m and p.

Another main tool is the modified Vitali Covering Lemma:

Lemma 3.4. *(Cf. [2].) Assume that C and D are measurable sets. $C \subset D \subset \Omega$ with Ω $(\delta, 1)$-Reifenberg flat, and that there exists an $\varepsilon > 0$ such that*

$$|C| < \varepsilon |B_1| \tag{4}$$

and for all $x \in B_1$ and for all $r \in (0, 1]$ with $|C \cap B_r(x)| \geqslant \varepsilon |B_r(x)|$,

$$B_r(x) \cap \Omega \subset D. \tag{5}$$

Then

$$|C| \leqslant \left(\frac{10}{1 - \delta} \right)^n \varepsilon |D|.$$

4. Regularity for elliptic equations

4.1. Interior estimates

In this section we investigate the interior $W^{1,p}$ estimates for a solution of

$$-\operatorname{div}\left(A(x) \nabla u\right) = \operatorname{div} f \quad \text{in } \Omega, \tag{6}$$

under the conditions as in Section 1.

$W^{1,p}$ estimate without discontinuity in A was done by S. Byun and L. Wang in [2]. Here we consider the case that A has discontinuity along the boundary of subdomains Ω^i's in Ω for $i = 1, \dots, K$.

The main result of this section is the following:

Theorem 4.1. *There is a constant N_1 so that for any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ such that for all $f \in L^2(B_4; \mathbb{R}^n)$ and for all A as in Section 2.2 with $R = 4$ and Ω are $(\delta, 9)$-flat, if u is a weak solution of $-\operatorname{div}(A \nabla u) = \operatorname{div} f$ in $\Omega \supset B_4$ and if*

$$\left| \left\{ x \in \Omega \colon \mathcal{M}\left(|\nabla u|^2\right)(x) > N_1^2 \right\} \cap B_r \right| \geqslant \varepsilon |B_r| \quad \textit{for all } r \in (0, 1],$$

then

$$B_r \subset \left\{ x \in \Omega \colon \mathcal{M}\left(|\nabla u|^2\right)(x) > 1 \right\} \cup \left\{ x \in \Omega \colon \mathcal{M}\left(|f|^2\right)(x) > \delta^2 \right\}.$$

Definition 4.2. We say that $u \in H^1(B_R)$ $(R > 0)$ is a weak solution of (6) if

$$\int_{B_R} A\nabla u\nabla\varphi\, dx = -\int_{B_R} f\nabla\varphi\, dx \quad \forall\varphi \in H_0^1(B_R).$$

Lemma 4.3. *(Cf. [2].) Assume that u is a weak solution of* (6) *in* B_2. *Then*

$$\int_{B_2} \varphi^2|\nabla u|^2\, dx \leqslant C\left(\int_{B_2} \varphi^2|f|^2\, dx + \int_{B_2} |\nabla\varphi|^2|u|^2\, dx\right) \quad \text{for any } \varphi \in C_0^\infty(B_2). \tag{7}$$

We want to control the gradient of the weak solution of (6) using the gradient of the weak solution of the related homogeneous equation. The following lemma shows that one can bound the gradient of homogeneous solution by L^2-norm. The following is well known, we include the proof for the sake of completeness and using our notation here.

Lemma 4.4. *If* v *is a weak solution of* $\operatorname{div}(\bar{A}\nabla v(x)) = 0$ *in* B_1 *for a piecewise constant matrix* $\bar{A} = \overline{A^1}\chi_{B_1\cap\{x_n>a\}} + \overline{A^0}\chi_{B_1\cap\{x_n<a\}}$ *for any* $a \in (-1, 1)$, *then*

$$\|\nabla v\|_{L^\infty(B_{\frac{1}{2}})} \leqslant C\|v\|_{L^2(B_1)}.$$

Proof. First assume $a = 0$. Let $D_i^h v(x) = \frac{v(x+he_i)-v(x)}{h}$, for $h > 0$, $i = 1, \ldots, n-1$. Since the jump of the coefficient matrix $\bar{A}$ occurs across $\{x_n = 0\}$,

$$\operatorname{div}\bigl(\bar{A}\nabla D_i^h v(x)\bigr) = 0$$

for sufficiently small $h > 0$. Also

$$\int_{B_{\frac{1}{2}+\frac{1}{4}}} \left|\nabla D_i^h v(x)\right|^2 dx \leqslant C \int_{B_{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}} \left|D_i^h v(x)\right|^2 dx \tag{8}$$

$$\leqslant C \int_{B_{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}}} \left|\nabla v(x)\right|^2 dx \tag{9}$$

$$\leqslant C \int_{B_1} \left|v(x)\right|^2 dx \tag{10}$$

for $0 < h < \frac{1}{16}$. Here we used Lemma 4.3 for the first and the third inequality. So $v_{x_i} \in H^1(B_{\frac{3}{4}})$ for $i = 1, \ldots, n-1$. Similarly, we can apply this method to v_{x_i}, i.e. using $D_j^h v_{x_i}(x)$ for $i, j = 1, \ldots, n-1$. So $v_{x_i x_j} \in H^1(B_{\frac{1}{2}+\frac{1}{8}})$ for $i = 1, \ldots, n-1$. Let $S = [\frac{n}{2}] + 3$. For any tangential vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 0)$ such that $|\alpha| \leqslant S$, we can iterate $|\alpha|$ times and get

$$D^\alpha v(x) \in H^1(B_{\frac{1}{2}+\frac{1}{2^{S+1}}}).$$

Since $\operatorname{div}(\bar{A}\nabla D^\alpha v(x)) = 0$, we can use the De Giorgi–Nash theorem to say that $D^\alpha v$ is Hölder continuous. So there is a constant C such that

$$\left\| D^\alpha v \right\|_{L^\infty(B_{\frac{1}{2}+\frac{1}{2^{S+2}}})} \leqslant C \left\| D^\alpha v \right\|_{L^2(B_{\frac{1}{2}+\frac{1}{2^{S+1}}})} \tag{11}$$

$$\leqslant C \| v \|_{L^2(B_1)}. \tag{12}$$

Now consider the vertical direction. Define

$$g(x_1, x_2, \dots, x_n) := v(x_1, \dots, x_{n-1}, 0) \quad \text{in } B^+_{\frac{1}{2}+\frac{1}{2^{S+1}}}.$$

We can see that $g_{x_n} = 0$ and also by (11),

$$\begin{cases} D^\alpha g = D^\alpha v \in H^1\big(B^+_{\frac{1}{2}+\frac{1}{2^{S+1}}}\big), \\ \left\| D^\alpha g \right\|_{L^\infty(B_{\frac{1}{2}+\frac{1}{2^{S+2}}})} = \left\| D^\alpha v \right\|_{L^\infty(B_{\frac{1}{2}+\frac{1}{2^{S+2}}})} \leqslant C \| v \|_{L^2(B_1)} \end{cases}$$

for $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 0)$ such that $|\alpha| \leqslant S$. Let

$$\tilde{v}(x_1, \dots, x_n) := v(x_1, \dots, x_n) - g(x_1, \dots, x_n).$$

Note that $\tilde{v} \in H^1(B^+_{\frac{1}{2}+\frac{1}{2^{S+1}}})$ and $\tilde{v}|_{x_n=0} = 0$. Since $\operatorname{div}(\bar{A}\nabla(\tilde{v} + g)) = 0$,

$$\begin{aligned} \operatorname{div}(\bar{A}\nabla\tilde{v}) &= -\operatorname{div}(\bar{A}\nabla g) \\ &= -\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \bar{a}_{ij} g_{x_i} \right)_{x_j} \\ &= -\sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} \bar{a}_{ij} g_{x_i} \right)_{x_j} \in H^{S-1}\big(B^+_{\frac{1}{2}}\big) \\ &= H^{[\frac{n}{2}]+1}\big(B^+_{\frac{1}{2}}\big). \end{aligned}$$

Furthermore, by Theorem 5 in Section 6.3 and the Trace Theorem, see Section 5.5 in [5], also by Lemma 4.3,

$$\|\tilde{v}\|_{H^{S-1}(B^+_{\frac{1}{2}})} \leqslant C\big(\|v\|_{L^2(B_1)} + \|\tilde{v}\|_{L^2(B_{\frac{1}{2}})}\big) \leqslant C\|v\|_{L^2(B_1)}, \tag{13}$$

we can combine (13) and Sobolev inequality to get

$$\|\tilde{v}\|_{C^{S-[\frac{n}{2}]-2,\gamma}(B^+_{\frac{1}{2}})} \leqslant C\|\tilde{v}\|_{H^{S-1}(B^+_{\frac{1}{2}})} \leqslant C\|v\|_{L^2(B_1)}.$$

