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Effectivity on Continuous Functions in Topological Spaces

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Abstract

In this paper we investigate aspects of effectivity and computability on partial continuous functions in topological spaces. We use the framework of TTE, where continuity and computability on finite and infinite sequences of symbols are defined canonically and transferred to abstract sets by means of notations and representations. We generalize the representations introduced in [14] for the Euclidean case to computable T_0-spaces and computably locally compact Hausdorff spaces respectively. We show their equivalence and in particular, prove an effective version of the Stone-Weierstrass approximation theorem.

Keywords: Computable Analysis, TTE, Stone-Weierstrass approximation theorem, Representations of Continuous Functions

1 Introduction

Computable Analysis connects Computability/Computational Complexity with Analysis/Numerical Computation by combining concepts of approximation and of computation. During the last 70 years various mutually non-equivalent models of real number computation have been proposed (Chap. 9 in [14]). Among these

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models the representation approach (Type-2 Theory of Effectivity, TTE) proposed by Grzegorczyk and Lacombe [8,10] seems to be particularly realistic, flexible and expressive. So far the study of computability on sets of points, sets (open, closed, compact) and continuous functions has developed mainly bottom-up, i.e., from the real numbers to Euclidean space and metric spaces [18,2,16,14,19,1,20]. But often generalizations to more general spaces are needed (locally compact Hausdorff spaces [4], non-metrizable spaces [17], second countable T_0-spaces [11,7]).

In this article we investigate various representations of continuous functions in a general setting. In Section 2, we sketch some basic notions on TTE and provide some fundamental definitions and properties of representations of points and sets in computable T_0-spaces. In Section 3, we introduce some equivalent chacterizations of the representation η of the partial continuous functions on Cantor space. In Section 4 we introduce three multi-representations (open-open, via realization and via pointwise continuity) of the set of the partial continuous functions in computable T_0-spaces, we define the compact-open representation for total continuous functions in computably locally compact spaces and show their equivalences. In the last section we prove an effective version of the Stone-Weierstrass approximation theorem. For a compact space K we construct a notation ν_K of a dense subset $D_K \subseteq C(K)$ and a metric d_K such that the metric space $\mathbf{M}_K = (C(K), d_K, D_K, \nu_K)$ is semi-computable for κ-computable K, computable for κ_{mc}-computable K and the Cauchy representation is equivalent to the open-open representation.

2 Preliminaries

This section consists of two parts. In Section 2.1, we sketch some concepts of TTE. In Section 2.2, we introduce computable T_0-spaces and the underlying representations of points and sets.

2.1 Type-2 Theory of Effectivity (TTE)

In this article we use the framework of TTE (Type-2 theory of effectivity). For more details see [14]. We assume that Σ is a fixed finite alphabet containing the symbols 0 and 1 and consider computable functions on finite and infinite sequences of symbols Σ^* and Σ^ω, respectively, which can be defined, for example, by Type-2 machines, i.e., Turing machines reading from and writing on finite or infinite tapes. We use the "wrapping function" $\iota : \Sigma^* \to \Sigma^*$, $\iota(a_1a_2 \ldots a_k) := 110a_10a_20 \ldots a_k011$ for coding words such that $\iota(u)$ and $\iota(v)$ cannot overlap properly. We consider standard functions for finite or countable tupling on Σ^* and Σ^ω denoted by $\langle \cdot \rangle$. By "$\lhd$" we denote the subword relation. For $p \in \Sigma^\omega$ let $p^{<i} \in \Sigma^*$ be the prefix of p of length $i \in \mathbb{N}$. We write $x \sqsubseteq y$ if x is a prefix of y and $x \sqsubset y$ if x is a proper prefix of y.

We use the concept of multi-functions. A *multi-valued partial function*, or *multi-function* for short, from A to B is a triple $f = (A, B, \mathrm{R}_f)$ such that $\mathrm{R}_f \subseteq A \times B$ (the *graph* of f). Usually we will denote a multi-function f from A to B by $f :\subseteq A \rightrightarrows B$. For $X \subseteq A$ let $f[X] := \{b \in B \mid (\exists a \in X)(a, b) \in \mathrm{R}_f\}$ and for $a \in A$ define

