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A Class of Vertex-Transitive Digraphs, II

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(1). We determine the number of non-isomorphic classes of self-complementary circulant digraphs with pq vertices, where p and q are distinct primes. The non-isomorphic classes of these circulant digraphs with pq vertices are enumerated. (2). We also determine the number of non-isomorphic classes of self-complementary, vertex-transitive digraphs with a prime number p vertices, and the number of self-complementary strongly vertex-transitive digraphs with p vertices. The non-isomorphic classes of strongly vertex-transitive digraphs with p vertices are also enumerated.

1. INTRODUCTION

This is an extension of the results in [8]. Again, the digraphs (directed graphs) which we consider here are finite, simple, and without loops. A digraph X is said to be vertex-transitive if its group of automorphisms is transitive on its vertices, and X is said to be strongly vertex-transitive if its group of automorphisms acts regularly on its vertices. Also, a digraph on n vertices is said to be a circulant digraph if its group of automorphisms contains the cyclic group of order n. In [8], we showed that the vertex-transitive digraphs on a prime number, p, of vertices are precisely the circulant digraphs on p vertices, and we enumerated the non-isomorphic classes of these digraphs by using Pólya's theorem of enumeration.

By applying de Bruijn's generalization of Pólya's theorem, Read [12] presented a function which determines the number of distinct (non-isomorphic classes of) self-complementary digraphs with n vertices. Here, by using a method similar to Read's, we, in Section 3, determine the number of

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distinct self-complementary circulant digraphs with pq vertices, where p and q are distinct primes. In Section 2, we enumerate the circulant digraphs with pq vertices. Our proof depends on a recent result of Alspach and Parsons in [3] concerning Ádám's conjecture in [1]. On p. 306 in [10], Elspas and Turner remarked that there was no convenient enumeration of strongly vertex-transitive digraphs with p vertices. Here, in Section 4, by using the idea of the principle of inclusion and exclusion, we enumerate the strongly vertex-transitive digraphs with p vertices, as well as those vertex-transitive digraphs with p vertices which are not strongly vertex-transitive. We also determine the number of strongly vertex-transitive, self-complementary digraphs with p vertices. Some of our results concerning vertex-transitive digraphs with p vertices coincide with the ones obtained by Chia and Lim in [9]. However, the methods are different.

2. An Enumeration

Let p and q be distinct primes and G be the cyclic group generated by the permutation $(0\ 1\ 2 \cdots (pq-1))$. We apply Schur's algorithm as described in [6–8], to G, and we obtain all of the digraphs with pq vertices, each of whose group of automorphisms contains or is G. Namely, each of these digraphs on pq vertices is a combination (union) of the basic digraphs D_1, $D_2, \ldots, D_{pq-1}$, where the vertex set, $V(D_i)$, of D_i is $H = \{0, 1, \ldots, (pq-1)\}$ and the directed edge set, $E(D_i)$ of D_i is $\{[j, j+i]; j = 0, 1, \ldots, (pq-1)\}$ for $i = 1, 2, \ldots, pq-1$. Or we may consider each of these digraphs on pq vertices to be a Cayley digraph $X_{H,K}$, where K is a subset of H such that the identity of H does not belong to K, i.e., $V(X_{H,K}) = H$ and $E(X_{H,K}) = \{[j, j+i]; j \in H$ and $i \in K\}$. The following is an extension of Theorem 4 in [8]:

Lemma 1. *The circulant digraphs $X_{H,K}$ and $X_{H,K'}$ with pq vertices, where p and q are distinct primes, are isomorphic if and only if there exists a σ in $A(H)$ such that σ maps K onto K', where $A(H)$ is the group of automorphisms of the additive group $H = \{0, 1, \ldots, (pq-1)\}$, modulo pq.*

Proof. We assume that there exists a σ in $A(H)$ which maps K onto K'. Then $i\sigma \in K'$ for every $i \in K$. Hence $[j, j+i] \in E(X_{H,K})$ if and only if $[j, j+i\sigma] \in E(X_{H,K})$. That is, $X_{H,K}$ and $X_{H,K'}$ are isomorphic.