Thus $\tilde{v}$ is $C^{1,\gamma}$ Hölder continuous. Finally we can say that $|\nabla\tilde{v}|$ is bounded in $\overline{B^+_{\frac{1}{2}}}$. Similarly $|\nabla\tilde{v}|$ is also bounded in $\overline{B^-_{\frac{1}{2}}}$. So $|\nabla\tilde{v}| = |\nabla v - \nabla\tilde{g}|$ is bounded in $B_{\frac{1}{2}}$. Thus

$$\|\nabla v\|_{L^\infty(B_{\frac{1}{2}})} \leqslant C\|v\|_{L^2(B_1)}. \tag{14}$$

Assume $|a| > \frac{3}{4}$. Then $\bar{A}$ has no discontinuity in $B_{\frac{3}{4}}$. So there is a constant C such that

$$\|\nabla v\|_{L^\infty(B_{\frac{1}{2}})} \leqslant C\|v\|_{L^2(B_{\frac{3}{4}})} \leqslant C\|v\|_{L^2(B_1)}. \tag{15}$$

Assume $0 < |a| < \frac{3}{4}$. Say $L := \{x \in \mathbb{R}^n\colon\ x_n = a\}$.

For any $x \in B_{\frac{3}{4}} \cap L$, $B_{\frac{1}{4}}(x) \subset B_1$. By above case for $a = 0$, there exists a constant C such that

$$\|\nabla v\|_{L^\infty(\{x\in B_{\frac{1}{2}}\colon \operatorname{dist}(x,L)<\frac{1}{8}\})} \leqslant \sup_{x\in B_{\frac{3}{4}}\cap L} \|\nabla v\|_{L^\infty(B_{\frac{1}{8}}(x))} \tag{16}$$

$$\leqslant C\|v\|_{L^2(B_{\frac{1}{4}}(x))} \leqslant C\|v\|_{L^2(B_1)}. \tag{17}$$

For any $x \in \{x \in B_{\frac{1}{2}}\colon\ \operatorname{dist}(x, L) \geqslant \frac{1}{8}\}$, $B_{\frac{1}{8}}(x) \subset B_1$ and $\bar{A}$ has no discontinuity in $B_{\frac{1}{8}}(x)$. So there exists a constant C such that

$$\sup_{\{x\in B_{\frac{1}{2}}\colon \operatorname{dist}(x,L)\geqslant\frac{1}{8}\}} \|\nabla v\|_{L^\infty(B_{\frac{1}{16}}(x))} \leqslant C\|v\|_{L^2(B_{\frac{1}{8}}(x))} \leqslant C\|v\|_{L^2(B_1)}. \tag{18}$$

By taking the maximum C in (14)–(16) and (18), we are done. □

Lemma 4.5. *For any $\varepsilon > 0$, there is a small $\delta = \delta(\varepsilon) > 0$ such that for any weak solution u of (6) in B_2 where for any $l, m = 0, \dots, K$ and any $|a| < 2$,*

$$B_2 \cap \{x_n > a + \delta\} \subset \Omega_2^l \subset B_2 \cap \{x_n > a - \delta\}, \tag{19}$$

$$B_2 \cap \{x_n < a - \delta\} \subset \Omega_2^m \subset B_2 \cap \{x_n < a + \delta\} \tag{20}$$

and

$$\frac{1}{|B_2|}\int_{B_2} |\nabla u|^2\,dx \leqslant 1, \tag{21}$$

$$\frac{1}{|B_2|}\int_{B_2} \left(|f|^2 + |A - \tilde{A}_{B_2}|^2\right)dx \leqslant \delta^2, \tag{22}$$

where $\tilde{A}_{B_2} = \sum_i \overline{A^i}_{\Omega_2^i}\chi_{\Omega_2^i}$, there exists a piecewise constant matrix $\tilde{A^b}_{B_2}$ as $\tilde{A^b}_{B_2} = \overline{A^l}_{\Omega_2^l}\chi_{B_2\cap\{x_n>a\}} + \overline{A^m}_{\Omega_2^m}\chi_{B_2\cap\{x_n<a\}}$ and for a corresponding weak solution v of

$$-\operatorname{div}\left(\tilde{A^b}_{B_2}\nabla v\right) = 0 \quad \textit{in } B_2 \tag{23}$$

such that

$$\int_{B_2} |u - v|^2\,dx \leqslant \varepsilon^2.$$

Proof. If not, there exists $\varepsilon_0 > 0$, $\{A_k\} = \{\sum_{i=0}^{K} A_k^i \chi_{\Omega^{i,k}}\}, \{u_k\}, \{f_k\}, \{\Omega_2^{l,k}\}$ and $\{(\Omega^{m,k})_2\}$ for some $l, m = 0, \dots, K$ and some $|a| < 2$ such that u_k is a weak solution of

$$-\operatorname{div}(A_k \nabla u_k) = \operatorname{div} f_k \quad \text{in } B_2 \tag{24}$$

with

$$B_2 \cap \left\{x_n > a + \frac{1}{k}\right\} \subset \left(\Omega^{l,k}\right)_2 \subset B_2 \cap \left\{x_n > a - \frac{1}{k}\right\},$$
$$B_2 \cap \left\{x_n < a - \frac{1}{k}\right\} \subset \left(\Omega^{m,k}\right)_2 \subset B_2 \cap \left\{x_n < a + \frac{1}{k}\right\}$$

but

$$\int_{B_2} |u_k - v_k|^2 \, dx > \varepsilon_0^2 \tag{25}$$

for any weak solution v_k of

$$-\operatorname{div}\left(\tilde{A}^b_{k\,B_2} \nabla v_k\right) = 0 \quad \text{in } B_2 \tag{26}$$

where $\tilde{A}^b_{k\,B_2} = \overline{A^l_k}_{(\Omega^{l,k})_2} \chi_{B_2 \cap \{x_n > a\}} + \overline{A^m_k}_{(\Omega^{m,k})_2} \chi_{B_2 \cap \{x_n < a\}}$.

By (21), $\{u_k - \overline{u_k}_{B_2}\}_{k=1}^{\infty}$ is bounded in $H^1(B_2)$, and so $\{u_k - \overline{u_k}_{B_2}\}$ has a subsequence, which we denote as $\{u_k - \overline{u_k}\}$, such that

$$u_k - \overline{u_k} \rightharpoonup u_0 \quad \text{in } H^1(B_2), \qquad u_k - \overline{u_k} \to u_0 \quad \text{in } L^2(B_2). \tag{27}$$

Since $\tilde{A}^b_{k\,B_2}$ is bounded in L^∞, there is a subsequence $\{\tilde{A}^b_k\}$ such that

$$\left\| \tilde{A}^b_k - A_0 \right\|_\infty \to 0 \quad \text{as } k \to \infty, \tag{28}$$

for some piecewise constant matrix A_0. Since $\tilde{A}^b_k - \tilde{A}_{k\,B_2} \to 0$ in $L^2(B_2)$ and $\tilde{A}_{k\,B_2} - A_k \to 0$ in $L^2(B_2)$. Thus $A_k \to A_0$ in $L^2(B_2)$.