$f(a) := f[\{a\}]$. Notice that f is well-defined by the values $f(a) \subseteq B$ for all $a \in A$. We define $\mathrm{dom}(f) := \{a \in A \mid f(a) \neq \emptyset\}$. In the applications we have in mind, for a multi-function $f :\subseteq A \rightrightarrows B$, $f(a)$ is interpreted as the set of all results which are "acceptable" on input $a \in A$. Any concrete computation will produce on input $a \in \mathrm{dom}(f)$ some element $b \in f(a)$, but usually there is no method to select a specific one. In accordance with this interpretation the "functional" composition $g \circ f :\subseteq A \rightrightarrows D$ of $f :\subseteq A \rightrightarrows B$ and $g :\subseteq C \rightrightarrows D$ is defined by $\mathrm{dom}(g \circ f) := \{a \in A \mid a \in \mathrm{dom}(f)$ and $f(a) \subseteq \mathrm{dom}(g)\}$ and $g \circ f(a) := g[f(a)]$ (in contrast to "non-deterministic" or "relational" composition gf defined by $g\,f(a) := g[f(a)]$ for *all* $a \in A$). A partial function from X to Y, denoted by $f :\subseteq X \to Y$, is a single-valued multi-function.

Notations $\nu :\subseteq \Sigma^* \to M$ and representations $\delta :\subseteq \Sigma^\omega \to M$ are used for introducing relative continuity and computability on "abstract" sets M. For a representation $\delta :\subseteq \Sigma^\omega \to M$, if $\delta(p) = x$ then the point $x \in M$ can be identified by the "name" $p \in \Sigma^\omega$. We will have applications where a sequence $p \in \Sigma^\omega$ contains information about a point x, which is sufficient for some computation, although p does not identify x. We arrive at the concept of *multi-notation* $\nu :\subseteq \Sigma^* \rightrightarrows M$ and *multi-representation* $\delta :\subseteq \Sigma^\omega \rightrightarrows M$. A multi-representation can be considered as a naming system for the points of a set M where each name can encode many points. It can be interpreted also as a naming system of an *attribute* on M. We generalize the concept of realization of a function or multi-function w.r.t. (single-valued) naming systems [14] to "naming systems", i.e., multi-notations or multi-representations, as follows [15]:

Definition 2.1 For naming systems $\gamma_i :\subseteq Y_i \rightrightarrows M_i$ $(i = 0, \ldots, k)$, abbreviate $Y := Y_1 \times \ldots \times Y_k$, $M := M_1 \times \ldots \times M_k$, and $\gamma(y_1, \ldots, y_k) := \gamma_1(y_1) \times \ldots \times \gamma_k(y_k)$. Then a function $h :\subseteq Y \to Y_0$ is a (γ, γ_0)-realization of a multi-function $f :\subseteq M \rightrightarrows M_0$, iff for all $p \in Y$ and $x \in M$,

(1) $x \in \gamma(p) \cap \mathrm{dom}(f) \Longrightarrow f(x) \cap \gamma_0 \circ h(p) \neq \emptyset\,.$

The multi-function f is
– (γ, γ_0)-continuous, if it has a continuous (γ, γ_0)-realization,
– (γ, γ_0)-computable, if it has a computable (γ, γ_0)-realization.
(We will say $(\gamma_1, \ldots \gamma_k, \gamma_0)$-computable instead of (γ, γ_0)-computable, etc.)

Fig. 1 illustrates the definition. Whenever p is a γ-name of $x \in \mathrm{dom}(f)$, then $h(p)$ is a γ_0-name of some $y \in f(x)$.

We introduce reduction and equivalence [13,12].

Definition 2.2 [reducibility, equivalence] For multi-representations $\gamma :\subseteq Y \rightrightarrows M$ and $\gamma' :\subseteq Y' \rightrightarrows M'$ $(Y, Y' \in \{\Sigma^*, \Sigma^\omega\})$, let $\gamma \leq_t \gamma'$ (t-reducible) and $\gamma \leq \gamma'$ (reducible), iff the identity $\mathrm{id} : a \mapsto a$ $(a \in M)$ is (γ, γ')-continuous and (γ, γ')-computable, respectively. Define t-equivalence and equivalence by
$\gamma \equiv_t \gamma' \iff (\gamma \leq_t \gamma'$ and $\gamma' \leq_t \gamma)$ and $\gamma \equiv \gamma' \iff (\gamma \leq \gamma'$ and $\gamma' \leq \gamma)$, respectively.