Now we assume that $X_{H,K}$ and $X_{H,K'}$ are isomorphic. Alspach and Parsons in [3] proved that Ádám's conjecture holds for circulant digraphs with pq vertices; i.e., for every pair of isomorphic circulant digraphs $X_{H,K}$ and $X_{H,K'}$ with pq vertices, there exists an integer u relatively prime to pq such that $K' = uK$, where the product is taken modulo pq. Since u is relatively prime to pq, there exists a σ in $A(H)$ which maps K onto K'.

For our application of Pólya's theorem, it is convenient to think of the digraph $X_{H,K}$ as a combination of the basic digraphs $D_1, D_2, \ldots, D_{pq-1}$. For each σ in $A(H)$, σ induces a permutation $\bar{\sigma}$ on the set $\{D_1, D_2, \ldots, D_{pq-1}\}$; i.e., for every $\sigma \in A(H)$, and for every $i \in K$ with $i\sigma = iu$, $D_{i\sigma} = D_{iu}$. Let $\bar{A}(H)$ be the set of permutations $\bar{\sigma}$ on $\{D_1, D_2, \ldots, D_{pq-1}\}$ induced by all σ in $A(H)$. Clearly, $\bar{A}(H)$ is a permutation group and is isomorphic to $A(H)$. By Lemma 1, two circulant digraphs $X_{H,K}$ and $X_{H,K'}$ with pq vertices are isomorphic if and only if there exists a $\bar{\sigma}$ in $\bar{A}(H)$ such that $\bar{\sigma}$ maps the basic digraphs in $X_{H,K}$ onto the basic digraphs in $X_{H,K'}$. Applying Pólya's theorem with the domain consisting of the basic digraphs $D_1, D_2, \ldots, D_{pq-1}$, the range consisting of 0 and 1 with the weight function w such that $w(0) = 1$ and $w(1) = x^{pq}$, and $\bar{A}(H)$ acting on the domain, we obtain the following function of enumeration:

THEOREM 1. *The non-isomorphic classes of circulant digraphs with pq vertices, where p and q are distinct primes, are enumerated by the function*

$$P_{\bar{A}(H)}(1 + x^{pq}, 1 + x^{2pq}, \ldots),$$

where $P_{\bar{A}(H)}(x_1, x_2, \ldots)$ is the cycle index of $\bar{A}(H)$ and the coefficient of x^n is the number of non-isomorphic digraphs having n directed edges.

The cycle index $P_{\bar{A}(H)}(x_1, x_2, \ldots)$ is relatively simple, since $\bar{A}(H)$ is isomorphic to $A(H)$ and since $A(H)$ is isomorphic to the direct product of $A(C_p)$ and $A(C_q)$, where C_n is the cyclic group or order n. For the case $q = 2$, we have $\bar{A}(H) \simeq A(H) \simeq C_{p-1}$, which yields the following enumeration formula for the circulant digraphs on $2p$ vertices:

$$\frac{1 + x^{2p}}{p-1} \sum_{d} \phi(d)(1 + x^{2pd})\,\frac{2(p-1)}{d}, \tag{2}$$

where ϕ is Euler's ϕ function and the summation is taken over the divisors d of $p - 1$.

We list the enumerations for the cases $pq = 6$, 10, 14, and 15:

$pq = 6$, $\quad 1 + 3x^6 + 6x^{12} + 6x^{18} + 3x^{24} + x^{30}$.

$pq = 10$, $\quad 1 + 3x^{10} + 10x^{20} + 22x^{30} + 34x^{40} + 34x^{50} + 22x^{60} + 10x^{70} + 3x^{80} + x^{90}$.

