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Tame minimal simple groups of finite Morley rank

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Abstract

We consider tame minimal simple groups of finite Morley rank and of odd type. We show that the Prüfer 2-rank of such a group is bounded by 2. We also find all potential nonalgebraic configurations; there are essentially four of them, and we delineate them with some precision.

1. Introduction

The role of groups of finite Morley rank in model theory was first seen in the work of Zilber on $\aleph_1$-categorical theories ([33], cf. [35]). Motivated by a sense that most interesting structures occur "in nature," Cherlin and Zilber independently proposed:

Classification Conjecture. *A simple infinite group of finite Morley rank is isomorphic as an abstract group to an algebraic group over an algebraically closed field.*

To date there have been three fruitful lines of attack on this problem. First of all, one may simply attempt to mimic the theory of algebraic groups. The second line of attack is to embed the problem in model theory proper. The third line, taken here and in numerous related recent articles, is to see what can be done by the methods of finite group theory, consisting of local geometrical analysis and some considerations involving involutions

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(i.e., elements of order 2). These methods may serve to limit the Sylow 2-subgroup structure severely.

In the classification of the finite simple groups, it was noticeable that quite indirect and subtle methods are usually required for the classification of "small" simple groups, whereas "generic" or "large" simple groups can be handled by more direct and elementary methods. This holds with a vengeance in the case of groups of finite Morley rank. Accordingly work on simple groups of finite Morley rank has tended to focus on those which are large in some sense. Here we take up the problem from the other end, and attempt to bring some order into the study of *minimal* simple groups of finite Morley rank:

Definition 1.1. A *minimal* simple group is a connected simple group of finite Morley rank in which every proper definable connected subgroup is solvable.

Examples of such groups were encountered in the earliest work in this area, in an extreme form:

Definition 1.2. A *bad* group is a simple group of finite Morley rank for which every proper definable connected subgroup is nilpotent.

The structure of Sylow 2-subgroups in a bad group is dramatically trivial:

Fact 1.3 [10,14,22]. *A simple bad group has no involutions.*

Minimal simple groups were already considered in [21] (where they were called *FT-groups*) as a possible generalization of bad groups. The task we set ourselves here is to determine the Sylow 2-subgroup structure of tame non-algebraic minimal simple groups of finite Morley rank as tightly as we can. The role of tameness in this enterprise will be discussed further below. Ideally one would like to eliminate involutions entirely, reducing the problem to the analog of the Feit–Thompson theorem, whose proof would clearly require other methods entirely; but it is well known that there are some other configurations, such as cyclic or quasicyclic Sylow 2-subgroups, which offer little scope for internal geometric analysis. As we will explain below, we encountered some additional configurations in Prüfer 2-rank 2 with a similar flavor, but using tameness we are able to exclude higher Prüfer 2-ranks, and at the same time severely limit the structure of the Sylow 2-subgroups in Prüfer 2-ranks 1 and 2.

In general, the *connected component* of a Sylow 2-subgroup S of a group of finite Morley rank is defined as $S^\circ = S \cap d(S)^\circ$, where $d(S)$ denotes the definable closure of S, i.e., the smallest definable subgroup containing S. With this definition, one can say a good deal about the Sylow 2-subgroup structure in an arbitrary group of finite Morley rank:

Fact 1.4 [11]. *Let G be a group of finite Morley rank. Then its Sylow 2-subgroups are conjugate. The connected component of a Sylow 2-subgroup is nilpotent, and is a central product, with finite intersection, of a 2-unipotent subgroup U and a 2-torus T.*

In this connection a *p-unipotent* subgroup is a definable connected p-subgroup of bounded exponent, and a *p-torus* is a divisible abelian p-group. The terminology is motivated by the situation in algebraic groups, in which a Sylow 2-subgroup is a finite extension of a 2-torus in characteristic not equal to 2, and is 2-unipotent in both the algebraic and model theoretic senses when the characteristic is 2. Accordingly, the following terminology has been adopted.

Definition 1.5. Let G be a group of finite Morley rank, and S the connected component of a Sylow 2-subgroup of G. Then G is said to be:

(1) of *degenerate* type if $S = 1$;
(2) of *odd* type if S is a nontrivial 2-torus;
(3) of *even* type if S is a nontrivial 2-unipotent group;
(4) of *mixed* type if S is a central product of a nontrivial 2-unipotent group and a nontrivial 2-torus.

Work on the structure of simple groups of finite Morley rank implies that there are no minimal simple groups of finite Morley rank of mixed type, and none of even type other than the algebraic group $SL_2(K)$, with K an algebraically closed field of characteristic 2. These results have been proved in considerably greater generality, using the notion of a K^*-group, which is a group G of finite Morley rank such that every infinite definable proper simple section of G is algebraic. This class would include any counterexample to the main conjecture of minimal rank, as well as all the minimal simple groups of finite Morley rank.

Building on earlier work in [2] about tame K^*-groups, it is shown in [19]:

Fact 1.6 [2,19]. *Let G be a simple infinite K^*-group of finite Morley rank. Then G is not of mixed type.*

In addition, work in course of publication shows that all K^*-groups of even type are algebraic; in any case it is easy to deduce from [3] that a minimal simple group of finite Morley rank of even type is isomorphic to $SL_2(K)$ with K an algebraically closed field of characteristic 2.

Hence, for the determination of minimal simple groups of finite Morley rank, it remains to deal with the degenerate and odd type cases. The degenerate case is of substantial interest, and while the connected component of a Sylow 2-subgroup is trivial in that case, this does not sufficiently limit the Sylow structure, and one would hope eventually to limit the 2-rank severely. Extreme forms of minimal simple groups, without involutions, are also studied in [21]. However, we turn our attention here to the odd type case, in which case the connected component of a Sylow 2-subgroup is a 2-torus S, whose structure is entirely determined by its so-called Prüfer 2-rank, which can be defined as the dimension over F_2 of the subgroup $\Omega_1(S) = \{x \in S\colon x^2 = 1\}$, or more informatively as the number of quasicyclic factors in a direct product decomposition of S (this number is finite according to [11]). We will denote the Prüfer 2-rank by $\mathrm{Pr}_2(S)$, or $\mathrm{Pr}_2(G)$ if G is the ambient group. Under the assumption of tameness, we prove that the Prüfer 2-rank is at most 2, and we delineate the troublesome configurations with some precision.

Tameness is defined as follows.

Definition 1.7. A *bad field* is a structure $\langle F, T; \ldots\rangle$ of finite Morley rank in which F carries the structure of an algebraically closed field and T is an infinite proper subgroup of the multiplicative group of F. A group of finite Morley rank is *tame* if it does not interpret a bad field naturally. Here a natural interpretation of the bad field $\langle F, T; \ldots\rangle$ in the group G consists of a pair of definable sections A, B of G, with B acting naturally on A (the action being induced by conjugation in G) so that

$$\langle A, B; \cdot_A, \cdot_B, \text{action}\rangle \simeq \langle F, T; \cdot_F, \cdot_T, \text{multiplication}\rangle.$$

Work on groups of odd type has emphasized the tame case in the past, primarily because of difficulties with signalizer functor theory, recently reworked by Jeff Burdges in [12]. We need the tameness restriction for other reasons, as we are very much concerned with the structure of tori in our groups. This hypothesis is used quite heavily throughout the present paper.

The main result of this paper is that the Prüfer 2-rank of a tame minimal simple group of finite Morley rank is at most 2. For the remaining cases, in which the Prüfer rank is 1 or 2, we analyze the groups from various points of view, notably in terms of the structure of *Borel* subgroups, i.e., the maximal proper definable connected (solvable) subgroups of the ambient minimal simple group. We obtain in particular the following theorem.

Theorem 1.8. *Let G be a tame minimal simple group of finite Morley rank and of odd type. Let S be a Sylow 2-subgroup of G, $A = \Omega_1(S^\circ)$, $T = C_G^\circ(S^\circ)$, $C = C_G^\circ(A)$, and $W = N_G(T)/T$, which is called the* Weyl *group. Then $\mathrm{Pr}_2(G) \leqslant 2$ and one has the following two possibilities*:

(1) $\mathrm{Pr}_2(G) = 1$:
   - (a) *If C is not a Borel subgroup of G, then G is of the form $PSL_2(K)$ with K an algebraically closed field of characteristic different from 2.*
   - (b) *If C is a Borel subgroup of G and if $W \neq 1$, then $C = T$ is 2-divisible abelian, $|W| = 2$, W acts by inversion on T, and $N_G(T)$ splits as $T \rtimes \mathbb{Z}_2$. All involutions in G are conjugate.*

(2) $\mathrm{Pr}_2(G) = 2$:
   Then $T = C = C_G(A)$ is nilpotent, $|W| = 3$, all involutions of G are conjugate, and G interprets an algebraically closed field of characteristic 3. Furthermore:
   - (a) *If C is not a Borel subgroup of G, then T is divisible abelian, and for each involution i in S°, the subgroup $B_i = C_G^\circ(i)$ is a Borel subgroup of G of the form $O(B_i) \rtimes T$, where $O(B_i)$ is inverted by the two involutions in T different from i.*
   - (b) *Otherwise, T is a nilpotent Borel subgroup of G.*

And without tameness? Burdges recently developed a new abstract notion of unipotence, leading to a robust signalizer functor theory without the tameness assumption [12]. This allows one to prove a Trichotomy Theorem [7]: a simple K^*-group of odd type is either a Chevalley group, or has small Sylow 2-subgroups, or has a proper “2-generated core.” In the third case, the ambient group has recently been shown, without tameness, to be minimal simple in [8], provided it has large enough Sylow 2-subgroups. Thus, the problem of the limitation of the Prüfer 2-rank of a potential nonalgebraic simple K^*-group of odd type reduces to the case of minimal simple groups without tameness. Assuming tameness, our result gives thus an absolute bound: 2. Unfortunately, tameness is used very intensively in our proof. On the one hand, it is used heavily to analyze the intersections of Borel subgroups. On the other hand, it is used in a critical arithmetical argument at the end of our proof that $\mathrm{Pr}_2(G) \leqslant 2$. Without tameness, such a bound remains a major open problem. To be continued, thus.

The paper is organized as follows. In Section 2 we review known results (and some direct corollaries) needed here. Our main reference for the theory of groups of finite Morley rank is [5] and our notations generally follow [5]; the reader can also refer to [27] for a more model theoretic introduction to the subject. In Section 3 we derive some additional, less familiar, results of a general nature. Notably, we prove in Proposition 3.11 the important consequences of tameness for intersections of Borel subgroups which are used heavily throughout the paper.

After these preparations we prove our main results in Sections 4–7. We deal with the case of Prüfer rank 1 in Sections 4 and 5, and with the case of Prüfer rank at least 2 in Sections 6 and 7. The treatment is parallel in the two cases; in particular, the division into two subcases is the same in each case, and there are other parallels throughout. On the other hand, the case of Prüfer rank 1 is much briefer than the case of Prüfer rank at least 2, which works out similar themes on a substantially larger scale. In particular, Section 6 is quite elaborate.

In Section 4, dealing with a minimal simple group G of finite Morley rank of Prüfer 2-rank 1 and in which C is not a Borel subgroup, we prove part (1a) of Theorem 1.8. This is Theorem 4.1.

In Section 5 we assume that $\mathrm{Pr}_2(G) = 1$ and that C is a Borel subgroup of G, and we prove statement (1b) of Theorem 1.8. We first suppose that the Borel subgroup C is nonnilpotent in Section 5.1, showing that the Weyl group W is trivial in that case, and then we consider the case in which C is nilpotent, in Section 5.2. In this case we also analyze the geometry of involutions in G, at the end of Section 5.2.

In Section 6 we assume that G has Prüfer 2-rank at least 2 and that C is not a Borel subgroup. We show that $\mathrm{Pr}_2(G) = 2$ (Proposition 6.3), and prove part (2a) of Theorem 1.8 in Theorem 6.6. Then we show that W acts faithfully on A (Corollary 6.18), obtaining, in particular, $|W| = 1, 2, 3$, or 6. We show that the cases $|W| = 2, 6$, and 1 do not occur, in Sections 6.1–6.3, respectively. We end the proof of the main statement of part (2) of Theorem 1.8 in Section 6.4 (the remaining case: $|W| = 3$), and we also analyse the geometry of involutions in this case.

In Section 7 we assume that $\mathrm{Pr}_2(G) \geqslant 2$ and that C is a Borel subgroup. We then show easily that C is nilpotent in Section 7.1 (Theorem 7.1). In Section 7.2, with $C = T$ nilpotent, we obtain a very good description of G and prove part (2) of Theorem 1.8. In this case, we find that G has Prüfer 2-rank 2 at the very end of the analysis in Section 7.2 (Proposition 7.29), completing the proof of our main result that $\mathrm{Pr}_2(G) \leqslant 2$ in all cases.

We use the following notation throughout: if X is any subset of a group G, then $I(X)$ denotes the set of involutions in X, and $X^\#$ denotes the set of nontrivial elements of X.

To describe Borel subgroups, we will also use the notation $\mathfrak{B}$ to denote a certain class of Borel subgroups in Sections 5.2, 6, and 7.2. The definition of $\mathfrak{B}$ will be slightly different in Section 6, but we adopt the same terminology throughout as Borel subgroups from $\mathfrak{B}$ will always play the same role in the different cases considered.

2. Toolbox

The proofs of most of the following facts can be found in [5].

2.1. Generalities

Fact 2.1 [13]. *A group of finite Morley rank is connected if and only if its Morley degree is one.*

Fact 2.2 ([32], [5, Corollary 5.29]). *Let H be a definable connected subgroup of a group of finite Morley rank G. Then the subgroup $[H, X]$ is definable and connected for any subset X of G.*

Fact 2.3 [5, Corollary 5.13]. *Let G be a connected group of finite Morley rank and X a definable subset of G. If X is generic in G, then $G = X \cdot X$.*

If X is a subset of a group of finite Morley rank, then its *definable closure*, denoted by $d(X)$, is the smallest definable subgroup of G containing X.

Fact 2.4 [5, Exercise 2, p. 92]. *Let G be a group of finite Morley rank and X a subset of G. Then $C_G(X) = C_G(d(X))$.*

Fact 2.5 [9]. *Let H be a group of finite Morley rank and N a normal definable subgroup of H. If h is an element of H such that $\bar{h}$ is a p-element of $\overline{H} = H/N$ (p a prime), then the coset hN contains a p-element.*

2.2. Nilpotent groups

Fact 2.6 [5, Lemma 6.3]. *Let G be a nilpotent group of finite Morley rank. If $H < G$ is a definable subgroup of infinite index in G, then $N_G(H)/H$ is infinite.*

Fact 2.7 [5, Exercise 5, p. 98]. *Let G be a nilpotent group of finite Morley rank. If H is a normal infinite subgroup of G, then $H \cap Z(G)$ is infinite.*

Fact 2.8 [26]. *Let G be a nilpotent group of finite Morley rank. Then G is a central product $D * C$ where D and C are two definable characteristic subgroups, D is divisible and C is of bounded exponent. If T is the set of torsion elements of D, then T is central in D and $D = T \times N$ where N is a divisible subgroup. Furthermore, C is the direct sum of its Sylow p-subgroups.*

Fact 2.9 [11]. *Let P be a locally finite p-subgroup of a group of finite Morley rank. Then P has the following properties*:

(i) *P° is nilpotent and $P^\circ = B * T$ is the central product of a nilpotent subgroup B of bounded exponent and a p-torus T.*
(ii) *$Z(P) \neq 1$ and P satisfies the normalizer condition*: *for $Q < P$, we have $Q < N_P(Q)$.*
(iii) *If P is infinite and of finite exponent, then P is nilpotent and its center contains infinitely many elements of order p.*

The following result is called *rigidity* of p-tori in groups of finite Morley rank.

Fact 2.10 [11]. *If T is a p-torus in a group of finite Morley rank G, then $[N_G(T) : C_G(T)]$ is finite.*

Fact 2.11 [31, p. 146]. $\mathrm{Aut}(\mathbb{Z}_{2^n})$ *is a* 2-*group for every positive integer n.*

2.3. Solvable groups

Fact 2.12 [5, Theorem 9.29]. *Let G be a connected solvable group of finite Morley rank. Then the Sylow p-subgroups of G are connected.*

If π is a set of prime numbers, then we call any maximal π-subgroup of a solvable group G a *Hall* π-subgroup of G.

Fact 2.13 [4]. *Let G be a solvable group of finite Morley rank. If π is a set of prime numbers, then the Hall π-subgroups of G are conjugate in G.*

Fact 2.14 ([4], [1, Fact 2.30]). *Let G be a solvable group of finite Morley rank and N a definable normal subgroup of G. If π is a set of prime numbers, then a Hall π-subgroup of G/N is of the form HN/N for a Hall π-subgroup H of G.*

For every group H of finite Morley rank, its *Fitting* subgroup, denoted by $F(H)$, is the maximal normal nilpotent subgroup of H. It is well-defined and definable in H (see [25]).

Fact 2.15 [24]. *Let H be a connected solvable group of finite Morley rank. Then $H/F^\circ(H)$ is divisible abelian.*

The preceding fact has the following corollary.

Corollary 2.16 [2, Fact 2.36]. *Let H be a connected solvable group of finite Morley rank, p a prime number, and U_p a p-unipotent subgroup of H. Then $U_p \leqslant F^\circ(H)$. In particular, H contains a unique maximal p-unipotent subgroup, which is nilpotent and characteristic in H.*

The following useful fact has been proved by several people; a simple proof, due to B. Poizat, can be found in [21].

Fact 2.17. *Let H be a nontrivial connected solvable group of finite Morley rank. Then any element of H has an infinite centralizer in H.*

Corollary 2.18. *Let G be a nontrivial connected group of finite Morley rank with a definable connected solvable subgroup H such that $\bigcup_{g\in G} H^g$ is generic in G. Then any element of G has an infinite centralizer.*

Proof. If $g \in G$ has a finite centralizer, then its conjugacy class is generic in G and g is in a conjugate of H by Fact 2.1, a contradiction to Fact 2.17. □

A subgroup of a group G which is nilpotent and selfnormalizing in G will be called a *Carter* subgroup of G.

Fact 2.19 [16,29]. *Let H be a connected solvable group of finite Morley rank. Then H contains Carter subgroups. Furthermore*:

(i) *If C is a definable nilpotent subgroup of H of finite index in its normalizer in H, then C is a Carter subgroup of H.*
(ii) *Carter subgroups of H are H-conjugate.*
(iii) *If C is a Carter subgroup of H, then $H = F^\circ(H)C$.*

The following corollary is due to O. Frécon.

Corollary 2.20 [17]. *Let H be a connected solvable group of finite Morley rank of odd type with an element x of prime order p. If $F^\circ(H)$ contains no nontrivial p-unipotent subgroup, then x centralizes a Sylow 2-subgroup of H.*

Proof. We first claim that if T_q is a maximal q-torus of H (q a prime), then T_q is contained in a Carter subgroup of H. For, let C be a Carter subgroup of $C^\circ_H(T_q)$. Then $T_q \leqslant C$ and T_q is the maximal q-torus of C as in Fact 2.8. Now Fact 2.10 shows that $N^\circ_H(C) \leqslant N^\circ_H(T_q) = C^\circ_H(T_q)$, thus $N^\circ_H(C) \leqslant N^\circ_{C^\circ_H(T_q)}(C) = C$. Hence C is a Carter subgroup of H containing T_q, which proves the claim.

By our assumption about H, Facts 2.9, 2.12, and 2.16 show that a Sylow q-subgroup of H is a q-torus for $q = 2$ and $q = p$. Thus, x is in a maximal p-torus of H, which is in a Carter subgroup of H by the claim. Similarly, a Sylow 2-subgroup of H is in a Carter subgroup of H. We can now conclude by conjugacy of Carter subgroups (Fact 2.19(ii)) and Fact 2.8. □

We note that the first half of the above proof has recently been generalized by Frécon and Jaligot in the following way: if G is any group of finite Morley rank, and T is a maximal direct sum of q-tori of G (q varies), then T is contained in a nilpotent definable connected subgroup of G of finite index in its normalizer.

Fact 2.21 [16, Corollaire 5.20]. *Let H be a connected solvable group of finite Morley rank and C a Carter subgroup of H. Let N be a* (*not necessarily definable*) *normal subgroup* *of H. Then CN/N is a Carter subgroup of H/N and every Carter subgroup of H/N has this form.*

If H is any group, we denote by $H_{\mathcal{N}}$ the intersection of all normal subgroups H_1 of H such that H/H_1 is nilpotent. $H_{\mathcal{N}}$ is obviously a characteristic subgroup of H.

Fact 2.22 [16, Corollary 7.7 and remarks following]. *Let H be a connected solvable group of finite Morley rank and C a Carter subgroup of H. Assume that H is solvable of class* 2*. Then $H_{\mathcal{N}}$ is definable in H and $H = H_{\mathcal{N}} \rtimes C$.*

If H is a group and U a subset of H, then the *generalized centralizer* of U in H, denoted by $E_H(U)$, is defined as

$$E_H(U) = \bigcap_{u \in U} \left(\bigcup_{n \in \mathbb{N}} \left\{ h \in H\colon\ (\mathrm{ad}_u)^n(h) = 1 \right\} \right),$$

where ad_u is the map

$$\mathrm{ad}_u : H \longrightarrow H, \quad h \longmapsto [h, u].$$

Fact 2.23 [16, Théorème 1.2, Corollaire 5.17 and 7.4]. *Let H be a connected solvable group of finite Morley rank and let U be a nilpotent subgroup of H. Then $E_H(U)$ is a definable connected subgroup of H which contains a Carter subgroup of H, and $U \leqslant F(E_H(U))$.*

Corollary 2.24. *Let H be a connected solvable group of finite Morley rank of the form $U \rtimes C$, where C is a Carter subgroup of H and U is a nontrivial definable connected nilpotent subgroup normal in H. Let X be a nilpotent subgroup of H. If $E_H(X)$ is not a Carter subgroup of H, then $C_U^\circ(X) \neq 1$.*

Proof. By Fact 2.23, $E_H(X)$ contains a Carter subgroup of H, that is C^u for some $u \in U$ by Fact 2.19. By assumption we have thus $E_H(X) = U_1 \rtimes C^u$, where $U_1 = E_H(X) \cap U$ is nontrivial and connected (Facts 2.1 and 2.23). As $U_1 \trianglelefteq E_H(X)$, $U_1 \leqslant F(E_H(X))$ and U_1 contains infinitely many elements in the center of $F(E_H(X))$ by Fact 2.7. But $X \leqslant F(E_H(X))$ by Fact 2.23, thus $1 \neq C_{U_1}^\circ(X) \leqslant C_U^\circ(X)$. $\square$

2.4. Torsion and automorphisms

Fact 2.25 [23]. *Let G be a group of finite Morley rank with a definable involutive automorphism σ. If σ fixes only finitely many elements in G, then G has a definable* (*abelian*) *normal subgroup inverted by σ and of finite index in G.*

Fact 2.26 [5, Exercise 14, p. 73]. *Let H be a group of finite Morley rank without involutions and with a definable involutory automorphism σ. If H^- denotes the set of elements of H inverted by σ, then H^- is a* 2*-divisible subset of H, $H = C_H(\sigma)H^-$, and each coset* *of $C_H(\sigma)$ contains a unique element of H^-. In particular, $C_H(\sigma)$ is connected if H is connected.*

Fact 2.27 [5, Exercise 10, p. 98]. *Let G be a group of finite Morley rank, $U \lhd G$ a connected definable nilpotent subgroup, and ϕ a definable automorphism of G stabilizing U and centralizing finitely many elements of U. Then $U = \{[u, \phi]\colon u \in U\}$. Furthermore, if $[G, \phi] \subseteq U$, then $G = UC_G(\phi)$.*

We give now a stronger form of Fact 2.25.

Fact 2.28 [23, Proposition 4.1]. *Let H be a group of finite Morley rank such that H/H° is of order 2 and such that the elements of $H \setminus H^\circ$ are generically of order 2. Then H splits as $H = H^\circ \rtimes \langle i\rangle$ for some involution i which inverts H°.*

Proof. Let $X = \{x \in H \setminus H^\circ\colon x^2 = 1\}$, $i \in X$, and $A = iX$. By assumption X is generic in the coset iH°, and $A = iX$ is generic in H°. Note that i inverts by conjugation every element of A: for if $a \in A$, then $ia \in iA = X$, so $(ia)^2 = 1$ and $a^i = a^{-1}$. We claim that $A \subseteq Z(H^\circ)$. If $g \in A$ and $h \in A \cap g^{-1}A$, then i inverts g, h, and gh, which shows that g commutes with h. Thus g commutes with $A \cap g^{-1}A$. But $A \cap g^{-1}A$ is generic in H° (by genericity of A and Fact 2.1), which implies that $H^\circ = (A \cap g^{-1}A)^2$ by Fact 2.3. Thus $g \in Z(H^\circ)$ and $A \subseteq Z(H^\circ)$ as claimed. Now, as i inverts A, it also inverts $A \cdot A$, i.e., H° by Fact 2.3. □

The following result provides a partial generalization of the foregoing for arbitrary primes.

Fact 2.29 [18, Corollary 16]. *Let H be a group of finite Morley rank such that H° is solvable. Assume that there is a prime p and a coset xH° of H° ($x \in H \setminus H^\circ$) of order p modulo H°, such that the elements of the coset xH° are generically of order p. Then H° is nilpotent.*

Fact 2.29 has the following special case.

Fact 2.30 ([30, Theorem 2.4.7], [5, Exercise 14, p. 79]). *Let H be a connected solvable group of finite Morley rank with a definable automorphism of prime order which centralizes only finitely many elements. Then H is nilpotent.*

We also prove here a lemma about automorphisms of order 2 of 2-tori of Prüfer 2-rank 2.

Lemma 2.31. *Let T_0 be a 2-torus of Prüfer 2-rank 2 and α an involutive automorphism of T_0 which fixes only one involution z of the three involutions of T_0. Then $T_0 = C_{T_0}(\alpha)T_0^-$ where T_0^- is the set of elements of T_0 inverted by α. Furthermore, the two factors in this product are two 2-tori of Prüfer 2-rank one and they intersect exactly in the subgroup of order 2 generated by z.*

Proof. Let z_1 and $z_2 = \alpha(z_1)$ be the two involutions of T_0 distinct from z. Let T_1 be a 2-torus in T_0 of Prüfer 2-rank one containing z_1, and $T_2 = \alpha(T_1)$. Then $I(T_1 \cap T_2) = \emptyset$ and $T_1 \cap T_2 = 1$, so $T_0 = T_1 \times T_2$. Now it is easy to see that

$$C_{T_0}(\alpha) = \{t_1\alpha(t_1)\colon t_1 \in T_1\} \quad \text{and that} \quad T_0^- = \{t_1\alpha(t_1)^{-1}\colon t_1 \in T_1\},$$

where both subgroups are isomorphic to T_1. As $T_1 \cong \mathbb{Z}_{2^\infty}$ is 2-divisible, we find $T_0 = T_1 \times T_2 = C_{T_0}(\alpha)T_0^-$, which proves our lemma. □

2.5. Fusion

Fact 2.32 [5, Proposition 10.2]. *Let G be a group of finite Morley rank and let i, j be two involutions of G. Then i and j are $d(ij)$-conjugate or they both commute with an involution in $d(ij)$.*

As we will work only with groups of odd type, we will apply the following fact only in the case in which $S^\circ = T$ is both the connected component of a Sylow 2-subgroup and a maximal 2-torus of the ambient group.

Fact 2.33 [5, Lemma 10.22]. *Let G be a group of finite Morley rank, S a Sylow 2-subgroup of G, and T the maximal 2-torus of S°. If X and Y are two subsets of S° with $X = Y^g$ for some $g \in G$, then $X = Y^h$ for some $h \in N_G(T)$ (that is, $N_G(T)$* controls fusion *in S°).*

Lemma 2.34. *Let G be a group of finite Morley rank of odd type and of Prüfer 2-rank one, S a Sylow 2-subgroup of G, and i the unique involution of S°. Then $C_G(S^\circ) \cap i^G = \{i\}$.*

Proof. If j is an involution in $C_G(S^\circ) \cap i^G$, then $j = i^g$ for some $g \in G$. Furthermore, S° and $S^{\circ g}$ are both contained in $C_G^\circ(j)$, so they are conjugate in $C_G^\circ(j)$. As the Prüfer 2-rank is one, this implies that i and j are conjugate in $C_G^\circ(j)$, thus $i = j$. □

A proper definable subgroup M of a group G of finite Morley rank is said to be *strongly embedded* in G if M has an involution and $M \cap M^g$ has no involution for every $g \in G \setminus M$.

Fact 2.35 [5, Theorem 10.19]. *Let G be a group of finite Morley rank with a strongly embedded subgroup M. Then involutions of G and M are respectively G-conjugate and M-conjugate.*

Fact 2.36 [20, Lemme 2.13]. *Let G be a simple infinite group of finite Morley rank and M a proper definable subgroup of G. Then $\mathrm{rk}(x^G \cap M) < \mathrm{rk}(x^G)$ for every nontrivial element x of G.*

As this last fact is not so well-known, we give the proof.

Proof. The intersection of the conjugates of M is a proper normal subgroup of G, hence trivial. Hence, by the descending chain condition on definable subgroups, some finite intersection $M^{g_1} \cap \cdots \cap M^{g_k} = 1$. On the other hand, x^G has Morley degree 1, as this conjugacy class can be identified with $G/C(x)$.