Next we will show that u_0 is a weak solution of

$$-\operatorname{div}(A_0 \nabla u_0) = 0 \quad \text{in } B_2. \tag{29}$$

To do this, fix any $\varphi \in H_0^1(B_2)$. Then by (24),

$$\int_{B_2} A_k \nabla u_k \nabla \varphi \, dx = -\int_{B_2} f_k \nabla \varphi \, dx. \tag{30}$$

Since $\nabla u_k \rightharpoonup \nabla u_0$ and $A_k \to A_0$ in $L^2(B_2)$, $A_k \nabla u_k \rightharpoonup A_0 \nabla u_0$ in $L^2(B_2)$. Then by letting $k \to \infty$,

$$\int_{B_2} A_0 \nabla u_0 \nabla \varphi \, dx = 0. \tag{31}$$

This shows (29). Note that

$$\begin{aligned}-\operatorname{div}\big(\tilde{A}_k^b\nabla u_0\big) &= -\operatorname{div}\big(\big(\tilde{A}_k^b - A_0\big)\nabla u_0\big) - \operatorname{div}(A_0\nabla u_0)\\ &= -\operatorname{div}\big(\big(\tilde{A}_k^b - A_0\big)\nabla u_0\big)\end{aligned}$$

in B_2. Let h_k be the weak solution of

$$\begin{cases} -\operatorname{div}\big(\tilde{A}_k^b\nabla h_k\big) = \operatorname{div}\big(\big(\tilde{A}_k^b - A_0\big)\nabla u_0\big) & \text{in } B_2,\\ h_k = 0 & \text{on } \partial B_2. \end{cases} \tag{32}$$

Then $u_0 - h_k$ is a weak solution of

$$\operatorname{div}\big(\tilde{A}_k^b\nabla(u_0 - h_k)\big) = 0 \quad \text{in } B_2. \tag{33}$$

Furthermore, by (32),

$$\begin{aligned}\|h_k\|_{L^2(B_2)} &\leqslant C\|\nabla h_k\|_{L^2(B_2)} \leqslant C\big\|\big(\tilde{A}_k^b - A_0\big)\nabla u_0\big\|_{L^2(B_2)}\\ &\leqslant C\big\|\big(\tilde{A}_k^b - A_0\big)\big\|_{L^\infty}\|\nabla u_0\|_{L^2(B_2)}\\ &\leqslant C\big\|\big(\tilde{A}_k^b - A_0\big)\big\|_{L^\infty(B_2)}.\end{aligned}$$

So now

$$\begin{aligned}\big\|u_k - (u_0 + \overline{u_k} - h_k)\big\|_{L^2(B_2)} &\leqslant \|u_k - \overline{u_k} - u_0\|_{L^2(B_2)} + \|h_k\|_{L^2(B_2)}\\ &\leqslant \|u_k - \overline{u_k} - u_0\|_{L^2(B_2)} + C\big\|\big(\tilde{A}_k^b - A_0\big)\big\|_{L^\infty(B_2)}.\end{aligned}$$

This estimate, (27) and (28) imply that

$$\big\|u_k - (u_0 + \overline{u_k} - h_k)\big\|_{L^2(B_2)} \to 0 \quad \text{as } k \to \infty.$$

But this is a contradiction to (25) by (32). □

Corollary 4.6. *For any $\varepsilon > 0$, there is a small $\delta = \delta(\varepsilon) > 0$ such that for any weak solution u of (6) in B_2 where for any $l, m = 0, \dots, K$ and any $|a| < 2$,*

$$B_2 \cap \{x_n > a + \delta\} \subset \Omega_2^l \subset B_2 \cap \{x_n > a - \delta\}, \tag{34}$$

$$B_2 \cap \{x_n < a - \delta\} \subset \Omega_2^m \subset B_2 \cap \{x_n < a + \delta\} \tag{35}$$

and

$$\frac{1}{|B_2|}\int_{B_2} |\nabla u|^2\,dx \leqslant 1, \tag{36}$$

$$\frac{1}{|B_2|}\int_{B_2} \big(|f|^2 + |A - \tilde{A}_{B_2}|^2\big)\,dx \leqslant \delta^2, \tag{37}$$

where $\tilde{A}_{B_2} = \sum_i \overline{A^i}_{\Omega_2^i} \chi_{\Omega_2^i}$, *there exists a piecewise constant matrix* $\tilde{A^b}_{B_2}$ *as* $\tilde{A^b}_{B_2} = \overline{A^l}_{\Omega_2^l} \chi_{B_2 \cap \{x_n > a\}} + \overline{A^m}_{\Omega_2^m} \chi_{B_2 \cap \{x_n < a\}}$ *and for a corresponding weak solution* v *of*

$$-\operatorname{div}\bigl(\tilde{A^b}_{B_2} \nabla v\bigr) = 0 \quad \text{in } B_2 \tag{38}$$

such that

$$\int_{B_{\frac{4}{3}}} \bigl|\nabla(u-v)\bigr|^2 \, dx \leqslant \varepsilon^2.$$

Proof. By Lemma 4.5, for any $\eta > 0$, there exists $\delta = \delta(\eta) > 0$, a piecewise constant matrix $\tilde{A^b}_{B_2} = \overline{A^l}_{\Omega_2^l} \chi_{B_2 \cap \{x_n > a\}} + \overline{A^m}_{\Omega_2^m} \chi_{B_2 \cap \{x_n < a\}}$ and a corresponding weak solution v of $-\operatorname{div}(\tilde{A^b}_{B_2} \nabla v) = 0$ in B_2 such that

$$\int_{B_2} |u-v|^2 \, dx \leqslant \eta^2.$$

First we see that $u - v \in H^1(B_2)$ is a weak solution of

$$-\operatorname{div}\bigl(A\nabla(u-v)\bigr) = \operatorname{div}\bigl(f + \bigl(A - \tilde{A^b}_{B_2}\bigr)\nabla v\bigr) \quad \text{in } B_2. \tag{39}$$

Now, by (7),

$$\int_{B_{\frac{4}{3}}} \bigl|\nabla(u-v)\bigr|^2 \leqslant C\biggl(\int_{B_{\frac{3}{2}}} \bigl|f + \bigl(A - \tilde{A^b}_{B_2}\bigr)\nabla v\bigr|^2 + |u-v|^2 \, dx\biggr) \tag{40}$$

$$\leqslant C\biggl(\int_{B_{\frac{3}{2}}} |f|^2 \, dx + \int_{B_{\frac{3}{2}}} \bigl|\bigl(A - \tilde{A^b}_{B_2}\bigr)\nabla v\bigr|^2 \, dx + \int_{B_{\frac{3}{2}}} |u-v|^2 \, dx\biggr) \tag{41}$$

$$\leqslant C\biggl(\int_{B_2} |f|^2 + \int_{B_2} \bigl|A - \tilde{A^b}_{B_2}\bigr|^2 \, dx + \int_{B_2} |u-v|^2 \, dx\biggr). \tag{42}$$

Here we used the fact that v is Lipschitz, which we showed in Lemma 4.4, and (36). Also,

$$\int_{B_2} |f|^2 + \bigl|A - \tilde{A^b}_{B_2}\bigr|^2 \, dx \leqslant 2\int_{B_2} \bigl(|f|^2 + |A - \tilde{A}_{B_2}|^2\bigr) + \bigl|\tilde{A}_{B_2} - \tilde{A^b}_{B_2}\bigr|^2 \tag{43}$$

$$\leqslant 2\bigl(|B_2|\delta^2 + C(\Lambda)\delta\bigr) \tag{44}$$

$$\leqslant C\delta \quad \text{for a small } \delta. \tag{45}$$

So $\|\nabla(u-v)\|^2_{L^2(B_2)} \leqslant C(\delta + \eta^2) = \varepsilon^2$ by taking η and δ satisfying the last identity. This completes our proof. □

We can control the measure of the set where $|\nabla u|$ is quite big as the following lemma.

Lemma 4.7. *(Cf. [2].) There is a constant $N_1 > 0$ so that for any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ such that for all A as in Section 2.2 with $R = 4$ and for any $l, m = 0, \ldots, K$ and any $|a| < 4$ in appropriate coordinate system*

$$B_4 \cap \{x_n > a + \delta\} \subset \Omega_4^l \subset B_4 \cap \{x_n > a - \delta\}, \tag{46}$$

$$B_4 \cap \{x_n < a - \delta\} \subset \Omega_4^m \subset B_4 \cap \{x_n < a + \delta\}, \tag{47}$$

and if u is a weak solution of $-\operatorname{div}(A\nabla u) = \operatorname{div} f$ in $\Omega \supset B_4$ and if

$$\{x \in B_1\colon \mathcal{M}(|\nabla u|^2) \leqslant 1\} \cap \{x \in B_1\colon \mathcal{M}(|f|^2) \leqslant \delta^2\} \neq \emptyset, \tag{48}$$

then

$$\left|\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_1\right| < \varepsilon |B_1|. \tag{49}$$

Proof. By (48), there is a point $x_0 \in B_1$ such that for all $r > 0$,

$$\frac{1}{|B_r|} \int\limits_{B_r(x_0)\cap\Omega} |\nabla u|^2\,dx \leqslant 1, \qquad \frac{1}{|B_r|} \int\limits_{B_r(x_0)\cap\Omega} |f|^2\,dx \leqslant \delta^2. \tag{50}$$