$pq = 14$, $\quad 1 + 3x^{14} + 14x^{28} + 50x^{42} + 123x^{56} + 217x^{70} + 292x^{84} + 292x^{98} + 217x^{112} + 123x^{126} + 50x^{140} + 14x^{154} + 3x^{168} + x^{182}$.

$pq = 15, \qquad 1 + 3x^{15} + 15x^{30} + 50x^{45} + 137x^{60} + 263x^{75} + 395x^{90} + 444x^{105} + 395x^{120} + 263x^{135} + 137x^{150} + 50x^{165} + 15x^{180} + 3x^{195} + x^{210}.$

3. SELF-COMPLEMENTARY DIGRAPHS

Let X be a digraph. A digraph X^c is said to be a complementary graph of X if $V(X^c) = V(X)$, and a directed edge belongs to $E(X^c)$ if and only if it does not belong to $E(X)$. X is said to be self-complementary if it is isomorphic to X^c. Here, we shall use de Bruijn's generalization of Pólya's theorem to obtain the following.

THEOREM 2. *The number of non-isomorphic classes of self-complementary circulant digraphs with pq vertices is*

$$P_{\bar{A}(H)}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \ldots\right) e^{2(z_2+z_4+z_6+\cdots)} \tag{3}$$

evaluated at $z_1 = z_2 = z_3 = \cdots = 0$, *where p and q are distinct primes,* $\bar{A}(H)$ *is the permutation group, as stated in Theorem* 1, *acting on the set of basic digraphs,* $\{D_1, D_2, \ldots, D_{pq-1}\}$, *and* $P_{\bar{A}(H)}(\partial/\partial z_1, \partial/\partial z_2, \partial/\partial z_3, \ldots)$ *is the cycle index of* $\bar{A}(H)$.

Proof. Let $D = \{D_1, D_2, \ldots, D_{pq-1}\}$, $R = \{0, 1\}$, $\bar{A}(H)$ act on D and the identity group act on R. Then, by Theorem 5.4 on p. 172 in [5] the number

$$P_{\bar{A}(H)}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \cdots\right) e^{2(z_1+z_2+z_3+\cdots)} \tag{4}$$

evaluated at $z_1 = z_2 = z_3 = \cdots = 0$ is the number of non-isomorphic classes of circulant digraphs with pq vertices.

If in addition, we let $T = \{(0)(1), (01)\}$ act on R, then we have placed each circulant digraph with pq vertices in the same equivalence class with its complementary graph, and the number is

$$P_{\bar{A}(H)}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots\right) P_T(e^{z_1+z_2+\cdots}, e^{2(z_2+z_4+\cdots)}, \ldots)$$
$$= P_{\bar{A}(H)}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots\right) \frac{1}{2}\left(e^{2(z_1+z_2+\cdots)} + e^{2(z_2+z_4+\cdots)}\right)$$

evaluated at $z_1 = z_2 = z_3 = \cdots = 0.$ (5)

Doubling (5) which counts each distinct self-complementary digraph twice, and subtracting (4) give the number of distinct self-complementary circulant digraphs with pq vertices, i.e.,

$$2P_{\bar{A}(H)}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2},\ldots\right)\frac{1}{2}\left(e^{2(z_1+z_2+\cdots)}+e^{2(z_2+z_4\cdots)}\right)$$

$$-P_{\bar{A}(H)}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2},\ldots\right)e^{2(z_1+z_2+\cdots)}$$

$$=P_{\bar{A}(H)}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2},\ldots\right)e^{2(z_2+z_4+\cdots)}$$

evaluated at $z_1=z_2=z_3=\cdots=0$ which is (3).

We note that since the complete digraph on n vertices has $n(n-1)$ edges, a self-complementary digraph must have $\frac{1}{2}n(n-1)$ edges. Therefore there are no self-complementary circulant digraphs on $2p$ vertices. For $pq=15$, there are 20 self-complementary circulant digraphs; for $pq=21$, there are 88; for $pq=35$, there are 5560. Also since there are $\frac{1}{4}n(n-1)$ edges in a self-complementary undirected graph, with n vertices, none of the 20 self-complementary circulant digraphs with 15 vertices can be a self-complementary undirected graph, nor can any of the 5560 self-complementary circulant digraphs with 35 vertices be a self-complementary undirected graph.