If $\mathrm{rk}(M \cap x^G) = \mathrm{rk}(x^G)$, then $M \cap x^G = x^G$ modulo sets of lower rank, so $x^G = (M \cap x^G)^{g_1} \cap \cdots \cap (M \cap x^G)^{g_k} = \{1\}$ modulo sets of lower rank, and $x = 1$, a contradiction. □

2.6. Generation

We call any elementary abelian 2-group of order 4 a *four-group*.

Fact 2.37 [6, Theorem 5.14]. *Let H be a group of finite Morley rank such that H° is solvable and without involutions. If V is a four-subgroup of H, then $H^\circ = \langle C^\circ_{H^\circ}(v)\colon v \in V^\#\rangle$.*

2.7. Tame solvable groups

Fields appear in connected solvable groups of finite Morley rank via the following fundamental result, called here *Zilber's Field Theorem*.

For its statement, recall that a subgroup A of a group H of finite Morley rank is said to be *H-minimal* if it is infinite, definable, normal in H, and minimal with respect to these properties. Note that A is then connected and abelian by Fact 2.2. Note also that if H is connected and solvable, then $A \leqslant Z(F(H))$ by Fact 2.7.

Fact 2.38 (Zilber's Field Theorem [5, Theorem 9.1]). *Let $G = A \rtimes H$ be a group of finite Morley rank where A and H are two infinite definable abelian subgroups, A is H-minimal and $C_H(A) = 1$. Then*

(i) *The subring $K = \mathbb{Z}[H]/\mathrm{ann}_{\mathbb{Z}[H]}(A)$ of the set $\mathrm{End}(A)$ of endomorphisms of A is a definable algebraically closed field; in fact, there exists an integer l such that each element of K can be represented by an endomorphism of the form $\sum_{i=1}^{l} h_i$, for some elements $h_i \in H$.*

(ii) *$A \cong K^+$, H is isomorphic to a subgroup T of $K^\times$, and H acts on A by multiplication, i.e.,*

$$G = A \rtimes H \cong \left\{\begin{pmatrix} t & a \\ 0 & 1 \end{pmatrix}\colon\ t \in T,\ a \in K\right\}.$$

(iii) *In particular, H acts freely on A, $K = T + \cdots + T$ (l times) and (with additive notation) $A = \{\sum_{i=1}^{l} h_i a\colon\ h_i \in H\}$ for each $a \in A^\#$.*

Zilber's Field Theorem has the following important corollary.

Corollary 2.39 [34]. *Let H be a solvable nonnilpotent connected group of finite Morley rank. Then H interprets an algebraically closed field K. More precisely, a definable section of $F(H)$ is isomorphic to K^+ and a definable section of $H/F(H)$ is isomorphic to an infinite definable subgroup of $K^\times$.*

The following fact is also a direct corollary of Zilber's Field Theorem.

Fact 2.40 [16]. *Let H be a connected solvable group of finite Morley rank and A an H-minimal subgroup of H. Then $C_H(a) = C_H(A)$ for every nontrivial element $a \in A$.*

For any group H of finite Morley rank we denote by $O(H)$ its maximal normal definable connected subgroup without involutions. (Note that $O(H)$ is well-defined by Fact 2.5.)

Lemma 2.41. *Let H be a connected solvable group of finite Morley rank of odd type which does not interpret a bad field. If U is a definable connected subgroup of H without involutions, then $U \leqslant O(F(H)) = O(H)$.*

Proof. First note that, as H does not interpret a bad field, $O(H)$ is nilpotent by Corollary 2.39 and Fact 2.14, thus $O(H) = O(F(H))$. Note also that the assumption about bad fields implies that $U \leqslant F^\circ(H)$ (else Fact 2.15 and Corollary 2.39 would imply that $F^\circ(H)U$ interprets an algebraically closed field of characteristic different from 2 as H is of odd type, forcing a nontrivial 2-torus into U by Fact 2.14).

It remains to show that $U \leqslant O(F^\circ(H)) = O(F(H))$. But the normalizer condition in nilpotent groups of finite Morley rank (Fact 2.6) implies the existence of a finite sequence $U = U_0 \lhd U_1 \lhd \cdots \lhd U_{k-1} \lhd U_k = F^\circ(H)$ of definable connected subgroups U_i $(0 \leqslant i \leqslant k)$, and we have clearly $U \leqslant O(U_1) \leqslant \cdots \leqslant O(U_{k-1}) \leqslant O(F^\circ(H))$. $\square$

2.8. *Around Zsigmondy's theorem*

We will use in the sequel a purely arithmetical result. If a and n are integers greater than 1, then a prime p is called a *Zsigmondy prime* for $\langle a, n\rangle$ if p does not divide a and a has order n modulo p, and p is called a *large* Zsigmondy prime for $\langle a, n\rangle$ if, in addition, $|a^n - 1|_p > n + 1$.

Couples $\langle a, n\rangle$ without a large Zsigmondy prime were classified by W. Feit. For $a = 2$ this gives:

Fact 2.42 [28, Theorem 6]. *Let $n > 1$ be an integer. Then there exists a large Zsigmondy prime for $\langle 2, n\rangle$ except exactly in the following cases*: $n = 2, 4, 6, 10, 12,$ *or* 18.

Corollary 2.43. *Let $n \geqslant 1$ be an integer such that $2^n - 1$ divides $d^n - 1$ for all integers d relatively prime to $2^n - 1$. Then $n = 1, 2, 4, 6,$ or 12.*

Proof. Let n be as in the statement. We first claim:

$$\text{if } p^k = \left|2^n - 1\right|_p > 1, \quad \text{then } p^{k-1}(p-1) \text{ divides } n. \tag{1}$$

So let $p^k = |2^n - 1|_p > 1$. The subgroup of invertible elements modulo p^k has order $p^{k-1}(p-1)$ and as p is odd, it is well known that it is cyclic. Thus there exists d of order $p^{k-1}(p-1)$ modulo p^k, and we may furthermore assume by the Chinese Remainder

Theorem that d is relatively prime to $2^n - 1$. But now $2^n - 1$ divides $d^n - 1$ by assumption, thus $d^n = 1$ modulo p^k. It follows that the order $p^{k-1}(p-1)$ of d modulo p^k divides n, and our first claim is proved. Now we claim:

$$\text{there is no large Zsigmondy prime for } \langle 2, n\rangle. \tag{2}$$

If p is a Zsigmondy prime for $\langle 2, n\rangle$, then 2 has order n modulo p and it follows that n divides $p-1$. Let now $p^k = |2^n - 1|_p$. Then $p^{k-1}(p-1)$ divides n by (1). Thus $k = 1$, $n = p - 1$, and $p^k = p = n + 1$. Therefore, p cannot be large and our claim (2) is proved.

We are now in a position to apply Fact 2.42, thus $n = 1, 2, 4, 6, 10, 12$, or 18, and it suffices to eliminate the cases $n = 10$ and 18. But $2^{10} - 1 = 31 \cdot 11 \cdot 3$ and the prime 31 violates (1), and $2^{18} - 1 = 262143 = 73 \cdot 19 \cdot 7 \cdot 3^3$ and the prime 73 violates (1). $\square$

2.9. Recognition

We use the following result to recognize $PSL_2(K)$ in the odd type setting.

Definition 2.44. A doubly transitive permutation group G is:

(1) a *Zassenhaus group* if the stabilizer of any three points is trivial;
(2) *split* if the stabilizer of two points $G_{x,y}$ has a normal complement in the stabilizer of one point G_x.

Fact 2.45 ([5, Theorem 11.89], [15]). *Let G be an infinite split Zassenhaus group of finite Morley rank. If a two point stabilizer T contains an involution, then $G \simeq PSL_2(K)$ for some algebraically closed field of characteristic not* 2.

3. General principles

In this section we will present some general results of a more specialized nature, useful for the analysis of Borel subgroups of tame minimal simple groups of odd type. Recall that *Borel* subgroups of a given group of finite Morley rank are defined as the maximal definable connected solvable subgroups. If the ambient group is minimal simple, then Borel subgroups are exactly the maximal proper definable connected subgroups.

3.1. Solvable groups of odd type

We begin with two lemmas about the structure of connected solvable groups of finite Morley rank of odd type.

Lemma 3.1. *Let H be a connected solvable group of finite Morley rank of odd type. Then the Sylow* 2*-subgroup of $F(H)$ is in $Z(H)$.*

Proof. Let $F(H) = D * C$ be the decomposition of $F(H)$ into a central product of definable characteristic subgroups as in Fact 2.8, where D is divisible and C of bounded exponent. As D is divisible, it is in particular connected and it contains a unique maximal 2-torus by Facts 2.12 and 2.8 again, which is central in H by Fact 2.10. Fact 2.8 also shows that C contains a unique Sylow 2-subgroup S, which is finite as H is of odd type. So H acts by conjugation on this finite Sylow 2-subgroup S, and H centralizes S as H is connected. $\square$

Lemma 3.2. *Let H be a connected solvable group of finite Morley rank of odd type. If $O(H)=1$, then H is divisible abelian.*

Proof. Let $F=F^\circ(H)$. As $O(H)=1$, we have $O(F)=1$ and Fact 2.8 shows that F contains no nontrivial p-unipotent subgroups for any prime $p>2$, and in fact for any prime p as F is of odd type. Thus, Fact 2.8 again shows that F is divisible and $F=\mathrm{Tor}(F)\times U$ where $\mathrm{Tor}(F)$ denotes the subgroup of torsion elements of F, which is central in F, and U is a torsion free subgroup. Note that $\mathrm{Tor}(F)$ is the product of p-tori (p varies) which are characteristic in H, thus central in H by rigidity of tori (Fact 2.10). It follows that $F'\leqslant U$, and as F' is definable and connected by Fact 2.2, it must be trivial as $O(F)=1$. So F is abelian and divisible.

To conclude it suffices to show that F is central in H, because then H is nilpotent by Fact 2.15, and thus equal to F. For this, it suffices to show that $[h,F]=1$ for any $h\in H$. But if $h\in H$, then $[h,F]\simeq F/C_F(h)$ is torsion free by Fact 2.14 (with π the set of all primes), since $\mathrm{Tor}(F)$ is central in H. Thus Fact 2.2 again shows that $[h,F]\leqslant O(F)=1$. $\square$

3.2. Genericity

Lemmas 3.3 and 3.4 will be applied to suitable Borel subgroups B of the ambient group G.

Lemma 3.3. *Let G be a connected group of finite Morley rank and B a definable subgroup of G of finite index in its normalizer. Assume that there is a definable subset X of B, not generic in B, such that $B\cap B^g\subseteq X$ whenever $g\in G\setminus N_G(B)$. Then $\bigcup_{g\in G}B^g$ is generic in G.*

Proof. An element of $B\setminus X$ cannot belong to a conjugate of B distinct from B. Thus

$$\mathrm{rk}\Big(\bigcup_{g\in G}(B\setminus X)^g\Big)\geqslant \mathrm{rk}\big(G/N_G(B)\big)+\mathrm{rk}(B\setminus X).$$

But B is of finite index in its normalizer, so

$$\mathrm{rk}\big(G/N_G(B)\big)+\mathrm{rk}(B\setminus X)=\mathrm{rk}(G)$$

and $\bigcup_{g\in G}B^g$ is generic in G. $\square$

Lemma 3.4. *Let G be a connected group of finite Morley rank and B a proper definable connected subgroup of finite index in its normalizer in G such that $\bigcup_{g\in G} B^g$ is generic in G. Assume that $x \in N_G(B) \setminus B$ is of order $n > 1$ modulo B, and let $\langle x\rangle B$ be the union $xB \cup x^2B \cup \cdots \cup x^{n-1}B \cup B$. Then the definable subset*

$$X_1 = \{x_1 \in xB\colon\ x_1 \in (\langle x\rangle B)^g \text{ for some } g \in G \setminus N_G(B)\}$$

of xB is generic in xB.

Proof. Assume that X_1 is not generic in xB. Then $xB \setminus X_1$ is generic in xB. So we have that

$$\mathrm{rk}\big((xB \setminus X_1)^G\big) \geqslant \mathrm{rk}(G) - \mathrm{rk}\big(N_G(B)\big) + \mathrm{rk}(xB \setminus X_1) = \mathrm{rk}(G) - \mathrm{rk}\big(N_G(B)\big) + \mathrm{rk}(B),$$

and as B is of finite index in its normalizer, $\mathrm{rk}((xB \setminus X_1)^G) = \mathrm{rk}(G)$. But $(xB \setminus X_1)^G$ is disjoint from $\bigcup_{g\in G} B^g$, thus G cannot be connected by Fact 2.1, a contradiction. □

The following important lemma was proved by O. Frécon.

Lemma 3.5 [17]. *Let H be a connected solvable group of finite Morley rank and C a Carter subgroup of H. Then $\bigcup_{h\in H\setminus C} (C \cap C^h)$ is not generic in C.*

Proof. Assume toward a contradiction that H is a counterexample of minimal rank, so that

$$\bigcup_{h\in H\setminus C} (C \cap C^h) = \Big(\bigcup_{h\in H\setminus CA} (C \cap C^h)\Big) \cup \Big(\bigcup_{h\in CA\setminus C} (C \cap C^h)\Big)$$

is generic in C, where A is an H-minimal subgroup of H. Let also the notation "$^{-}$" denote the quotient by A.

As

$$\overline{\bigcup_{h\in H\setminus CA} (C \cap C^h)} \subseteq \bigcup_{\bar h\in \overline{H}\setminus \overline{C}} (\overline{C} \cap \overline{C}^{\bar h}),$$

and as $\overline{C}$ is a Carter subgroup of $\overline{H}$ (Fact 2.21), then the minimality implies that $\bigcup_{h\in H\setminus CA} (C \cap C^h)$ is not generic in C. It follows that $\bigcup_{h\in CA\setminus C} (C \cap C^h)$ is generic in C and the minimality again implies that $H = CA$.

Note that $A \nleqslant C$, as otherwise $H = C$. So $C_C(A) < C$ and, in particular, $C_C(A)$ is not generic in C. It is thus enough to show that

$$\bigcup_{h\in CA\setminus C} (C \cap C^h) \subseteq C_C(A)$$

to get a final contradiction.

So let $C_1 = C \cap C^h$ for some $h \in CA \setminus C$. As C is selfnormalizing and nilpotent, we have $C_1 \leqslant C < \langle C, C^h \rangle \leqslant E_H(C_1)$, where $E_H(C_1)$ is the generalized centralizer of C_1 in H. So the subgroup $A_1 = A \cap E_H(C_1)$ is nontrivial. But A_1 is normal in $E_H(C_1)$, so $A_1 \leqslant F(E_H(C_1))$. It follows that there exists a nontrivial element $a \in A_1 \cap Z(F(E_H(C_1)))$. But $C_1 \leqslant F(E_H(C_1))$ by Fact 2.23, so $C_1 \leqslant C_C(a) = C_C(A)$ by Fact 2.40. The proof is now complete. $\square$

3.3. *Automorphisms and torsion*

Lemma 3.6. *Let H be a group of finite Morley rank such that H° is abelian. If x is an element in $H \setminus H^\circ$ such that the elements of the coset xH° are generically of order n for some integer $n > 1$, then every element in xH° is of order n.*

Proof. Let X be a generic definable subset of H° such that every element of xX is of order n. We may assume that x is of order n, and as $H^\circ = X \cdot X$ by Fact 2.3, it suffices to show that $(xx_1x_2)^n = 1$ for all elements $x_1, x_2 \in X$. But if x_1 and x_2 are such elements, then

$$(xx_1x_2)^n = x^n(x_1x_2)^{x^{n-1}}(x_1x_2)^{x^{n-2}}\dots(x_1x_2),$$

that is

$$(xx_1x_2)^n = x_1^{x^{n-1}}x_2^{x^{n-1}}x_1^{x^{n-2}}x_2^{x^{n-2}}\dots x_1x_2$$

as $x^n = 1$. As H° is abelian, we have thus

$$(xx_1x_2)^n = \left(x_1^{x^{n-1}}x_1^{x^{n-2}}\dots x_1\right)\left(x_2^{x^{n-1}}x_2^{x^{n-2}}\dots x_2\right).$$

But

$$\left(x_1^{x^{n-1}}x_1^{x^{n-2}}\dots x_1\right) = x^n\left(x_1^{x^{n-1}}x_1^{x^{n-2}}\dots x_1\right) = (xx_1)^n = 1,$$

so the first factor in the product is trivial and similarly the second factor is trivial. Thus $(xx_1x_2)^n = 1$. $\square$

Lemma 3.7. *Let H be a group of finite Morley rank such that H° is nilpotent, H/H° is of prime order p, and the elements of each coset of H° distinct from H° are generically of order p. If some element $x \in H \setminus H^\circ$ has an infinite centralizer in H°, then H° contains a nontrivial p-unipotent subgroup.*

Proof. Suppose that H is a counterexample of minimal rank and let $x \in H \setminus H^\circ$ such that $C := C^\circ_{H^\circ}(x)$ is nontrivial. We claim that the minimality of H implies that $C \leqslant Z^\circ(H^\circ)$. Assume that $C \not\leqslant Z^\circ(H^\circ)$ and let the notation "$\overline{}$" denote the quotients by $Z^\circ(H^\circ)$. Then the elements of the cosets of $\overline{H}^\circ$ in $\overline{H}$, distinct from $\overline{H}^\circ$, are still generically of order p and $\overline{C} = \overline{C}^\circ$ is a nontrivial subgroup of the centralizer of $\bar{x}$ in $\overline{H}^\circ$. As $Z^\circ(H^\circ) \neq 1$

by Fact 2.7, $\mathrm{rk}(\overline{H}) < \mathrm{rk}(H)$ and the minimality implies that $\overline{H}^\circ$ contains a nontrivial p-unipotent subgroup, hence also H° (Facts 2.14 and 2.8). This contradiction proves that $C \leqslant Z^\circ(H^\circ)$. This implies that $C \leqslant Z(H)$.

The coset xH° is partitioned by the definable equivalence relation "being in the same coset of $Z^\circ(H^\circ)$," so there is $x_1 \in xH^\circ$ such that the elements of the coset $x_1 Z^\circ(H^\circ)$ are generically of order p, and then each element of $x_1 Z^\circ(H^\circ)$ is of order p by Lemma 3.6. As $C \leqslant Z(H)$, we then have that $c^p = x_1^p c^p = (x_1 c)^p = 1$ for every $c \in C$. Thus C is a connected elementary abelian p-subgroup of $Z^\circ(H^\circ)$, a contradiction. □

Lemma 3.8. *Let H be a group of finite Morley rank of odd type and S a Sylow 2-subgroup of H. Assume that $H^\circ \leqslant C_H(S^\circ)$ (which is the case in particular if H° is nilpotent). Assume also that for each element $x \in H \setminus H^\circ$ there is an integer $n > 1$ such that the elements of the coset xH° are generically of order bounded by n. Then $C_H(S^\circ) = H^\circ$.*

Proof. First note that if H° is nilpotent, then $H^\circ \leqslant C_H(S^\circ)$ by Fact 2.8. Suppose that $H^\circ < C_H(S^\circ)$. Then there is an element $x \in H \setminus H^\circ$ which centralizes S°, hence also $d(S^\circ)$ by Fact 2.4, and there is an integer n such that the elements of the coset xH° are generically of order bounded by n. But xH° is definably partitioned by the equivalence relation of "being in the same coset of $d(S^\circ)$," so we can find $x_1 \in xH^\circ$ such that the elements of the coset $x_1 d(S^\circ)$ are generically of order bounded by n. As $\langle x_1\rangle d(S^\circ)$ is abelian, Lemma 3.6 shows that each element of $x_1 d(S^\circ)$ is of order bounded by n, and hence $d(S^\circ)$ has bounded exponent, a contradiction. □

Lemma 3.9. *Let H be a group of finite Morley rank where H° is solvable, of odd type, and has Prüfer 2-rank one. Assume that H/H° is of prime order p and assume also that there is a finite subgroup T_0 of H° without involutions, disjoint from $F^\circ(H^\circ)$, such that the definable subset*

$$\left\{x_1 \in xH^\circ\colon\ x_1^p \in T_0^{F(H^\circ)}\right\}$$

of xH° is generic in xH° for each $x \in H \setminus H^\circ$. Then $p = 2$ and H splits as $H^\circ \rtimes \langle x\rangle$ for some involution $x \in H$ which inverts H°.

Proof. Let S be a Sylow 2-subgroup of H°, that is a 2-torus of Prüfer rank 1. We first show that $p = 2$.

The subgroup $[S, H^\circ]$ is definable and connected (Fact 2.2) and normalized by H°. By a Frattini argument, $H = H^\circ N_H(S)$. Hence, $[S, H^\circ]$ is normal in H. We claim that $[S, H^\circ]$ contains no involutions. If $S \leqslant F^\circ(H^\circ)$, then S is central in H° by Lemma 3.1, and $[S, H^\circ] = 1$. Otherwise, as S has Prüfer rank 1, we have $S \cap F^\circ(H^\circ) = 1$ by Fact 2.12, and again $[S, H^\circ] \leqslant F^\circ(H^\circ)$ (Fact 2.15) contains no involutions.

Let "$\overline{}$" denote quotients by $[S, H^\circ]$. As $[S, H^\circ]$ contains no involutions, $\overline{H}^\circ$ has Prüfer 2-rank 1. For $\bar{x} \notin \overline{H}^\circ$, the elements of the coset $\bar{x}\overline{H}$ are generically of order bounded by $p|T_0|$. By Lemma 3.8, we have $C_{\overline{H}}(\bar{S}) = \overline{H}^\circ$, and it follows that $\overline{H}/\overline{H}^\circ \cong \mathbb{Z}_p$ embeds into $\mathrm{Aut}(\mathbb{Z}_{2^\infty})$. By Fact 2.11, this forces $p = 2$.

Now let X_1 be the generic subset of xH° consisting of elements x_1 such that $x_1^2 \in T_0^f$ for some $f \in F(H^\circ)$. We claim that $x_1^2 = 1$ for $x_1 \in X_1$.

For the remainder of the argument we use the bar notation "$\overline{}$" to denote quotients modulo $F^\circ(H^\circ)$. Note that $\overline{H}^\circ$ is divisible abelian by Fact 2.15.

First we show that $\overline{x_1}$ has a finite centralizer in $\overline{H}^\circ$. Let $\overline{C}$ denote the connected component of its centralizer in $\overline{H}^\circ$. One can find $x_2 \in X_1$ such that the elements of the coset $\overline{x_2}\overline{C}$ are generically of order bounded by $p|T_0|$, and Lemma 3.6 implies that each element in $\overline{x_2}\overline{C}$ has an order bounded by $p|T_0|$. As $\langle\overline{x_2}\rangle\overline{C}$ is abelian, this implies that $\overline{C}$ is of bounded exponent and as $\overline{H}^\circ$ is divisible, $\overline{C}$ is trivial.

Now $\overline{x_1}$ induces by conjugacy an involutory automorphism of $\overline{H^\circ}$ and Fact 2.25 shows that $\overline{x_1}$ inverts $\overline{H^\circ}$. So $\overline{x_1}^2$ is equal to its inverse as it is both centralized and inverted by $\overline{x_1}$. But $\overline{x_1}^2 \in \overline{T_0}$ which has no involutions by assumption; thus $\overline{x_1}^2 = \overline{1}$ and $x_1^2 \in T_0^{F(H^\circ)} \cap F^\circ(H^\circ) = 1$. We have shown that the elements of the coset xH° are generically of order 2, and we may conclude by invoking Fact 2.28. □

3.4. Borel subgroups

The next result shows that in a tame minimal simple group of odd type, the connected components of centralizers of maximal 2-tori behave like tori in algebraic groups.

Lemma 3.10. *Let G be a tame minimal simple group of odd type and S a Sylow 2-subgroup of G. Then $C_G^\circ(S^\circ)$ is nilpotent and of finite index in its normalizer. In particular, $C_G^\circ(S^\circ)$ is a Carter subgroup of any connected definable proper subgroup L of G containing $C_G^\circ(S^\circ)$.*

Proof. First note that $d(S^\circ)$ is central in $C_G^\circ(S^\circ)$ by Fact 2.4. Facts 2.12 and 2.14 show that $C_G^\circ(S^\circ)/d(S^\circ)$ has no involution and it is thus nilpotent by Lemma 2.41, as G interprets no bad field. So $C_G^\circ(S^\circ)$ is central-by-nilpotent and it is nilpotent. We have also that it is of finite index in its normalizer by Fact 2.10 and the fact that $N_G(C_G^\circ(S^\circ)) \leqslant N_G(S^\circ)$. The last statement then follows from Fact 2.19. □

The next proposition, together with Lemma 2.41, will be used intensively in our analysis based on the tameness assumption. We are not able to prove it without tameness. Nevertheless, there are weak analogs that may be useful in the absence of tameness.

Proposition 3.11. *Let G be a tame minimal simple group of odd type.*

(i) *Assume that B_1 and B_2 are two distinct Borel subgroups of G such that $O(B_1) \neq 1$ and $O(B_2) \neq 1$. Then $F(B_1) \cap F(B_2) = 1$.*
(ii) *In particular, any nontrivial connected definable subgroup without involutions U of G is contained in a unique Borel subgroup of G.*

Proof. The second statement follows from the first one: by Lemma 2.41, $U \leqslant F(B)$ for any Borel subgroup B containing U.

We prove now the first statement. We first show that $(O(B_1) \cap O(B_2))^\circ = 1$. Assume that B_1 and B_2 are as in the statement and that

$$X := \bigl(O(B_1) \cap O(B_2)\bigr)^\circ \neq 1$$

is of maximal rank. Let B_3 be a Borel subgroup of G containing $N_G^\circ(X)$. If $X < O(B_1)$, then we can look at $N_{O(B_1)}^\circ(X)$, which contains X as a subgroup of infinite index by the normalizer condition (Fact 2.6), and the maximality of $\mathrm{rk}(X)$ together with Lemma 2.41 shows that $B_1 = B_3$, and as $B_1 \neq B_2$, we then have for the same reason that $O(B_2) = X \leqslant O(B_1)$. But now $N_{O(B_1)}^\circ(O(B_2)) \leqslant O(N_G^\circ(O(B_2))) = O(B_2)$ by Lemma 2.41, and Fact 2.6 shows that $O(B_1) = O(B_2)$, and thus $B_1 = N_G^\circ(O(B_1)) = N_G^\circ(O(B_2)) = B_2$, a contradiction. We have proved that $X = O(B_1)$. Symmetrically we also have that $X = O(B_2)$, thus $O(B_1) = X = O(B_2)$, which implies as just seen that $B_1 = B_2$, a contradiction. So $(O(B_1) \cap O(B_2))^\circ = 1$ whenever B_1 and B_2 are as in the first statement of the proposition.

We now end the proof of the proposition. Assume that there is a nontrivial element $f \in F(B_1) \cap F(B_2)$. Let B_3 be a Borel subgroup of G containing $C_G^\circ(f)$. Fact 2.7 and Lemma 2.41 show that $(O(B_1) \cap O(B_3))^\circ$ is nontrivial, as well as $(O(B_2) \cap O(B_3))^\circ$, thus what we have shown before implies that $B_1 = B_3 = B_2$, a final contradiction. □

Lemma 3.12. *Let G be a tame minimal simple group of odd type and B a Borel subgroup of G. Then $C_G^\circ(f) \leqslant B$ for each $f \in F(B)^\#$.*

Proof. If $O(B) = 1$ then B is abelian by Lemma 3.2, so $B = C_G^\circ(f)$.

Assume $O(B) \neq 1$. Then $O(B) = O(F(B))$ by Lemma 2.41, and Fact 2.7 shows that $C_{O(B)}^\circ(f)$ is nontrivial. By Proposition 3.11, B is the unique Borel subgroup containing $C_{O(B)}^\circ(f)$, so B contains $C_G^\circ(f)$. □

To conclude this section, we remark that if G is a tame minimal simple group of *degenerate* type, then its Borel subgroups are without involutions by Fact 2.12 and are nilpotent by the proof of Lemma 2.41. Thus G is a bad group and it again satisfies Proposition 3.11 and Lemma 3.12 by the well-known structural properties of bad groups (cf. [5, Chapter 13]).

4. $\mathrm{Pr}_2(G) = 1$ and $C_G^\circ(A)$ not a Borel

In this section, as well as in the next ones, we assume that G is a tame minimal simple group of odd type and we fix the notations as in Theorem 1.8:

$$S \text{ is a fixed Sylow 2-subgroup of } G,$$

$$A = \bigl\langle I(S^\circ)\bigr\rangle, \qquad C = C_G^\circ(A), \qquad T = C_G^\circ(S^\circ), \quad \text{and} \quad W = N_G(T)/T.$$

In this section we assume furthermore,

$$\mathrm{Pr}_2(G) = 1 \text{ and } C_G^\circ(A) \text{ is not a Borel subgroup of } G,$$

and we will prove part (1a) of Theorem 1.8.

Theorem 4.1. *Assume that* $\mathrm{Pr}_2(G) = 1$ *and that* C *is not a Borel subgroup of* G. *Then* $G \cong PSL_2(K)$ *for some algebraically closed field* K *of characteristic different from* 2.

We embark now on the proof of Theorem 4.1. We let i denote the unique involution of A, so that $A = \langle i \rangle$. We will compute the rank of G and eventually show that G is a split Zassenhaus group.

Lemma 4.2. $F(B)$ *has no involution for any Borel subgroup* B *of* G.