Since $B_2(0) \subset B_3(x_0)$, we have by (50),

$$\frac{1}{|B_2|}\int\limits_{B_2} |f|^2\,dx \leqslant \frac{|B_3|}{|B_2|}\frac{1}{|B_3|}\int\limits_{B_3(x_0)} |f|^2\,dx \leqslant \left(\frac{3}{2}\right)^n \delta^2. \tag{51}$$

Similarly, we see that

$$\frac{1}{|B_2|}\int\limits_{B_2} |\nabla u|^2\,dx \leqslant \left(\frac{3}{2}\right)^n. \tag{52}$$

In view of (51) and (52), and from the assumption on A, we can apply Corollary 4.6 with u replaced by $(\frac{2}{3})^n u$ and f replaced by $(\frac{2}{3})^n f$, respectively, to find that for any $\eta > 0$, there exists a small $\delta(\eta)$ and a corresponding weak solution v of

$$-\operatorname{div}(\tilde{A^b}_{B_2}\nabla v) = 0 \tag{53}$$

in B_2 such that

$$\int\limits_{B_{\frac{4}{3}}} \left|\nabla(u - v)\right|^2 dx \leqslant \eta^2, \tag{54}$$

provided that

$$\frac{1}{|B_2|}\int\limits_{B_2} \left(|f|^2 + |A - \tilde{A}_{B_2}|^2\right) dx \leqslant \delta^2. \tag{55}$$

By the interior $W^{1,\infty}$ regularity that we proved in Lemma 4.4, we can find a constant N_0 such that

$$\|\nabla v\|_{L^\infty(B_{\frac{3}{2}})} \leqslant N_0. \tag{56}$$

Now we will show that

$$\left\{x \in B_1\colon \mathcal{M}|\nabla u|^2 > N_1^2\right\} \subset \left\{x \in B_1\colon \mathcal{M}_{B_2}\left|\nabla(u-v)\right|^2 > N_0^2\right\} \tag{57}$$

for $N_1^2 := \max\{5^n, 4N_0^2\}$. To do this, suppose that

$$x_1 \in \left\{x \in B_1\colon \mathcal{M}_{B_2}\left(\left|\nabla(u-v)\right|\right)^2(x) \leqslant N_0^2\right\}. \tag{58}$$

For $r \leqslant \frac{1}{2}$, $B_r(x_1) \subset B_{\frac{3}{2}}$, and by (56) and (58), we have

$$\frac{1}{|B_r|}\int_{B_r(x_1)} |\nabla u|^2\,dx \leqslant \frac{2}{|B_r|}\int_{B_{\frac{3}{2}}} \left(\left|\nabla(u-v)\right|^2 + |\nabla v|^2\right) \leqslant 4N_0^2. \tag{59}$$

For $r > \frac{1}{2}$, $B_r(x_1) \subset B_{5r}(x_0)$, and by (50), we have

$$\frac{1}{|B_r|}\int_{B_r(x_1)} |\nabla u|^2\,dx \leqslant \frac{5^n}{|B_{5r}|}\int_{B_{5r}(x_0)\cap\Omega} |\nabla u|^2\,dx \leqslant 5^n. \tag{60}$$

Then (59) and (60) show

$$x_1 \in \left\{x \in B_1\colon \mathcal{M}\left(|\nabla u|\right)^2 \leqslant N_1^2\right\}. \tag{61}$$

Thus assertion (57) follows from (58) and (61).

By (57), weak 1–1 estimates and (54), we obtain

$$\begin{aligned}\left|\left\{x \in B_1\colon \mathcal{M}\left(|\nabla u|\right)^2 > N_1^2\right\}\right| &\leqslant \left|\left\{x \in B_1\colon \mathcal{M}_{B_2}\left(\left|\nabla(u-v)\right|\right)^2 > N_0^2\right\}\right| \\ &\leqslant \frac{C}{N_0^2}\int_{B_{\frac{4}{3}}} \left|\nabla(u-v)\right|^2\,dx \\ &\leqslant \frac{C}{N_0^2}\eta^2 = \varepsilon|B_1|,\end{aligned}$$

by taking small η satisfying the last identity above. Now Corollary 4.6 gives the desired δ. □

Corollary 4.8. *There is a constant $N_1 > 0$ so that for any $\varepsilon, r \in (0,1]$, there exists a small $\delta = \delta(\varepsilon) > 0$ such that for all A as in Section 2.2 with $R = 4$ and for any $l, m = 0, \ldots, K$ and any $|a| < 4r$ in appropriate coordinate system*

$$B_{4r} \cap \{x_n > a + \delta r\} \subset \Omega_{4r}^l \subset B_{4r} \cap \{x_n > a - \delta r\}, \tag{62}$$

$$B_{4r} \cap \{x_n < a - \delta r\} \subset \Omega_{4r}^m \subset B_{4r} \cap \{x_n < a + \delta r\} \tag{63}$$

and if u is a weak solution of $-\operatorname{div}(A\nabla u) = \operatorname{div} f$ *in* $\Omega \supset B_{4r}$ *and if*

$$\{x \in B_r\colon \mathcal{M}(|\nabla u|^2) \leqslant 1\} \cap \{x \in B_r\colon \mathcal{M}(|f|^2) \leqslant \delta^2\} \neq \emptyset, \tag{64}$$

then

$$\left|\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_r\right| < \varepsilon |B_r|. \tag{65}$$

Proof. The proof is given by Lemma 4.7 and a scaling argument. □

To use the modified Vitali Covering Lemma, we need to show Theorem 4.1 holds for any ball $B_r(x)$ for $r \in (0,1]$ and $x \in \Omega$. If $B_r(x)$ intersects with only one subdomain Ω^l then the proof of Theorem 4.1 comes directly from Lemma 4.8 for $l = m$. If $B_r(x)$ intersects with two subdomains Ω^l and Ω^0, then the proof of Theorem 4.1 also comes directly from Lemma 4.8 for $m = 0$.

Then next natural question would be how many subdomains can intersect with $B_r(x)$ for $r \in (0,1]$ and $x \in \Omega$ when $\partial\Omega^i$'s are flat enough. Next lemma will be used to show that a ball can intersect with at most three subdomains.

Lemma 4.9. *H_i's for $i = 1, \ldots, K$ are half spaces. If $\{H_i \cap B_2\}_i$ are disjoint. Then at most two half spaces can intersect with B_1.*

Proof. Assume there are three half spaces, say H_1, H_2 and H_3 such that $B_2 \cap H_i$'s are disjoint and $H_i \cap B_1 \neq \emptyset$ for $i = 1,2,3$. Let $p_i \in H_i \cap B_1$ for $i = 1,2,3$. Note that since half spaces are disjoint in B_2 these points are not collinear. Let $\mathcal{T}$ be the two dimensional plane containing p_1, p_2, p_3. For $j = 1,2$ let $\mathcal{D}_j = \mathcal{T} \cap B_j$ which are indeed two dimensional balls. Let $r_j =$ radius of $\mathcal{D}_j$ for $j = 1,2$. Note that $r_2 \geqslant 2r_1$.

Let $h_i := \mathcal{T} \cap H_i$ and $l_i := \mathcal{T} \cap \partial H_i = \partial h_i$. We have

(1) $p_i \in l_i \cap \mathcal{D}_1$ for $i = 1,2,3$;
(2) $h_i \cap \mathcal{D}_2$'s are disjoint for $i = 1,2,3$.

Pushing l_i's into h_i by $\delta_i > 0$, we may assume that l_i's are tangent to the $\mathcal{D}_1$ and $p_i \in \partial\mathcal{D}_1$ for $i = 1,2,3$. Let also A_i and B_i be the points where l_i intersects $\partial\mathcal{D}_2$ for $i = 1,2,3$. Let $h_i \cap \partial D_2 = \overset{\frown}{A_iB_i}$.

Note that $\overset{\frown}{A_iB_i}$ for $i = 1,2,3$ are disjoint on ∂D_2. Since $r_2 \geqslant 2r_1$ and l_i's are tangent to D_1,

$$\frac{\text{length of } \overset{\frown}{A_iB_i}}{\text{length of } \partial D_2} \geqslant \frac{1}{3}, \quad \text{for } i = 1,2,3. \tag{66}$$

The above is a strict inequality if $r_2 > 2r_1$, which is a contradiction to the fact that $\overset{\frown}{A_iB_i}$'s are disjoint on ∂D_2. If $r_2 = 2r_1$, (66) is an equality. In this case l_i's end points meet each other. So we cannot push l_i outward from h_i which means $\delta_i = 0$ for $i = 1,2,3$. □

So now we consider the case that a ball intersect with three subdomains Ω^l, Ω^0 and Ω^m for any $l, m = 1, \ldots, K$. To prove Theorem 4.1 for this case, our goal is to show Lemma 4.7 holds for this case as well. Roughly there can be two different cases; The first case is when Ω^l and Ω^m are quite close and the second case is when Ω^l and Ω^m are not so close.