For the prime case, we take H to be the additive group $\{0, 1,\ldots (p-1)\}$. Then $A(H)$ is the cyclic group of order $p-1$. We take D to be the set consisting of the basic digraphs $D_1, D_2,\ldots D_{p-1}$, R to be $\{0, 1\}$, $\bar{A}(H)$ to be the group which is isomorphic to $A(H)$ and which acts on D, and T to be $\{(0)(1), (01)\}$ acting on R. By using the above technique, we have

THEOREM 3. *The number of non-isomorphic classes of self-complementary, vertex-transitive digraphs with a prime number p vertices is*

$$\frac{1}{p-1}\left(\sum_d \phi(d)\frac{\partial^{(p-1)/d}}{\partial z_d^{(p-1)/d}}\right)e^{2(z_2+z_4+\cdots)} \tag{6}$$

evaluated at $z_1=z_2=\cdots=0$, *where* ϕ *is Euler's* ϕ *function and the summation is taken over the divisors* d *of* $p-1$.

4. Strongly Vertex-Transitive Digraphs

A vertex-transitive digraph with a prime number p of vertices is said to be strongly vertex-transitive if its group of automorphisms is the cyclic group of order p. Here we shall present the enumerating functions for strongly vertex-transitive digraphs and for vertex-transitive digraphs which are not strongly vertex-transitive with p vertices, and determine the number of non-isomorphic strongly vertex-transitive self-complementary digraphs with p vertices.

In order to enumerate the strongly vertex-transitive digraphs, we shall first enumerate the vertex-transitive digraphs which are not strongly vertex-transitive. Let X be a non-complete and non-null vertex-transitive digraph with p vertices. Our Theorem 3 in [8] states that if the group of automorphisms, $G(X)$, of X is not cyclic, then $G(X) = \langle S, \sigma \rangle$ with the defining relations $S^p = e$, $\sigma^n = e$, and $\sigma^{-1} S \sigma = S^r$, where e is the identity of $G(X)$, n divides $p-1$ and $r^n \equiv 1 \bmod p$. Moreover, $\langle \sigma \rangle \subseteq A(H)$, which is the group of automorphisms of the additive group mod p, $H = \{0, 1, \dots, (p-1)\}$. Since $A(H)$ is a cyclic group of order $p-1$, the stabilizer of 0, $G(X)_0 = \langle \sigma \rangle$, is a cyclic group of order n which divides $p-1$, i.e., each element in $\langle \sigma \rangle$ fixes 0, and permutes the other $p-1$ vertices in orbits of length n. Let $A(H) = \langle \tau \rangle$ with $\tau^{p-1} = e$. Say $nq = p-1$. Then $\sigma = \tau^q$. Let $D_1, D_2, \dots, D_{p-1}$ be the basic digraphs relative to the group $\langle S \rangle = \langle (01 \cdots (p-1)) \rangle$; i.e., if we apply Schur's algorithm to $\langle S \rangle$, then we obtain all of the vertex-transitive digraphs with p vertices each of whose group of automorphisms $\supseteq \langle S \rangle$, and each of which is a combination of basic digraphs $D_1, D_2, \dots, D_{p-1}$. Now if we apply Schur's algorithm to the permutation group $\langle S, \sigma \rangle = \langle S, \tau^q \rangle$ acting on $H = \{0, 1, \dots, (p-1)\}$, then we obtain all of the vertex-transitive digraphs with p vertices each of whose group of automorphisms $\supseteq \langle S, \tau^q \rangle$, and each of which is a combination of the basic digraphs relative to $\langle S, \tau^q \rangle$. The basic digraphs relative to $\langle S, \tau^q \rangle$ consist of q digraphs each of which is of the type $D_{i_1} D_{i_2} \cdots D_{i_n}$, the union of $D_{i_1}, D_{i_2}, \dots, D_{i_n}$, relative to $\langle S \rangle$, where $i_1, i_2, \dots, i_n$ are distinct, $1 \leqslant i_1, i_2, \dots, i_n \leqslant n$, and they belong to the same orbit of $\langle \tau^q \rangle = \langle \sigma \rangle$.