Two representations induce the same continuity or computability, iff they are

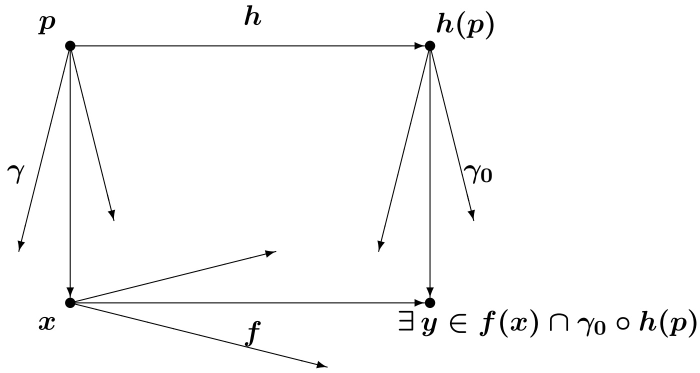

Fig. 1. $h(p)$ is a name of some $y \in f(x)$, if p is a name of $x \in \mathrm{dom}(f)$.

t-equivalent or equivalent, respectively. If multi-functions on represented sets have realizations, then their composition is realized by the composition of the realizations. In particular, the computable multi-functions on represented sets are closed under composition. Much more generally, the computable multi-functions on multi-represented sets are closed under flowchart programming with indirect addressing [15]. This result allows convenient informal construction of new computable functions on multi-represented sets from given ones.

For multi-representations $\delta :\subseteq \Sigma^\omega \rightrightarrows M$ and $\delta' :\subseteq \Sigma^\omega \rightrightarrows M'$ define $[\delta, \delta'] :\subseteq \Sigma^\omega \rightrightarrows M \times M'$ by $[\delta, \delta']\langle p, p'\rangle := \delta(p) \times \delta(p')$, $\delta \wedge \delta :\subseteq \Sigma^\omega \rightrightarrows M \cap M'$ by $\delta \wedge \delta'\langle p, p'\rangle := \delta(p) \cap \delta'(p')$ and $[\delta]^\omega :\subseteq \Sigma^\omega \rightrightarrows M^\omega$ by $[\delta]^\omega\langle p_0, p_1, p_2, \ldots\rangle := (\delta(p_0) \times \delta(p_1) \times \delta(p_2) \times \ldots)$.

Let ρ be the standard representation of $\mathbb{R}$. A $\rho^<$-name represents a real number by lower rational bounds. $\rho^<(p) = x$, if p is a list of all rational numbers $a < x$ and η^{ab} a standard representation of F^{ab}, the partial continuous functions $f :\subseteq \Sigma^a \to \Sigma^b$ with open or G_δ domain, if $b = *$ or $b = \omega$ respectively, with properties $\mathrm{utm}(\eta^{ab})$ and $\mathrm{smn}(\eta^{ab})$.

A computable metric space [13,14] is a tuple $\mathbf{M} = (M, d, A, \alpha)$ such that (M, d) is a metric space and $\alpha :\subseteq \Sigma^* \to A$ is a notation with recursive domain of a (countable) dense subset $A \subseteq M$ such that the distance on A, $(a, b) \mapsto d(a, b)$, is (α, α, ρ)–computable. If d_K is $(\nu_K, \nu_K, \rho_>)$-computable, then $\mathbf{M}$ is called semi-computable. The canonical representation of a semi-computable metric space is the Cauchy representaion defined by $\delta_C(p) = x$, iff $p = \iota(u_0)\iota(u_1)\iota(u_2)\ldots$ such that $(\forall i) d(x, \alpha(u_i)) \le 2^{-i}$. In particular, the elments $a \in A$ are δ_C–computable.

2.2 *Representations of Points and Sets in Computable* T_0*-spaces*

In this Section we introduce computable T_0-spaces together with some fundamental representations of points and sets ([9]).

A topological space $\mathbf{X} = (X, \tau)$ is a T_0-space, if for all $x, y \in X$ such that $x \ne y$, there is an open set $O \in \tau$ such that $x \in O \iff y \notin O$. In a T_0-space, every point can be identified by the set of its neighborhoods $O \in \tau$. $\mathbf{X}$ is called second-countable, if it has a countable base [5].

In the following we consider only second countable T_0-spaces. For introducing concepts of effectivity we assume that some notation ν of a base β with recursive domain is given. The notation is partial as well as in many applications.