Proof. If a Borel subgroup B has an involution, then one can assume, by conjugacy of Sylow 2-subgroups and Fact 2.12, that this involution is i. If $i \in F(B)$, then Lemma 3.1 shows that $B = C_G^\circ(i)$, a contradiction to our assumption that $C_G^\circ(i)$ is not a Borel subgroup. □

Corollary 4.3. $B_1 \cap F^\circ(B_2)$ *is finite and* $F(B_1) \cap F(B_2) = 1$ *for every pair of distinct Borel subgroups* B_1 *and* B_2 *of* G.

Proof. This follows from Lemma 2.41 and Proposition 3.11. □

Fix B a Borel subgroup of G containing $C = C_G^\circ(i)$. Note then that $S^\circ \leqslant T \leqslant C < B$, and that S° is a Sylow 2-subgroup of B by Fact 2.12. Let also $M = N_G(B)$. Then $(i^G \setminus M)$ is generic in i^G by Fact 2.36, so

$$\mathrm{rk}\big(i^G \setminus M\big) = \mathrm{rk}\big(i^G\big) = \mathrm{rk}(G) - \mathrm{rk}(C).$$

We define an equivalence relation $\sim$ on $i^G \setminus M$ by $w_1 \sim w_2$ if and only if w_1 and w_2 are in the same coset of B. Let

$$p : \big(i^G \setminus M\big) \longrightarrow \big(i^G \setminus M\big)/\sim$$

be the natural (definable) projection, and for $0 \leqslant k \leqslant \mathrm{rk}(B)$, let

$$X_k = \big\{w \in \big(i^G \setminus M\big) \colon \mathrm{rk}\big(p^{-1}\big(p(w)\big)\big) = k\big\}.$$

As $i^G \setminus M$ is partitioned by the (finite number of) X_k's, there exists k_0 such that X_{k_0} is generic in $i^G \setminus M$, and such a k_0 is unique, since the definable set $(i^G \setminus M)$ has degree 1.

Lemma 4.4. $k_0 \geqslant 1$.

Proof. If $k_0 = 0$, then $\mathrm{rk}(G) - \mathrm{rk}(C_G(i)) = \mathrm{rk}(X_0/\sim) \leqslant \mathrm{rk}(G) - \mathrm{rk}(B)$, so $\mathrm{rk}(B) \leqslant \mathrm{rk}(C)$ and $B = C$, contradicting our assumption. □

For every involution w in $i^G \setminus M$, let

$$T(w) = \{ww_1\colon\ w_1 \in (i^G \cap wB)\}.$$

Lemma 4.5. *If $w \in X_{k_0}$, then $T(w)$ is an infinite definable abelian subgroup of B which intersects $F^\circ(B)$ trivially, and contains a unique B-conjugate of S°.*

Proof. Let T_w be the set of all elements of B inverted by w. Corollary 4.3 and the fact that $w \notin N_G(B)$ shows that $T_w \cap F^\circ(B)$ is trivial. As $\langle T_w\rangle'$ is included in $F^\circ(B)$ (by Fact 2.15) and normalized by w, Corollary 4.3 again shows that $\langle T_w\rangle'$ must be trivial as $w \notin N_G(B)$. Thus T_w is an abelian subgroup of B. It is also obviously definable, and infinite as it contains $T(w)$.

We claim that $T(w) = T_w$. For this it suffices to show that each involution of wT_w is T_w-conjugate to w. Let wt be such an involution for some $t \in T_w$. It suffices to show that T_w is 2-divisible, as then $wt = wt'^2 = t'^{-1}wt'$ for some element $t' \in T_w$ such that $t'^2 = t$.

Claim 4.6. *T_w is 2-divisible.*

Proof of claim. First note that T_w is definably isomorphic to a subgroup of $B/F^\circ(B)$ as it is disjoint from $F^\circ(B)$. Facts 2.8 and 2.15 show that $T_w = T_w^\circ * C$, where T_w° is divisible and C is a direct product of finite p-groups for some prime numbers p. As $T_w^\circ \neq 1$ is disjoint from $F^\circ(B) = O(B)$ (Lemmas 2.41 and 4.2), one sees with the same kind of arguments as in the proof of Lemma 2.41, given the absence of bad fields, that T_w° contains a Sylow 2-subgroup of B. Thus a Sylow 2-subgroup of T_w is in T_w° and one can assume that C is the direct product of finite p-groups for some prime numbers $p > 2$. It follows that C is 2-divisible and T_w is also 2-divisible. □

We have now that $T(w) = T_w$ is an infinite definable abelian subgroup of B disjoint from $F^\circ(B)$. The fact that it contains a B-conjugate of S° has been shown in the proof of the claim. This conjugate is unique as it is a Sylow 2-subgroup of the abelian group $T(w)$, ending the proof of Lemma 4.5. □

Lemma 4.7. *$C = T$ is abelian.*

Proof. Pick an element $w \in X_{k_0}$ (as $X_{k_0} \neq \emptyset$!) which inverts S°. Then w centralizes i, so w normalizes C as well as its commutator subgroup C' which is contained in $F^\circ(B)$ (Fact 2.15), and must then be trivial by Corollary 4.3. So C is abelian and as $S^\circ \leqslant T \leqslant C$, we have that $C = T$. □

Corollary 4.8. *$F^\circ(B)$ is inverted by i and $B = F^\circ(B) \rtimes T$.*

Proof. If $C^\circ_{F^\circ(B)}(i) \neq 1$, then $1 \neq O(C^\circ_G(i)) \leqslant F(B)$ by Corollary 4.3 and Lemma 2.41 and if we pick $w \in X_{k_0}$ which inverts S°, then $w \in N_G(B)$ by Corollary 4.3, a contradiction. Thus $C^\circ_{F^\circ(B)}(i) = 1$ and i inverts $F^\circ(B)$ by Fact 2.25.

One sees then easily that $C_{B/F^\circ(B)}(i) = C_B(i)F^\circ(B)/F^\circ(B)$. As $B/F^\circ(B)$ is abelian, this gives that $B = F^\circ(B)C_B(i) = F^\circ(B) \rtimes C_B(i)$ and the connectedness of B implies that $B = F^\circ(B) \rtimes C^\circ_B(i) = F^\circ(B) \rtimes C$. $\square$

At this point, we can conclude the proof of Theorem 4.1 as follows: we take a B-minimal subgroup U of $F(B)$ and we remark that $C^\circ_C(U)$ has no involution (as the unique involution i of C inverts U by the preceding corollary). So $F^\circ(B)C^\circ_C(U)$ is nilpotent and included in $F^\circ(B) = O(B)$ by Lemma 2.41, and $C^\circ_C(U) \leqslant C \cap F^\circ(B) = 1$. So we can apply the result of [20], without further use of the assumption on bad fields.

To keep this text self-contained, we may proceed as follows, first embarking on the rank computation of the group G. As $T \leqslant B$ and T is of finite index in $N_G(S^\circ)$ by rigidity of S° (Fact 2.10), the equivalence classes of the definable equivalence relation $\approx$ on $X_{k_0}/\sim$, defined by $(w_1/\sim) \approx (w_2/\sim)$ if and only if w_1 and w_2 invert the same B-conjugate of S°, are all finite. So

$$\mathrm{rk}(X_{k_0}/\sim) \leqslant \mathrm{rk}(B) - \mathrm{rk}\big(N_B(S^\circ)\big) = \mathrm{rk}(B) - \mathrm{rk}(T).$$

Finally, as

$$\mathrm{rk}(G) - \mathrm{rk}(C) = \mathrm{rk}(X_{k_0}) = k_0 + \mathrm{rk}(X_{k_0}/\sim),$$

we get that

$$\mathrm{rk}(G) \leqslant k_0 + \mathrm{rk}(B) - \mathrm{rk}(T) + \mathrm{rk}(C),$$

and Lemma 4.7 shows that

$$\mathrm{rk}(G) \leqslant \mathrm{rk}(B) + k_0.$$

Corollary 4.9. $\mathrm{rk}(F^\circ(B)) \leqslant k_0$.

Proof. Pick an element $g \in G \setminus M$. As $B \cap F^\circ(B)^g$ is finite, we have that $\mathrm{rk}(B) + \mathrm{rk}(F^\circ(B)^g) \leqslant \mathrm{rk}(G)$. So $\mathrm{rk}(F^\circ(B)) = \mathrm{rk}(F^\circ(B)^g) \leqslant k_0$. $\square$

Let U be a B-minimal subgroup of B. Then $Z := C_T(U)$ is finite by Corollary 4.8 and Lemma 2.41. So we have that

$$U \rtimes (T/Z) \cong K^+ \rtimes K^*$$

for some algebraically closed field K by the field theorem (Fact 2.38) and the absence of bad fields. Thus

$$\mathrm{rk}(T) = \mathrm{rk}(U) \leqslant \mathrm{rk}\big(F^\circ(B)\big) \leqslant k_0.$$

So $\mathrm{rk}(T) = k_0$, and T is entirely inverted by an involution in X_{k_0} by connectedness. We also have that $k_0 = \mathrm{rk}(U) \leqslant \mathrm{rk}(F^\circ(B)) \leqslant k_0$, so $F^\circ(B) = U$. Note now that $Z(B) = Z$ is inverted by an involution in X_{k_0}, so it must be trivial (otherwise this involution would normalize $C_G^\circ(Z) = B$).

To summarize, we have that $B = K^+ \rtimes K^*$ and $F(B) = F^\circ(B)$.

Lemma 4.10. *$F(B)^g \cap M = 1$ for every element $g \in G \setminus M$.*

Proof. $F(B)^g \cap M$ is finite by Corollary 4.3. If it is nontrivial, then K must be of characteristic $p > 0$. If y is an element of order p in this intersection, then $C_{M^\circ}^\circ(y) \leqslant (F(B)^g \cap M^\circ)^\circ$ by Corollary 4.3, thus $C_{M^\circ}^\circ(y)$ is trivial by the same corollary. As y normalizes B, Fact 2.30 implies that M° is nilpotent, a contradiction. Thus $F(B)^g \cap M$ is trivial. □

Lemma 4.11. *$M = B$ and $G = B \sqcup F(B)wB$, where w is an involution of $G \setminus B$ which inverts T.*

Proof. If g is in $G \setminus M$, then the map $(f, m) \mapsto fgm$ from $F(B) \times M$ to $F(B)gM$ is an interpretable bijection by the preceding lemma. Its image, of rank $3k_0$, is generic in G, so it must be of degree one, as well as $F(B) \times M$. In particular, M is connected and thus equal to B. By connectedness again, $G = B \sqcup F(B)gB$. □

Proof of Theorem 4.1. To conclude the proof of Theorem 4.1, it remains to show that G is a split Zassenhaus group and to apply Fact 2.45. The group G, acting by left multiplication on the left coset space of B, is a split doubly transitive group; the stabilizer of B and wB is $T = C = B \cap B^w$. This stabilizer T contains an involution. It remains to show that the stabilizer of three points is trivial: if $t \in T$ stabilizes a third point fwB, where f is a nontrivial element of $F(B)$, then $fwB = tfwB$ and $t^f \in T^f \cap B^w \leqslant T^f \cap B \cap B^w \leqslant T^f \cap T = 1$. Theorem 4.1 is proved. □

5. $\mathrm{Pr}_2(G) = 1$ and $C_G^\circ(A)$ a Borel

In this section we assume that G is fixed as in Theorem 1.8, and we adopt all the associated notation from the statement of that theorem. We assume furthermore,

$$\mathrm{Pr}_2(G) = 1 \text{ and } C = C_G^\circ(A) \text{ is a Borel subgroup of } G.$$

We will prove part (1b) of Theorem 1.8. As in the last section, we let i denote the unique involution generating A. Notice that $I(C) = \{i\}$ by Fact 2.12, as $\mathrm{Pr}_2(G) = 1$.

5.1. *Case: $C_G^\circ(A)$ a nonnilpotent Borel subgroup*

We assume here that C is a nonnilpotent Borel subgroup of G and we will show that $C_G(i) = C$ and that $W = N_G(T)/T = 1$ in that case.

Lemma 5.1. *$O(B) = F^\circ(B)$ $(\neq 1)$ for every Borel subgroup B of G.*

Proof. If $F^\circ(B)$ has an involution for some Borel subgroup B of G, then $F^\circ(B)$ contains an infinite Sylow 2-subgroup which is a conjugate of S° by Fact 2.12, as $\mathrm{Pr}_2(G) = 1$. This conjugate of S° is characteristic in B by Fact 2.8, and central in B by Fact 2.10. This shows that $C_G^\circ(S^\circ)$ is a Borel subgroup of G, thus equal to $C_G^\circ(A)$. But $C_G^\circ(S^\circ)$ is nilpotent by Lemma 3.10, a contradiction to our assumption, which shows that $F^\circ(B)$ has no involutions. Thus $F^\circ(B) = O(B)$ by Lemma 2.41. □

Lemma 5.2. *There is a finite subgroup T_0 of odd order of C, disjoint from $F^\circ(C)$, and such that $C \cap C^g$ is $F(C)$-conjugate to a subgroup of T_0 for every $g \in G \setminus N_G(C)$. Furthermore, $C_{F(C)}(t_0)$ is finite for every nontrivial element t_0 belonging to $C \cap C^g$ for some $g \in G \setminus N_G(C)$.*

Proof. Let $g \in G \setminus N_G(C)$ and assume that $T_g := C \cap C^g$ is nontrivial. If T_g has an involution, then it is the unique involution i of C and i^g of C^g, respectively, so $i = i^g$, a contradiction to our assumption that $g \notin N_G(C)$. Thus T_g has no involutions and $T_g^\circ = O(T_g)$ must be trivial by Lemma 2.41 and Proposition 3.11. The family of subgroups T_g of G is thus a uniformly definable family of finite subgroups. It follows that there is a uniform bound n on the order of each T_g, by elimination of infinite quantifiers (cf. [27, Introduction]).

We now claim that T_g intersects trivially $F(C)$, as well as $F(C^g)$. If $t \in T_g^\#$ is in $F(C)$, then $C_G^\circ(t) \leqslant C$ by Lemma 3.12 (as $O(C) = F^\circ(C) \neq 1$ by the preceding lemma) and $C_{C^g}^\circ(t) \leqslant C^g \cap C$ is finite, a contradiction to Fact 2.17. Thus T_g intersects $F(C)$ trivially, and we get in the same way that $T_g \cap F(C^g)$ is trivial.

Let t be a nontrivial element of T_g. If $C_{F(C)}^\circ(t) \neq 1$, then Lemma 5.1 shows that $C_G^\circ(t) \leqslant C$ by Proposition 3.11(ii). Thus $C_{C^g}^\circ(t) \leqslant T_g^\circ = 1$, a contradiction to Fact 2.17. Thus any nontrivial element of T_g has a finite centralizer in $F(C)$.

Let now π be the set of prime numbers dividing $|T_g|$ for some $g \in G \setminus N_G(C)$. The preceding, together with Facts 2.8, 2.10, and 2.9 shows that the Hall π-subgroup of $F^\circ(C)$ is trivial. Let now S_π be a Hall π-subgroup of C. Note that S_π is a direct product of p-tori ($p \in \pi$), disjoint from $F^\circ(C)$. Each T_g is, after conjugacy by an element of $F^\circ(C)$ if necessary (Fact 2.13), in S_π. Let T_0 be the subgroup of S_π generated by all these conjugates of the T_g's. As S_π is divisible abelian and the exponent of the T_g's is uniformly bounded, T_0 is the product of finitely many conjugates of the T_g's, and T_0 satisfies all the required properties. □

The preceding lemma allows us to apply Lemma 3.3 and to get the following corollary.

Corollary 5.3. *$\bigcup_{g \in G} C^g$ is generic in G.*

Corollary 5.4. *If x is an element of $N_G(C) \setminus C$ and is of order n modulo C, for some integer $n > 1$, then the condition $x_1^n \in T_0^{F(C)}$ is satisfied for every x_1 in a definable generic subset X_1 of xC.*

Proof. Let X_1 be the definable subset of xC of elements $x_1 \in xC$ such that $x_1 \in (\langle x\rangle C)^g$ for some $g \in G \setminus N_G(C)$. Then X_1 is generic in xC by Lemma 3.4 and if $x_1 \in X_1$, then $x_1 \in (\langle x\rangle C)^g$ for some $g \in G \setminus N_G(C)$ and $x_1^n \in C \cap C^g \subseteq T_0^{F(C)}$ by Lemma 5.2. □

Corollary 5.5. *$C_G(i)$ is connected (in particular, $S = S^\circ$).*

Proof. Use the preceding corollary, Lemma 3.9, and the fact that C is nonnilpotent. □

Corollary 5.6. *The Weyl group $W = N_G(T)/T$ is trivial.*

Proof. T is a Carter subgroup of C by Lemma 3.10, so it is selfnormalizing in C. But $N_G(T) \leqslant C_G(i) = C$ by the preceding corollary, so $N_G(T) = N_C(T) = T$ and $W = 1$. □

5.2. Case: $C_G^\circ(A)$ a nilpotent Borel subgroup

We assume here that $C = C_G^\circ(A)$ is a nilpotent Borel subgroup of G. As $S^\circ \leqslant C_G^\circ(A)$, Fact 2.8 then shows that $C = C_G^\circ(A) = C_G^\circ(S^\circ) = T$. We will show that the Weyl group $W = N_G(T)/T$ is either trivial or of order 2 (Corollary 5.13 below) and that involutions in G are all conjugate (Lemma 5.14). If $|W| = 2$, then we will show in Corollary 5.15 that $N_G(T)$ splits as $T \rtimes \mathbb{Z}_2$, proving the statement (1b) of Theorem 1.8. We will also obtain a good algebraic description of G in Lemma 5.11 and Corollaries 5.16 and 5.17. After all that, we will finally analyze the geometry of involutions in G.

Lemma 5.7. *$T \cap T^g = 1$ for each $g \in G \setminus N_G(T)$.*

Proof. Assume that $T \cap T^g \neq 1$ for some $g \in G$. Proposition 3.11 then shows that $O(T) = O(T^g) = 1$. But then Lemma 3.2 implies that T is abelian, thus $T, T^g \leqslant C_G^\circ(T \cap T^g)$ and $T = T^g = C_G^\circ(T \cap T^g)$ as T is a Borel subgroup of G. Thus $g \in N_G(T)$. □

Corollary 5.8. *$\bigcup_{g\in G} T^g$ is generic in G.*

Proof. This follows immediately from the preceding lemma. □

Corollary 5.9. *If x is in $N_G(T) \setminus T$ and is of order n modulo T, for some integer $n > 1$, then the elements of the coset xT are generically of order n.*

Proof. It suffices to apply the preceding corollary and Lemma 3.4, and to remark that an element $x \in N_G(T) \setminus T$ of order n modulo T and such that $x \in (\langle x\rangle T)^g$ for some $g \in G \setminus N_G(T)$ satisfies $x^n \in T \cap T^g = 1$. □

Corollary 5.10. *$C_G(S^\circ) = T$.*

Proof. This follows from Corollary 5.9 and Lemma 3.8. □

We now detail the general structure of G. Let $\mathfrak{B}$ be the set of Borel subgroups of G nonconjugate to T and having a nontrivial Sylow 2-subgroup, that is a conjugate of S° by Fact 2.12, as $\mathrm{Pr}_2(G) = 1$.

The same notation $\mathfrak{B}$ will be introduced in Section 6 (before Lemma 6.22) and in Section 7.2 (before Lemma 7.10), but with a different definition in Section 6. Nevertheless, Borel subgroups in each version of $\mathfrak{B}$ will all have analogous properties, as will be seen throughout the paper.

Lemma 5.11. *$\mathfrak{B}$ is nonempty and every Borel subgroup of G nonconjugate to T is in $\mathfrak{B}$. If $B \in \mathfrak{B}$ contains the involution $i \in A^\#$, then $B = F(B) \rtimes C_B(i)$, $F(B) = O(B)$ is inverted by i, and $C_B(i)$ is a connected divisible abelian subgroup of T containing S°. Furthermore*

$$G = \left(\bigcup_{g \in G} N_G(T)^g \right) \cup \left(\bigcup_{B \in \mathfrak{B}} N_G(B) \right).$$

Proof. We first show that G contains no Borel subgroups without involutions. Suppose that B is such a Borel subgroup of G. Then $B = O(B)$ is nilpotent by Lemma 2.41, and Proposition 3.11 shows that two distinct conjugates of B have a trivial intersection. Thus $\bigcup_{g \in G} B^g$ is generic in G by Lemma 3.3, as well as $\bigcup_{g \in G} T^g$. But then there exists an element $b \in B^\#$ which is in a conjugate of T by Fact 2.1. In particular, b centralizes a conjugate of S°. This is a contradiction because $C_G^\circ(b) \leqslant B$ (Lemma 3.12) has no involutions. Thus every Borel subgroup of G has an involution. If every such Borel subgroup is conjugate to T, then G is a simple bad group, and it cannot have involutions by Fact 1.3, a contradiction which ends the proof of our first sentence.

Let now B be a Borel subgroup in $\mathfrak{B}$ containing the involution $i \in A^\#$. If k is an involution in $F(B)$, then $k \in Z(B)$ by Lemma 3.1. But k is in a Sylow 2-subgroup of B which is connected by Fact 2.12, thus in $S^{\circ g}$ for some $g \in G$. So $B, T^g \leqslant C_G^\circ(k)$, and $B = T^g$ by maximality, a contradiction to the definition of $\mathfrak{B}$, which shows that $F(B)$ has no involutions. Notice then that B is in particular nonnilpotent, and that $F^\circ(B) = O(B)$ by Lemma 3.2. If $C_{O(B)}^\circ(i) \neq 1$, then as this is a subgroup of T, Proposition 3.11(ii) implies that $T = B$, a contradiction. Thus $C_{O(B)}^\circ(i)$ is trivial and Fact 2.25 shows that $O(B)$ is inverted by i. As $B/O(B)$ is abelian by Fact 2.15, we conclude that $B = O(B) \rtimes C_B(i)$ with Fact 2.27. It follows then from Fact 2.1 that $C_B(i)$ is connected and contained in $C_G^\circ(i) = T$. As $C_B(i)$ is isomorphic to $B/F(B)$, it is also divisible abelian by Fact 2.15. We now show that $O(B) = F(B)$. If $O(B) < F(B)$, then the finite group $C_B(i) \cap F(B)$ is nontrivial and it contains an element t of prime order p. As $C_B(i)$ is divisible, Fact 2.12 shows that t is in a p-torus of $C_B(i)$; so it is in a p-torus of T and t is central in T by Fact 2.10. Thus $T \leqslant C_G^\circ(t) \leqslant B$ by Lemma 3.12 and $T = B$ by maximality, a contradiction which shows that $O(B) = F(B)$.

It remains to show that $G = (\bigcup_{g \in G} N_G(T)^g) \cup (\bigcup_{B \in \mathfrak{B}} N_G(B))$. If g is any element in G, then g has an infinite centralizer by Corollaries 5.8 and 2.18, that is $C_G^\circ(g) \neq 1$. If $C_G^\circ(g)$ contains an involution, then it contains a nontrivial 2-torus by Fact 2.12, so it contains an involution i^h for some element $h \in G$. Then $g \in N_G(C_G^\circ(i^h)) \leqslant$

$N_G(T)^h$. If $C^\circ_G(g)$ has no involutions, then it is in a unique Borel subgroup B of G by Proposition 3.11(ii), and $g \in N_G(B)$. $\square$

We now look at the structure of the finite group $N_G(T)/T$. In what follows the notation "$\bar{\ }$" denotes the quotients by T.

Lemma 5.12. *$\overline{N_G(T)}$ is trivial or $\overline{N_G(T)} = \overline{w}$ for some involution $w \in G$ which inverts T and $wT = w^T$.*

Proof. Assume that $\overline{N_G(T)}$ is nontrivial. Then $\overline{N_G(T)}$ embeds into a finite subgroup of $\mathrm{Aut}(S^\circ)$ by Corollary 5.10. But finite subgroups of $\mathrm{Aut}(S^\circ) \cong \mathrm{Aut}(\mathbb{Z}_{2^\infty})$ are 2-groups by Fact 2.11, thus $\overline{N_G(T)}$ is a 2-group.

Assume that $w \in N_G(T) \setminus T$ is such that $\overline{w}$ is an involution. Then elements of the coset wT are generically of order 2 by Corollary 5.9, and Fact 2.28 shows that w is an involution which inverts T. If $\overline{w'}$ is another involution of $\overline{N_G(T)}$, then w' is also an involution which inverts T, and $ww' \in C_G(S^\circ) = T$ by the preceding lemma, that is $\overline{w} = \overline{w'}$. This shows that $\overline{N_G(T)}$ is a 2-group with a unique involution.

To show that $\overline{N_G(T)}$ is cyclic of order 2, it remains to show that it cannot contain an element of order 4. Assume that $\bar{x}$ is an element of order 4 in $\overline{N_G(T)}$, for some $x \in N_G(T)$. Let Y be the subgroup of elements $t \in T$ such that $t^4 = 1$. Y is cyclic of order 4, thus $Y = \{1, y, i, y^{-1}\}$ for some generator y such that $y^2 = i$. As x acts by conjugation on y, we have $y^x = y$ or $y^x = y^{-1}$. In any case, x^2 centralizes the generator y of Y. But x^2 has an image of order 2 in $\overline{N_G(T)}$, so it is an involution which inverts T by the preceding remarks and it must in particular invert y. Thus the element y of order 4 is both centralized and inverted by x^2, a contradiction.

This shows that $\overline{N_G(T)} = \langle \overline{w} \rangle$ for some involution $\overline{w}$, and w is an involution of G which inverts T. Furthermore, $wT = w^T$ because T is 2-divisible. $\square$

Corollary 5.13. *$C_G(A)$ is connected or $C_G(A) = T \rtimes \langle w \rangle$ where w is an involution which inverts T and such that $wT = w^T$.*

Lemma 5.14. *All involutions in G are conjugate.*

Proof. Lemma 5.12 shows that $S = S^\circ$ or $S = S^\circ \rtimes \langle w \rangle$, where w inverts S°. In the first case we have nothing to prove because then each involution of G is conjugate to i which is the unique involution of S°. So we assume now that $S = S^\circ \rtimes \langle w \rangle$; Lemma 5.12 also tells us that involutions of the coset wS° are all S°-conjugate as S° is 2-divisible. The conjugacy of Sylow 2-subgroups in G then shows that G possesses at most two conjugacy classes of involutions: i^G and w^G. It suffices thus to show that $w^G = i^G$.

Suppose, in order to get a contradiction, that $w^G \neq i^G$. Notice then that w is never in the connected component of a Borel subgroup of G, by Fact 2.12 and our assumption that $\mathrm{Pr}_2(G) = 1$. Notice also that $C^\circ_G(w) \neq 1$, as otherwise G would be abelian by Fact 2.25. If $C^\circ_G(w)$ has an involution, then it contains a conjugate $S^{\circ g}$ of S° for some $g \in G$, by Fact 2.12 and our assumption that $\mathrm{Pr}_2(G) = 1$. But then $S^{\circ g}\langle w \rangle = S^{\circ g} \times \langle w \rangle$ (as $\mathrm{Pr}_2(G) = 1$ and $w^g \neq i^g$) is in a Sylow 2-subgroup S^h of G, for some $h \in G$.

As $\mathrm{Pr}_2(G) = 1$ again, $S^{\circ g} = (S^h)^\circ$ and w inverts $S^{\circ g}$ by Lemma 5.12, a contradiction which shows that $C_G^\circ(w)$ has no involution. Proposition 3.11(ii) then shows that $C_G^\circ(w)$ is contained in a unique Borel subgroup B of G. If $B = T^g$ for some $g \in G$, then w is not in T^g, so w inverts T^g, a contradiction as $C_G^\circ(w) \leqslant T^g$. Thus B is not conjugate to T and it is in $\mathfrak{B}$ by Lemma 5.11. It is in particular clear from the proof of Lemma 5.11 that B is nonnilpotent.

We now claim that $i^G \subseteq N_G(B)$, which will contradict the simplicity of G. Let $j = i^g$ for some $g \in G$. If $[j, w] = 1$, then j normalizes $C_G^\circ(w)$ and $j \in N_G(B)$. Assume now that $[j, w] \neq 1$. As j and w are not conjugate, there is a third involution z of G which commutes with both j and w by Fact 2.32. Notice that z is not conjugate to j, as otherwise it is equal to j which then commutes with w. Thus $z = w^h$ for some $h \in G$ and $C_G^\circ(z)$ is in particular without involutions. As z normalizes $C_G^\circ(w)$, it also normalizes B, and $z \in N_G(B) \setminus B$. As B is nonnilpotent, Fact 2.25 shows that $C_B^\circ(z) \neq 1$. But $C_B^\circ(z)$ has no involution, as it is conjugate to a subgroup of $C_G^\circ(w)$, and is in a unique Borel subgroup B_1 of G. Now Proposition 3.11(ii) shows that $B = B_1$, and $C_G^\circ(z) \leqslant O(B)$. As j normalizes $C_G^\circ(z)$, it also normalizes B, and we are done. □

Corollary 5.15. *$C_G(A)$ is connected or $C_G(A) = T \rtimes \langle w \rangle$ where w is an involution conjugate to i which inverts T and such that $wT = w^T$.*

We can now refine Lemma 5.11.

Corollary 5.16. $G = \{1\} \sqcup (\bigcup_{g \in G} T^g)^\# \sqcup (\bigcup_{B \in \mathfrak{B}} O(B))^\#$.

Proof. Corollary 5.15 tells us that $\bigcup_{g \in G} N_G(T)^g = \bigcup_{g \in G} T^g$. If a nontrivial element $f \in G$ is in $O(B)$ for some $B \in \mathfrak{B}$, then $C_G^\circ(f) \leqslant B$ by Lemmas 5.11 and 3.12 and $O(B) \leqslant C_G^\circ(f)$. But $C_G^\circ(f)$ has no involution by Lemma 5.11 again, so $C_G^\circ(f) = O(B)$ by Lemma 2.41. In particular, f cannot be in a conjugate of T, so the second union in the statement of the corollary is disjoint.