Lemma 4.10. *There exists a constant $N_1 > 0$ so that for any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ and for all $\Omega \supset B_4$ and subdomain Ω^i for all $i = 1, \ldots, K$ and Ω are $(\delta, 9)$-flat and for all A as in Section 2.2 with $R = 9$, and if u is a weak solution of* $-\operatorname{div}(A\nabla u) = \operatorname{div} f$ *in* $\Omega \supset B_4$ *and if*

$$\{x \in B_1\colon \mathcal{M}(|\nabla u|^2) \leqslant 1\} \cap \{x \in B_1\colon \mathcal{M}(|f|^2) \leqslant \delta^2\} \neq \emptyset, \tag{67}$$

then

$$\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > N_1^2\right\} \cap B_1\right| < \varepsilon |B_1|. \tag{68}$$

Proof. If B_4 intersects with two subdomains, then we are done by Lemma 4.7.

Suppose B_4 intersects with three subdomains, say Ω^l, Ω^0 and Ω^m. First assume that $\operatorname{dist}(\Omega^l, \Omega^m) < \gamma$ in B_1 for some small $\gamma > 0$. Since $\operatorname{dist}(\Omega^l, \Omega^m) < \gamma$ in B_1, there exist $p_l \in \partial\Omega^l \cap B_1$ and $p_m \in \partial\Omega^m \cap B_1$ such that $\operatorname{dist}(p_l, p_m) < \gamma$. Also assume that Ω^l, Ω^m are $(\delta, 9)$-Reifenberg flat for a δ with $\gamma < \delta \ll 1$. So for each p_i, $i = l, m$, there exist $(n-1)$ dimensional hyper plane $\mathcal{P}_i$ such that

$$D\left[\partial\Omega^i \cap B_9(p_i), \mathcal{P}_i \cap B_9(p_i)\right] \leqslant 9\delta, \quad \text{for } i = l, m \tag{69}$$

where D denotes the Hausdorff distance. In other words, the boundary of Ω^i is squeezed between $\mathcal{P}_i$ and $\mathcal{P}_i^{9\delta}$ which is the translation of $\mathcal{P}_i$ by 9δ in the normal direction of $\mathcal{P}_i$ inward Ω^i for $i = l, m$. We can choose a coordinate system such that the normal direction of $\mathcal{P}_l^{9\delta}$ is the x_n axis. Let us say y_i is the intersection point between $\mathcal{P}_i^{9\delta}$ and vertical line of $\mathcal{P}_i^{9\delta}$ passing through p_i for $i = l, m$. Then the distance between y_m and $\mathcal{P}_l^{9\delta}$ is less than $\gamma + 18\delta < 19\delta$ by (69). Since $\mathcal{P}_l^{9\delta} \cap \mathcal{P}_m^{9\delta} \cap B_4 = \emptyset$, on $\mathcal{P}_m^{9\delta}$

$$\left|\frac{\partial x_n}{\partial x_i}\right| < \frac{\gamma + 18\delta}{3 - \gamma - 18\delta} < \frac{19\delta}{3 - 19\delta} < 7\delta \quad \text{for any } \gamma < \delta \ll 1, \text{ and } i = 1, \ldots, n-1.$$

So $\max_{y \in \mathcal{P}_m^{9\delta} \cap B_4} \operatorname{dist}(y, \mathcal{P}_l^{9\delta} \cap B_4) < C\delta + \gamma$ where C depends on the dimension n.

The above is nothing but Harnack Inequality. Since distance function between $\mathcal{P}_l^{9\delta}$ and $\mathcal{P}_m^{9\delta}$ in B_4 is nonnegative harmonic, we can apply Harnack Inequality:

$$\max_{y \in \mathcal{P}_m^{9\delta} \cap B_1} \operatorname{dist}\left(\mathcal{P}_l^{9\delta}, y\right) < C_1 \min_{y \in \mathcal{P}_m^{9\delta} \cap B_1} \operatorname{dist}\left(\mathcal{P}_l^{9\delta}, y\right) < C \operatorname{dist}(y_l, y_m) = C(19\delta + \gamma) \tag{70}$$

where C depends on the dimension n.

Since the Hausdorff distance between $\mathcal{P}_l^{9\delta}, \mathcal{P}_m^{9\delta}$ is less than $C(\delta + \gamma)$, we can choose small δ_0 and γ_0 such that $C(\delta_0 + \gamma_0)$ is less than δ in Lemma 4.7. By Lemma 4.7, we can conclude.

Now suppose $\operatorname{dist}(\partial\Omega^l, \partial\Omega^m) > \gamma_0$ in B_1 for above γ_0. If $y \in S_1 = \{x \in B_1 \mid x \in \partial\Omega^l \cap \partial\Omega^m\}$, then $B_{\gamma_0}(y)$ has only two subdomains. From (67), there exists $x_0 \in B_1$ such that

$$\mathcal{M}\left(|\nabla u|^2\right)(x_0) \leqslant 1 \quad \text{and} \quad \mathcal{M}\left(|f|^2\right)(x_0) \leqslant \delta^2.$$

For any $y \in S_1$, by weak 1–1 estimate in Theorem 3.2,

$$\begin{aligned}\left|\left\{x \in B_{\frac{\gamma_0}{4}}(y)\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > \lambda_1\right\}\right| &\leqslant \frac{C}{\lambda_1} \int\limits_{B_2(x_0)} |\nabla u|^2\, dx \\ &\leqslant \frac{C}{\lambda_1}\left|B_2(x_0)\right| < \frac{1}{2}\left|B_{\frac{\gamma_0}{4}}(y)\right|\end{aligned}$$

when $\lambda_1 > \frac{C2^{3n+1}}{\gamma_0^n}$. Similarly for this λ_1,

$$\left|\left\{x \in B_{\frac{\gamma_0}{4}}(y)\colon \mathcal{M}\left(|f|^2\right)(x) > \delta^2\lambda_1\right\}\right| \leqslant \frac{C}{\delta^2\lambda_1}\int_{B_2(x_0)} |f|^2\,dx \leqslant \frac{C}{\delta^2\lambda_1}\left|B_2(x_0)\right| < \frac{1}{2}\left|B_{\frac{\gamma_0}{4}}(y)\right|.$$

From above two inequalities, one can find an $x_y \in B_{\frac{\gamma_0}{4}}(y)$ such that

$$\mathcal{M}\left(|\nabla u|^2\right)(x_y) \leqslant \lambda_1 \quad \text{and} \quad \mathcal{M}\left(|f|^2\right)(x_y) \leqslant \delta^2\lambda_1.$$

By Lemma 4.8, there is a constant N_1 so that for any $\varepsilon > 0$

$$\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > \lambda_1 N_1^2\right\} \cap B_{\frac{\gamma_0}{4}}(y)\right| < \varepsilon\left|B_{\frac{\gamma_0}{4}}(y)\right|. \tag{71}$$

If $y \in S_2 = \{x \in B_1 \mid \min_{i=l,m} \operatorname{dist}(x, \partial\Omega^i) > \frac{\gamma_0}{4\times 5}\}$, $B_{\frac{\gamma_0}{20}}(y) \subset \Omega^i$ for $i = 0, l, m$. Similarly as above, there is an $x_y \in B_{\frac{\gamma_0}{80}}(y)$ such that

$$\mathcal{M}\left(|\nabla u|^2\right)(x_y) \leqslant \lambda_2 \quad \text{and} \quad \mathcal{M}\left(|f|^2\right)(x_y) \leqslant \delta^2\lambda_2,$$

when $\lambda_2 > \frac{C2^{5n+1}5^n}{\gamma_0^n}$. By Lemma 4.8, there is a constant N_1 so that for any $\varepsilon > 0$

$$\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > \lambda_2 N_1^2\right\} \cap B_{\frac{\gamma_0}{80}}(y)\right| < \varepsilon\left|B_{\frac{\gamma_0}{80}}(y)\right|. \tag{72}$$