Definition 2.3 [computable T_0-space]

A computable T_0-space is a tuple $\mathbf{X} = (X, \tau, \beta, \nu)$ such that (X, τ) is a second countable T_0-space and $\nu :\subseteq \Sigma^* \to \beta$ is a notation of a base β of τ with recursive domain, $U \neq \emptyset$ for $U \in \beta$ and $\mathbf{X}$ has computable intersection: there is a computable function $h :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$ such that for all $u, v \in \mathrm{dom}(\nu)$,

$$(2) \quad \nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid w \in \mathrm{dom}(\nu) \text{ and } \iota(w) \lhd h(u,v)\}\,.$$

Call two computable T_0-spaces $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$ recursively related, if and only if there are computable functions $g, g' :\subseteq \Sigma^* \to \Sigma^\omega$ such that

$$(3) \quad \nu_1(u) = \bigcup_{\iota(w) \lhd g(u)} \nu_2(w) \quad \text{and} \quad \nu_2(v) = \bigcup_{\iota(w) \lhd g'(v)} \nu_1(w)\,.$$

We are interested in computability concepts which are "robust", that is, which do not change if a space is replaced by a recursively related one.

In the following let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space. Now we introduce the standard representation of X.

Definition 2.4 [standard representation δ of X] Define the standard representation $\delta :\subseteq \Sigma^\omega \to X$ as follows: $\delta(p) = x$ iff

- $u \in \mathrm{dom}(\nu)$ if $\iota(u) \lhd p$
- $\{u \in \mathrm{dom}(\nu) \mid x \in \nu(u)\} = \{u \mid \iota(u) \lhd p\}$.

A δ-name p of an element $x \in X$ is a list of all words u such that $x \in \nu(u)$. The definition of δ corresponds to the definition of $\delta'_{\mathbf{S}}$ in Lemma 3.2.3 of [14], in particular, δ is admissible with final topology τ (Sec. 3.2 in [14]).

Definition 2.5 [union representation of open and closed sets]

(i) Define the union representation $\theta :\subseteq \Sigma^\omega \to \tau$ of the set of open subsets by

$$\mathrm{dom}(\theta) := \{q \in \Sigma^\omega | u \in \mathrm{dom}(\nu) \text{ if } \iota(u) \lhd q\} \text{ and } \theta(p) := \bigcup_{\iota(u) \lhd p} \nu(u)\,.$$

(ii) Define the union representation $\psi :\subseteq \Sigma^\omega \to \tau^c$ of the set of closed subsets by $\psi(p) := X \setminus \theta(p)$.

Thus, $\theta(p)$ is the union of all $\nu(u)$ such that u is listed by p. The union representation of the closed sets is defined by the union representation of their complements.

For technical reasons we define a notation $\nu^* :\subseteq \Sigma^* \to \{M \subseteq \beta \mid M \text{ is finite}\}$ of all finite sets of base elements by $\mathrm{dom}(\nu^*) := \{w \in \Sigma^* \mid u \in \mathrm{dom}(\nu) \text{ if } \iota(u) \lhd w\}$ and

$$\nu^*(w) := \{\nu(u) \mid \iota(u) \lhd w\}$$

and a notation $\theta^* : \subseteq\Sigma^* \to \tau^{fin}$ of all open sets that can be written as the union of finitely many base elements by

$$\theta^*(w) := \bigcup \nu^*(w).$$

The representations δ and θ are not only very natural, but they can be characterized up to equivalence as maximal elements among representations for which the element relation is open or r.e., respectively. Furthermore "$O \neq \emptyset$" is θ-r.e., countable union on τ is $([\theta]^\omega, \theta)$-computable and intersection is (θ, θ, θ)-computable. Also finite intersection on the base is computable:

(4) $\bigcap \nu^* \leq \theta$.

where $\bigcap \nu^*$ is a notation of the set of all open sets which equal to an intersection of finitely many base elements ([9]). Since $w \in \Sigma^*$ is a prefix of some $p \in \mathrm{dom}(\delta)$ iff $\bigcap \nu^*(w) \neq \emptyset$, the set of all finite prefixes of δ-names

(5) $P := \{w \in \Sigma^* \mid (\exists\, p \in \mathrm{dom}(\delta))\, w \sqsubset p\}$ is r.e..