In order to apply Pólya's theorem, we let $\overline{D_q}$ be the set of the q basic digraphs relative to $\langle S, \tau^q \rangle$, $R = \{0, 1\}$ and w_q be the weight function such that $w_q(0) = 1$ and $w_q(1) = x^{np}$, where $qn = p-1$. For each $\bar{\theta} \in A(H)/\langle \tau^q \rangle$, $\bar{\theta} = \langle \tau^q \rangle \theta$. Since $\sigma = \tau^q$ is an automorphism of every basic digraph in $\overline{D_q}$, we may define $(D_{i_1} D_{i_2} \cdots D_{i_n}) \bar{\theta} = (D_{i_1 \theta} D_{i_2 \theta} \cdots D_{i_n \theta})$. Let $\overline{H}_q$ be the set of all permutations acting on $\overline{D_q}$ induced by all $\bar{\theta}$ in $A(H)/\langle \tau^q \rangle$. Then $\overline{H_q}$ is a permutation group and is isomorphic to $A(H)/\langle \tau^q \rangle$.

Each of the vertex-transitive digraphs with p vertices whose group of automorphisms $\supseteq \langle S, \tau^q \rangle$ is a combination of the basic digraphs in $\overline{D_q}$. Clearly, it is a Cayley digraph $X_{H,K}$. We claim that any two such digraphs $X_{H,K}$ and $X_{H,K'}$ are isomorphic if and only if there exists an element in $\overline{H}_q$ corresponding to an element $\bar{\theta} = \langle \tau^q \rangle \theta$ in $A(H)/\langle \tau^q \rangle$ such that θ maps K onto K'. This follows from our Theorem 4 in [8]; i.e., if $X_{H,K}$ and $X_{H,K'}$ are isomorphic, by Theorem 4 in [8], there exists a $\theta \in A(H)$ which maps K

onto K'. Then $\langle\tau^q\rangle\theta=\bar{\theta}\in A(H)/\langle\tau^q\rangle$. By the isomorphism of $A(H)/\langle\tau^q\rangle$ and $\bar{H}_q$, there is such an element in $\bar{H}_q$. Conversely, if there is an element in $\bar{H}_q$ corresponding to $\bar{\theta}=\langle\tau^q\rangle\theta$ in $A(H)/\langle\tau^q\rangle$ such that θ maps K onto K', then, by Theorem 4 in [8], $X_{H,K}$ and $X_{H,K'}$ are isomorphic. Applying Pólya's theorem, we have

THEOREM 4. *The non-isomorphic classes of vertex-transitive digraphs with a prime number p vertices whose group of automorphisms* $\supseteq\langle S,\tau^q\rangle$, *where* $S=(0\ 1\ \cdots\ (p-1))$ *and* τ *is a generator of the group* $A(H)$ *of automorphisms of* $H=\{0,1,\ldots,(p-1)\}$, *are enumerated by the function*

$$P_{\bar{H}_q}(1+x^{np},1+x^{2np},\ldots). \tag{7}$$

Here $nq=p-1$, $P_{\bar{H}_q}(x_1,x_2,\ldots)$ *is the cycle index of* $\bar{H}_q$, *and the coefficient of* x^m *is the number of non-isomorphic digraphs having m directed edges. In fact, the enumerating function* (7) *is*

$$\frac{1}{q}\sum_d \phi(d)(1+x^{dnp})^{q/d}, \tag{8}$$

where the sum is taken over the divisors d of q and ϕ *is Euler's* ϕ *function.*

Since $\bar{H}_q\simeq A(H)/\langle\tau^q\rangle$ and $A(H)$ is a cyclic group of order $p-1$ and $\langle\tau^q\rangle$ is of order n, $\bar{H}_q$ is a cyclic group of order q acting on $\bar{D}_q$ whose cardinality is q. Hence, (8) follows from (7).