Let now B be a Borel subgroup in $\mathfrak{B}$ containing the involution i, as in Lemma 5.11. Note that $N_G(B) = N_{N_G(B)}(S^\circ)B$ by the Frattini argument, that is $N_G(B) = C_{N_G(B)}(i)B$. Then Lemma 5.11 shows that $N_G(B) = C_{N_G(B)}(i)O(B)$ and as i inverts $O(B)$, the product is semidirect. If a nontrivial element $f \in O(B)$ centralizes a nontrivial element $c \in C_G(i) = N_G(T)$, then f is in the normalizer of a conjugate T^h of T which contains c, thus in a conjugate of T, a contradiction. This shows, with Fact 2.27, that $cO(B) = (c)^{O(B)}$, so elements of $N_G(B) \setminus O(B)$ are all in conjugates of T. Our statement follows by conjugacy of Sylow 2-subgroups. □

We can also obtain some additional information on Borel subgroups in $\mathfrak{B}$:

Corollary 5.17. *If $B \in \mathfrak{B}$ contains the involution i, then $C_{N_G(B)}(i) < N_G(B)$ is a Frobenius group with $O(B)$ as a Frobenius kernel, and $C_{N_G(B)}(i) \leqslant T$. In particular, i is the unique involution in $C_{N_G(B)}(i)$ and $I(N_G(B)) = iO(B)$. We also have that $\mathrm{rk}(O(B)) \leqslant \mathrm{rk}(T)$.*

Proof. We know from the proof of Corollary 5.16 that $N_G(B) = O(B) \rtimes C_{N_G(B)}(i)$. If z is an involution in $C_{N_G(B)}(i)$ different from i, then there is an involution z' in the elementary abelian 2-group $\langle i, z\rangle$ of order 4 with an infinite centralizer in $O(B)$ by Fact 2.37. Then $B = C_G^\circ(z')$ by Proposition 3.11(ii) as z' is conjugate to i, a contradiction. Thus i is the unique involution of $C_{N_G(B)}(i)$, and Corollary 5.15 shows that $C_{N_G(B)}(i) \leqslant T$. If $f \in O(B)$ and $C_{N_G(B)}(i) \cap C_{N_G(B)}(i)^f$ is nontrivial, then $f \in N_G(T) \cap O(B) = 1$.

It remains to show the last point. Assume that $\mathrm{rk}(T) < \mathrm{rk}(O(B))$. Then $\mathrm{rk}(G/O(B)) < \mathrm{rk}(i^G)$, and by Fact 2.36, there is an involution $w \in G \setminus N_G(B)$ such that $wO(B)$ contains infinitely many involutions. Then $w \in N_G(B)$, a contradiction. □

We now analyze the geometry of involutions of G. Let

$$D = \{(j,k) \in i^G \times i^G \colon [j,k] \neq 1\}.$$

If $C_G(A)$ is connected, then $I(C_G(A)) = I(T) = \{i\}$, so in that case D is simply the set of pairs of distinct involutions of G. Notice that, in any case, D is generic in $i^G \times i^G$, as otherwise there would be an involution j commuting with a generic subset of i^G, which is impossible by Fact 2.36. Let ψ be the definable map

$$\psi : D \longrightarrow G, \quad (j,k) \longmapsto jk.$$

By Corollary 5.16, we have a definable partition of D into definable subsets D_1 and D_2, that is $D = D_1 \sqcup D_2$, where

$$D_1 = \{(j,k) \in D \colon jk \in O(B) \text{ for some } B \in \mathfrak{B}\} \quad \text{and}$$
$$D_2 = \{(j,k) \in D \colon jk \in T^g \text{ for some } g \in G\}.$$

Lemma 5.18. *$D_1 \neq \emptyset$ and $(j,k) \in D$ is in D_1 if and only if $j, k \in N_G(B)$ for some Borel subgroup $B \in \mathfrak{B}$. In particular, $\psi(D_1) = \cup_{B\in\mathfrak{B}} O(B)$.*

Proof. Obvious from Lemma 5.11 and Corollaries 5.16 and 5.17. □

Lemma 5.19. *$D_2 \neq \emptyset$ if and only if $C_G(A)$ is not connected. Then $(j,k) \in D$ is in D_2 if and only if $j, k \in C_G(z)$ for a third involution $z \in i^G$. In particular, $\psi(D_2) = \bigcup_{g\in G} T^g$ when $C_G(A)$ is not connected.*

Proof. Obvious from Lemma 5.11 and Corollaries 5.16 and 5.17. □

Lemma 5.20. *If $C_G(A)$ is not connected, then D_2 is generic in D (and, thus, in $i^G \times i^G$).*

Proof. If $(j,k) \in D_1$, then $jk \in O(B)$ for a unique $B \in \mathfrak{B}$ and we claim that $\psi^{-1}(jk) = \{(jf, jfjk) \colon f \in O(B)\}$. If $(j',k') \in \psi^{-1}(jk)$, then j' and k' invert $j'k' = jk$, so j' and k' normalize $C_G^\circ(jk) = O(B)$ and $j', k' \in N_G(B)$. Thus $(j',k') = (jf, jf')$ where f and f' are in $O(B)$ by Corollary 5.17. Then $(j',k') = (jf, jf(jfjf')) = (jf, jf(j'k')) = (jf, jfjk)$, which proves the claim. In particular, $\mathrm{rk}(\psi^{-1}(jk)) = \mathrm{rk}(O(B))$.

Let $\psi(D_1) = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_s$ be a finite partition of $\psi(D_1)$ into definable sets $U_{s'}$, such that the fibers of ψ are of constant rank s' in each $U_{s'}$, and let s_0 such that $\psi^{-1}(U_{s_0})$ is generic in D_1. Note then that $s_0 = \mathrm{rk}(O(B))$ for some $B \in \mathfrak{B}$. By additivity of the rank, we have

$$\mathrm{rk}\Bigl(\bigcup_{B\in\mathfrak{B}} O(B)\Bigr) = \mathrm{rk}\bigl(\psi(D_1)\bigr) \geqslant \mathrm{rk}\bigl(\psi\bigl(\psi^{-1}(U_{s_0})\bigr)\bigr) = \mathrm{rk}\bigl(\psi^{-1}(U_{s_0})\bigr) - s_0$$
$$= \mathrm{rk}(D_1) - \mathrm{rk}\bigl(O(B)\bigr).$$

We can also compute $\mathrm{rk}(\bigcup_{g\in G} T^g)$ using D_2. If $(j,k) \in D_2$, then $(j',k') \in D_2$ satisfies $\psi(j',k') = jk$ if and only if $(j',k') = (jt, jtjk)$ where t varies over the conjugate of T which contains jk. So the fibers of ψ restricted to D_2 have a constant rank equal to $\mathrm{rk}(T)$. Thus we have $\mathrm{rk}(\bigcup_{g\in G} T^g) = \mathrm{rk}(D_2) - \mathrm{rk}(T)$.

As $\mathrm{rk}(\bigcup_{B\in\mathfrak{B}} O(B)) < \mathrm{rk}(\bigcup_{g\in G} T^g)$ by Corollary 5.16, we get that $\mathrm{rk}(D_1) - \mathrm{rk}(O(B)) < \mathrm{rk}(D_2) - \mathrm{rk}(T)$, that is $\mathrm{rk}(D_1) - \mathrm{rk}(D_2) < \mathrm{rk}(O(B)) - \mathrm{rk}(T)$. But Lemma 5.17 shows that $\mathrm{rk}(O(B)) - \mathrm{rk}(T) \leqslant 0$, so $\mathrm{rk}(D_1) - \mathrm{rk}(D_2) < 0$ and $\mathrm{rk}(D_1) < \mathrm{rk}(D_2)$. □

6. $\mathrm{Pr}_2(G) > 1$ and $C_G^\circ(A)$ not a Borel

In this section we again assume that G is fixed as in Theorem 1.8, and we adopt all the associated notation from the statement of that theorem. We assume furthermore,

$$\mathrm{Pr}_2(G) > 1 \text{ and } C = C_G^\circ(A) \text{ is not a Borel subgroup of } G.$$

Note that $|A| = 2^{\mathrm{Pr}_2(G)} \geqslant 4$ in the case considered. We will prove part (2a) of Theorem 1.8. We will first prove that $\mathrm{Pr}_2(G) = 2$ in this case (Proposition 6.3 below). Then we will show part (2a) of Theorem 1.8 in Lemma 6.4 and Theorem 6.6 below. After that, the main point will be to show that W acts faithfully on A (Proposition 6.17 below), obtaining in particular $|W| = 1, 2, 3$, or 6 (Corollary 6.18 below). The cases $|W| = 2, 6$, and 1 will be removed from the horizon in Section 6.1 (Theorem 6.29), Section 6.2 (Theorem 6.43), and Section 6.3 (Theorem 6.63), respectively. After this lengthy analysis, the remaining statements of part (2) of Theorem 1.8 will be shown in Section 6.4.

Lemma 6.1. *Assume that there are two distinct Borel subgroups B_1 and B_2 of G, each containing a conjugate of S°, and with a nontrivial intersection. Then* $\mathrm{Pr}_2(G) = 2$.

Proof. Fix two distinct Borel subgroups B_1 and B_2 of G so that $X := B_1 \cap B_2$ is nontrivial and of maximal rank.

We first claim

$$X \text{ is infinite.}$$

Suppose the contrary, and pick an element x of prime order p in X. We will eventually apply Corollary 2.20 to x in both B_1 and B_2.

We show that $F^\circ(B_1)$ has no nontrivial p-unipotent subgroup. Suppose on the contrary that the maximal (normal) p-unipotent subgroup U_p of $F^\circ(B_1)$ (Corollary 2.16) is nontrivial. Then $C^\circ_{U_p}(x)$ is nontrivial (Fact 2.9) and if B_3 is a Borel subgroup of G containing $C^\circ_G(x)$, then $B_3 = B_1$ by Proposition 3.11(ii). We then get that $C^\circ_{B_2}(x) \leqslant B_2 \cap B_1$ is finite, a contradiction to Fact 2.17. Thus p-unipotent subgroups of $F^\circ(B_1)$ and, similarly, $F^\circ(B_2)$ are trivial and we can apply Corollary 2.20 to see that $C^\circ_G(x)$ contains a Sylow° 2-subgroup of both B_1 and B_2. If B_3 is now a Borel subgroup of G containing $C^\circ_G(x)$, then we get that $B_1 = B_3 = B_2$ by the maximality of $\mathrm{rk}(X)$. This final contradiction proves that X is infinite.

If $O(X) \neq 1$, then $B_1 = B_2$ by Proposition 3.11(ii). Thus $O(X) = 1$ and X° is abelian divisible by Lemma 3.2. Let S_X be the (nontrivial) maximal 2-torus of X, and let S°_1 (respectively S°_2) be a Sylow° 2-subgroup of B_1 (respectively B_2) such that $S_X \leqslant S^\circ_1$ (respectively $S_X \leqslant S^\circ_2$). If S_X is not a Sylow° 2-subgroup of G, then we can consider a Borel subgroup B_3 of G containing $N^\circ_G(d(S_X))$; it contains X, as well as S°_1 $(> S_X)$ and S°_2 $(> S_X)$, thus the maximality of $\mathrm{rk}(X)$ implies $B_1 = B_3 = B_2$, a contradiction which shows that $S^\circ_1 = S_X = S^\circ_2$.

We now claim that $O(B_1) \neq 1$ and $O(B_2) \neq 1$. If these are both trivial, then B_1 and B_2 are abelian by Lemma 3.2, thus included in $C^\circ_G(X)$ and equal, a contradiction. We may assume therefore that $O(B_1) \neq 1$. If $O(B_2) = 1$, then by Fact 2.37 one can find an involution $i \in S_X$ such that $C^\circ_{O(B_1)}(i) \neq 1$; but $C^\circ_{O(B_1)}(i) \leqslant C^\circ_G(i) = B_2$ as B_2 is abelian by Lemma 3.2, a contradiction to Lemma 2.41, as $O(B_2) = 1$.

Proposition 3.11(ii) shows that any involution in S_X cannot have an infinite centralizer both in $O(B_1)$ and $O(B_2)$. Thus any such involution inverts $O(B_1)$ or $O(B_2)$ by Fact 2.25. We can now conclude that the Prüfer 2-rank of S_X is two. Suppose on the contrary that S_X contains an elementary abelian 2-subgroup of order eight, that is seven distinct involutions. This is then the union of two sets of involutions, those which invert $O(B_1)$ and those which invert $O(B_2)$, and neither set contains three linearly dependent elements; but this is impossible. □

Corollary 6.2. *Suppose* $\langle C^\circ_G(i)\colon i \in A^\#\rangle = G$. *Then* $\mathrm{Pr}_2(G) = 2$.

Proof. The hypothesis implies that there are involutions $i, j \in A^\#$ such that $C^\circ(i)$ and $C^\circ(j)$ are contained in distinct Borel subgroups, so the preceding lemma applies. □

Proposition 6.3. $\langle C^\circ_G(i)\colon i \in A^\#\rangle = G$. *In particular,* $\mathrm{Pr}_2(G) = 2$ *by Corollary 6.2.*

Proof. Suppose $\langle C^\circ_G(i)\colon i \in A^\#\rangle < G$. Let B be a Borel subgroup of G containing $\langle C^\circ_G(i)\colon i \in A^\#\rangle$. As $C < B$, there is an involution $i \in A^\#$ such that $C^\circ_G(i) < B$. In particular, B is not abelian, and thus $O(B) \neq 1$ by Lemma 3.2.

Let $T(w)$ denote the set $\{ww_1\colon w_1 \in i^G \cap wB\}$ for each $w \in i^G \setminus N_G(B)$. Note that $\mathrm{rk}(i^G \setminus N_G(B)) = \mathrm{rk}(i^G) = \mathrm{rk}(G/C^\circ_G(i))$ by Fact 2.36. As $C^\circ_G(i) < B$, we have $\mathrm{rk}(G/B) < \mathrm{rk}(G/C^\circ_G(i)) = \mathrm{rk}(i^G \setminus N_G(B))$. Thus there is a coset of B disjoint from $N_G(B)$ containing infinitely many involutions of i^G, and if w is such an involution, then $T(w)$ is infinite. As $w \notin N_G(B)$, $F(B) \cap F(B)^w$ is trivial by Proposition 3.11 and one sees as in the proof of Lemma 4.5 that $d(T(w))' \leqslant F(B) \cap F(B)^w = 1$. Thus $d(T(w))$ is an infinite abelian subgroup of B inverted by w, and it is necessarily disjoint from $F(B)$. Notice that, conjugating by an element of B if necessary, we may assume without loss of generality that the Sylow 2-subgroup of $d(T(w))$ is contained in S°. If $d(T(w))$ contains a four-subgroup of A, then there is by Fact 2.37 an involution $k \in A$ such that $C^\circ_{O(B)}(k) \neq 1$, and then $1 \neq O(C^\circ_G(k)) \leqslant O(B)$ by Lemma 2.41 and Proposition 3.11(ii) and as $k^w = k$, $w \in N_G(B)$ by Proposition 3.11(ii), a contradiction. Thus $d(T(w))$ has at most one involution and has Prüfer 2-rank at most 1. On the other hand, $O(d(T(w))) \leqslant O(B) \cap O(B)^w = 1$ by Lemma 2.41. Thus $\mathrm{Pr}_2(d(T(w))^\circ) = 1$.

Let j be the unique involution of $d(T(w))^\circ$. As $w \in C_G(j)$, w acts by conjugacy on $C^\circ_G(j) \leqslant B$. If $O(C^\circ_G(j)) \neq 1$, then, as this is normalized by w, Proposition 3.11(ii) would show that $B = B^w$, a contradiction. Thus $O(C^\circ_G(j)) = 1$ and $C^\circ_G(j)$ is abelian by Fact 3.2. But then S° is the unique Sylow 2-subgroup of $C^\circ_G(j)$, and w acts by conjugacy on $I(S^\circ) = A^\#$. As above there is $k \in A^\#$ such that $1 \neq O(C^\circ_G(k)) \leqslant O(B)$. As $k^w \in A$, we have $1 \neq O(C^\circ_G(k^w)) \leqslant O(B)^w \cap O(B)$ by the definition of B. Thus $B = B^w$ by Proposition 3.11(ii), a final contradiction. □

Now let i_1, i_2, and $i_3 = i_1 i_2$ be the three involutions of $A^\#$.

Lemma 6.4. *$O(C) = 1$ and $T = C$ is abelian divisible.*

Proof. If $O(C) \neq 1$, and B_{i_1}, B_{i_2}, and B_{i_3} are Borel subgroups of G containing $C^\circ_G(i_1)$, $C^\circ_G(i_2)$, and $C^\circ_G(i_3)$, respectively, then Proposition 3.11(ii) implies that $B_{i_1} = B_{i_2} = B_{i_3}$. Thus $\langle C^\circ_G(i)\colon i \in A^\# \rangle < G$, a contradiction. So $O(C) = 1$, and C is abelian divisible by Lemma 3.2. As $S^\circ \leqslant C^\circ_G(S^\circ) = T \leqslant C$, $T = C$. □

Lemma 6.5. *If a Borel subgroup B of G contains T, then $O(B)$ is nontrivial and is inverted by an involution of A. Furthermore $B = O(B) \rtimes T$.*

Proof. If $O(B) = 1$, then B is abelian by Lemma 3.2, so $B = C$ is a Borel subgroup of G, a contradiction to our assumption. Thus $O(B) \neq 1$. If $C^\circ_{O(B)}(k) \neq 1$ for each involution $k \in I(A)$, then $C^\circ_G(k) \leqslant B$ for each $k \in I(A)$ by Proposition 3.11(ii), a contradiction. Thus there is an involution $k_0 \in I(A)$ such that $C^\circ_{O(B)}(k_0) = 1$, and k_0 inverts $O(B)$ by Fact 2.25.

It remains to show that $B = O(B) \rtimes T$. As T is nilpotent and of finite index in its normalizer by Lemma 3.10, it is a Carter subgroup of B by Fact 2.19. As $O(B/O(B)) = 1$, $B/O(B)$ is abelian by Lemma 3.2 and as $O(B)$ is also abelian by the preceding, B is solvable of class 2. Thus $B = B_{\mathcal{N}} \rtimes T$ by Fact 2.22 and it suffices now to show that $B_{\mathcal{N}} = O(B)$. But $B_{\mathcal{N}} \leqslant O(B)$ as $B/O(B)$ is abelian and thus $O(B) = B_{\mathcal{N}} \rtimes (T \cap O(B))$. As $O(B)$ is connected, this shows that $(T \cap O(B))$ is connected. Then Lemma 6.4 shows that $(T \cap O(B)) \leqslant O(T) = 1$, and $O(B) = B_{\mathcal{N}}$. □

Theorem 6.6. *For each $k \in I(A)$, $C^\circ_G(k) = O(C^\circ_G(k)) \rtimes T$ is a Borel subgroup of G, where $O(C^\circ_G(k))$ is nontrivial and inverted by the two involutions in $I(A) \setminus \{k\}$.*

The proof will depend on the three following lemmas.

Lemma 6.7. *There is an involution $k \in I(A)$ such that $C_G^\circ(k)$ is a Borel subgroup of G and $O(C_G^\circ(k)) \neq 1$.*

Proof. By Lemma 6.5, it suffices to show that there is a Borel subgroup B of G containing T and such that an involution $k \in S^\circ$ centralizes $O(B)$. Assume toward a contradiction that $C_{O(B)}^\circ(k) < O(B)$ for each Borel subgroup B of G containing T and each $k \in I(A)$, and fix such a Borel subgroup B.

By Lemma 6.5, there is an involution $k_0 \in I(A)$ which inverts $O(B)$. As $O(B)$ is in particular abelian, we have $O(B) = C_{O(B)}^\circ(k_1) \times C_{O(B)}^\circ(k_2)$ by Fact 2.26, where k_1 and $k_2 = k_0k_1$ are the two other involutions in $I(A)$. Our assumption shows that the two factors in the product are proper in $O(B)$ and nontrivial. Thus $C_G^\circ(k_1)$ and $C_G^\circ(k_2)$ are both contained in B by Proposition 3.11(ii). Thus $C_G^\circ(k_0) \nleqslant B$ by Proposition 6.3. Let B_0 be a Borel subgroup of G containing $C_G^\circ(k_0)$. Note that $O(B_0) \neq 1$ by Lemma 6.5. As $B_0 \neq B$, we have $C_{O(B_0)}^\circ(k_1) = C_{O(B_0)}^\circ(k_2) = 1$ by Proposition 3.11(ii). But k_1 and k_2 are in B_0, so they normalize $O(B_0)$ and they invert $O(B_0)$ by Fact 2.25. Thus $k_0 = k_1k_2$ centralizes $O(B_0)$, as well as $B_0 = O(B_0) \rtimes T$ (Lemma 6.5). Now k_0 is central in a Borel subgroup and our claim is proved. $\square$

To prove Theorem 6.6, we can now assume, in view of the preceding lemma, that

$$C_G^\circ(i_1) \text{ is a Borel subgroup of } G. \tag{$*$}$$

Let B_1 denote this Borel subgroup. There is an involution $k \in I(A)$ such that $C_{O(B_1)}^\circ(k) = 1$, as otherwise $\langle C_G^\circ(k)\colon k \in I(A)\rangle \leqslant B_1$ by Proposition 3.11(ii). Then this involution k inverts $O(B_1)$ by Fact 2.25, as does i_1k. Thus i_2 and i_3 invert $O(B_1)$.

If $O(C_G^\circ(i_2)) = 1$ and $O(C_G^\circ(i_3)) = 1$, then $C_G^\circ(i_2)$ and $C_G^\circ(i_3)$ are abelian by Lemma 3.2, thus equal to T and contained in B_1, a contradiction. Thus for the proof of Theorem 6.6, we may suppose that

$$O\bigl(C_G^\circ(i_2)\bigr) \neq 1.$$

By Proposition 3.11(ii), $C_G^\circ(i_2)$ is contained in a unique Borel subgroup B_2 of G. Note that if $C_{O(B_2)}^\circ(i_1)$ is nontrivial, then $B_1 = B_2$ by Proposition 3.11(ii), and $O(C_G^\circ(i_2)) \leqslant O(B_1)$ by Lemma 2.41, a contradiction as i_2 inverts $O(B_1)$. Thus, as i_1 normalizes B_2, i_1 inverts $O(B_2)$ by Fact 2.25.

Lemma 6.8. $C_G^\circ(i_2) = B_2$.

Proof. Suppose that $C_G^\circ(i_2) < B_2$. Then $C_{O(B_2)}^\circ(i_2) < O(B_2)$ by Lemma 6.5. As $O(B_2)$ is inverted by i_1, it is abelian and Fact 2.26 implies that

$$O(B_2) = C_{O(B_2)}^\circ(i_2) \times C_{O(B_2)}^\circ(i_3),$$

where both factors in the product are nontrivial. Then Proposition 3.11(ii) shows that $C_{O(B_2)}^\circ(i_3)$ is contained in a unique Borel subgroup B_3, and that $B_2 = B_3$.

As $C_G^\circ(i_2) < B_2$, we have $\mathrm{rk}(G/B_2) < \mathrm{rk}(i_2^G)$ and there is a coset wB_2 of B_2, for some $w \in i_2^G \setminus N_G(B_2)$, containing infinitely many involutions of i_2^G. Let then $T(w) = \{ww'\colon w' \in i_2^G \cap wB_2\}$. We can see as in the proof of Proposition 6.3 that $d(T(w))$ is an infinite abelian subgroup of B_2 disjoint from $F(B_2)$. Furthermore $O(d(T(w))) = 1$ and $d(T(w))$ contains a nontrivial 2-torus T_1 by Fact 2.12, which is inverted by w. Now $T_1 \rtimes \langle w \rangle$ is in a Sylow 2-subgroup S_1 of G, and $w \in S_1 \setminus S_1^\circ$ (as connected components of Sylow 2-subgroups of G are abelian). Thus there is an involution $w' \in S \setminus S^\circ$ which is conjugate to i_2 and which inverts a nontrivial 2-torus $T_{w'}$ of S°.

We claim now that $w' \in N_G(B_2) \setminus B_2$. As we assume that $C_G^\circ(i_2)$ is not a Borel subgroup of G, i_2 is not conjugate to i_1 and $i_2^{w'}$ is equal to i_2 or to i_3. But $O(C_G^\circ(i_2))$ and $O(C_G^\circ(i_3))$ are both contained in $B_2 = B_3$ by Proposition 3.11(ii). With Proposition 3.11(ii) again, we find $w' \in N_G(B_2)$ in each case. Furthermore $w' \notin B_2$ as Sylow 2-subgroups of B_2 are abelian by Fact 2.12.

Now w' normalizes $O(B_2)$ and in fact w' inverts $O(B_2)$: else $C_{O(B_2)}^\circ(w') \neq 1$ by Fact 2.25, which shows that $C_G^\circ(w') \leqslant B_2$ by Proposition 3.11(ii), and as $w' \in C_G^\circ(w')$, this is a contradiction.

Now as w' also inverts $d(T_{w'})$, it inverts $O(B_2) \rtimes d(T_{w'})^\circ$ (Fact 2.25) which is therefore abelian, and is normal in B_2 by Lemma 6.5. In particular, $d(T_{w'})^\circ \leqslant F^\circ(B_2)$ and $F^\circ(B_2)$ contains an involution which is central in B_2 by Lemma 3.1. As i_1 inverts $O(B_2)$, this involution is either i_2 or i_3, a final contradiction. □

Lemma 6.9. $T < C_G^\circ(i_3)$.

Proof. Assume that $T = C_G^\circ(i_3)$. Then $C_G^\circ(i_3)$ is a proper subgroup of B_1 by Lemma 6.5, and one can see as in the preceding lemma that there is an involution $w \in i_3^G \setminus N_G(B_1)$ such that $T(w) = \{ww'\colon w' \in i_3^G \cap wB_1\}$ is infinite and $d(T(w))$ is an abelian subgroup of B_1 inverted by w and containing a nontrivial 2-torus. As before, we can find an involution $w' \in S \setminus S^\circ$ which is conjugate to i_3 and which inverts a nontrivial 2-torus $T_{w'}$ in S°.

We claim that $\mathrm{Pr}_2(C_{d(S^\circ)}^\circ(w')) = 1$. First we show that $C_{d(S^\circ)}^\circ(w') \neq 1$: otherwise w' inverts $d(S^\circ)$ by Fact 2.25, so w' centralizes i_1 and i_2, and it normalizes $O(B_1)$ and $O(B_2)$. As w' is conjugate to i_3, we have $O(C_G^\circ(w')) = 1$ by Lemma 6.4, thus $C_{O(B_1)}^\circ(w') = C_{O(B_2)}^\circ(w') = 1$ by Lemma 2.41 and w' inverts $O(B_1)$ and $O(B_2)$ by Fact 2.25. As w' also inverts $d(S^\circ)$, it inverts $O(B_1) \rtimes d(S^\circ)$ (Fact 2.25) which is therefore abelian and contained in $F(B_1)$ by Lemma 6.4 and Lemma 6.5. This shows that $S^\circ \leqslant F(B_1)$ is central in B_1 by Lemma 3.1, and $i_3 \in Z(B_1)$, a contradiction. Thus $C_{d(S^\circ)}^\circ(w') \neq 1$ and $O(C_{d(S^\circ)}^\circ(w')) \leqslant O(C_G^\circ(w')) = 1$. Thus $C_{d(S^\circ)}^\circ(w')$ contains a nontrivial 2-torus by Fact 2.12. If the Prüfer 2-rank of $C_{d(S^\circ)}^\circ(w')$ is two, then the 2-torus involved is S°, a contradiction as w' inverts the nontrivial 2-torus $T_{w'} \leqslant S^\circ$.

We now show that w' centralizes A. Let T_1 be the 2-torus of Prüfer 2-rank one of $C_{d(S^\circ)}^\circ(w')$. We have $T_1 \leqslant C_G^\circ(w')$ and as w' is conjugate to i_3, w' is the only involution of $C_G^\circ(w')$ whose centralizer is not a Borel subgroup of G. Thus $I(T_1) \neq \{i_3\}$, as otherwise $w' = i_3 \in S^\circ$, a contradiction as $w' \in S \setminus S^\circ$. Therefore $I(T_1) = \{i_1\}$ or $I(T_1) = \{i_2\}$ and as i_3 is conjugate to neither i_1 nor i_2, w' centralizes A.

So w' normalizes $O(B_1)$ and $O(B_2)$. As $O(C_G^\circ(w')) = 1$, Fact 2.25 and Lemma 2.41 show that w' inverts $O(B_1)$ and $O(B_2)$. As w' also inverts $d(T_{w'}) \leqslant T$, w' inverts $O(B_1) \rtimes d(T_{w'})$ and $O(B_2) \rtimes d(T_{w'})$ by Fact 2.25. These subgroups are therefore abelian and contained in $F(B_1)$ and $F(B_2)$, respectively, by Lemma 6.5. Thus $d(T_{w'}) \leqslant F(B_1) \cap F(B_2)$ and as $O(B_1)$ and $O(B_2)$ are both nontrivial, Proposition 3.11 shows that $B_1 = B_2$, a contradiction. □

Proof of Theorem 6.6. The statement of Theorem 6.6 is proved for i_1 and i_2 by Lemmas 6.5, 6.7, and 6.8, and it remains only to prove that $C_G^\circ(i_3)$ is a Borel subgroup of G.