Definition 2.6 [inner representation of closed sets] Define the inner representation $\psi^< : \subseteq\Sigma^\omega \to \tau_c$ as follows: $\psi^<(p) = A$ iff

- $u \in \mathrm{dom}(\nu)$ if $\iota(u) \lhd p$,
- $\{w \mid \iota(w) \lhd p\} = \{w \mid \nu(w) \cap A \neq \emptyset\}$.

A topological space is a T_2-space (also called Hausdorff space), if for all $x, y \in X$ such that $x \neq y$, there are disjoint open sets $O, O' \in \tau$ such that $x \in O$ and $y \in O'$. A subset $K \in X$ of a Hausdorff space (X, τ) is compact, if every open cover of X by elements of the base has a finite subcover. Let

$$\mathcal{K}(\mathbf{X}) := \{K \subseteq X \mid K \text{ compact}\}$$

denote the set of all compact subsets of a Hausdorff space (X, τ). We write $\mathcal{K}$ instead of $\mathcal{K}(\mathbf{X})$, if there is no need to specify the space or if it's obvious which space we refer to.

The following definitions are generalizations of the cover representations of the compact sets in $\mathbb{R}$ introduced in [14], see also [7] and [9].

Definition 2.7 [representations of compact sets] Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space and let (X, τ) be a Hausdorff space.

(i) Define the *cover representation* $\kappa : \subseteq\Sigma^\omega \to \mathcal{K}$ as follows: $K = \kappa(p)$ iff
  - $w \in \mathrm{dom}(\theta^*)$ if $\iota(w) \lhd p$,
  - $\{w \in \Sigma^* \mid \iota(w) \lhd p\} = \{w \in \Sigma^* \mid K \subseteq \theta^*(w)\}$.

(ii) Define the *minimal cover representation* $\kappa^{mc} : \subseteq \Sigma^\omega \to \mathcal{K}$ as follows: $K = \kappa^{mc}(p)$ iff
  - $w \in \mathrm{dom}(\theta^*)$ if $\iota(w) \lhd p$,
  - $\{w \in \Sigma^* \mid \iota(w) \lhd p\} = \{w \in \Sigma^* \mid K \subseteq \theta^*(w)$ and $(\forall \iota(u) \lhd w)\, \nu(u) \cap K \neq \emptyset\}$.

For compact K and open O, "$K \subseteq O$" is (κ, θ)-r.e., on compact sets, union is (κ, κ, κ)-computable and countable intersection is $([\kappa]^\omega, \kappa)$-computable and for closed A and compact K the mapping $(A, K) \to A \cap K$ is (ψ, κ, κ)-computable. Finally, $\kappa^{mc} \equiv \kappa \wedge \psi^<$.

Next we introduce an effective version of the Hausdorff property and an effective version of locally compactness.

Definition 2.8 [computably Hausdorff] A computable T_0-space $\mathbf{X} = (X, \tau, \beta, \nu)$ is called *computably Hausdorff* if there exists an r.e. set $H \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ such that

(6) $$(\forall (u, v) \in H) \quad \nu(u) \cap \nu(v) = \emptyset,$$

(7) $$(\forall x, y \in X \text{ with } x \neq y)\, (\exists (u, v) \in H)\, x \in \nu(u) \wedge y \in \nu(v).$$

It can be shown that $\mathbf{X}$ is computably Hausdorff iff

$$\{(x, y) \in X \times X \mid x \neq y\} \text{ is } (\delta, \delta) - \text{r.e..}$$

Furthermore, for computably Hausdorff spaces it is $\kappa \leq \psi$.

A topological space (X, τ) is called locally compact, if for every point $x \in X$, there exists a neighborhood O of x such that the closure $\bar{O}$ is compact. Next we introduce an effective version of locally compactness by means of the representation κ of the compact subsets of a Hausdorff space.

Definition 2.9 [computably locally compact] A computable T_0-space $\mathbf{X}' = (X, \tau, \beta', \nu')$ is called *computably locally compact* if (X, τ) is a Hausdorff space and there is some computable T_0-space $\mathbf{X} = (X, \tau, \beta, \nu)$ such that $\mathrm{CLS} : \beta \to \mathcal{K}(\mathbf{X})$ defined by $\mathrm{CLS}(U) := \bar{U}$ is (ν, κ)-computable and $\mathbf{X}'$ and $\mathbf{X}$ are recursively related.