THEOREM 5. (a) *The enumerating function of the non-isomorphic classes of strongly vertex-transitive digraphs with a prime number p vertices is*

$$\sum_{d_i}\mu(d_i)\,P_{\bar{H}_{q_i}}(1+x^{d_ip},1+x^{2d_ip},\ldots), \tag{9}$$

where $d_iq_i=p-1$, *the summation is taken over all divisors* d_i *of* $p-1$, $P_{\bar{H}_{q_i}}(x_1,x_2,\ldots)$ *is the cycle index of* $\bar{H}_{q_i}$, *the coefficient of* x^m *is the number of non-isomorphic digraphs having m directed edges and* μ *is the Möbius function.*

(b) *The number of non-isomorphic classes of strongly vertex-transitive, self-complementary digraphs with p vertices is*

$$\sum_{d_i}\mu(d_i)\left(P_{\bar{H}_{q_i}}\left(\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2},\ldots\right)e^{2(z_2+z_4+\cdots)}\right) \tag{10}$$

evaluated at $z_1=z_2=\cdots=0$, *where* $d_iq_i=p-1$, *the summation is taken over all divisors* d_i *of* $p-1$, $P_{\bar{H}_{q_i}}(x_1,x_2\cdots)$ *is the cycle index of* $\bar{H}_{q_i}$, *and* μ *is the Möbius function.*

Proof. (a) Let $p-1=p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t}$ be the prime power decomposition of $p-1$. Beginning with the enumeration of all the vertex-transitive digraphs with p vertices each of whose group of automorphisms $\supseteq\langle S\rangle$, we subtract those that are not strongly vertex-transitive each of whose group of automorphisms $\supseteq\langle S, \tau^{p_i}\rangle$ for $i=1, 2,..., t$. Since we have twice subtracted each of those whose group of automorphisms $\supseteq\langle S, \tau^{p_ip_j}\rangle$, where $i\neq j$ and i, $j=1, 2,..., t$, we must compensate by adding these once, and so on. In this way, the Möbius function, $\mu(d)$, gives us exactly what is needed for our enumeration as d runs through the divisors of $p-1$. Thus, (9) follows from using (7) repeatedly.

(b) Apply de Bruijn's theorem to obtain (10).

EXAMPLE. Let $p=7$, $S=(0123456)$ and $H=\{0, 1, 2, 3, 4, 5, 6\}$. Then $A(H)=\langle\tau\rangle$, where $\tau=(0)(132645)$ and $\tau^6=e$.

(a) Applying Schur's algorithm to the group $\langle S,\tau^6\rangle=\langle S\rangle$, we obtain the basic digraphs D_1D_2, D_3, D_4, D_5, D_6 relative to $\langle S\rangle$, where $V(D_i)=H$ and $E(D_i)=\{[j, j+i];\ j=0,\ 1,..., 6\}$ for $i=1, 2,..., 6$. $\bar{H}_6=\bar{A}(H)$ acts on $\{D_1, D_2, D_3, D_4, D_5, D_6\}$. In fact, $\bar{H}_6=\langle(D_1D_2D_3D_6D_4D_5)\rangle$. By (7), the enumerating function of vertex-transitive digraphs with seven vertices each of whose group of automorphisms $\supseteq\langle S\rangle$ is

$$\begin{aligned} P_{\bar{H}_6}(x_1, x_2,...) &= \frac{1}{6}\sum_{d\mid 6}\phi(d)(1+x^{7d})^{6/d} \\ &= 1+x^7+3x^{14}+4x^{21}+3x^{28}+x^{35}+x^{42}. \end{aligned} \tag{11}$$