Note that $O(C_G^\circ(i_3)) \neq 1$, as otherwise $C_G^\circ(i_3) = T$ by Lemma 3.2, which contradicts Lemma 6.9. Hence Lemma 6.8 applies to i_3 in place of i_2. □

This proves the statement of part (2a) of Theorem 1.8. We will now analyze the Weyl group $W = N_G(T)/T$. Note that $T \leqslant C_G(A) \leqslant N_G(A) = N_G(T)$ as $T = C_G^\circ(A)$ by Lemma 6.4. Note also that $N_G(A)/C_G(A)$ acts faithfully on A, so embeds into S_3, and $|N_G(A)/C_G(A)| = 1$, 2, 3, or 6. Our target is now to show that $T = C_G(A)$, i.e., that $W = N_G(A)/C_G(A)$, which will be obtained in Proposition 6.17 below.

We set $B_l = C_G^\circ(i_l)$ for $l = 1, 2, 3$; these are three distinct Borel subgroups.

Lemma 6.10. *There is a definable nongeneric subset X of T such that $T \cap T^g \subseteq X$ for each $g \in G \setminus N_G(T)$.*

Proof. For each $g \in N_G(T) \setminus T$, let $T_g = T \cap T^g$. If $T_g \neq 1$, then $\langle T, T^g \rangle \leqslant C_G^\circ(T_g)$. Note that $O(C_G^\circ(T_g))$ is nontrivial, as otherwise $C_G^\circ(T_g)$ is abelian by Lemma 3.2 and then $S^\circ = S^{\circ g}$ and $g \in N_G(T)$. As $A \leqslant C_G^\circ(T_g)$, there is by Fact 2.37 an involution $k \in A^\#$ with an infinite centralizer in $O(C_G^\circ(T_g))$. Now Theorem 6.6 and Proposition 3.11(ii) show that $C_G^\circ(T_g) \leqslant C_G^\circ(k)$. So T and T^g are two Carter subgroups of $C_G^\circ(k)$ and one can assume that $g \in C_G^\circ(k) \setminus T$ by Fact 2.19. We have shown that

$$T_g \subseteq \bigcup_{l=1}^{3} \Bigg[\bigcup_{h \in B_l \setminus T} \big(T \cap T^h\big) \Bigg].$$

It suffices now to apply Lemma 3.5. □

Corollary 6.11. $\bigcup_{g \in G} T^g$ *is generic in G.*

Proof. We apply the preceding lemma and Lemma 3.3. □

Lemma 6.12. *If w is an involution in $S \setminus S^\circ$, then $w \notin C_G^\circ(w)$. In particular, $(S \setminus S^\circ) \cap I(S^\circ)^G = \emptyset$.*

Proof. Suppose toward a contradiction that $w \in I(S)$, $w \notin S^\circ$, but $w \in C_G^\circ(w)$. Then w centralizes an involution $i_l \in A^\#$, for some $l = 1$, 2, or 3. We will show that $w \in B_l$, which gives a contradiction: S° is a Sylow 2-subgroup of B_l by Fact 2.12, and w normalizes S°, so $w \in S^\circ$.

If w does not invert $O(B_l)$ then w has an infinite centralizer in $O(B_l)$ by Fact 2.25, and $C_G^\circ(w) \leqslant B_l$ by Proposition 3.11(ii), so $w \in B_l$. So suppose

$$w \text{ inverts } O(B_l).$$

Then w does not invert S°, as otherwise w would invert $O(B_l) \rtimes d(S^\circ)$ by Fact 2.25. As $O(d(S^\circ)) = 1$ by Lemma 6.4, it follows that $C_{S^\circ}^\circ(w)$ is nontrivial. We may suppose that $i_l \in C_{S^\circ}^\circ(w)$.

Let $P \supseteq C_{S^\circ}^\circ(w)$ be a Sylow 2-subgroup of $C_G^\circ(w)$ containing $\langle w, C_{S^\circ}^\circ(w)\rangle$. Then $w \in P \leqslant B_l$, as claimed. □

Lemma 6.13. *Let $l \in \{1, 2, 3\}$ and assume that x is an element in $N_G(T) \setminus T$. Then the definable set $X_l = \{y \in xT\colon\ C_{O(B_l)}^\circ(y) = 1\}$ is generic in xT.*

Proof. As xT has Morley degree one, we may assume toward a contradiction that $Y_l = xT \setminus X_l$ is generic in xT. It follows that $P_l = Y_l O(B_l)$ is also generic in $x(T \ltimes O(B_l)) = xB_l$. As $O(B_l)$ is abelian, any element of P_l has an infinite centralizer in $O(B_l)$.

If $g_1, g_2 \in G$ are such that $g_1 N_G(B_l) \neq g_2 N_G(B_l)$, then $P_l^{g_1} \cap P_l^{g_2}$ is empty: otherwise an element x in this intersection would have an infinite centralizer in both $O(B_l)^{g_1}$ and $O(B_l)^{g_2}$, and thus $B_l^{g_1} = B_l^{g_2}$ by Proposition 3.11(ii). It follows that $\mathrm{rk}(P_l^G) \geqslant \mathrm{rk}(P_l) + \mathrm{rk}(G) - \mathrm{rk}(N_G(B_l)) = \mathrm{rk}(G)$. Now Corollary 6.11 together with Fact 2.1 shows that there is an element $y \in P_l \cap T^g$ for some $g \in G$. Then $y \in T^g \leqslant C_G^\circ(y) \leqslant B_l$ (Proposition 3.11(ii)). Thus $xB_l \subseteq B_l$ and $x \in N_{B_l}(T) = T$, a contradiction. □

Corollary 6.14. *Assume that x is an element in $N_G(T) \setminus T$. Then the definable set*

$$X = \bigl\{y \in xT\colon\ C_{O(B_1)}^\circ(y) = C_{O(B_2)}^\circ(y) = C_{O(B_3)}^\circ(y) = 1\bigr\}$$

is generic in xT.

Proof. This follows from the preceding lemma and the fact that xT has Morley degree one. □

Lemma 6.15. *If $C_G(A) \cap C_G(A)^g$ is nontrivial, with $g \in G$, then $A \cap A^g$ is also nontrivial.*

Proof. Suppose that x is a nontrivial element of $C_G(A) \cap C_G(A)^g$. Note that $C_G^\circ(x) \neq 1$ by Corollary 2.18 and the genericity of $\bigcup_{g\in G} T^g$, and that $A, A^g \leqslant C_G(x)$. If the maximal 2-torus T_1 of $F(C_G^\circ(x))$, which is characteristic in $C_G(x)$, is nontrivial, then it has Prüfer 2-rank 1 or 2. If $\mathrm{Pr}_2(T_1) = 2$, then $A = \Omega_1(T_1) = A^g$, by Lemma 6.12. If $\mathrm{Pr}_2(T_1) = 1$, then A and A^g have in common the unique involution of T_1, by Lemma 6.12 again. So we can assume that $F^\circ(C_G^\circ(x))$ has no involution by Fact 2.12, and by Fact 2.37 there are involutions $k \in A$ and $k' \in A^g$ such that $C_{F(C_G^\circ(x))}^\circ(k)$ and $C_{F(C_G^\circ(x))}^\circ(k')$ are both nontrivial. Now $C_G^\circ(k) = C_G^\circ(k')$ by Theorem 6.6 and Proposition 3.11(ii), and Theorem 6.6 shows that $k = k'$. □

Lemma 6.16. *Assume $x \in C_G(A) \setminus T$ and let X be the definable generic subset of xT as in Corollary* 6.14*. Fix $y \in X$. Then there is a finite subset F_y of $\bigcup_{l=1}^{3} O(B_l)$, depending only on y, with the property that for every $y_1 \in X$ and $g \in G$, $y = y_1^g$ implies that $T^g = T^f$ for some $f \in F_y$.*

Proof. We show that the set $F_y = \bigcup_{l=1}^{3} C_{y,l}$, where

$$C_{y,l} = \bigl\{ f \in O(B_l)\colon\ f^2 \in C_{O(B_l)}(y) \bigr\},$$

has the required properties. First, remark that F_y is finite: for each l, $C_{O(B_l)}(y)$ is finite (by definition of X, as $y \in X$), and as any element of the abelian group $O(B_l)$ (Theorem 6.6) has at most one square root, $C_{y,l}$ is also a finite subgroup of $O(B_l)$.

Suppose now that $y_1 \in X$ and $g \in G$ satisfy $y = y_1^g$. Then $y \in C_G(A) \cap C_G(A)^g$ and $A \cap A^g$ is nontrivial by Lemma 6.15. If $A = A^g$, then $T^g = T = T^1$, and $1 \in F_y$. Assume now $A \neq A^g$. Then $A \cap A^g = \langle i_l \rangle$ for some $l \in \{1, 2, 3\}$, and T and T^g are two Carter subgroups of $C_G^\circ(i_l) = B_l$, i.e., $T^g = T^f$ for some $f \in O(B_l)$ by Theorem 6.6. It suffices now to show that such an f necessarily belongs to $C_{y,l}$. For, notice that $C_G(A)$ is characteristic in $N_G(T)$, thus $C_G(A)^g = C_G(A)^f$ and $y = y_1^g \in C_G(A)^f$ centralizes A and A^f. In particular, y centralizes $i_{l'}$ and $i_{l'}^f$ where $l' \in \{1, 2, 3\} \setminus \{l\}$; but $i_{l'}^f = i_{l'} f^2$ by Theorem 6.6, thus y centralizes f^2 and $f \in C_{y,l}$. □

Proposition 6.17. $C_G(A) = T$.

Proof. Assume toward a contradiction that x is an element in $C_G(A) \setminus T$ and let X be the definable generic subset of xT as in Corollary 6.14. Consider the definable map

$$\Psi : X \times G \longrightarrow G, \quad (y, g) \longmapsto y^g.$$

For $y \in X$ and $g \in G$, we claim that

$$\Psi^{-1}\bigl(y^g\bigr) \subseteq \bigcup_{f \in F_y} \bigl\{ \bigl(y^{f^{-1}t^{-1}}, tfg\bigr)\colon\ t \in N_G(T) \bigr\}, \qquad (*)$$

where F_y is the finite subset of $\bigcup_{l=1}^{3} O(B_l)$ depending only on y as in Lemma 6.16. So let (y_1, g_1) be in the fiber of y^g. Then $y = y_1^{g_1 g^{-1}}$ and $T^{g_1 g^{-1}} = T^f$ for some $f \in F_y$ by Lemma 6.16. Then the element $t = g_1 g^{-1} f^{-1}$ is in $N_G(T)$ and $g_1 = tfg$, $y_1 = y^{g g_1^{-1}} = y^{f^{-1}t^{-1}}$, which proves inclusion $(*)$.

Clearly, each member in the finite union of the right side of inclusion $(*)$ has a rank equal to $\mathrm{rk}(N_G(T)) = \mathrm{rk}(T)$, thus $\mathrm{rk}(\Psi^{-1}(y^g)) \leqslant \mathrm{rk}(T)$. We have shown that the fibers of elements of the image of Ψ have a rank uniformly bounded by $\mathrm{rk}(T)$. It follows that $\mathrm{rk}(X \times G) \leqslant \mathrm{rk}(\Psi(X \times G)) + \mathrm{rk}(T)$, i.e.,

$$\mathrm{rk}\bigl(X^G\bigr) = \mathrm{rk}\bigl(\Psi(X \times G)\bigr) \geqslant \mathrm{rk}(X \times G) - \mathrm{rk}(T) = \mathrm{rk}(X) + \mathrm{rk}(G) - \mathrm{rk}(T) = \mathrm{rk}(G)$$

as $\mathrm{rk}(X) = \mathrm{rk}(xT) = \mathrm{rk}(T)$. Thus X^G is generic in G.

Now, by Fact 2.1 and Corollary 6.11, there exists $x \in X$ and $g \in G$ such that $x \in T^g$. But then $T^g \leqslant C_G^\circ(i_l)$ for some $l \in \{1, 2, 3\}$ (Lemma 6.15). As $x \in T^g \leqslant B_l$, $x \in N_{B_l}(T) = T$, a contradiction. □

Corollary 6.18. *$W = N_G(T)/T = N_G(A)/C_G(A)$ acts faithfully on A and $|W| = 1, 2, 3,$ or* 6.

Lemma 6.19. *If $x \in N_G(T) \setminus T$ is of order* 2 *modulo T, then $xT = wT$ for some involution $w \in I(G) \setminus I(S^\circ)^G$. For such a w, the subgroup T^- of elements of T inverted by w is connected and $I(wT) = w^T$. Furthermore, if w centralizes the involution i_l of T, then $w^G \cap C_G(i_l) = w^{B_l}$.*

Proof. First note that we can apply Lemma 2.31 to S° by Corollary 6.18. By Fact 2.5, xT contains a 2-element y. Now $y^2 \in C_{S^\circ}(y)$, thus $y^2 = s^2$ for some $s \in C_{S^\circ}(y)$ by Lemma 2.31. Then $w = ys^{-1} = (ys^{-1})^{-1} \in xT \cap I(G) \setminus I(S^\circ)^G$ by Lemma 6.12.

By Lemma 2.31, the Sylow 2-subgroup of T^- is connected and thus in $(T^-)^\circ$. Then $T^-/(T^-)^\circ$ has odd order by Fact 2.5. But if $t \in T^-$, then $t^2 = [w, t] \in [w, T] \leqslant (T^-)^\circ$ (Fact 2.2); thus T^- is connected. In particular, it is 2-divisible and $I(wT) = w^T$.

Assume now that w centralizes $i_l \in I(T)$. Note that $C_G(i_l) = B_l \rtimes \langle w \rangle$ by the Frattini argument. If $w' \in w^G \cap C_G(i_l)$, then $w' \in N_G(S^\circ)^f = N_G(T)^f$ for some $f \in O(B_l)$, and $w' \in ((N_G(T) \setminus T) \cap C_G(i_l))^f$, thus $w' \in I(wT)^f = (w^T)^f \subseteq w^{B_l}$. □

Corollary 6.20. *The structure of S and the conjugacy classes of involutions are the following*:

(a) *If $|W| = 1$ or 3, then $S = S^\circ$ and*
  (i) *if $|W| = 1$, then $I(G) = i_1^G \sqcup i_2^G \sqcup i_3^G$;*
  (ii) *if $|W| = 3$, then $I(G) = i_1^G$.*
(b) *If $|W| = 2$ or 6, then there is an involution $w \in N_G(A) \setminus C_G(A)$ and $S = S^\circ \rtimes \langle w \rangle$. In that case we may assume, changing indices if necessary, that w centralizes i_1. Then*
  (iii) *if $|W| = 2$, then $I(G) = i_1^G \sqcup i_2^G \sqcup w^G$ (here $i_2^G = i_3^G$);*
  (iv) *if $|W| = 6$, then $I(G) = i_1^G \sqcup w^G$.*

Proof. Everything is clear from Fact 2.33 and Lemmas 6.12 and 6.19. □

After these investigations of the structure of W, we now push further the analysis of Borel subgroups of G. First note that we can compare the ranks of the B_i's even if they are not conjugate:

Lemma 6.21. $\mathrm{rk}(B_1) = \mathrm{rk}(B_2) = \mathrm{rk}(B_3)$ *and* $\mathrm{rk}(O(B_1)) = \mathrm{rk}(O(B_2)) = \mathrm{rk}(O(B_3))$.

Proof. The second equality follows from the first one by Theorem 6.6.

Assume toward a contradiction that $\mathrm{rk}(B_l) < \mathrm{rk}(B_{l'})$ for some $l, l' \in \{1,2,3\}$. Then $\mathrm{rk}(G/B_{l'}) < \mathrm{rk}(i_l^G)$ and by Lemma 2.36 there exists $\alpha \in i_l^G \setminus N_G(B_{l'})$ such that

$$T(\alpha) := \{\alpha\alpha_1\colon \alpha_1 \in i_l^G \cap B_{l'}\}$$

is infinite. As α normalizes $d(T(\alpha))$, we have $[d(T(\alpha)), d(T(\alpha))] \leqslant F(B_{l'}) \cap F(B_{l'})^\alpha = 1$ (Proposition 3.11), thus $d(T(\alpha))$ is an abelian group inverted by α, and $d(T(\alpha)) \cap F(B_{l'}) = 1$ by the same argument as before. Now the maximal 2-torus T_1 of $d(T(\alpha))^\circ$ is nontrivial (Lemma 2.41). But $T_1 \rtimes \langle\alpha\rangle \leqslant S^{\circ g}$ for some $g \in G$ (Lemma 6.12) and α centralizes T_1, a contradiction. □

Let $\mathfrak{B}$ be the set of Borel subgroups of G nonconjugate to B_l for all $l \in \{1,2,3\}$. Note that $\mathfrak{B}$ might be empty here. We will see that $\mathfrak{B}$ is not empty only at the very end of the analysis of our final configuration, in Lemma 6.73.

This definition of $\mathfrak{B}$ is different from the one in Section 5.2 (before Lemma 5.11), but we will see throughout this section that Borel subgroups in $\mathfrak{B}$ have the same kind of behavior as those in Section 5.2.

Lemma 6.22. *If $B \in \mathfrak{B}$, then $F(B) = O(B) < B$ and B contains an involution k conjugate to i_l for some $l \in \{1,2,3\}$. Furthermore, k inverts $O(B)$, $B = O(B) \rtimes C_B(k)$, and $\mathrm{Pr}_2(C_B(k)) = 1$.*

Proof. If $B = O(B)$, then $\bigcup_{g\in G} B^g$ is generic in G (Lemma 2.41 and Proposition 3.11), so there is by Fact 2.1 a nontrivial element $t \in T \cap B^g$ for some $g \in G$. Now $S^\circ \leqslant C_G^\circ(t) \leqslant B^g$ by Lemma 3.12, a contradiction. This shows that $O(B) < B$. Let now S_1 be a Sylow 2-subgroup of B. As S_1 is connected, $S_1 \leqslant S^{\circ g}$ for some $g \in G$ and S_1 contains an involution $k = i_l^g$ for some $l \in \{1,2,3\}$. If $F(B)$ has an involution j, then $B = C_G^\circ(j)$ by Lemma 3.1 and thus $j \in S_1$, so $j = i_s^g$ for some $s \in \{1,2,3\}$ and $B = B_s^g$, a contradiction. Thus $F(B)$ has no involution; in particular, Lemma 2.41 implies that $F^\circ(B) = O(B) < B$. We will show later that $F(B) = O(B)$.

If an involution k' in S_1 has an infinite centralizer in $O(B)$, then $B = C_G^\circ(k')$ by Proposition 3.11(ii), a contradiction. Thus $\mathrm{Pr}_2(S_1) = 1$ and k is the unique involution in S_1 by Fact 2.37. Furthermore k inverts $O(B)$ by Fact 2.25. Facts 2.15 and 2.27 also show that $B = O(B) \rtimes C_B(k)$, and it follows also that $C_B(k)$ is divisible abelian.

It remains to show that $F(B) = O(B)$, i.e., that $F(B)$ is connected. If $O(B) < F(B)$, then the finite group $C_{F(B)}(k)$ contains an element t of prime order $p \neq 2$. As $C_B(k)$ is divisible, t is in the maximal p-torus T_p of $C_B(k)$, and we have $T_p \leqslant C_G^\circ(k)$. We claim that T_p centralizes a conjugate of S°: by Theorem 6.6 and Fact 2.10, the maximal p-torus of $O(C_G^\circ(k))$ is trivial and it follows that T contains a maximal p-torus of $C_G^\circ(k)$. Thus T_p is in a conjugate of T, which proves our claim that T_p centralizes $S^{\circ h}$ for some $h \in G$. In particular, $S^{\circ h} \leqslant C_G^\circ(t)$. But $t \in F(B)$, so $C_{O(B)}^\circ(t) \neq 1$ by Fact 2.7 and $C_G^\circ(t) \leqslant B$ by Proposition 3.11(ii). This is a contradiction as $\mathrm{Pr}_2(B) = 1$. □

Lemma 6.23. $T = d(S^\circ)$. *For any involution $i \in A$, there is a definable connected subgroup T_i of T such that $S_i = T_i \cap S^\circ$ is a 2-torus of Prüfer rank 1, $i \in S_i$, and*

$T_i = d(S_i)$. For $i, j \in A$ distinct involutions, $T = T_i \times T_j$, and T_i, T_j are definably isomorphic.

Proof. Let M_l be a T-minimal subgroup of $O(B_l)$. Let $T_l^+ = C_T(M_l)$, $S_l = (T_l^+ \cap S^\circ)^\circ$, $T_l = (T_l^+)^\circ$. Then T/T_l^+ is isomorphic to $K_l^\times$ for some algebraically closed field K_l of characteristic not 2, and in particular S_l has Prüfer 2-rank equal to 1.

Now T_1^+ acts faithfully on M_2, as otherwise we again have $x \in T^\#$ with $M_1, M_2 \leqslant C(x)$, leading to $B_1 = B_2$, a contradiction. By tameness, $T_1 \simeq K_2^\times$, and T_1 has no infinite proper definable subgroups. Thus $T_1 = d(S_1)$. Similarly $T_2 = d(S_2)$. Looking at the action of T_2 on M_1, we find $T_1^+ \times T_2 = T$ and $T_1^+ = T_1$ by connectedness. Thus $T = T_1 \times T_2 = d(S_1) \times d(S_2) \leqslant d(S^\circ)$.

Changing notation, so that $T_i = T_l$ if $i = i_l$, the remaining statements are simply a paraphrase of the foregoing. The definable isomorphisms come from isomorphisms of, e.g., T_2 and T_3 with $K_1^\times$. Note however that we have not made any claims of "canonicity" as far as the groups T_i and S_i are concerned. □

Corollary 6.24. *If R is an infinite proper definable subgroup of T, then* $\mathrm{rk}(T) = 2\,\mathrm{rk}(R)$.

Proof. By the proof of Lemma 6.23, we have $T = T_1 \times T_2$ for two definably isomorphic definable subgroups T_1 and T_2, each having no infinite proper definable subgroups. If $R \cap T_i$ is infinite for some i, then $T_i \leqslant R < T_i \times T_j$ and T_i has a finite index in R, proving our lemma in that case. Thus we may assume $R \cap T_i$ finite. Then $T = T_i R$ and again $\mathrm{rk}(T) = \mathrm{rk}(T_i) + \mathrm{rk}(R)$, i.e., $\mathrm{rk}(R) = \mathrm{rk}(T_i)$. □

Lemma 6.25. *The following properties are satisfied*:

(1) *T is isomorphic to the product of* 2 *split* 1*-dimensional tori, i.e.,* 2 *copies of the multiplicative group of some algebraically closed field, of characteristic $p \neq 2$.*
(2) *If $p > 0$, then $O(B_l)$ is p-unipotent for $l = 1, 2, 3$.*
(3) *If $p = 0$, then $O(B_l)$ is torsion-free for $l = 1, 2, 3$.*

Proof. The first claim was seen in the proof of Lemma 6.23.

Observe that the divisible part of $O(B_l)$ is torsion free, as a maximal q-torus in $O(B_l)$ would have to be central in B_l, which is impossible by Theorem 6.6.

Suppose that the maximal q-unipotent subgroup U_q of B_l is nontrivial. Then in the notation of the proof of Lemma 6.23, we may take $M_l \leqslant U_q$, and hence $q = p$. Similarly, in the event that the divisible part of $O(B_l)$ is nontrivial, $p = 0$. Since $O(B_l) \neq 1$ for each l, and the value of p is determined by the structure of T, all claims follow. □

Notation 6.26. Let $p = \operatorname{char} T$ denote the characteristic of the algebraically closed field K such that $T \cong K^\times \times K^\times$ as in Lemma 6.25.

Lemma 6.27. *If $B \in \mathfrak{B}$, then $O(B)$ is a p-group* (*i.e., torsion-free if $p = 0$*).

Proof. Let k be an involution in B as in Lemma 6.22. By conjugacy, we may assume that $k = i_l$ for some $l = 1$, 2, or 3. Let $T_k = C_B(k)$ and M be a T_k-minimal subgroup of $O(B)$. As $O(C_{T_k}(M)) \leqslant O(B) \cap T_k = 1$ and the unique involution k in T_k inverts M, $C_{T_k}(M)$ is finite of odd order. By tameness, we have $T_k/C_{T_k}(M) \cong K^\times$ for some algebraically closed field K of characteristic not 2. Thus the torsion subgroup T_1 of T_k contains a nontrivial q-torus for every $q \neq \mathrm{char}(K)$. On the other hand, $T_1 \leqslant C_G^\circ(k) = B_l$ and $T_1 \cap O(B_l)$ must be finite by Lemma 6.25 and the fact that the divisible part of $O(B_l)$ is torsion free. Thus, by Theorem 6.6, T contains a nontrivial q-torus for every $q \neq \mathrm{char}(K)$.

Assume now toward a contradiction that $\mathrm{char}(K) \neq p$. If $p > 0$, then T contains a nontrivial p-torus, a contradiction to Lemma 6.25. Thus $p = 0$ and $\mathrm{char}(K) > 0$. By conjugacy, we may assume $T_1 \leqslant T$. Then, by tameness, $T_k = d(T_1) \leqslant T$. This is a contradiction as infinite definable subgroups of T must contain a nontrivial $\mathrm{char}(K)$-torus by Lemmas 6.23, 6.25, tameness, and Fact 2.5. □

We will now consider the different cases for the value of $|W|$. The following lemma will be useful.

Lemma 6.28. *If $t \in T^\#$ is inverted by an involution $j \in {A^\#}^G$, then $t \in I(T)$.*

Proof. If $j \in N_G(T)$, then $j \in T$ and $t = t^j = t^{-1}$, so $t \in I(T)$. Assume now $j \notin N_G(T)$. Then $T, T^j \leqslant C_G^\circ(t)$. $O(C_G^\circ(t)) \neq 1$, as otherwise $C_G^\circ(t) = T = T^j$ by Lemma 3.2, and $C_G^\circ(t) \leqslant B_l$ for some $l = 1$, 2, or 3 by Fact 2.37 and Proposition 3.11(ii). So $j \in N_G(B_l)$ by Proposition 3.11(ii) and $j \in B_l$ by Lemma 6.12.

Computing modulo $O(B_l)$, one sees that j inverts t and centralizes t, thus $t \in I(T)$ by Theorem 6.6. □

6.1. Case: $|W| = 2$

We will eliminate this case.

Theorem 6.29. $|W| \neq 2$.

So we assume now toward a contradiction that $|W| = 2$ and we fix the notations as in Corollary 6.20(iii): $w \in I(S \setminus S^\circ)$ centralizes i_1 and $I(G) = i_1^G \sqcup i_2^G \sqcup w^G$. Let also

$$S_1 = C_{S^\circ}(w).$$

By Lemma 2.31, $i_1 \in C_{S^\circ}(w) \cong \mathbb{Z}_{2^\infty}$.

To prove Theorem 6.29, we will get a contradiction by computing the rank of G in two different manners, using the Thompson Rank Formula in each case (see [3] for a general discussion about this formula), and then by looking at the distribution of involutions in cosets of B_1. We need the following preliminaries.

Lemma 6.30. $C_G(w) \cap i_2^G = \emptyset$.

Proof. If $C_G(w) \cap i_2^G$ is nonempty, then there are $g, h \in G$ such that the four-group $\langle i_2^h, w^g \rangle$ is in S. By Lemma 6.12, $w^g \notin S^\circ$ and $i_2^h \in S^\circ$. Thus $i_2^h \in I(S_1) = \{i_1\}$ and $i_2^h = i_1$, a contradiction. □

Lemma 6.31. $C_G^\circ(w) \nleqslant B_1$.

Proof. Assume toward a contradiction that $C_G^\circ(w) \leqslant C_G^\circ(i_1)$. As w inverts a nontrivial 2-torus in S° (Lemma 2.31), $C_G^\circ(w) < B_1$. Thus, by Fact 2.36, there is $w' \in w^G \setminus C_G(i_1)$ such that $T(w') = \{w'w''\colon\ w'' \in w'B_1 \cap w^G\}$ is infinite. Now w' normalizes $[d(T(w')), d(T(w'))] \leqslant F(B_1) \cap F(B_1)^{w'} = 1$ (Fact 2.15 and Proposition 3.11), thus $d(T(w'))$ is an infinite subgroup of B_1 inverted by w'. Now $O(d(T(w'))) = 1$ (as $O(d(T(w'))) \leqslant F(B_1) \cap F(B_1)^{w'} = 1$ by Lemma 2.41), thus $d(T(w'))$ contains a 2-torus of Prüfer 2-rank 1. Its involution i $(\in I(S^\circ)^G)$ is centralized by w', thus $i \notin i_2^G$ by Lemma 6.30 and $i \in i_1^G \cap C_G^\circ(i_1) = \{i_1\}$ (Theorem 6.6). So $w' \in C_G(i_1)$, a contradiction. □

Corollary 6.32. *If $i' \in i_1^G$ and $w' \in w^G$, then $O(C_G^\circ(i', w')) = 1$.*

Proof. We may assume $i' = i_1$. Now the statement follows from Proposition 3.11(ii), Lemma 6.19, and the preceding lemma. □

Lemma 6.33. $F^\circ(C_G^\circ(w)) = O(C_G^\circ(w))$.