The definition of computably locally compactness ensures its robustness. In the following if $\mathbf{X} = (X, \tau, \beta, \nu)$ is a computably locally compact space, we suppose CLS to be (ν, κ)-computable (without changing the base or its notation).

If $\mathbf{X}$ is a computably locally compact space, then it is locally compact since the closure of each base element is compact. Therefore $\mathbf{X}$ is Tychonoff, thus regular (and a Hausdorff space) and even metrizable since $\mathbf{X}$ is second countable ([5]).

For computably locally compact spaces, "$\bar{U} \subseteq O$" is (ν, θ)-r.e., "$\bar{O} \subseteq O'$" is (θ^*, θ)-r.e and the multi-function $\mathrm{F} : \subseteq \mathcal{K} \times \tau \rightrightarrows \tau$ defined by

$$U \in \mathrm{F}(K, O) :\Longleftrightarrow K \subseteq U \subseteq \bar{U} \subseteq O$$

is $(\kappa, \theta, \theta^*)$-computable. For more details see [9].

3 Representations of $\mathrm{F}^{\omega\omega}$

In [14], Def. 2.3.10 a representation $\eta^{\omega\omega} : \Sigma^\omega \to \mathrm{F}^{\omega\omega}$ of the set $F^{\omega\omega}$ of the partial continuous functions $f : \subseteq \Sigma^\omega \to \Sigma^\omega$ with G_δ-domain (i.e., $\mathrm{dom}(f)$ is a countable intersection of open sets) is introduced, which can be considered as a Type-2 version of an "admissible Gödel numbering" of the computable number functions. In the following let $\eta := \eta^{\omega\omega}$. Since its definition is too abstract for some applications we introduce other equivalent ones.

Definition 3.1 [representations of $\mathrm{F}^{\omega\omega}$]

(i) Let M be a Type-2 machine which on input $\langle p, q\rangle$, $p, q \in \Sigma^\omega$ works in stages $n = 0, 1, \ldots$ as follows: In Stage 0 it does nothing. For $n \geq 1$ let z_{n-1} be the word on the output tape before Stage n. Then in Stage n the machine M searches for the first subword $\iota\langle y, z\rangle$, $y, z \in \Sigma^*$, of p with $y \sqsubset q$ and $z_n \sqsubset z$ and extends z_{n-1} on the output tape to z. (If there are no such words y, z then the machine remains in Stage n forever) Define a representation $\bar\eta : \Sigma^\omega \to \mathrm{F}^{\omega\omega}$ by $\bar\eta_p(q) := f_M\langle p, q\rangle$.

(ii) Let η' be $\bar\eta$ restricted to those $r \in \Sigma^\omega$ listing some consistent set $V \subseteq \Sigma^\omega \times \Sigma^\omega$, where consistent means

$$\Big((u, v) \in V \wedge (u', v') \in V \wedge u \sqsubseteq u'\Big) \implies v \sqsubseteq v'$$

(iii) Let $\hat\eta$ be $\bar\eta$ restricted to those $r \in \Sigma^\omega$ that list the graph of some monotone total function $h : \Sigma^* \to \Sigma^*$.

By [14], Thm. 2.3.7, $f \in \mathrm{F}^{\omega\omega}$ iff there is some monotone function $h : \Sigma^* \to \Sigma^*$ such that $f = h_\omega$, i.e., $f(p) = q \iff q = \sup\{h(w) \mid w \sqsubset p\}$. If p lists the graph of some monotone total function $h : \Sigma^* \to \Sigma^*$, then $\bar\eta_r = h_\omega$.

Lemma 3.2 $\eta \equiv \hat\eta \equiv \eta' \equiv \bar\eta$.

Proof of Lemma 3.2 In the following let $\xi_x := \xi_x^{\omega\omega} :\subseteq \Sigma^\omega \to \Sigma^\omega$ be the function computed by the Type-2 machine with canonical code $x \in \Sigma^*$.