By (6), the number of non-isomorphic self-complementary, vertex-transitive digraphs with seven vertices each of whose group of automorphisms $\supseteq\langle S\rangle$ is

$$\begin{aligned} &\left[\frac{1}{6}\left(\sum_{d\mid 6}\phi(d)\frac{\partial^{6/d}}{\partial z_d^{6/d}}\right)e^{2(z_2+z_4+\cdots)}\right]_{z_1=z_2=\cdots=0} \\ &= \left[\frac{1}{6}\left(\frac{\partial^6}{\partial z_1^6}+\frac{\partial^3}{\partial z_2^3}+2\frac{\partial^2}{\partial z_3^2}+2\frac{\partial}{\partial z_6}\right)e^{2(z_2+z_4+z_6+\cdots)}\right]_{z_1=z_2=\cdots=0} \\ &= \frac{1}{6}(2^3+2\cdot 2)=2. \end{aligned} \tag{12}$$

(b) Applying Schur's algorithm to the group $\langle S,\tau^3\rangle$, we obtain the basic digraphs D_1D_6, D_2D_5, D_3D_4 relative to $\langle S,\tau^3\rangle$, where $V(D_iD_k)=H$ and $E(D_iD_k)=\{[j, j+i];\ j=0, 1,... 6\}\cup\{[j, j+k];\ j=0, 1,... 6\}$. $\bar{H}_3$ acts on $\{D_1D_6=d_1, D_2D_5=d_2, D_3D_4=d_3\}$ and $\bar{H}_3=\langle(d_1d_2d_3)\rangle$.

By (7), the enumerating function of vertex-transitive digraphs with seven vertices each of whoese group of automorphisms $\supseteq\langle S, \tau^3\rangle$ is

$$P_{\bar{H}_3}(x_1, x_2,...) = \frac{1}{3}\sum_{d|3} \phi(d)(1+x^{14d})^{3/d} = 1 + x^{14} + x^{28} + x^{42}. \quad (13)$$

By (6), the number of self-complementary, vertex-transitive digraphs with seven vertices each of whose group of automorphisms $\supseteq\langle S, \tau^3\rangle$ is

$$\left[\frac{1}{3}\left(\sum_{d|3} \phi(d)\frac{\partial^{3/d}}{\partial z_d^{3/d}}\right) e^{2(z_2+z_4+\cdots)}\right]_{z_1=z_2=\cdots=0}$$
$$= \left[\frac{1}{3}\left(\frac{\partial^3}{\partial z_1^3} + 2\frac{\partial}{\partial z_3}\right) e^{2(z_2+z_4+\cdots)}\right]_{z_1=z_2=\cdots=0} = 0. \quad (14)$$

(c) Applying Schur's algorithm to the group $\langle S, \tau^2\rangle$, we obtain the basic digraphs $D_1D_2D_4$, $D_3D_5D_6$, relative to $\langle S, \tau^2\rangle$, where $V(D_iD_kD_m) = H$ and $E(D_iD_RD_m) = \{[j, j+i];\ j = 0, 1,..., 6\} \cup \{[j, j+k];\ j = 0, 1,... 6\} \cup \{[j, j+m];\ j = 0, 1,... 6\}$. $\bar{H}_2$ acts on $\{D_1D_2D_4 = e_1, D_3D_5D_6 = e_2\}$ and $\bar{H}_2 = \langle(e_1e_2)\rangle$.