So $U = \{B_r(y) \mid r = \frac{\gamma_0}{4\times 5},\ \ y \in S_1\} \cup \{B_r(y) \mid r = \frac{\gamma_0}{80\times 5},\ \ y \in S_2\}$ covers B_1. Then by Vitali Covering Lemma, there exist disjoint balls $\{B_{r_i}(y_i)\}_{i=1}^{\infty} \subset U \subset B_2$ such that $B_1 \subset \bigcup_i B_{5r_i}(y_i)$. Let N_1 to be $\max(\sqrt{\lambda_1}N_1, \sqrt{\lambda_2}N_1)$. Then by (71) and (72),

$$\begin{aligned}
&\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > N_1^2\right\} \cap B_1\right| \\
&\quad < \sum_i \left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > N_1^2\right\} \cap B_{5r_i}(y_i)\right| \\
&\quad < \varepsilon \sum_i \left|B_{5r_i}(y_i)\right| < \varepsilon 5^n \sum_i \left|B_{r_i}(y_i)\right| \\
&\quad < \varepsilon 5^n |B_2| < \varepsilon (10)^n |B_1|.
\end{aligned}$$

Since Ω^i's for $i = 0, \ldots, n$ are $(\delta, 9)$-flat, B_4 does not intersect more than three subdomains. To see that, assume that B_4 intersects with $\Omega^0, \Omega^1, \Omega^2, \Omega^3$. For any $p_i \in \partial\Omega^i \cap B_4$, for $i = 1, 2, 3$, there exists a hyperplane $\mathcal{P}_i$ such that $\partial\Omega^i \cap B_9$ is between $\mathcal{P}_i$ and $\mathcal{P}_i^{9\delta}$ where $\mathcal{P}_i^{9\delta}$ is translation of $\mathcal{P}_i$ into Ω^i in the normal direction by 9δ since Ω^i's for $i = 0, \ldots, n$ are $(\delta, 9)$-flat. Then for any $\delta < \frac{1}{18}$, on the plane $\mathcal{T}$ containing p_1, p_2, p_3, H_i for $i = 1, 2, 3$ intersect with $B_{\frac{9}{2}}$ but they are disjoint in B_9, which is a contradiction to Lemma 4.9. □

Proof of Theorem 4.1. The proof follows from Lemma 4.10 and scaling argument. □

The following is an interior regularity theorem.

Theorem 4.11. *Let $1 < p < \infty$ be a real number. There is a small $\delta = \delta(\lambda, p, n, R)$ so that for all $\Omega = \bigcup_{i=0}^{K} \Omega^i$ where Ω^i's for $i = 1, \dots, K$ and Ω are $(\delta, 9)$-flat and A as in Section 2.2 with $R = 9$ and for all $f \in L^p(B_4; \mathbb{R}^n)$, if u is a weak solution of the elliptic PDE (1) in B_4, then u belong to $W^{1,p}(B_1)$ with the estimate*

$$\|\nabla u\|_{L^p(B_1)} \leqslant C\big(\|u\|_{L^p(B_4)} + \|f\|_{L^p(B_4)}\big),$$

where the constant C is independent of u and f.

Proof. The proof follows from the global regularity theory in the next section with u replaced by ϕu for an appropriately chosen cutoff function ϕ. □

Remark 4.12. We can change the ball B_4 in Theorem 4.11 to any ball B_R for $R > 1$.

4.2. Global estimates

Definition 4.13. We say that $u \in H_0^1(\Omega)$ is a weak solution of (1) if

$$-\int_\Omega A\nabla u \nabla\varphi\, dx = \int_\Omega f\nabla\varphi\, dx \quad \forall\varphi \in H_0^1(\Omega). \tag{73}$$

In this section our interest is the following case:

$$\Omega_R \supset T_R \quad \text{with } D(\Omega_R, T_R) \text{ small},$$

where D denotes the Hausdorff distance. We consider weak solution of

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla u(x)\big) = \operatorname{div} f & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial_w\Omega_R, \end{cases} \tag{74}$$

under the conditions as in Section 1.

Definition 4.14. $u \in H^1(\Omega_R)$ is a weak solution of (74) in Ω_R if

$$\int_{\Omega_R} A\nabla u\nabla\varphi\, dx = -\int_{\Omega_R} f\nabla\varphi\, dx \quad \text{for any } \varphi \in H_0^1(\Omega_R)$$

and u's 0-extension is in $H^1(B_R)$.

In [2], the following lemmas were proven for A without discontinuity.

Lemma 4.15. *(Cf. [2].) There is a constant $N_1 > 0$ so that for any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ with A uniformly elliptic and $(\delta, 4)$-vanishing, and if $u \in H_0^1(\Omega)$ is a weak solution of (74) with $B_4^+ \subset \Omega_4 \subset B_4 \cap \{x_n > -\delta\}$ and*

$$\big\{x \in \Omega_1\colon \mathcal{M}\big(|\nabla u|^2\big) \leqslant 1\big\} \cap \big\{x \in \Omega_1\colon \mathcal{M}\big(|f|^2\big) \leqslant \delta^2\big\} \neq \emptyset, \tag{75}$$

then

$$\big|\big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big)(x) > N_1^2\big\} \cap B_1\big| < \varepsilon|B_1|. \tag{76}$$

Corollary 4.16. *(Cf. [2].) There is a constant $N_1 > 0$ so that for any $\varepsilon, r > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ with A uniformly elliptic and $(\delta, 4r)$-vanishing, and if $u \in H_0^1(\Omega)$ is a weak solution of (74) with $B_{4r}^+ \subset \Omega_{4r} \subset B_{4r} \cap \{x_n > -\delta r\}$ and*

$$\{x \in \Omega_r \colon \mathcal{M}(|\nabla u|^2) \leqslant 1\} \cap \{x \in \Omega_r \colon \mathcal{M}(|f|^2) \leqslant \delta^2\} \neq \emptyset, \tag{77}$$

then

$$\left|\{x \in \Omega \colon \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_r\right| < \varepsilon |B_r|. \tag{78}$$

Now we consider how to control the measure of the set where $|\nabla u|$ is big for the case that A has big discontinuity along the subdomains.

Lemma 4.17. *There is a constant $N_1 > 0$ so that for any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ with A as in Section 2.2 with $R = 9$ and Ω and Ω^i's are $(\delta, 9)$-flat for $i = 1, \dots, K$, and if $u \in H_0^1(\Omega)$ is a weak solution of (74) with $B_4^+ \subset \Omega_4 \subset B_4 \cap \{x_n > -4\delta\}$ and*

$$\{x \in \Omega_1 \colon \mathcal{M}(|\nabla u|^2) \leqslant 1\} \cap \{x \in \Omega_1 \colon \mathcal{M}(|f|^2) \leqslant \delta^2\} \neq \emptyset, \tag{79}$$

then

$$\left|\{x \in \Omega \colon \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_1\right| < \varepsilon |B_1|. \tag{80}$$

Proof. If B_4 intersects with only Ω^0, then this lemma is nothing but what Lemma 4.15 says. Note that B_4 cannot intersect with more than two subdomains by the same argument in the proof of Lemma 4.10 (considering Ω^c as $(\delta, 9)$-flat for any sufficiently small δ). Assume that B_4 intersects with Ω^0 and Ω^l for any $l = 1, \dots, K$.

First suppose $\operatorname{dist}(\partial\Omega^l, \partial\Omega) < \gamma$ in B_4 for some $\gamma > 0$. Then there exist $p_l \in \partial\Omega^l \cap B_4$ and $p \in \partial\Omega \cap B_4$ such that $\operatorname{dist}(p, p_l) < \gamma$. Since Ω^l are $(\delta, 9)$-flat, $\mathcal{P}_l^{9\delta}(p_l) \cap B_4 \subset \Omega^l$ where $\mathcal{P}_l^{\delta}(p_l)$ is the $(n-1)$ dimensional plane which is translated hyperplane at p_l by δ along the normal direction toward Ω^l. Let us say y_l is the intersection point between $\mathcal{P}_l^{9\delta}$ and vertical line of $\mathcal{P}_l^{9\delta}$ passing through p_l. Then the $\operatorname{dist}(y_l, \{x \in B_4 \colon x_n = -4\delta\}) < 9\delta + \gamma + 4\delta = 13\delta + \gamma$. Note that $\mathcal{P}_l^{9\delta} \cap B_4 \subset \Omega^l$. Since distance function between $\mathcal{P}_l^{9\delta} \cap B_4$ and $\{x \in B_4 \colon x_n = -4\delta\}$ is nonnegative harmonic, we can apply Harnack Inequality:

$$\begin{aligned}
&\max_{y \in \mathcal{P}_l^{9\delta} \cap B_4} \operatorname{dist}(y, \{x \in B_4 \colon x_n = -4\delta\}) \\
&\quad \leqslant C \min_{y \in \mathcal{P}_l^{9\delta} \cap B_4} \operatorname{dist}(y, \{x \in B_4 \colon x_n = -4\delta\}) \\
&\quad \leqslant C \operatorname{dist}(y_l, \{x \in B_4 \colon x_n = -4\delta\}) \\
&\quad = C(13\delta + \gamma)
\end{aligned}$$

where C depends on the dimension n. One can choose small γ_0 and δ_0 so that $C(13\delta_0 + \gamma_0) < \delta$ for δ in Lemma 4.15. We conclude by Lemma 4.15.