Proof. By Lemma 2.41, it suffices to show that $F^\circ(C_G^\circ(w))$ has no involutions, so assume toward a contradiction the contrary. Then $F^\circ(C_G^\circ(w))$ contains a nontrivial 2-torus T_1. As $C_G^\circ(w)$ has Prüfer 2-rank at most 1 by Lemma 6.12 and Proposition 6.17, it follows that this 2-torus is maximal in $C_G^\circ(w)$. So $T_1 = S_1$, and by Fact 2.10, $C_G^\circ(w) \leqslant C_G^\circ(T_1) = C_G^\circ(S_1) \leqslant B_1$, a contradiction to Lemma 6.31. □

Corollary 6.34. *$C_G^\circ(w) \leqslant B$ for some unique Borel subgroup $B \in \mathfrak{B}$. In particular, i_1 inverts $O(B) = F(B)$.*

Proof. By Proposition 3.11(ii), $C_G^\circ(w) \leqslant B$ for some unique Borel subgroup B. If $B = B_l^g$ for some $g \in G$, then $i_l^g \notin i_1^G$ by Proposition 3.11(ii) and Corollary 6.32. But w centralizes i_l^g, a contradiction to Lemma 6.30. Thus $B \in \mathfrak{B}$ and everything follows now from Lemma 6.22. □

Lemma 6.35. $C_G^\circ(w) = O(C_G^\circ(w)) \rtimes C_T^\circ(w)$ *and* $C_T^\circ(w) = C_{B_1}^\circ(w)$.

Proof. Let B be the Borel subgroup containing $C_G^\circ(w)$, as in Corollary 6.34. By Lemma 6.22, $B = O(B) \rtimes C_B(i_1)$. By tameness, one sees as in Lemma 6.27 that $C_B(i_1)$ has no infinite proper definable subgroups. But $S_1 \leqslant C_G^\circ(w) \cap C_G^\circ(i_1)$, so $S_1 \leqslant C_B^\circ(i_1)$ and $C_B^\circ(i_1) \leqslant C_T^\circ(w)$. In particular, $B = O(B)C_T^\circ(w)$. If $C_B^\circ(i_1) < C_T^\circ(w)$, then $C_T^\circ(w) \cap O(B) \neq 1$ and a nontrivial element f in this intersection is such that $C_G^\circ(f) \leqslant B$

(Lemma 3.12), implying $T \leqslant B$, a contradiction. Thus $C^\circ_B(i_1) = C^\circ_T(w)$ and $B = O(B) \rtimes C^\circ_T(w)$. Now $O(B) = C_{O(B)}(w) \times O(B)^-$ where $O(B)^-$ is the subgroup of elements of $O(B)$ inverted by w (Fact 2.26) and the members in the product are connected. Thus $O(C^\circ_G(w)) = C_{O(B)}(w)$ and $C^\circ_G(w) = O(C^\circ_G(w)) \rtimes C^\circ_T(w)$.

It remains to show that $C^\circ_T(w) = C^\circ_{B_1}(w)$, so assume toward a contradiction that $C^\circ_T(w) < C^\circ_{B_1}(w)$. Then $C^\circ_{B_1}(w) = U \rtimes C^\circ_T(w)$ where $U = C^\circ_{B_1}(w) \cap C_{O(B)}(w)$ is nontrivial and connected. Then $B_1 = B$ by Proposition 3.11(ii), a contradiction. □

Lemma 6.36. $C_G(w) \cap I(S^\circ)^G = i_1 O(C^\circ_G(w))$.

Proof. Let B be the unique Borel subgroup containing $C^\circ_G(w)$, as in Corollary 6.34. We have $C_G(w) \leqslant N_G(B)$. Notice that there is no involution of $I(S^\circ)^G$ in $N_G(B) \setminus B$: otherwise $N_G(B)$ would contain a conjugate of A, a contradiction as then $B \notin \mathfrak{B}$ by Fact 2.37 and Proposition 3.11(ii). Thus $I(S^\circ)^G \cap C_G(w) = I(S^\circ)^G \cap C_B(w)$. But it is clear from the proof of Lemma 6.35 that $C_B(w) = C^\circ_B(i_1) \ltimes C_{O(B)}(w)$, and that $C_{O(B)}(w) = O(C^\circ_G(w))$, so $I(C_B(w)) = i_1 C_{O(B)}(w) = i_1 O(C^\circ_G(w))$. □

We are now ready to embark on a first computation of $\mathrm{rk}(G)$.

Lemma 6.37. *If $i' \in i_1^G$ and $w' \in w^G$, then $d(i'w')$ contains a unique involution z. Furthermore $z \in w^G$.*

Proof. Fact 2.32 shows that the elementary abelian 2-subgroup X of $d(i'w')$ is nontrivial. As w' inverts $d(i'w')$, $X^\# \cap i_2^G = \emptyset$ by Lemma 6.30.

We claim also that $X^\# \cap i_1^G = \emptyset$: for if $i'' \in X^\# \cap i_1^G$, then $[i'', i'] = 1$ implies that $i' = i''$ (as $i_1^G \cap S = i_1$), thus i' ($\in d(i'w')$) is centralized by w' and $X^\# = \{i'w'\} \subseteq w^G$ (Lemma 6.19), a contradiction as we assumed $X^\# \cap i_1^G \neq \emptyset$.

Thus $X^\# \subseteq w^G$ and if $X^\#$ contains two distinct involutions z and z', then $zz' \in X^\# \cap C^\circ(i')$ (Lemma 6.19), a contradiction. □

Consider the definable map

$$\Psi : i_1^G \times w^G \longrightarrow w^G, \quad (i', w') \longmapsto z,$$

where z is the unique involution in $d(i'w')$.

Lemma 6.38. *If $w_0 \in w^G$, then* $\mathrm{rk}(\Psi^{-1}(w_0)) = 2\,\mathrm{rk}(O(C^\circ_G(w)))$.

Proof. We may take $w_0 = w$. We will show that

$$\Psi^{-1}(w) = \bigl\{(i_1 f, w i_1 f') \colon (f, f') \in O\bigl(C^\circ_G(w)\bigr)^2\bigr\}.$$

The inclusion from right to left is clear: if f, $f' \in O(C^\circ_G(w))$, then $i_1 f w i_1 f' = w f^{i_1} f' = w f^{-1} f'$ (Corollary 6.34) and $(w f^{-1} f')^2 = (f^{-1} f')^2 \in O(C^\circ_G(w))$, thus

$d(i_1 f w i_1 f') = d(w f^{-1} f')$ contains a 2-element of $\langle w\rangle \times O(C_G^\circ(w))$ (Fact 2.5) which is necessarily w. Thus $\Psi(i_1 f, w i_1 f') = w$.

We have now to prove the inclusion from left to right, so let $(i', w') \in i_1^G \times w^G$ be such that $\Psi(i', w') = w$. Then i', $w' \in C_G(w)$. By Lemma 6.36, $i' = i_1 f$ for some $f \in O(C_G^\circ(w))$. Note that $w' \neq w$: otherwise $i'w' = w'$ and $i' = 1$. Thus $ww' \in C_G(w) \cap i_1^G$ (Lemmas 6.19 and 6.30), so $ww' = i_1 f'$ for some $f' \in O(C_G^\circ(w))$ by Lemma 6.36 and $w' = w i_1 f'$. □

Corollary 6.39. $\mathrm{rk}(G) = \mathrm{rk}(B_1) + 2\,\mathrm{rk}(O(C_G^\circ(w)))$.

Proof. By conjugacy, $\mathrm{Im}(\Psi) = w^G$, thus $\mathrm{rk}(i_1^G \times w^G) = \mathrm{rk}(w^G) + 2\,\mathrm{rk}(O(C_G(w)))$, and the corollary follows. □

We embark now on our second computation of $\mathrm{rk}(G)$.

Lemma 6.40. *If $j' \in i_2^G$ and $w' \in w^G$, then $d(j'w')$ contains a unique involution z. Furthermore $z \in i_1^G$.*

Proof. By Fact 2.32, the elementary abelian 2-subgroup X of $d(j'w')$ is nontrivial. As w' and j' invert $d(j'w')$, $X^\# \subseteq i_1^G$ by Lemma 6.30. But two distinct involutions in i_1^G cannot commute (Lemma 6.12), so $|X^\#| = 1$. □

Consider the definable map

$$\Psi\colon\, i_2^G \times w^G \longrightarrow i_1^G, \quad (j', w') \longmapsto z,$$

where z is the unique involution in $d(j'w')$.

Lemma 6.41. *If $i \in i_1^G$, then* $\mathrm{rk}(\Psi^{-1}(i)) = \mathrm{rk}(O(B_1)) + \mathrm{rk}(B_1) - \mathrm{rk}(C_{B_1}^\circ(w))$.

Proof. By conjugacy, Ψ has fibers of constant rank, so we just have to compute the rank of $\Psi^{-1}(i_1)$. For any $j' \in i_2^G \cap C_G(i_1)$ and $w' \in w^G \cap C_G(i_1)$, the unique involution of $d(j'w')$ is necessarily i_1, as $C_G(i_1) \cap i_1^G = \{i_1\}$. Thus $\Psi^{-1}(i_1) = (i_2^G \cap C_G(i_1)) \times (w^G \cap C_G(i_1))$.

By Lemma 6.12 and Theorem 6.6, $i_2^G \cap C_G(i_1) = i_2 O(B_1) \sqcup i_3 O(B_1)$, thus $\mathrm{rk}(i_2^G \cap C_G(i_1)) = \mathrm{rk}(O(B_1))$. On the other hand, $w^G \cap C_G(i_1)$ has rank $\mathrm{rk}(B_1) - \mathrm{rk}(C_{B_1}^\circ(w))$ by Lemma 6.19. Thus we get $\mathrm{rk}(\Psi^{-1}(i_1)) = \mathrm{rk}(O(B_1)) + \mathrm{rk}(B_1) - \mathrm{rk}(C_{B_1}^\circ(w))$. □

Corollary 6.42. $\mathrm{rk}(G) = \mathrm{rk}(B_2) + \mathrm{rk}(C_G(w)) + \mathrm{rk}(O(B_1)) - \mathrm{rk}(C_{B_1}(w))$.

Proof. As in Corollary 6.39, we get that

$$\mathrm{rk}\big(i_2^G \times w^G\big) = \mathrm{rk}\big(i_1^G\big) + \mathrm{rk}\big(O(B_1)\big) + \mathrm{rk}(B_1) - \mathrm{rk}\big(C_{B_1}(w)\big),$$

thus it follows that

$$\mathrm{rk}(G) = \mathrm{rk}(B_2) + \mathrm{rk}\big(C_G^\circ(w)\big) + \mathrm{rk}\big(O(B_1)\big) - \mathrm{rk}\big(C_{B_1}^\circ(w)\big). \qquad \square$$

Proof of Theorem 6.29. As $\mathrm{rk}(B_1) = \mathrm{rk}(B_2)$ by Lemma 6.21, Corollaries 6.39 and 6.42 give the equality

$$2\,\mathrm{rk}\big(O\big(C_G^\circ(w)\big)\big) = \mathrm{rk}\big(C_G^\circ(w)\big) + \mathrm{rk}\big(O(B_1)\big) - \mathrm{rk}\big(C_{B_1}^\circ(w)\big).$$

Thus, by Lemma 6.35 we get $\mathrm{rk}(O(C_G^\circ(w))) = \mathrm{rk}(O(B_1))$. By Lemma 6.35 again, we get $\mathrm{rk}(C_G^\circ(w)) = \mathrm{rk}(O(B_1) \rtimes C_T^\circ(w))$ and, as $C_T(w) < T$, we have

$$\mathrm{rk}\big(C_G(w)\big) < \mathrm{rk}(B_1).$$

It follows that $\mathrm{rk}(G/B_1) < \mathrm{rk}(w^G)$. Now, by Fact 2.36, there exists $w_1 \in w^G \setminus N_G(B_1)$ such that $T(w_1) = \{w_1 w_2\colon\ w_2 \in w_1 B_1 \cap w^G\}$ is infinite. As usual, $d(T(w_1))$ is an infinite group and $d(T(w_1))^\circ$ contains a nontrivial 2-torus T_1. If k is an involution in T_1, then $k \in i_1^G$ (Lemmas 6.12 and 6.30), thus $k = i_1$ (as $C_G(i_1) \cap i_1^G = \{i_1\}$), and $w_1 \in C_G(i_1) = N_G(B_1)$, a contradiction which ends the proof of Theorem 6.29. □

6.2. Case: $|W| = 6$

We will eliminate this case.

Theorem 6.43. $|W| \neq 6$.

So we assume now toward a contradiction that $|W| = 6$ and we fix the notations as in Corollary 6.20(iv): $w \in I(S \setminus S^\circ)$ centralizes i_1 and $I(G) = i_1^G \sqcup w^G$. Let also $S_1 = C_{S^\circ}(w)$. By Lemma 2.31, $i_1 \in C_{S^\circ}(w) \cong \mathbb{Z}_{2^\infty}$.

To prove Theorem 6.43, we will compute the rank of G with the Thompson Rank Formula, and get a contradiction by looking at the distribution of involutions in cosets of $C_G^\circ(w)$.

Lemma 6.44. *If* $\mathrm{rk}(C_G^\circ(w)) < \mathrm{rk}(B_1)$, *then* $\mathrm{rk}(G) \leqslant \mathrm{rk}(B_1) + \mathrm{rk}(O(B_1)) + \mathrm{rk}(C_G^\circ(w)) - \mathrm{rk}(C_{B_1}^\circ(w))$.

Proof. By assumption, $\mathrm{rk}(G/B_1) < \mathrm{rk}(w^G) = \mathrm{rk}(w^G \setminus N_G(B_1))$. For $w_1 \in w^G \setminus N_G(B_1)$, let $T(w_1) = \{w_1\alpha\colon\ \alpha \in w_1 B_1 \cap I(G)\}$. Let also

$$C_1 = \big\{w_1 \in w^G \setminus N_G(B_1)\colon\ T(w_1) \text{ is finite}\big\} \quad \text{and}$$
$$C_2 = \big\{w_1 \in w^G \setminus N_G(B_1)\colon\ T(w_1) \text{ is infinite}\big\}.$$

Then C_2 is generic in $w^G \setminus N_G(B_1)$.

If $w' \in C_2$, then, as usual, $d(T(w'))$ is an infinite abelian group inverted by w'. Let now M be a B_1-minimal subgroup in $O(B_1)$. If $t \in d(T(w'))^\#$, then $C_M(t) = 1$: otherwise M, $M^w \leqslant C_G^\circ(t)$ by Fact 2.40 and $w' \in N_G(B_1)$ by Proposition 3.11(ii), a contradiction. Thus

$d(T(w')) \cap C_{B_1}(M) = 1$. On the other hand, $B_1/C_{B_1}(M)$ has no infinite proper definable subgroup by Fact 2.38 and tameness. Thus $B_1 = C_{B_1}(M) \rtimes d(T(w'))$. In particular,

$$d\big(T(w')\big) \text{ is connected and divisible}$$

(Facts 2.1, 2.8, and 2.15). It follows also that $\mathrm{rk}(T) = 2\,\mathrm{rk}(d(T(w')))$ by Corollary 6.24.

If $w' \in C_2$, then $i_1 \notin d(T(w'))$ and $d(T(w'))$ has Prüfer 2-rank 1, so its unique involution j is in $i_2 O(B_1) \cup i_3 O(B_1)$ (Theorem 6.6). We have shown that

$$C_2 \subseteq \bigcup_{j \in (i_2 O(B_1) \cup i_3 O(B_1))} \big(C_G(j) \cap w^G\big).$$

But $\mathrm{rk}(C_G(j) \cap w^G) = \mathrm{rk}(B_1) - \mathrm{rk}(C^\circ_{B_1}(w))$ by Lemma 6.19, thus

$$\mathrm{rk}(G) - \mathrm{rk}\big(C^\circ_G(w)\big) = \mathrm{rk}(C_2) \leqslant \mathrm{rk}\big(O(B_1)\big) + \mathrm{rk}(B_1) - \mathrm{rk}\big(C^\circ_{B_1}(w)\big). \qquad \square$$

Lemma 6.45. $C^\circ_G(w) \nleqslant B_1$.

Proof. Assume $C^\circ_G(w) \leqslant B_1$. Then $\mathrm{rk}(C^\circ_G(w)) = \mathrm{rk}(C^\circ_{B_1}(w)) < \mathrm{rk}(B_1)$ and the preceding lemma gives $\mathrm{rk}(G) \leqslant \mathrm{rk}(B_1) + \mathrm{rk}(O(B_1)) = \mathrm{rk}(B_1 B_2) \leqslant \mathrm{rk}(G)$, i.e., $\mathrm{rk}(G) = \mathrm{rk}(B_1) + \mathrm{rk}(O(B_1))$.

With the notations of the previous proof, if we pick $w' \in C_2$, then

$$\bigsqcup_{f \in O(B_1)} \big(w' d\big(T(w')\big)\big)^f \subseteq C_2.$$

(The union is disjoint: if $f \in O(B_1)$ normalizes $I(w'B_1)$, then f is in the normalizer in $O(B_1)$ of $d(T(w'))$, and the latter subgroup is trivial.) Thus $\mathrm{rk}(C_2) \geqslant \mathrm{rk}(O(B_1)) + (1/2)\,\mathrm{rk}(T)$ and the projection of C_2 over G/B_1 is generic in G/B_1 (as $\mathrm{rk}(d(T(w'))) = \mathrm{rk}(T(w')) = (1/2)\,\mathrm{rk}(T)$ by the proof of the previous lemma).

Now the same argument as in Lemma 6.21 shows that cosets of B_1 distinct from B_1 contain only finitely many involutions in i_1^G, thus the projection of i_1^G over G/B_1 is also generic in G/B_1. As G/B_1 has Morley degree 1, there exists $w' \in C_2$ and $j \in i_1^G \cap w'B_1$. Thus $w'j \in d(T(w'))$ and as the latter subgroup is 2-divisible, w' and j are conjugate, a contradiction. $\square$

Corollary 6.46. *If $i' \in i_1^G$ and $w' \in w^G$, then $O(C^\circ_G(i', w')) = 1$.*

Proof. As in Corollary 6.32. $\square$

Lemma 6.47. $F^\circ(C^\circ_G(w)) = O(C^\circ_G(w))$.

Proof. As in Lemma 6.33. $\square$

Corollary 6.48. *$C_G^\circ(w) \leqslant B$ for some unique Borel subgroup $B \in \mathfrak{B}$. In particular, i_1 inverts $O(B) = F(B)$.*

Proof. As in Corollary 6.34. □

Lemma 6.49. *$C_G^\circ(w) = O(C_G^\circ(w)) \rtimes C_T^\circ(w)$ and $C_T^\circ(w) = C_{B_1}^\circ(w)$.*

Proof. As in Lemma 6.35. □

Lemma 6.50. *$C_G(w) \cap I(S^\circ)^G = i_1 O(C_G^\circ(w))$.*

Proof. As in Lemma 6.36. □

Corollary 6.46 also has the following corollary.

Corollary 6.51. *$F^\circ(B_1) = O(B_1) \times T^-$, where T^- is the subgroup of elements of T inverted by w, and $F^\circ(B_1)$ is inverted by w (and in particular is abelian).*

Proof. $C_{O(B_1)}^\circ(w) = 1$ by Corollary 6.46, so w inverts $O(B_1)$ by Fact 2.25. Now w has a finite centralizer in $O(B_1) \rtimes T^-$, so w inverts $O(B_1) \rtimes T^-$ by Fact 2.25 again (recall from Lemma 6.19 that T^- is connected), so $(O(B_1) \times T^-) \leqslant F^\circ(B_1)$ by Theorem 6.6. If the containment is proper, then $T \leqslant F^\circ(B_1)$ by Corollary 6.24, a contradiction. □

We embark now on the computation of $\mathrm{rk}(G)$.

Lemma 6.52. *If $i' \in i_1^G$ and $w' \in w^G$, then $d(i'w')$ contains a unique involution z.*

Proof. The statement is obvious if $[i', w'] = 1$, so we assume $[i', w'] \neq 1$. In particular, i', $w' \notin d(i'w')$. By Fact 2.32, it suffices to show that $|I(d(i'w'))| \leqslant 1$.

We first claim that $|d(i'w') \cap w^G| \leqslant 1$: otherwise we find two distinct involutions w_1 and $w_2 \in d(i'w') \cap w^G$. Then the three distinct involutions w_1, w_2, and w' are in $(S \setminus S^\circ)^h$ for some $h \in G$ and commute, hence centralize some $j \in I(A)^h$. We have $w_1 = w_2 s$ for some $s \in S^{\circ h}$ inverted by w_2. As $[w_1, w_2] = 1$, s is also centralized by w_2, so $s = j$. By the same argument, $w_2 = w'j$. Thus $w' = w_2 j = w_1$, a contradiction which proves our first claim.

Secondly, we claim that $|d(i'w') \cap i_1^G| \leqslant 1$: otherwise, by Lemma 6.12, $A^h \leqslant d(i'w')$ for some $h \in G$. Then $w' \in C_G(A)^h = T^h$, a contradiction to Lemma 6.12 again.

Thus $|I(d(i'w'a))| \leqslant 2$ and hence $|I(d(i'w'))| = 1$. □

Consider the definable map

$$\Psi : i_1^G \times w^G \longrightarrow i_1^G \sqcup w^G, \quad (i', w') \longmapsto z,$$

where $\{z\} = I(d(i'w'))$. Let

$$D_i = \{(i', w') \in i_1^G \times w^G\colon \Psi(i', w') \in i_1^G\} \quad \text{and}$$
$$D_w = \{(i', w') \in i_1^G \times w^G\colon \Psi(i', w') \in w^G\}.$$

Then $i_1^G \times w^G = D_i \sqcup D_w$ and as $\Psi(i_1, w) \in w^G$ and $\Psi(i_2, w) = i_1 \in i_1^G$, D_i and D_w are both nonempty. By conjugacy, the fibers are of constant rank on D_i and D_w.

Lemma 6.53. $\mathrm{rk}(\Psi^{-1}(w)) = 2\,\mathrm{rk}(O(C_G^\circ(w)))$.

Proof. As in Lemma 6.38, using Lemma 6.50. □

Corollary 6.54. $\mathrm{rk}(D_w) = \mathrm{rk}(G) + \mathrm{rk}(O(C_G^\circ(w))) - (1/2)\,\mathrm{rk}(T)$.

Proof. We have $\mathrm{rk}(D_w) = \mathrm{rk}(G) - \mathrm{rk}(C_G^\circ(w)) + 2\,\mathrm{rk}(O(C_G^\circ(w)))$, and it suffices to apply Corollary 6.24 and Lemma 6.49. □

Lemma 6.55. $\mathrm{rk}(\Psi^{-1}(i_1)) = 2\,\mathrm{rk}(O(B_1)) + (1/2)\,\mathrm{rk}(T)$.

Proof. We have here, in some sense, to refine the proof of Lemma 6.41. For this we show that

$$\Psi^{-1}(i_1) = \{(jf, (wt)^{f'})\colon\ j \in \{i_2, i_3\},\ f, f' \in O(B_1),\ t \in T^-\},$$

where T^- is the subgroup of elements of T inverted by w. Note that $T^- = Z(B_1)$.

Inclusion from right to left: if $(i', w') = (jf, (wt)^{f'})$, then $i'w' = jff'^{-1}wtf'$. By Corollary 6.51, w inverts $O(B_1) \times T^-$, so $i'w' = jwf'f^{-1}tf' = jwtf'^2f^{-1}$. If we put $f_1 = f'^2f^{-1}$ $(\in O(B_1))$, then

$$(i'w')^2 = (jwtf_1)^2 = jwtf_1jf_1^{-1}t^{-1}w = jwf_1tjt^{-1}f_1^{-1}w = jwf_1jf_1^{-1}w,$$

that is

$$(i'w')^2 = jwf_1^2jw = jwf_1^2wk = jf_1^{-2}k = jkf_1^2 = i_1f_1^2,$$

where $k = j^w$. As i_1 is the unique 2-element in $\langle i_1\rangle \times O(B_1)$, Fact 2.5 shows that $i_1 \in d((i'w')^2) \leqslant d(i'w')$, i.e., $\Psi(i', w') = i_1$.

Inclusion from left to right: if $\Psi(i', w') = i_1$, then $i' \in C_G(i_1) \cap i_1^G$ and $w' \in C_G(i_1) \cap w^G$. Thus $i' = jf$ where $j \in \{i_2, i_3\}$ and $f \in O(B_1)$ by Lemma 6.12 (note that $i' \neq i_1$, as otherwise $i'w' \in w^G$, i.e., $\Psi(i', w') \neq i_1$). By the proof of Lemma 6.19, w' has the desired form.

If $(wt)^f = (wt_1)^{f_1}$, where $t, t_1 \in T^-$ and $f, f_1 \in O(B_1)$, then $wt_1 = (wt)(ff_1^{-1})^2$ as wt inverts $O(B_1) \times T^-$, thus $t^{-1}t_1 = (ff_1^{-1})^2 \in T \cap O(B_1) = 1$ and $t = t_1$, $f = f_1$. This shows that $\mathrm{rk}(\Psi^{-1}(i_1)) = 2\,\mathrm{rk}(O(B_1)) + \mathrm{rk}(T^-)$ and it suffices now to apply Corollary 6.24. □

Corollary 6.56. $\mathrm{rk}(D_i) = \mathrm{rk}(G) + \mathrm{rk}(O(B_1)) - (1/2)\,\mathrm{rk}(T)$.

Proof. We have $\operatorname{rk}(D_i) = \operatorname{rk}(G) - \operatorname{rk}(B_1) + 2\operatorname{rk}(O(B_1)) + (1/2)\operatorname{rk}(T)$, so it suffices to apply Theorem 6.6. □

Lemma 6.57. $\operatorname{rk}(O(B_1)) < \operatorname{rk}(O(C_G^\circ(w)))$.

Proof. As $i_1^G \times w^G = D_i \sqcup D_w$ has degree 1, Corollaries 6.54 and 6.56 show that $\operatorname{rk}(O(B_1)) \neq \operatorname{rk}(O(C_G^\circ(w)))$, so it suffices to show that $\operatorname{rk}(O(B_1)) \leqslant \operatorname{rk}(O(C_G^\circ(w)))$.

So assume toward a contradiction that $\operatorname{rk}(O(B_1)) > \operatorname{rk}(O(C_G^\circ(w)))$. Then $\operatorname{rk}(F(B_1)) > \operatorname{rk}(C_G^\circ(w))$ (Corollaries 6.24, 6.51, and Lemma 6.49), so $G/F(B_1)$ has rank strictly less than $\operatorname{rk}(w^G)$. As usual, Fact 2.36 implies the existence of $w_1 \in w^G \setminus N_G(B_1)$ such that $w_1 F(B_1)$ contains infinitely many involutions, a contradiction as then $w_1 \in N_G(B_1)$ by Proposition 3.11. □

Corollary 6.58. $\operatorname{rk}(G) = \operatorname{rk}(B_1) + 2\operatorname{rk}(O(C_G^\circ(w)))$.

Proof. By the preceding lemma, D_w is generic in $i_1^G \times w^G$, thus $\operatorname{rk}(i_1^G) + \operatorname{rk}(w^G) = \operatorname{rk}(w^G) + \operatorname{rk}(\Psi^{-1}(w))$ and $\operatorname{rk}(G) = \operatorname{rk}(B_1) + 2\operatorname{rk}(O(C_G^\circ(w)))$ by Lemma 6.53. □

Lemma 6.59. *If B is any Borel subgroup in G, then* $\operatorname{rk}(B) \leqslant \operatorname{rk}(B_1)$. *In particular,* $\operatorname{rk}(C_G^\circ(w)) \leqslant \operatorname{rk}(B_1)$.

Proof. Otherwise, $\operatorname{rk}(G/B) < \operatorname{rk}(i_1^G)$ and by Fact 2.36 there exists $j \in i_1^G \setminus N_G(B)$ such that $T(j) = \{jj_1\colon\ j_1 \in i_1^G \cap jB\}$ is infinite. As usual, $d(T(j))$ is an abelian group inverted by j. Also, $O(d(T(j))) \leqslant F(B) \cap F(B)^j = 1$, thus j inverts a nontrivial 2-torus T_1, a contradiction as $T_1 \rtimes \langle j \rangle \leqslant S^{\circ g}$ for some $g \in G$ by Lemma 6.12. □

Lemma 6.60. $\operatorname{rk}(C_G^\circ(w)) = \operatorname{rk}(B_1)$.

Proof. By the preceding lemma, we may assume toward a contradiction that $C_G^\circ(w)$ has rank strictly less than $\operatorname{rk}(B_1)$. Then $\operatorname{rk}(G) \leqslant \operatorname{rk}(B_1) + \operatorname{rk}(O(B_1)) + \operatorname{rk}(C_G^\circ(w)) - \operatorname{rk}(C_{B_1}^\circ(w))$ by Lemma 6.44. Now Lemmas 6.49 and 6.57 give

$$\operatorname{rk}(G) \leqslant \operatorname{rk}(B_1) + \operatorname{rk}\bigl(O(B_1)\bigr) + \operatorname{rk}\bigl(O\bigl(C_G^\circ(w)\bigr)\bigr) < \operatorname{rk}(B_1) + 2\operatorname{rk}\bigl(O\bigl(C_G^\circ(w)\bigr)\bigr),$$

a contradiction to Corollary 6.58. □

We now look at the distribution of involutions in $G/C_G^\circ(w)$ (left cosets). Let B be the Borel subgroup of G containing $C_G^\circ(w)$, as in Corollary 6.48. By the preceding two lemmas, $B = C_G^\circ(w)$. Let also π denote the natural projection of G over $G/C_G^\circ(w)$.