By (7), the enumerating function of vertex-transitive digraphs with seven vertices each of whose group of automorphisms $\supseteq\langle S, \tau^2\rangle$ is

$$P_{\bar{H}_2}(x_1, x_2,...) = \frac{1}{2}\sum_{d|2} \phi(d)(1+x^{21d})^{2/d} = 1 + x^{21} + x^{42}. \quad (15)$$

By (6), the number of self-complementary, vertex-transitive digraphs with seven vertices each of whose group of automorphisms $\supseteq\langle S, \tau^2\rangle$ is

$$\left[\frac{1}{2}\left(\sum_{d|2} \phi(d)\frac{\partial^{2/d}}{\partial z_d^{2/d}}\right) e^{2(z_2+z_4+\cdots)}\right]_{z_1=z_2=\cdots=0}$$
$$= \left[\frac{1}{2}\left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial}{\partial z_2}\right) e^{2(z_2+z_4+\cdots)}\right]_{z_1=z_2=\cdots=0} = 1. \quad (16)$$

(d) Applying Schur's algorithm to the group $\langle S, \tau\rangle$ we obtain the basic digraph $D_1D_2D_3D_4D_5D_6$ relative to $\langle S, \tau\rangle$.

It is the complete digraph with seven vertices. $\bar{H}_1$ acts on $\{D_1D_2D_3D_4D_5D_6\}$ and $\bar{H}_1$ is the identity group. By (7), the enumerating function of vertex-transitive digraphs with seven vertices each of whose group of automorphisms $\supseteq\langle S, \tau\rangle$ is

$$P_{\bar{H}_1}(x_1, x_2,...) = 1 + x^{42}. \quad (17)$$

By (6), the number of self-complementary, vertex-transitive digraphs with seven vertices each of whose group of automorphisms $\supseteq\langle S, \tau^6\rangle$ is

$$\left[\left(\frac{\partial}{\partial z_1}\right) e^{2(z_2+z_4+\cdots)}\right]_{z_1=z_2=\cdots=0} = 0. \tag{18}$$

By (9), we know that the enumerating function of strongly vertex-transitive digraphs with seven vertices is

$$\begin{aligned}
&\mu(1)\, P_{\bar{H}_6} + \mu(2)\, P_{\bar{H}_3} + \mu(3)\, P_{\bar{H}_2} + \mu(6)\, P_{\bar{H}_1} \\
&\quad = (1 + x^7 + 3x^{14} + 4x^{21} + 3x^{28} + x^{35} + x^{42}) - (1 + x^{14} + x^{28} + x^{42}) \\
&\qquad - (1 + x^{21} + x^{42}) + (1 + x^{42}) \\
&\quad = x^7 + 2x^{14} + 3x^{21} + 2x^{28} + x^{35},
\end{aligned} \tag{19}$$

where (11), (13), (15) and (17) are used. The digraphs X_{72}, X_{73}, X_{74}, X_{76}, X_{78}, X_{79} on p. 254 in [8] and X_{72}^c, x_{73}^c, x_{74}^c constitute representatives of the nine non-isomorphic classes of (19).

By (10), we know that the number of strongly vertex-transitive, self-complementary digraphs with seven vertices is

$$\begin{aligned}
&\left[\sum_d \mu(d_i)\, P_{\bar{H}_{q_i}}\left(\frac{\partial}{\partial z_1}, \frac{2}{\partial z_2}, \ldots\right) e^{2(z_2+z_4+\cdots)}\right)\Bigg]_{z_1=z_2=\cdots=0} \\
&\quad = 2 - 0 - 1 + 0 = 1,
\end{aligned} \tag{20}$$

where (12), (14), (16) and (18) are used.

In conclusion we observe that, as we showed in our last paper [8], any digraph with p vertices is vertex-transitive if and only if its group is of the form $\langle S, \tau^q\rangle$, where p, S, τ and q are defined here in Theorem 4. Hence, in this theorem, we consider all digraphs with a prime number of vertices whose group is of the form $\langle S, \tau^q\rangle$ for each possible q.

Also, the technique used here to enumerate circulant digraphs with pq vertices is a generalization of the technique we used in [8] to enumerate circulant (all vertex-transitive) digraphs with a prime number of vertices. This technique could be extended to enumerate circulant digraphs with n vertices, provided that n is square-free and that Ádám's conjecture holds. We have not seen a conclusive proof that Ádám's conjecture holds for all square-free n, although this seems to be the case.

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