Now suppose $\operatorname{dist}(\partial\Omega^l, \partial\Omega) \geqslant \gamma_0$ in B_4 for the γ_0 above. For any $y \in S_1 = \{x \in B_1 \mid x \in \partial\Omega^l\}$, $B_{\gamma_0}(y)$ has two subdomains and $B_{\gamma_0}(y) \cap \partial\Omega = \emptyset$. From (79), there exists $x_0 \in \Omega_1$ such that

$$\mathcal{M}(|\nabla u|^2)(x_0) \leqslant 1 \quad \text{and} \quad \mathcal{M}(|f|^2)(x_0) \leqslant \delta^2.$$

As we showed in the proof of Lemma 4.10, there is a constant N_1 so that for any $\varepsilon > 0$, there exists $\delta > 0$ so that

$$\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > \lambda_1 N_1^2\right\} \cap B_{\frac{\gamma_0}{4}}(y)\right| < \varepsilon \left|B_{\frac{\gamma_0}{4}}(y)\right|, \tag{81}$$

where $\lambda_1 > \frac{C2^{3n+1}}{\gamma_0^n}$. Also for any $y \in S_2 = \{x \in B_1 \mid x \in \partial\Omega\}$, $B_{\gamma_0}^+ \subset \Omega^0 \subset B_{\gamma_0} \cap \{x_n > -\gamma_0\delta\}$ in appropriate coordinate system. By applying Corollary 4.16, there is a constant N_1 so that for any $\varepsilon > 0$, there exists $\delta > 0$ so that

$$\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > \lambda_1 N_1^2\right\} \cap B_{\frac{\gamma_0}{4}}(y)\right| < \varepsilon \left|B_{\frac{\gamma_0}{4}}(y)\right|. \tag{82}$$

For any $y \in T = \{x \in B_1 \mid \min(\operatorname{dist}(x, \partial\Omega^l), \operatorname{dist}(x, \partial\Omega)) > \frac{\gamma_0}{4\times 5}\}$, $B_{\frac{\gamma_0}{20}}(y) \subset \Omega^i$ for $i = 0, l$. Then by Lemma 4.7 there is a constant N_1 so that for any $\varepsilon > 0$, there exists $\delta > 0$ so that

$$\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > \lambda_2 N_1^2\right\} \cap B_{\frac{\gamma_0}{20\times 4}}(y)\right| < \varepsilon \left|B_{\frac{\gamma_0}{80}}(y)\right| \tag{83}$$

where $\lambda_2 > \frac{C2^{5n+1}5^n}{\gamma_0^n}$.

Since $B \subset U := \{B_r(y) \mid r < \frac{\gamma_0}{4\times 5},\ y \in S_1 \cup S_2\} \cup \{B_r(y) \mid r < \frac{\gamma_0}{80\times 5},\ y \in T\}$, by Vitali Covering Lemma, there are disjoint set $\{B_{r_i}(y_i)\}_{i=1}^{\infty} \subset U \subset B_2$ s.t. $B_1 \subset \bigcup_i B_{5r_i}(y_i)$

$$\begin{aligned}
&\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > N_1^2\right\} \cap B_1\right| \\
&\quad < \sum_i \left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > N_1^2\right\} \cap B_{5r_i}(y_i)\right| \\
&\quad < \varepsilon \sum_i \left|B_{5r_i}(y_i)\right| < \varepsilon 5^n \sum_i \left|B_{r_i}(y_i)\right| \\
&\quad < \varepsilon 5^n |B_2| < \varepsilon (10)^n |B_1|.
\end{aligned}$$

Here we used (81)–(83). □

Corollary 4.18. *There is a constant $N_1 > 0$ so that for any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ with A as in Section 2.2 with $R = 9$ and Ω, Ω^i's are $(\delta, 9)$-flat for $i = 1, \dots, K$, and if $u \in H_0^1(\Omega)$ is a weak solution of (74) with $B_{4r}^+ \subset \Omega_{4r} \subset B_{4r} \cap \{x_n > -4\delta r\}$ and*

$$\left\{x \in \Omega_r\colon \mathcal{M}\left(|\nabla u|^2\right) \leqslant 1\right\} \cap \left\{x \in \Omega_r\colon \mathcal{M}\left(|f|^2\right) \leqslant \delta^2\right\} \neq \emptyset, \tag{84}$$

then

$$\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > N_1^2\right\} \cap B_r\right| < \varepsilon |B_r|. \tag{85}$$

Proof. Then proof is given by Lemma 4.17 and scaling argument. □

The following lemma shows that same result of Lemma 4.17 holds for any ball intersecting with Ω.

Lemma 4.19. *There is a constant $N_1 > 0$ so that for any $\varepsilon > 0$ and $0 < r < 1$, there exists a small $\delta = \delta(\varepsilon) > 0$ for all $\Omega = \bigcup_{i=0}^{K} \Omega^i$ where Ω and Ω^i's for $i = 1, \dots, K$ are $(\delta, 45)$-flat and for any A as in Section 2.2 with $R = 45$, and if $u \in H_0^1(\Omega)$ is the weak solution of $-\operatorname{div}(A\nabla u) = \operatorname{div} f$ in $\Omega \supset B_{4r}$ and if the following property holds*:

$$\{x \in \Omega_r\colon \mathcal{M}(|\nabla u|^2) \leqslant 1\} \cap \{x \in \Omega_r\colon \mathcal{M}(|f|^2) \leqslant \delta^2\} \neq \emptyset, \tag{86}$$

then

$$\left|\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_r\right| < \varepsilon |B_r|. \tag{87}$$

Proof. If $B_{4r} \cap \partial\Omega = \emptyset$, then by an interior estimate Theorem 4.1 we can conclude. Assume that $B_{4r} \cap \partial\Omega \neq \emptyset$. Note that $B_r \subset B_{5r}(y)$ for some $y \in \partial\Omega$. By (86), there exists $x_0 \in B_r \subset B_{5r}(y)$ such that $\mathcal{M}(|\nabla u|^2)(x_0) \leqslant 1$ and $\mathcal{M}(|f|^2)(x_0) \leqslant \delta^2$. Since Ω is $(\delta, 45)$-Reifenberg flat, we have, in appropriate coordinate system,

$$B_{20r}^+ \subset \Omega_{20r} \subset B_{20r} \cap \{x_n > -20\delta r\}.$$

Here we use Corollary 4.18 to the ball $B_{5r}(y)$ with ε replaced by $\frac{\varepsilon}{5^n}$. Then

$$\begin{aligned}\left|\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_r\right| &< \left|\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2)(x) > N_1^2\} \cap B_{5r}(y)\right| \\ &< \frac{\varepsilon}{5^n}|B_{5r}| = \varepsilon |B_r|. \qquad \square\end{aligned}$$

Corollary 4.20. *(Cf. [2].) Suppose that $u \in H_0^1(\Omega)$ is the weak solution of $-\operatorname{div}(A\nabla u) = \operatorname{div} f$ in Ω. Assume $\Omega = \bigcup_{i=0}^{K} \Omega^i$ where Ω, Ω^i's for $i = 1, \dots, K$ are $(\delta, 45)$-flat and A as in Section 2.2 with $R = 45$. Assume that*

$$\left|\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2) > N_1^2\}\right| < \varepsilon |B_1|. \tag{88}$$

Let k be a positive integer and set $\varepsilon_1 = (\frac{10}{1-\delta})^n \varepsilon$. Then we have

$$\left|\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2) > N_1^{2k}\}\right| \tag{89}$$

$$\leqslant \sum_{i=1}^{k} \varepsilon_1^i \left|\{x \in \Omega\colon \mathcal{M}(|f|^2) > \delta^2 N_1^{2(k-i)}\}\right| + \varepsilon_i^k \left|\{x \in \Omega\colon \mathcal{M}(|\nabla u|)^2(x) > 1\}\right|. \tag{90}$$

Proof. We prove by induction on k. For the case $k = 1$, set

$$C = \{x \in \Omega\colon \mathcal{M}(|\nabla u|^2)(x) > N_1^2\}$$

and

$$D = \{x \in \Omega\colon \mathcal{M}(|f|^2)(x) > \delta^2\} \cup \{x \in \Omega\colon \mathcal{M}(|\nabla u|^2)(x) > 1\}.$$

Since Ω is $(\delta, 45)$-Reifenberg flat, Ω is $(\delta, 1)$-Reifenberg flat. Then in view of (88), Lemma 4.19 and Theorem 3.4, we see $|C| \leqslant \varepsilon_1 |D|$, and so our conclusion is valid for $k = 1$.