Lemma 6.61. $\pi(w^G \setminus N_G(B))$ *is generic in* $G/C_G^\circ(w)$.

Proof. By Fact 2.36, $\mathrm{rk}(w^G \setminus N_G(B)) = \mathrm{rk}(G) - \mathrm{rk}(C_G^\circ(w))$. There is an integer t and a definable generic subset C_t of $w^G \setminus N_G(B)$ such that $\mathrm{rk}(\pi^{-1}(\pi(w')) \cap w^G) = t$ for every $w' \in C_t$. It suffices now to show that $t = 0$, as then

$$\mathrm{rk}\big(G/C_G^\circ(w)\big) = \mathrm{rk}(C_t) = \mathrm{rk}\big(\pi(C_t)\big) \leqslant \mathrm{rk}\big(\pi\big(w^G \setminus N_G(B)\big)\big).$$

So assume toward a contradiction that $t \geqslant 1$. For $w' \in C_t$, let $T(w') = \{w'w''\colon w'' \in w^G \cap w'C_G^\circ(w)\}$. As usual, $d(T(w'))$ is an abelian group inverted by w' and disjoint from $F(B) = O(B)$, and it has Prüfer 2-rank 1. If T_1 denotes its maximal 2-torus and k the unique involution in T_1, then $w, w' \in C_G(k) = C_G(i_1)^g$ for some $g \in G$. Rephrasing Corollary 6.51, with i_1^g and w' instead of i_1 and w, one sees that $T_1 \leqslant F(B_1)^{\circ g}$. But $w' = w^h$ for some $h \in B_1^g$ by Lemma 6.19. As $T_1 \leqslant Z(B_1^g)$, w also inverts T_1, a contradiction as $T_1 \leqslant C_G^\circ(w)$. □

Lemma 6.62. *$i_1^G \cap \pi^{-1}(\pi(w^G \setminus N_G(B)))$ is generic in i_1^G.*

Proof. If $j \in i_1^G \setminus N_G(B)$, then the coset $jC_G^\circ(w)$ cannot contain infinitely many involutions. This can be seen as in the proof of Lemma 6.59: otherwise j would invert a nontrivial 2-torus. Thus, by Lemma 6.60 and Fact 2.36, there is a generic subset of cosets in $(G/C_G^\circ(w)) \setminus (G/N_G(B))$ which all contain an involution in i_1^G. As $G/C_G^\circ(w)$ has Morley degree 1, it suffices now to apply Lemmas 6.60 and 6.61. □

Proof of Theorem 6.43. Let I be the generic subset of i_1^G as in Lemma 6.62. We show the following inclusion:

$$I \subseteq \bigcup_{f \in O(C_G^\circ(w))} C_G^\circ(i_1)^f.$$

So let $i \in I$. Then $i \notin N_G(B)$ and there exists $w' \in w^G$ such that $iw' \in C_G^\circ(w)$. By Corollary 6.48 and Lemma 6.49, $C_T^\circ(w) = C_B(i_1)$ is a Carter subgroup of $C_G^\circ(w) = O(C_G^\circ(w)) \rtimes C_T^\circ(w)$. Note that $C^\circ_{O(C_G^\circ(w))}(iw') = 1$, as otherwise $1 \neq O(C_G^\circ(iw')) \leqslant O(B)$ and $i \in N_G(O(C_G^\circ(iw'))) \leqslant N_G(B)$ by Proposition 3.11(ii). Thus, by Corollary 2.24, $E_{C_G^\circ(w)}(\langle iw' \rangle)$ is a Carter subgroup of $C_G^\circ(w)$, and $E_{C_G^\circ(w)}(\langle iw' \rangle) = C_T^\circ(w)^f$ for some $f \in O(C_G^\circ(w))$ by Fact 2.19. In particular, $iw' \in C_T^\circ(w)^f \leqslant T^f$ and Lemma 6.28 shows that $iw' \in I(C_T^\circ(w)^f) = \{i_1^f\}$. Thus $i \in C_G(i_1^f)$ and $i \in C_G^\circ(i_1)^f$ by Lemma 6.12. Our inclusion is shown.

The previous inclusion implies that

$$\mathrm{rk}\big(i_1^G\big) \leqslant \mathrm{rk}\big(O\big(C_G^\circ(w)\big)\big) + \mathrm{rk}\big(i_1^G \cap C_G^\circ(i_1)\big) = \mathrm{rk}\big(O\big(C_G^\circ(w)\big)\big) + \mathrm{rk}\big(O(B_1)\big)$$

(Theorem 6.6). Thus

$$\mathrm{rk}(G) \leqslant \mathrm{rk}(B_1) + \mathrm{rk}\big(O\big(C_G^\circ(w)\big)\big) + \mathrm{rk}\big(O(B_1)\big) < \mathrm{rk}(B_1) + 2\,\mathrm{rk}\big(O\big(C_G^\circ(w)\big)\big)$$

by Lemma 6.57. This is a contradiction to Corollary 6.58 which ends the proof of Theorem 6.43. □

6.3. *Case:* $|W| = 1$

We will eliminate this case.

Theorem 6.63. $|W| \neq 1$.

So we assume now toward a contradiction that $W = 1$. Recall from Corollary 6.20 that, in the case $W = 1$, $S = S^\circ$ and $I(G) = i_1^G \sqcup i_2^G \sqcup i_3^G$. By the Frattini argument, it is also clear that the three B_i's are selfnormalizing.

Lemma 6.64. *Any left coset of B_1 disjoint from B_1 cannot contain infinitely many involutions.*

Proof. This is what we actually have shown in the proof of Lemma 6.21, for involutions in the connected component of a Sylow 2-subgroup of G. □

Corollary 6.65. *For $l = 1$, 2, and 3, $(i_l^G \setminus B_1)B_1$ is generic in G.*

Proof. By Fact 2.36, Lemma 6.21, and the preceding lemma, $\mathrm{rk}(G/B_l) = \mathrm{rk}(i_l^G) = \mathrm{rk}(i_l^G \setminus B_1)$, and $\mathrm{rk}((i_l^G \setminus B_1)B_1) = \mathrm{rk}(i_l^G \setminus B_1) + \mathrm{rk}(B_1) = \mathrm{rk}(G)$. □

As G/B_1 has Morley degree 1, we get the following corollary.

Corollary 6.66. $\bigcap_{l=1}^{3}(i_l^G \setminus B_1)B_1$ *is generic in G.*

Proof of Theorem 6.63. By Corollary 6.66, there exists j_1, j_2, and $j_3 \in G \setminus B_1$ such that $j_l \in i_l^G$ and $j_1 B_1 = j_2 B_1 = j_3 B_1$. Let $R = \langle j_1 j_2, j_1 j_3 \rangle$. As usual, j_1 inverts R which is an abelian subgroup of B_1. As $E_{B_1}(R)$ contains a Carter subgroup of B_1 by Fact 2.23, it contains T^f for some $f \in O(B_1)$ (Fact 2.19 and Theorem 6.6) and we claim that $E_{B_1}(R) = T^f$: otherwise $C^\circ_{O(B_1)}(R) \neq 1$ by Corollary 2.24 and $j_1 \in N_G(O(C_G(R))) \leqslant N_G(B_1)$ by Proposition 3.11(ii), a contradiction. Thus $E_{B_1}(R) = T^f$ as claimed and in particular $R \leqslant T^f$. Now, by Lemma 6.28, $j_1 j_2$ and $j_1 j_3 \in I(T)^f$, and $R = A^f$. As j_1 inverts R, $j_1 \in R \leqslant B_1$, a contradiction which ends the proof. □

6.4. *Case:* $|W| = 3$

By the preceding results we are necessarily in the case $|W| = 3$, in which case W acts transitively on $A^\#$ and $I(G) = i_1^G$ by Corollary 6.20. It is also clear by the Frattini argument that the three B_i's are selfnormalizing.

It is now time to lift elements of order 3 from W.

Lemma 6.67. *If $\sigma \in N_G(A) \setminus C_G(A)$ is an element of order* 3 *modulo $C_G(A)$, then $\sigma^3 = 1$ and $\sigma T = \sigma^T$.*

Proof. The set of elements $\sigma' \in \sigma T$ such that $\sigma' \in (\langle\sigma\rangle T)^g$ for some $g \in G \setminus N_G(T)$ is generic in σT by Lemma 3.4. For such an element σ' we have that $\sigma'^3 \in C_G(A) \cap C_G(A)^g$. We claim that $C_G(A) \cap C_G(A)^g = 1$. Otherwise, A and A^g have a common involution k by Lemma 6.15 (and only one such, as $g \notin N_G(T)$). Then $k^{\sigma'} \in k^{(\langle\sigma\rangle T)^g} \subseteq A^g$, so $k^{\sigma'} \in I(A \cap A^g) = \{k\}$, and k is centralized by σ', a contradiction.

We have shown that the elements of the coset σT are generically of order 3. Now, as T is divisible, Lemma 3.7 shows that each element of σT has a finite centralizer in T and it follows that these elements are all T-conjugate, by connectedness of T and Fact 2.1. $\square$

Recall from Notation 6.26 that $p = \operatorname{char}(T)$ denotes the characteristic of the algebraically closed field K such that $T \cong K^\times \times K^\times$, and that $O(B)$ is p-unipotent (i.e., torsion-free if $p = 0$) for every Borel subgroup B in G (Lemmas 6.25 and 6.27). We will show that $p = 3$. First we show that G is covered by its Borel subgroups; more precisely:

Lemma 6.68. $G = (\bigcup_{g\in G} B_1^g) \sqcup (\bigcup_{B\in\mathfrak{B}} O(B)^\#)$.

Proof. First remark that the union is disjoint: if $f \in O(B)^\# \cap B_1$ for some $B \in \mathfrak{B}$, then $C_G^\circ(f) = O(B)$ (Lemmas 2.41, 3.12, and 6.22), thus $1 \neq C_{B_1}^\circ(f) \leqslant B_1 \cap O(B)$ (Fact 2.17) and $B_1 = B$ by Proposition 3.11(ii), a contradiction.

For any $x \in G$, $C_G^\circ(x) \neq 1$ by Corollaries 2.18 and 6.11. If $O(C_G^\circ(x)) = 1$, then $x \in C_G(i_1)^g = B_1^g$ for some $g \in G$ as Sylow 2-subgroups of G are connected and B_1 is selfnormalizing. If $O(C_G^\circ(x)) \neq 1$, then $x \in N_G(B)$ where B is the unique Borel subgroup B of G which contains $C_G^\circ(x)$ (Proposition 3.11(ii)). If B is conjugate to B_1, then $x \in N_G(B) = B$, so we assume now $B \in \mathfrak{B}$. Note that $N_G(B) = O(B) \rtimes T_1$ by the Frattini argument and Lemma 6.22, where $T_1 = C_{N_G(B)}(k)$ and k is an involution of B of the form i_1^g for some $g \in G$. As $C_G(i_1) = B_1$, $T_1 \leqslant B_1^g$ and it suffices now to show that $t_1 O(B) = t_1^{O(B)}$ for any $t_1 \in T_1^\#$. For this it suffices to show that $C_{O(B)}(t_1)$ is finite and then to apply Fact 2.27. So assume now toward a contradiction that $C_{O(B)}^\circ(t_1) \neq 1$. Then $C_G^\circ(t_1) \leqslant B$ by Proposition 3.11(ii) and $C_G^\circ(t_1)$ has Prüfer 2-rank at most 1 by Lemma 6.22. On the other hand, $C_{O(B_1)^g}^\circ(t_1) = 1$ by Proposition 3.11(ii), thus, by Corollary 2.24, $E_{B_1^g}(\langle t_1\rangle)$ is a Carter subgroup of B_1^g. In particular, t_1 is in a conjugate of T and it centralizes a 2-torus of Prüfer 2-rank 2, a contradiction. $\square$

Fix σ an element of order 3 such that $N_G(T) = T \rtimes \langle\sigma\rangle$, as in Lemma 6.67.

Lemma 6.69. $\sigma \notin \bigcup_{g\in G} T^g$.

Proof. Assume $\sigma \in T^g$ for some $g \in G$. By Lemma 6.25, the elementary abelian 3-subgroup A_3 of T is isomorphic to $(\mathbb{Z}_3)^2$. By the proof of Lemma 6.23, there are three nontrivial elements σ_1, σ_2, and σ_3 of A_3 such that $C_{O(B_l)}^\circ(\sigma_l) \neq 1$ ($l = 1, 2, 3$). Furthermore, the three subgroups $\langle\sigma_l\rangle$ are pairwise disjoint by Proposition 3.11(ii).

Now σ cannot centralize a σ_l, as otherwise $\sigma \in N_G(O(C_G^\circ(\sigma_l))) \leqslant N_G(B_l) = B_l$ by Proposition 3.11(ii), a contradiction. Thus $C_{A_3}(\sigma) = \langle\sigma_0\rangle$ for some element $\sigma_0 \in A_3^\#$ such that $\langle\sigma_0\rangle$ is disjoint from the three $\langle\sigma_l\rangle$, and A_3 is covered by the pairwise disjoint $\langle\sigma_l\rangle$ ($l = 0, 1, 2, 3$).

Remark that $C_G^\circ(\sigma_0) = C_G^\circ(\sigma_0^{-1}) = T$: otherwise $O(C_G^\circ(\sigma_0)) \neq 1$ by Lemma 3.2, and $O(C_G^\circ(\sigma_0)) \leqslant O(B_l)$ for some $l = 1, 2$, or 3 (Fact 2.37 and Proposition 3.11(ii)) and $\sigma \in N_G(B_l) = B_l$ by Proposition 3.11(ii), a contradiction. In particular, $C_G(\sigma_0) = N_G(T) = T \rtimes \langle\sigma\rangle$.

We claim now that $C_G^\circ(\sigma) = T^g$: otherwise we have $O(C_G^\circ(\sigma)) \neq 1$ (Lemma 3.2), $O(C_G^\circ(\sigma)) \leqslant B_l^g$ for some $l = 1, 2$, or 3 (Fact 2.37 and Proposition 3.11(ii)) and $\sigma_0 \in N_G(B_l^g) = B_l^g$ by Proposition 3.11(ii). As $C_G^\circ(\sigma_0) = T$, Lemmas 6.4, 2.41, and Corollary 2.24 show that $E_{B_l^g}(\langle\sigma_0\rangle)$ is a Carter subgroup of B_l^g, i.e., T^{gf} for some $f \in O(B_l^g)$. In particular, $\sigma_0 \in T^{gf}$. Thus $T^{gf} \leqslant C_G^\circ(\sigma_0) = T$ and $T = T^{gf} \leqslant B_l^g$. Now $\sigma \in N_G(T) \cap T^g \leqslant N_G(T) \cap B_l^g$ and as T is a Carter subgroup of B_l^g, we get $\sigma \in T$, a contradiction. Thus $C_G^\circ(\sigma) = T^g$ as claimed.

We claim now that $\sigma_0 \notin T^g$: otherwise $\langle\sigma_0\rangle \leqslant A_3^g$ and as the only proper nontrivial subgroup X of A_3^g such that $O(C_G^\circ(X)) = 1$ is $\langle\sigma_0^g\rangle$, we get $\langle\sigma_0\rangle = \langle\sigma_0^g\rangle = \langle\sigma\rangle$ (as $O(C_G^\circ(\sigma)) = O(T^g) = 1$ by Lemma 6.4). Thus $\langle\sigma\rangle \leqslant T$ and $\sigma \in T$, a contradiction. Thus $\sigma_0 \notin T^g$ as claimed, and $N_G(T^g) = T^g \rtimes \langle\sigma_0\rangle$.

Our final argument is now inspired by [22]. By Lemma 6.67, σ_0 and $\sigma\sigma_0$ are T^g-conjugate, $\sigma\sigma_0$ and $\sigma\sigma_0^2$ are T-conjugate, and $\sigma\sigma_0^2$ and σ_0^2 are T^g-conjugate. Thus $\sigma_0^{-1} = \sigma_0^2 = \sigma_0^h$ for some $h \in G$, and $h \in N_G(\langle\sigma_0\rangle) \leqslant N_G(C_G^\circ(\sigma_0)) = N_G(T) \leqslant C_G(\sigma_0)$. Thus $\sigma_0^{-1} = \sigma_0$, a final contradiction. $\square$

Corollary 6.70. *$\sigma \in O(B)$ for some Borel subgroup B of G (here we do not know whether $B \in \mathfrak{B}$, or B is conjugate to B_1).*

Proof. By Lemma 6.68, we may assume toward a contradiction that we have $\sigma \in (B_1 \setminus O(B_1))^g$ for some $g \in G$. Then $T^g \cong B_1^g/O(B_1)^g$ contains an element of order 3. By Fact 2.5, $\mathrm{char}(T) \neq 3$, i.e., $p \neq 3$. By Lemma 6.25, the Sylow 3-subgroup of $O(B_1)$ is trivial, thus Hall $\{2, 3\}$-subgroups of B_1 are abelian (as $B_1{}' \leqslant O(B_1)$) and conjugate to the Hall $\{2, 3\}$-subgroup of T (Facts 2.5, 2.13, and 2.14). Thus σ is in a conjugate of T, a contradiction to Lemma 6.69. $\square$

Corollary 6.71. $p = 3$.

Proof. We apply the preceding corollary and Lemmas 6.25 and 6.27. $\square$

This ends the proof of part (2) of Theorem 1.8, and in fact much more, in the case “C not a Borel subgroup of G.”

To complete our analysis, we now look at the geometry of involutions. Let

$$D = \{(j, k) \in I(G)^2\colon [j, k] \neq 1\}.$$

By genericity and Fact 2.36, one sees as in the end of Section 5.2 that D is generic in $I(G)^2$.

By Lemma 6.68, we have a definable partition of D into definable subsets D_1 and D_2, that is $D = D_1 \sqcup D_2$, where

$$D_1 = \{(j,k) \in D\colon\ jk \in O(B) \text{ for some } B \in \mathfrak{B}\} \quad \text{and}$$
$$D_2 = \{(j,k) \in D\colon\ jk \in B_1^g \text{ for some } g \in G\}.$$

Lemma 6.72. *Let $(j,k) \in D$. Then $(j,k) \in D_2$ if and only if $(jk)^2 \in O(B_1)^g$ for some $g \in G$.*

Proof. Assume $(j,k) \in D_2$, i.e., $jk \in B_1^g$ for some $g \in G$. We claim that $j,k \in B_1^g$. If $C^\circ_{O(B_1)^g}(jk) \neq 1$, then $O(C^\circ_G(jk)) \leqslant O(B_1)^g$ and $j,k \in N_G(B_1)^g = B_1^g$ by Proposition 3.11(ii). So we may assume $C^\circ_{O(B_1)^g}(jk) = 1$ and the generalized centralizer of jk in B_1^g is then a Carter subgroup of B_1^g by Corollary 2.24; in particular, jk is in a conjugate of T and $jk \in I(G)$ by Lemma 6.28, a contradiction as j and k do not commute. Thus $j,k \in B_1^g$ as claimed and, computing in B_1^g modulo $O(B_1)^g$, one sees with Theorem 6.6 that $(jk)^2 \in O(B_1)^g$.

Suppose now $(jk)^2 \in O(B_1)^g$ for some $g \in G$. Then $O(C^\circ_G((jk)^2)) = O(B_1)^g$ (Lemma 2.41 and Proposition 3.11(ii)) and $j,k \in N_G(O(B_1)^g) = N_G(B_1^g) = B_1^g$. In particular, $jk \in B_1^g$ and (j,k) is in D_2. $\square$

Lemma 6.73. *D_1 is generic in D (and, thus, in $I(G)^2$). In particular, $\mathfrak{B}$ is nonempty.*

Proof. Assume toward a contradiction that D_2 is generic in D and, in particular, that D_2 has Morley degree 1 as $I(G)^2$ does. We will show that D_2 cannot have degree 1 and, thus, get a contradiction.

Consider the definable map

$$\psi : D_2 \longrightarrow i_1^G, \quad (j,k) \longmapsto z_{j,k},$$

where $z_{j,k}$ is the unique involution in the center of the unique conjugate of B_1 containing $(jk)^2$ as in the preceding lemma.

Notice that

$$\psi^{-1}(i_1) = \bigsqcup_{(l,l') \in \{2,3\}^2} \{(i_l f, i_{l'} f')\colon\ f, f' \in O(B_1),\ f \neq f'\}. \tag{$*$}$$

It is a routine matter to check equality $(*)$ once one has noticed that a couple of involutions $(i_l f, i_{l'} f')$ in B_1 (with $(l,l') \in \{2,3\}^2$ and $f, f' \in O(B_1)$) is noncommuting if and only if $f \neq f'$. By Theorem 6.6, this is clear if $l = l'$ and if $l \neq l'$, it follows from the following equivalent equalities:

$$[i_l f, i_{l'} f'] = 1, \qquad i_1 f i_{l'} f' = f' i_l f, \qquad i_1 f i_{l'} f' f^{-1} = f' i_l, \qquad i_1 i_{l'} f^{-1} f' f^{-1} = f' i_l,$$
$$i_l f^{-2} f' = f' i_l, \qquad f^{-2} f' = i_l f' i_l, \qquad f^2 = f'^2, \qquad f = f'.$$

The four pieces F_1, F_2, F_3, and F_4 in the decomposition $(*)$ of $\psi^{-1}(i_1)$ all have rank $2\,\mathrm{rk}(O(B_1))$ and degree 1, as $O(B_1)$ has degree 1. It follows that $\psi^{-1}(i_1)$ has Morley rank $2\,\mathrm{rk}(O(B_1))$ and Morley degree 4. On the other hand, one checks easily with Theorem 6.6 that the four pieces in the decomposition $(*)$ of $\psi^{-1}(i_1)$ are invariant under conjugation by elements of $B_1 = C_G(i_1)$. As involutions are conjugate,

$$D_2 = \bigsqcup_{\bar{g} \in G/B_1} \bigl(\psi^{-1}(i_1)^{\bar{g}}\bigr) = \bigsqcup_{\bar{g} \in G/B_1} \bigl(F_1^{\bar{g}} \sqcup F_2^{\bar{g}} \sqcup F_3^{\bar{g}} \sqcup F_4^{\bar{g}}\bigr),$$

thus $D_2 = \bigsqcup_{s=1}^{4}(\bigsqcup_{\bar{g} \in G/B_1} F_s^{\bar{g}})$. As these four definable pieces in this decomposition of D_2 have the same rank, D_2 cannot have degree 1, which gives the desired contradiction. $\square$

For $(j,k) \in D_1$, we have $jk \in O(B)$ for some Borel subgroup $B \in \mathfrak{B}$, thus jk is a 3-element as $p = 3$ and $O(B)$ is 3-unipotent. We finish our analysis by showing that, generically, jk has exponent greater than 3.

Lemma 6.74. *For (j,k) generic in D_1 (and, thus, in $I(G)^2$), jk is a 3-element of order at least 9.*

Proof. Assume toward a contradiction that the subset $D_1{}'$ of D_1, consisting of couples (j,k) such that jk has order 3, is generic in D_1. Let π_1 denote the first projection of $D_1{}'$ over $I(G)$. As involutions are conjugate, our genericity assumption implies that $\mathrm{rk}(\pi_1^{-1}(i)) = \mathrm{rk}(I(G))$ for every involution $i \in I(G)$. In particular, the set of involutions z such that each of the three products $i_l z$ has order 3 is generic in $I(G)$. But for such a z, if we let $x = i_1 z$, then $x^3 = 1$ and $(i_2 x)^3 = (i_3 z)^3 = 1$. Thus $[i_2, i_2^z]$ is equal to

$$\bigl[i_2, i_2^x\bigr] = i_2 x^{-1}(i_2 x i_2)x^{-1} i_2 x = i_2 x^{-1}\bigl(x^{-1} i_2 x^{-1}\bigr)x^{-1} i_2 x = i_2 x i_2 x i_2 x = 1.$$

On the other hand, $[i_2, i_2^z] = (i_2 z)^4 = i_2 z$, thus $i_2 z = 1$; but $i_2 z$ has order 3, a contradiction. $\square$

7. $\mathrm{Pr}_2(G) > 1$ and $C_G^\circ(A)$ a Borel

In this final section, G and the notations are fixed as always as in Theorem 1.8, and we consider the only remaining case:

$$\mathrm{Pr}_2(G) > 1 \text{ and } C = C_G^\circ(A) \text{ is a Borel subgroup of } G.$$

We will prove part (2b) of Theorem 1.8. We will also complete our proof that $\mathrm{Pr}_2(G) \leqslant 2$ at the end of this section; recall that the other case was treated already in Proposition 6.3. Notice that our assumption implies that $I(C) = A^\#$ by Fact 2.12.

7.1. *Case: $C_G^\circ(A)$ a nonnilpotent Borel subgroup*

We will eliminate this case (assuming, as always in this section, that the Prüfer rank is at least 2).

Theorem 7.1. *If C is a Borel subgroup of G, then it is nilpotent.*

So we assume toward a contradiction that $|A| \geqslant 4$ and that $C_G^\circ(A)$ is a nonnilpotent Borel subgroup.

Lemma 7.2. $O(C) \neq 1$.

Proof. This is a special case of Lemma 3.2, as C is nonnilpotent. □

Lemma 7.3. $C \cap C^g = 1$ *for each* $g \in G \setminus N_G(C)$.

Proof. Assume that $C \cap C^g \neq 1$ for some $g \in G \setminus N_G(C)$. As $I(C) \subseteq Z(C)$ and C is a Borel subgroup of G, the intersection $C \cap C^g$ has no involutions. If $(C \cap C^g)^\circ$ is nontrivial, then by Proposition 3.11(ii) we have $C = C^g$, a contradiction. Thus $(C \cap C^g)^\circ = 1$ and $C \cap C^g$ is finite.

Thus, there is an element x of prime order p in $C \cap C^g$. We claim now that $F^\circ(C)$ contains no nontrivial p-unipotent subgroup: else, it would contain a maximal p-unipotent subgroup U_p normal in C (Corollary 2.16), and $C_{U_p}^\circ(x) \neq 1$ (Fact 2.9(iii)), showing that $C_G^\circ(x) \leqslant C$ by Proposition 3.11(ii); but then $C_{C^g}^\circ(x) \leqslant (C \cap C^g)^\circ = 1$, which contradicts Fact 2.17. The claim is proved.

We can now apply Corollary 2.20 to x in C and in C^g; this implies that $C_G^\circ(x)$ contains a Sylow 2-subgroup of C, say S_1, as well as a Sylow 2-subgroup of C^g, say S_2. Let B_1 be a Borel subgroup of G containing $C_G^\circ(x)$. If B_1 is abelian, then $S_1 = S_2 \leqslant C \cap C^g$, which contradicts the preceding remarks. Thus B_1 is not abelian and Lemma 3.2 shows that $O(B_1) \neq 1$. As $A^\# = I(S_1^\circ)$ consists of at least three involutions, there is $k \in A^\#$ such that $C_{O(B_1)}^\circ(k) \neq 1$ by Fact 2.37. Then $C = B_1$ by Proposition 3.11(ii). By considering the action of A^g on $O(B_1)$, one sees in the same way that $C^g = B_1$. Thus again $C = C^g$, a contradiction. □

Corollary 7.4. $\bigcup_{g \in G} C^g$ *is generic in* G.

Corollary 7.5. *If x is in $N_G(C) \setminus C$ and x is of order n modulo C, for some integer n, then the elements of the coset xC are generically of order n.*

Proof. It suffices to apply the preceding corollary and Lemma 3.4, and to remark that an element $x_1 \in N_G(C) \setminus C$ of order n modulo C and such that $x_1 \in (\langle x \rangle C)^g$ for some $g \in G \setminus N_G(C)$ satisfies $x_1^n \in C \cap C^g = 1$. □

Proof of Theorem 7.1. We claim first that $N_G(C) = C$. If not, then there is an element $x \in N_G(C) \setminus C$ of prime order p. The preceding corollary shows that the elements of the coset xC are generically of order p. But then Fact 2.29 implies that C must be nilpotent, a contradiction to our assumption. Thus C is selfnormalizing as claimed.

Now Lemma 7.3 shows that C is strongly embedded in G and Fact 2.35 implies that C has only one conjugacy class of involutions. But as $I(C) \subseteq Z(C)$, we have that C has only one involution and $|A^{\#}| = 1$, which contradicts our assumption that the Prüfer 2-rank is at least 2. □

7.2. *Case: $C_G^\circ(A)$ a nilpotent Borel subgroup*

If C is a nilpotent Borel subgroup of G, then $T = C$ by Fact 2.8. We will show that $N_G(T)$ is strongly embedded in G (Corollary 7.14), that $|A| = 4$, and that the Weyl group $W = N_G(T)/T$ is cyclic of order 3 in Proposition 7.29. This will prove part (2) of Theorem 1.8 in this case "C a nilpotent Borel subgroup of G," and will complete our proof that $\mathrm{Pr}_2(G) \leqslant 2$. We will also obtain a detailed description of G in the course of an extended analysis.