Assume that the conclusion is valid for some positive integer $k \geqslant 2$. Set $u_1 = u/N_1$ and corresponding $f_1 = f/N_1$. Then u_1 is the weak solution of

$$\begin{cases} -\operatorname{div}(A\nabla u_1) = \operatorname{div} f_1 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega \end{cases} \tag{91}$$

and the following inequality holds:

$$\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u_1|^2\right)(x) > N_1^2\right\}\right| < \varepsilon |B_1|.$$

By the induction assumption and from a simple calculation, we deduce the following estimates:

$$\begin{aligned} &\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > N_1^{2(k+1)}\right\}\right| \\ &\quad = \left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u_1|^2\right)(x) > N_1^{2k}\right\}\right| \\ &\quad \leqslant \sum_{i=1}^{k} \varepsilon_1^i \left|\left\{x \in \Omega\colon \mathcal{M}\left(|f_1|^2\right)(x) > \delta^2 N_1^{2(k-i)}\right\}\right| \\ &\qquad + \varepsilon_1^k \left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u_1|^2\right)(x) > 1\right\}\right| \\ &\quad \leqslant \sum_{i=1}^{k+1} \varepsilon_1^i \left|\left\{x \in \Omega\colon \mathcal{M}\left(|f|^2\right)(x) > \delta^2 N_1^{2(k+1-i)}\right\}\right| \\ &\qquad + \varepsilon_1^{k+1} \left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right)(x) > 1\right\}\right|. \end{aligned}$$

This estimate in turn completes the induction on k. □

Finally we are ready to prove the main theorem.

Theorem 4.21. *Let $1 < p < \infty$ be a real number. Then there is a small $\delta = \delta(\Lambda, p, n, R) > 0$ so that for all $\Omega = \bigcup_{i=0}^{i=K} \Omega^i$ where Ω, Ω^i's for $i = 1, \ldots, K$ are (δ, R)-Reifenberg flat, for all A as in Section 2.2, and for all $f \in L^p(\Omega, \mathbb{R}^n)$, the Dirichlet problem (1) has a unique weak solution with the estimate*

$$\int_\Omega |\nabla u|^p \, dx \leqslant C \int_\Omega |f|^p \, dx, \tag{92}$$

where the constant C is independent of u and f.

Proof. First we will consider the case $p > 2$. The case $p = 2$ is classical and the case $1 < p < 2$ will be proved using duality. Without loss of generality, we assume that

$$\|f\|_{L^p(\Omega)} \quad \text{is small enough} \tag{93}$$

and

$$\left|\left\{x \in \Omega\colon \mathcal{M}\left(|\nabla u|^2\right) > N_1^2\right\}\right| < \varepsilon |B_1|$$

by multiplying the PDE (1) by a small constant depending on $\|f\|_{L^2(\Omega)}$ and $\|\nabla u\|_{L^2(\Omega)}$. Since $f \in L^p(\Omega)$, $\mathcal{M}(|f|^2) \in L^{p/2}(\Omega)$ by strong p–p estimates. In view of Lemma 3.3, there is a constant C

depending only on δ, p, and N_1 such that

$$\sum_{k=0}^{\infty} N_1^{pk}\big|\big\{x\in\Omega\colon \mathcal{M}\big(|f|^2\big)(x)>\delta^2 N_1^{2k}\big\}\big| \leqslant C\big\|\mathcal{M}\big(|f|^2\big)\big\|_{L^{p/2}(\Omega)}^{p/2}. \tag{94}$$

Then this estimate, strong p–p estimates, and (93) imply

$$\sum_{k=0}^{\infty} N_1^{pk}\big|\big\{x\in\Omega\colon \mathcal{M}\big(|f|^2\big)(x)>\delta^2 N_1^{2k}\big\}\big| \leqslant 1. \tag{95}$$

Now we will claim that $\mathcal{M}(|\nabla u|^2)\in L^{p/2}$ by using Lemma 3.3 when $f=\mathcal{M}(|\nabla u|^2)$ and $m=N_1^2$. Let us compute

$$\begin{aligned}
&\sum_{k=0}^{\infty} N_1^{pk}\big|\big\{x\in\Omega\colon \mathcal{M}\big(|\nabla u|^2\big)(x)>N_1^{2k}\big\}\big| \\
&\quad\leqslant \sum_{k=1}^{\infty} N_1^{pk}\left(\sum_{i=1}^{k}\varepsilon_1^i\big|\big\{x\in\Omega\colon \mathcal{M}\big(|f|^2\big)(x)>\delta^2 N_1^{2(k-i)}\big\}\big| + \varepsilon_1^k\big|\big\{x\in\Omega\colon \mathcal{M}\big(|\nabla u|^2\big)(x)>1\big\}\big|\right) \\
&\quad= \sum_{i=1}^{\infty}\big(N_1^p\varepsilon_1\big)^i\left(\sum_{k=i}^{\infty} N_1^{p(k-i)}\big|\big\{x\in\Omega\colon \mathcal{M}\big(|f|^2\big)(x)>\delta^2 N_1^{2(k-i)}\big\}\big|\right) \\
&\qquad+\sum_{k=1}^{\infty}\big(N_1^p\varepsilon_1\big)^k\big|\big\{x\in\Omega\colon \mathcal{M}\big(|\nabla u|^2\big)(x)>1\big\}\big| \\
&\quad\leqslant C\sum_{k=1}^{\infty}\big(N_1^p\varepsilon_1\big)^k<+\infty,
\end{aligned}$$

where we used Corollary 4.20 and (95). Also we can choose ε_1 so that $N_1^p\varepsilon_1<1$ since N_1 is a universal constant depending on the dimension and ellipticity. So we can take ε, and find the corresponding $\delta>0$, also ε_1. By this estimate and Lemma 3.3, $\mathcal{M}(|\nabla u|^2)\in L^{p/2}(\Omega)$. Thus $\nabla u\in L^p(\Omega)$.

Now suppose that $1<p<2$. For any $g\in L^q(\Omega,\mathbb{R}^n)$ and A^T, a transpose matrix of A, consider the following equation:

$$\begin{cases} -\operatorname{div}\big(A^T(x)\nabla v(x)\big)=\operatorname{div} g & \text{in } \Omega, \\ v=0 & \text{on } \partial\Omega. \end{cases} \tag{96}$$

Then

$$\begin{aligned}
\int_\Omega f\nabla v\,dx &= -\int_\Omega \operatorname{div} f\,v\,dx = \int_\Omega \operatorname{div}(A\nabla u)v\,dx \\
&= -\int_\Omega (A\nabla u)(\nabla v)\,dx = -\int_\Omega \nabla u\big(A^T\nabla v\big)\,dx \\
&= \int_\Omega u\operatorname{div}\big(A^T\nabla v\big)\,dx = \int_\Omega u(-\operatorname{div} g)\,dx = \int_\Omega \nabla u g\,dx.
\end{aligned}$$

By above, note that $\|\nabla v\|_{L^q} \leqslant C\|g\|_{L^q}$,

$$\|\nabla u\|_{L^p(\Omega)} = \sup_{0 \neq g \in L^q(\Omega)} \frac{|\int_\Omega \nabla u g|}{\|g\|_{L^q(\Omega)}} \leqslant \frac{|\int_\Omega \nabla v f|}{\|g\|_{L^q(\Omega)}}$$
$$\leqslant \frac{\|\nabla v\|_{L^q}\|f\|_{L^p}}{\|g\|_{L^q}} \leqslant C\|f\|_{L^p},$$

which completes the proof. □

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