Lemma 7.6. *$T \cap T^g = 1$ for each $g \in G \setminus N_G(T)$.*

Proof. Assume that $T \cap T^g \neq 1$, with $g \in G$. Proposition 3.11 then shows that $O(T) = O(T^g) = 1$. But then Lemma 3.2 implies that T is abelian, thus $T, T^g \leqslant C_G^\circ(T \cap T^g)$ and $T = T^g = C_G^\circ(T \cap T^g)$ as T is a Borel subgroup of G. Thus $g \in N_G(T)$. □

Corollary 7.7. *$\bigcup_{g\in G} T^g$ is generic in G.*

Corollary 7.8. *If x is in $N_G(T) \setminus T$ and x is of order n modulo T, then the elements of the coset xT are generically of order n.*

Proof. As in Corollary 7.5, using Lemma 7.6 and Corollary 7.7. □

Corollary 7.9. *$C_G(S^\circ) = T$.*

Proof. This follows from Corollary 7.8 and Lemma 3.8. □

We now detail the general structure of G. Let $\mathfrak{B}$ be the set of Borel subgroups of G nonconjugate to T and having a nontrivial Sylow 2-subgroup. This definition is different from the one in Section 6 (before Lemma 6.22), but the same as in Section 5.2 (before Lemma 5.11). In the next lemmas we will see that Borel subgroups in $\mathfrak{B}$ have the same kind of behavior as those in the previous sections.

Lemma 7.10. *$\mathfrak{B}$ is nonempty, and every Borel subgroup of G nonconjugate to T is in $\mathfrak{B}$. If $B \in \mathfrak{B}$ contains an involution $k \in A^{\#}$, then $B = F(B) \rtimes C_B(k)$, $F(B) = O(B)$ is inverted by k, and $C_B(k)$ is a connected divisible abelian subgroup of T such that* $\mathrm{Pr}_2(C_B(k)) = 1$.

Furthermore,

$$G = \left(\bigcup_{g\in G} N_G(T)^g\right) \cup \left(\bigcup_{B\in\mathfrak{B}} N_G(B)\right).$$

Proof. We first show that G contains no Borel subgroups without involutions. Suppose that B is such a Borel subgroup of G. Then $B = O(B)$ is nilpotent as it interprets no bad fields, and Proposition 3.11 shows that two distinct conjugates of B have a trivial intersection. Thus $\bigcup_{g\in G} B^g$ is generic in G by Lemma 3.3, as well as $\bigcup_{g\in G} T^g$. But then there exists an element $b \in B^\#$ which is in a conjugate of T by Fact 2.1. In particular, b centralizes a conjugate of S°. This is a contradiction because $C^\circ_G(b) \leqslant B$ (Lemma 3.12), and B has no involutions. Thus every Borel subgroup of G has an involution. If every such Borel subgroup is conjugate to T, then G is a simple bad group, and it cannot have involutions by Fact 1.3, a contradiction. Thus $\mathfrak{B}$ is nonempty.

Now let B be a Borel subgroup in $\mathfrak{B}$ containing an involution $k \in A^\#$. If $k \in F(B)$ then $k \in Z(B)$ by Lemma 3.1. But k is in a Sylow 2-subgroup of B which is connected by Fact 2.12, thus in $S^{\circ g}$ for some $g \in G$. So B, $T^g \leqslant C^\circ_G(k)$, and $B = T^g$ by maximality, a contradiction to the definition of $\mathfrak{B}$, which shows that $F(B)$ has no involutions. In particular, B is nonnilpotent, and $F^\circ(B) = O(B)$ by Lemma 3.2. As $C^\circ_{O(B)}(k)$ is a subgroup of T, if $C^\circ_{O(B)}(k) \neq 1$ then Proposition 3.11(ii) implies that $T = B$, a contradiction. Thus $C^\circ_{O(B)}(k)$ is trivial and Fact 2.25 shows that $O(B)$ is inverted by k. As $B/O(B)$ is abelian by Fact 2.15, we conclude that $B = O(B) \rtimes C_B(k)$ by Fact 2.27. It follows then from Fact 2.1 that $C_B(k)$ is connected and contained in $C^\circ_G(k) = T$. As $C_B(k)$ is isomorphic to $B/F(B)$, it is also divisible abelian by Fact 2.15. We now show that $O(B) = F(B)$. If $O(B) < F(B)$, then the finite group $C_B(k) \cap F(B)$ is nontrivial and it contains an element t of prime order p. As $C_B(k)$ is divisible, Fact 2.12 shows that t is in a p-torus of $C_B(k)$; so it is in a p-torus of T and t is central in T by Fact 2.10. Thus $T \leqslant C^\circ_G(t) \leqslant B$ by Lemma 3.12 and $T = B$ by maximality, a contradiction which shows that $O(B) = F(B)$. If $C_B(k)$ contains an elementary abelian 2-subgroup A_1 of A order four, then each involution in A_1 inverts $O(B)$, a contradiction. So $\mathrm{Pr}_2(C_B(k)) = 1$.

It remains to show that $G = (\bigcup_{g\in G} N_G(T)^g) \cup (\bigcup_{B\in\mathfrak{B}} N_G(B))$. If g is any element in G, then g has an infinite centralizer by Corollaries 7.7 and 2.18, that is $C^\circ_G(g) \neq 1$. If $C^\circ_G(g)$ contains an involution, then it contains a nontrivial 2-torus by Fact 2.12, so it contains an element of the form k^h for some involution $k \in A^\#$ and some element $h \in G$. Then $g \in N_G(C^\circ_G(k^h)) \leqslant N_G(T)^h$. If $C^\circ_G(g)$ has no involutions, then it is in a unique Borel subgroup B of G by Proposition 3.11(ii), and $g \in N_G(B)$. □

We now look at the structure of the finite group $N_G(T)/T$, which acts faithfully on S°. In what follows the notation $\overline{}$ denotes the quotient by T.

Lemma 7.11. *$\overline{N_G(T)}$ is nontrivial.*

Proof. Otherwise Lemma 7.6 shows that T is strongly embedded in G, and hence has a single conjugacy class of involutions. But T centralizes A, so this would force $|A| = 2$. □

Lemma 7.12. *$\overline{N_G(T)}$ contains at most one involution $\overline{w}$. In that case $\overline{w}$ is the image of an involution $w \in G$ which inverts T, and $wT = w^T$.*

Proof. Assume that $w \in N_G(T) \setminus T$ is such that $\overline{w}$ is an involution. Then elements of the coset wT are generically of order 2 by Corollary 7.8, and Fact 2.28 shows that w is an involution which inverts T. In that case $wT = w^T$ because T is 2-divisible.

It remains now to show that such a hypothetical involution is unique. If $\overline{w'}$ is another involution, then w' also inverts T, and $ww' \in C_G(S^\circ) = T$ by Corollary 7.9, that is $\overline{w} = \overline{w'}$. □

Lemma 7.13. *$\overline{N_G(T)}$ is of odd order.*

Proof. Assume that there is an involution $w \in N_G(T) \setminus T$ which inverts T. We have two cases to consider, according as w is, or is not, conjugate to an involution of $A^\# = I(S^\circ)$.

Assume first that $w = i^g$ for some involution i of S° and some $g \in G$. We claim in this case that all involutions of A invert T^g, which provides a contradiction. Let $j \in A$. Then j centralizes $w = i^g$. Thus j normalizes T^g by Lemma 7.6. As $T \cap T^g$ is trivial by Lemma 7.6, we have $j \in N_G(T^g) \setminus T^g$. Then by Lemma 7.12 j inverts T^g.

It remains to treat the case in which w is not conjugate to an involution of S°, which we assume now. Notice that $C_G^\circ(w) \neq 1$, as otherwise G would be abelian by Fact 2.25. If w centralizes a nontrivial connected 2-subgroup of G, say S_1, then $\langle w\rangle S_1$ is in a Sylow 2-subgroup S_2 of G. As we assume $w \notin I(S^\circ)^G$, we have that $w \in S_2 \setminus S_2^\circ$ and w inverts S_2° by Lemma 7.12, a contradiction as w centralizes S_1. Thus $C_G^\circ(w)$ has no involution. Proposition 3.11(ii) then shows that $C_G^\circ(w) \leqslant B$ for a unique Borel subgroup B of G. In particular, $C_G(w) \leqslant N_G(B)$. As w inverts S°, w centralizes A and thus $A \leqslant N_G(B)$. Notice that B is not a conjugate of T, as otherwise Lemma 7.12 would show that w inverts B, a contradiction as $C_G^\circ(w) \neq 1$. Thus Lemma 7.10 shows that $F(B) = O(B)$. If k is any involution in A, then $C_{O(B)}^\circ(k) = 1$ by Proposition 3.11(ii), thus k inverts $F(B)$ by Fact 2.25. This contradicts our assumption that $|A| \geqslant 4$. □

Corollary 7.14. *$N_G(T)$ is strongly embedded in G (in particular, $N_G(T)$ acts transitively on $A^\#$).*

Proof. If $N_G(T) \cap N_G(T)^g$ contains an involution k for some $g \in G$, then k is in $T \cap T^g$, thus $T = T^g$ by Lemma 7.6, and $g \in N_G(T)$. So $N_G(T)$ is strongly embedded in G and Fact 2.35 shows that it acts transitively by conjugation on the set of its involutions, that is $A^\#$. □

Lemma 7.15. *Assume that t is a nontrivial element of $d(S^\circ)$ such that $T < C_G(t)$. Let $x \in C(t) \setminus T$. Then x has finite order modulo T, and if this order is n, then $t^n = 1$.*

Proof. Let t and x be as in the statement. As $C_G^\circ(t) = T$, we have $C_G(t) \leqslant N_G(T)$ and thus x has finite order modulo T. Let its order be n. The elements of the coset xT are generically of order n by Corollary 7.8, so as in the proof of Lemma 3.8, we can find an element $x_1 \in xT$ of order n such that the elements of the coset $x_1 d(S^\circ)$ are generically of order n. As $d(S^\circ)$ is divisible, it is the connected component of the definable group $d(S^\circ) \rtimes \langle x_1\rangle$, and we can apply Lemma 3.6 to get that the elements of the coset $x_1 d(S^\circ)$ are all of order n. In particular, $t^n = x_1^n t^n = (x_1 t)^n = 1$, which proves our lemma. □

Corollary 7.16. *$C_G(k)$ equals T for each involution $k \in A^\#$. In particular $N_G(T)/T$ acts regularly by conjugation on $A^\#$, and $|N_G(T)/T| = 2^{\mathrm{Pr}_2(G)} - 1$.*

Lemma 7.15 also allows us to make precise the structure of Borel subgroups in $\mathfrak{B}$, refining Lemma 7.10.

Corollary 7.17. *If $B \in \mathfrak{B}$ contains an involution $k \in A^\#$, then $C_{N_G(B)}(k) < N_G(B)$ is a Frobenius group with $O(B)$ as a Frobenius kernel, and $C_{N_G(B)}(k) \leqslant T$. In particular,*

$$N_G(B) = O(B) \sqcup \left(\bigcup_{u \in O(B)} C_{N_G(B)}(k)^{u\#} \right),$$

k is the unique involution in $C_{N_G(B)}(k)$, and $I(N_G(B)) = kO(B)$. We also have that $C_G(f) = O(B)$ for every nontrivial element f of $F(B) = O(B)$.

Proof. Let B and k be as in the statement. Lemma 7.10 tells us that $\mathrm{Pr}_2(B) = 1$. If T_k is a Sylow 2-subgroup of B containing k, $N_G(B) = N_{N_G(B)}(T_k)B$ by the Frattini argument, that is $N_G(B) = C_{N_G(B)}(k)B$. Then Lemma 7.10 shows that $N_G(B) = C_{N_G(B)}(k)O(B)$ and as k inverts $O(B)$, the product is semidirect. Corollary 7.16 tells us that $C_{N_G(B)}(k) \leqslant T$. If an element $u \in O(B)$ is such that $C_{N_G(B)}(k) \cap C_{N_G(B)}(k)^u$ is nontrivial, then $u \in N_G(T)$ by Lemma 7.6, so $u \in N_G(T \cap B) = C_B(k)$ and $u \in C_B(k) \cap O(B) = 1$. Thus $C_{N_G(B)}(k) < N_G(B)$ is a Frobenius group with $O(B)$ as a Frobenius kernel.

If z is an involution in $C_{N_G(B)}(k)$ distinct from k, then $z \in I(T) = A^\#$ by Corollary 7.16 and there is an involution z' in the elementary abelian 2-group $\langle k, z\rangle$ of order 4 with an infinite centralizer in $O(B)$ by Fact 2.37. Then $B = C_G^\circ(z')$ by Proposition 3.11(ii), a contradiction as $C_G(z') = T$ by Corollary 7.16. Thus k is the unique involution of $C_{N_G(B)}(k)$.

Let now f be a nontrivial element of $O(B)$. We get as in Corollary 5.16, using Lemma 7.10, that $C_G^\circ(f) = O(B)$. In particular, we have $C_G(f) \leqslant N_G(B) = O(B) \rtimes (T \cap N_G(B))$. As f is not in the Frobenius complement $(T \cap N_G(B))$ of $N_G(B)$, we have that $C_{(T \cap N_G(B))}(f) = 1$. Thus $C_G(f) = O(B)$. $\square$

Corollary 7.18. $G = \{1\} \sqcup (\bigcup_{g \in G} T^g)^\# \sqcup (\bigcup_{B \in \mathfrak{B}} O(B))^\#$.

Proof. First note that the union of nontrivial elements in the statement is disjoint: if $u \in O(B)^\#$ for some $B \in \mathfrak{B}$, then $C_G(u) = O(B)$ (Corollary 7.17) has no involution and u cannot be in a conjugate of T.

If g is a nontrivial element of G, then $C_G^\circ(g)$ is nontrivial by Corollaries 2.18 and 7.7. If $C_G^\circ(g)$ contains an involution, then this involution is in $S^{\circ h}$ for some $h \in G$ by Lemma 7.13 and $g \in T^h$ by Corollary 7.16. Suppose now that $C_G^\circ(g)$ has no involution. Then $C_G^\circ(g)$ is in a unique Borel subgroup B of G by Proposition 3.11(ii), and $g \in N_G(B)$. If $B \in \mathfrak{B}$, then $g \in O(B)$ or g is in a conjugate of T by Corollary 7.17. If $B \notin \mathfrak{B}$, then $B = T^h$ for some $h \in G$ by Lemma 7.10 and it remains to show that $g \in T^h$ in that case. So we assume now that $g \in N_G(T^h) \setminus T^h$ and we will get a contradiction.

By conjugation, we assume thus that $N_G(T) \setminus T$ contains an element x such that $C_G^\circ(x) \leqslant T$. There is an integer k such that x^k is of prime order p modulo T. Now $1 \neq C_G^\circ(x) \leqslant C_T^\circ(x^k)$. As cosets of T in $T\langle x^k\rangle$ (distinct from T) are generically of order p by Corollary 7.8, we can apply Lemma 3.7. So the maximal p-unipotent subgroup U_p of T (which is unique by Fact 2.8) is nontrivial. One can find by Lemma 3.4 an element $x_1 \in x^k T \cap (\langle x^k\rangle T)^l$ for some $l \in G \setminus N_G(T)$. Thus $x_1^p \in T \cap T^l = 1$ and as x_1 normalizes U_p and U_p^l, we have $C_{U_p}^\circ(x_1) \neq 1$ and $C_{U_p^l}^\circ(x_1) \neq 1$ (Fact 2.9). Then $1 \neq C_G^\circ(x_1) \leqslant T \cap T^l$ by Proposition 3.11(ii), and $l \in N_G(T)$ by Lemma 7.6, a final contradiction. □

We now give a strong form of Corollary 7.8.

Lemma 7.19. *If x is in $N_G(T) \setminus T$ and is of order n modulo T, for some integer n, then $xT = x^T$ and every element in the coset xT is of order n.*

Proof. By Corollary 7.8, it suffices to show that $xT = x^T$. If $x_1 \in xT$, then $C_T^\circ(x_1) = 1$; this can be seen as in the end of the proof of Corollary 7.18. So $\mathrm{rk}(x_1^T) = \mathrm{rk}(x_1 T)$. As this is valid for any $x_1 \in xT$, Fact 2.1 shows that $xT = x^T$. □

We will now use our assumption that G interprets no bad field in a critical manner.

Lemma 7.20. *Let $k \in I(A)$ and $S_k < S$ be a 2-torus of Prüfer 2-rank one containing k, and assume that there is a Borel subgroup B in $\mathfrak{B}$ containing S_k. Then B interprets an algebraically closed field K in such a way that $d(S_k)$ is interpretably isomorphic to $K^\times$. Furthermore proper definable subgroups of $d(S_k)$ are finite.*

Proof. Let U be a B-minimal subgroup of B in $O(B)$. Recall that $B = O(B) \rtimes C_B(k)$ where $O(B)$ and $C_B(k)$ are abelian (Lemma 7.10), so U is also $C_B(k)$-minimal. Corollary 7.17 shows that $C_G(U) = O(B)$, so the centralizer of U in $C_B(k)$ is trivial. By Fact 2.38 and the assumption that B interprets no bad field, $U \rtimes C_B(k)$ interprets an algebraically closed field K in such a manner that $U \cong K^+$, $C_B(k) \cong K^\times$, where both isomorphisms are interpretable, and proper definable subgroups of $C_B(k)$ are in particular finite. As $C_B(k)$ is definable and contains S_k, we have $d(S_k) \leqslant C_B(k)$, so $d(S_k) = C_B(k)$. □

Let $n = \mathrm{Pr}_2(G)$, and let $\{i_1, \ldots, i_{2^n-1}\}$ enumerate $I(A)$ in such a way that $\{i_1, \ldots, i_n\}$ generates A. Fix B a Borel subgroup in $\mathfrak{B}$ containing i_1. Let $T_{i_1} = B \cap T = C_B(i_1)$ and S_{i_1} be the 2-torus of T_{i_1} of Prüfer 2-rank one (Corollary 7.17). As $N_G(T)$ acts transitively by conjugation on $I(A)$, there are $2^n - 1$ distinct conjugates S_{i_s} of S_{i_1} in S, each one containing respectively i_s $(1 \leqslant s \leqslant 2^n - 1)$. If $s \neq s'$, then $S_{i_s} \cap S_{i_{s'}} = 1$, as otherwise $i_s = i_{s'}$. By considering the Prüfer 2-rank, we have thus

$$S = S^\circ = \bigoplus_{s=1}^{n} S_{i_s}. \tag{1}$$

It is then clear that

$$d(S) = \prod_{s=1}^{n} d(S_{i_s}). \tag{2}$$

We now apply Lemma 7.20 with i_1, S_{i_1}, and B, and we let K be the field interpreted by B. Let also

$$p = \operatorname{char}(K).$$

We will show later that $p > 0$.

Lemma 7.21. $\operatorname{Pr}_q(d(S^\circ)) = n$ *for every prime number* q *different from* p*, and if* $p \neq 0$*, then the Sylow* p*-subgroup of* $d(S^\circ)$ *is trivial.*

Proof. If $1 < s \leqslant n$, then i_s is the unique involution in the conjugate $d(S_{i_s})$ of $d(S_{i_1})$, and easily $i_s \notin \prod_{s'=1}^{s-1} d(S_{i_{s'}})$. Thus $\prod_{s'=1}^{s-1} d(S_{i_{s'}}) \cap d(S_{i_s})$ is a proper subgroup of $d(S_{i_s})$ and this intersection must be finite by Lemma 7.20.

The conjugates $d(S_{i_s})$ of $d(S_{i_1})$ are all isomorphic to $K^\times$. If q is now a prime number different from p, then it follows from the preceding and an induction over s varying between 1 and n that $\operatorname{Pr}_q(d(S^\circ)) = n$ by equality (2). If $p \neq 0$, then $d(S_{i_1}) \cong K^\times$ has a trivial Sylow p-subgroup, as well as $d(S^\circ)$ by equality (2) and Fact 2.5. $\square$

We eventually derive the following information from the preceding lemma.

Corollary 7.22. $O(B_1) = F(B_1)$ *is torsion free or* p*-unipotent for every* $B_1 \in \mathfrak{B}$*, depending on whether* $p = 0$ *or* $p > 0$.

Proof. First note that if $B_1 \in \mathfrak{B}$, then $O(B_1) = F(B_1)$ has trivial q-tori for every prime number $q > 2$, because such a maximal q-torus is both central in B_1 by Fact 2.10 and inverted by involutions in B_1 by Lemma 7.10. Thus Fact 2.8 shows that

$$O(B_1) = D \times U_{p_1} \times \cdots \times U_{p_l}$$

for finitely many prime numbers $p_1, \ldots, p_l$, where D is torsion free and U_{p_s} is p_s-unipotent for every p_s ($1 \leqslant s \leqslant l$).

Assume that $p = 0$. In that case we have to show that $O(B_1) = F(B_1)$ is torsion free, that is that the factors of bounded exponent in the decomposition as above are trivial. But if $U_{p_s} \neq 1$ for a prime number p_s, then U_{p_s} contains a B_1-minimal subgroup U (as $U_{p_s} \lhd B_1$), which is an elementary abelian p_s-subgroup. Of course we may assume without loss of generality that B_1 contains an involution $k \in I(A)$. Now the same analysis as in the proof of Lemma 7.20, with our assumption that B_1 interprets no bad field, shows that $C_{B_1}(k) \cong K_1^\times$ where K_1 is an interpretable algebraically closed field of characteristic p_s, and that $C_{B_1}(k) = d(S_k)$ where S_k is a 2-torus of Prüfer 2-rank one in S. Choosing a suitable minimal set of generators of A containing k, one can then carry out the same analysis as in the proof of Lemma 7.21 with B_1, K_1, and S_k instead of B, K and S_{i_1}, to get that the p_s-torus of $d(S^\circ)$ is trivial. This is a contradiction to Lemma 7.21. Similarly, if $p \neq 0$, then $U_q = 1$ for $q \neq p$.

Assume now $p \neq 0$ and let $B_1 \in \mathfrak{B}$ contain an involution $k \in I(A)$. If $O(B_1) = F(B_1)$ is not p-unipotent, then as before one can interpret an algebraically closed field, which is now of characteristic 0. Thus there are nontrivial q-tori in $d(S^\circ)$ for every prime q, again providing a contradiction to Lemma 7.21. □

The following lemma is inspired by [22].

Lemma 7.23. *Let q be the smallest prime divisor of $|W|$. Then no element of $N_G(T)$ representing an element of W of order q lies in a conjugate of T.*

Proof. Note that $q > 2$. Let $w = xT$ be an element of W of order q. Suppose that x lies in a conjugate T^g of T. By Lemma 7.19 x has order q. In particular T has a nontrivial Sylow q-subgroup, say S_q.

As $S_q \lhd T$ (Fact 2.8), x centralizes an element y of order q in $S_q \cap Z(T)$ (Facts 2.12, 2.7, and 2.9). Lemma 7.19 tells us that x, xy, and xy^2 are T-conjugate. On the other hand, $y \in N_G(T^g)$ as $[x, y] = 1$ (Lemma 7.6) and $y \notin T^g$ (as $T \neq T^g$). Thus y is of order q modulo T^g and Lemma 7.19 applied in T^g gives that y and xy are T^g-conjugate in the coset $T^g y$, and similarly y^2 and xy^2 are conjugate in the coset $T^g y^2$. As xy and xy^2 are T-conjugate, we conclude that y and y^2 are conjugate by some element h, and $h \in N_G(T)$ as $y, y^2 \in Z(T)$. As y is of order $q \neq 2$ and $h \notin T$, we have $\langle y\rangle = \langle y^2\rangle$ and $T \leqslant C_G(\langle y\rangle) < N_G(\langle y\rangle) \leqslant N_G(T)$. But $N_G(\langle y\rangle)/C_G(\langle y\rangle)$ embeds into $\mathrm{Aut}(\mathbb{Z}_q)$ and $|\mathrm{Aut}(\mathbb{Z}_q)| = q - 1$, so there is a prime number q' dividing $|N_G(T)/T|$ and $q - 1$. This contradicts the minimality of q. □

Lemma 7.24. *p is the smallest prime divisor of $|W|$ (in particular $p \neq 0$).*

Proof. Let q be the smallest prime divisor of $|W|$ and let $x \in N_G(T) \setminus T$ represent an element of order q in W. As x is not in a conjugate of T (Lemma 7.23), by Corollaries 7.18 and 7.22, x is a p-element. Hence $q = p$. □

Corollary 7.25. *The Sylow p-subgroup of T is trivial.*

Proof. Let $u \in N_G(T) \setminus T$ have order p modulo T. By Lemma 7.19 u has order p. By Corollaries 7.18 and 7.22, $u \in O(B_1)$ for some $B_1 \in \mathfrak{B}$.

Let S_p be the Sylow p-subgroup of T. Corollary 7.17 shows that $C_{S_p}(u) \leqslant C_G(u) = O(B_1)$. As $T \cap O(B_1) = 1$ by Corollary 7.18, we find $C_{S_p}(u) = 1$. By Fact 2.9, S_p is trivial. □

Corollary 7.26. *The centralizers of nontrivial p-elements of $N_G(T)/T$ are p-groups.*

Proof. Assume the contrary. Then $N_G(T)/T$ contains an element of order pq for an odd prime $q \neq p$. So $N_G(T) \setminus T$ also contains an element x of order pq by Corollary 7.8. Then

$x \in \bigcup_{g \in G} T^g$ by Corollaries 7.18 and 7.22, so x^q is of order p and in a conjugate of T, a contradiction to Corollary 7.25. □

We now dramatically reduce the size of $N_G(T)/T$.

Lemma 7.27. *$|N_G(T)/T| = 2^n - 1$ divides $l^n - 1$ for every integer $l > 1$ which is relatively prime to $2^n - 1$.*

Proof. By Dirichlet's theorem on primes in arithmetic progression, we may suppose that l is prime. Let $A_l = \{a \in d(S^\circ)\colon a^l = 1\}$. This is an elementary abelian l-group of rank n by Lemma 7.21, and by Lemma 7.15, as l is not a divisor of $|W|$, the action of W on A_l is semiregular. By Corollary 7.16 $|W| = 2^n - 1$, and our claim follows. □

In view of Corollary 2.43 we conclude:

Corollary 7.28. *Only one of the following four cases can occur*:

(a) $n = 2$ *and* $|N_G(T)/T| = 3$,
(b) $n = 4$ *and* $|N_G(T)/T| = 15 = 3 \cdot 5$,
(c) $n = 6$ *and* $|N_G(T)/T| = 63 = 3^2 \cdot 7$,
(d) $n = 12$ *and* $|N_G(T)/T| = 4095 = 3^2 \cdot 5 \cdot 7 \cdot 13$.

Finally we have the following proposition.

Proposition 7.29. *$n = 2$ and $N_G(T)/T$ is cyclic of order 3.*

Proof. By the preceding corollary, it suffices to eliminate the possibilities $n = 4, 6, 12$, with the order of $W = N_G(T)/T$ correspondingly:

$$3 \cdot 5; \quad 3^2 \cdot 7; \quad 3^2 \cdot 5 \cdot 7 \cdot 13.$$

By Lemma 7.24 and Corollary 7.26, the centralizer in W of an element of order 3 is a 3-group. By elementary group theory, this cannot hold in the three cases mentioned.

If the order of $F(W)$ is divisible by 3, then the same applies to $Z(F(W))$ and hence $F(W)$ is a 3-group. By the Feit–Thompson theorem (or direct examination), W is solvable, and hence by Fitting's lemma its Fitting subgroup $F(W)$ contains its own centralizer. Thus $W/F(W)$ acts faithfully as a group of automorphisms of $F(W)$. However this is a numerical impossibility: for example, in the second case it would force $|\operatorname{Aut}(F(W))|$ to be divisible by 7, with $F(W)$ either $(\mathbb{Z}/3\mathbb{Z})^2$, or $\mathbb{Z}/9\mathbb{Z}$.

On the other hand, if $|F(W)|_3 = 1$, then we get a similar contradiction by considering the action of a Sylow 3-subgroup of W on some Sylow subgroup of $F(W)$. □

Corollary 7.30. *If $B \in \mathfrak{B}$, then $F(B) = O(B)$ is 3-unipotent.*

Proof. This follows from the preceding proposition and Corollaries 7.24, 7.19, 7.22, and 7.25. □

Another way to handle the final analysis was suggested by Ron Solomon.

Proposition 7.31. *Let W be a group acting regularly on an elementary abelian 2-group A of rank n. Suppose that there is a prime divisor p of $2^n - 1$ such that for all elements $w \in W$ of order p, $C_W(w)$ is a p-group. Then $|W|$ is a Mersenne prime.*

Proof. As W has odd order and acts without fixed points on A, by a theorem of Burnside its Sylow subgroups are cyclic. (In particular, one may see that W is solvable without invoking Feit–Thompson.)

The main claim is:

no subgroup of W is a Frobenius group.

Suppose $F = RS$ is such a group with Frobenius kernel R and complement S. Then the faithful representation of F on A is a sum of irreducible constituents which are induced representations associated to irreducible R-modules. But the restriction of such an induced representation to S gives a free module, so S has fixed points in A, a contradiction.

If $|W|$ is not a prime power, let r, s be two primes dividing $|W|$, such that r is a divisor of $|F(W)|$, and either r or s is p. As the Sylow subgroups of W are cyclic, there is a unique subgroup R of $F(W)$ of order r, and R is normal in W. Let S be a subgroup of W of order s and consider RS. By our assumption on p, the group RS is nonabelian and is therefore a Frobenius group. As this is a contradiction, we find that

$$|W| = p^m = 2^n - 1$$

for some m. Now an elementary number theoretic argument shows $m = 1$. (n is a prime power l^k; $p = 2^l - 1$; $m = 1$.) □

However, we still need the appeal to Corollary 2.43 to complete the analysis.

Finally, we can then conclude, as at the end of Section 6.4.

Lemma 7.32. *For (j, k) generic in $I(G)^2$, we have $[j, k] \neq 1$ and jk is a 3-element (of $O(B)$ for some Borel subgroup $B \in \mathfrak{B}$) of order at least 9.*

Proof. Follow the line of the argument for Lemma 6.74. □

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