$\hat\eta \leq \eta'$: $\hat\eta_p = \eta'_p$ for all $p \in \mathrm{dom}(\hat\eta)$.
$\eta' \leq \bar\eta$: $\eta'_p = \bar\eta_p$ for all $p \in \mathrm{dom}(\eta')$.
$\bar\eta \leq \eta$: Let x be a codeword for the machine M defining $\bar\eta$, thus $f_M = \xi_x$, then

$$p \to \langle x, p\rangle$$

is a computable translation from $\bar\eta$ to η:

$$\bar\eta_p(q) = f_M\langle p, q\rangle \; = \xi_x\langle p, q\rangle = \eta_{\langle x,p\rangle}(q).$$

$\eta \leq \hat\eta$: We define a Type-2 machine N such that f_N translates η to $\hat\eta$. On input $\langle x, p\rangle$ with $x \in \Sigma^*, p \in \Sigma^\omega$ N works in stages $k = 0, 1, \ldots$ as follows: Let $v_k \in \Sigma^*$ be the output of the universal machine U of ξ after $|\nu_{\Sigma^*}(k)|$ steps on input $(x, \langle p, \nu_{\Sigma^*}(k)0^\omega\rangle)$. Then on stage k N writes $\iota\langle \nu_{\Sigma^*}(k), v_k\rangle$.
f_N is total and $f_N\langle x, p\rangle$ lists the graph of some monotone total function.

$$\eta_{\langle x,p\rangle}(q) = \hat\eta_{f_N\langle x,p\rangle}(q) \text{ for all } q \in \Sigma^\omega :$$

Let $\eta_{\langle x,p\rangle}(q) = s$. If $v \sqsubseteq s$ there exist $t, k \in \mathbb{N}$ such that U es v after t steps on input $(x, \langle p, \nu_{\Sigma^*}(k)0^\omega\rangle)$ with $t = |\nu_{\Sigma^*}(k)|$ and $\nu_{\Sigma^*}(k) \sqsubseteq q$. Therefore $\iota\langle \nu_{\Sigma^*}(k), v\rangle$ is a subword of $f_N\langle x, p\rangle$ and $v \sqsubseteq \hat\eta_{f_N\langle x,p\rangle}(q)$. If $q \notin \mathrm{dom}(\eta_{\langle x,p\rangle})$, then the output of the universal machine U of ξ on input $(x, \langle p, q\rangle)$ is not infinite. By definition of N there exist $n \in \mathbb{N}$ and $v \in \Sigma^*$ such that

$$\iota\langle u, v'\rangle \lhd f_N\langle x, p\rangle \implies v' = v$$

for all $u \in \Sigma^*$ with $u \sqsubset q$ and $|u| \geq n$. Therefore $q \notin \mathrm{dom}(\hat\eta_{f_N\langle x,p\rangle})$. □

4 Representations of Functions in Computable T_0-spaces

For computable T_0-spaces $\mathbf{X}$ and $\mathbf{X}'$ let

(8) $\mathrm{C}(\mathbf{X},\mathbf{Y}) := \{f : X \to Y \mid f \text{ continuous}\}$ and

(9) $\mathrm{C}_p(\mathbf{X},\mathbf{Y}) := \{f :\subseteq X \to Y \mid f \text{ continuous}\}$

be the set of all continuous total and partial functions, respectively, from X to Y. We will introduce three representations of $\mathrm{C}_p(\mathbf{X},\mathbf{Y})$ and compare them w.r.t. reducibility. The following representation has already been used in [13] (only for metric spaces) and in [7].

Definition 4.1 [open-open multi-representation] Let $\mathbf{X}$ and $\mathbf{X}'$ be computable T_0-spaces. Define the multi-representation $\delta_{\mathrm{oo}} :\subseteq \Sigma^\omega \rightrightarrows \mathrm{C}_p(\mathbf{X},\mathbf{X}')$ by

(10) $(r \in \mathrm{dom}(\delta_{\mathrm{oo}}) \text{ and } \iota\langle u,v\rangle \lhd r) \implies (u,v) \in \mathrm{dom}(\nu) \times \mathrm{dom}(\nu')\,,$

(11) $f \in \delta_{\mathrm{oo}}(r) :\iff (\forall\, v \in \mathrm{dom}(\nu'))f^{-1}[\nu'(v)] = \bigcup_{\iota\langle u,v\rangle \lhd r} \nu(u) \cap \mathrm{dom}(f)$

Notice that every continuous function $f :\subseteq X \to Y$ has at least one δ_{oo}-name and that every δ_{oo}-name p of f is also a δ_{oo}-name of each restriction of f.

Lemma 4.2 (robustness of δ_{oo}) *Suppose* $\mathbf{X} = (X,\tau,\beta,\nu), \mathbf{X}' = (X,\tau,\beta',\nu')$ $\mathbf{Y} = (Y,\sigma,\alpha,\mu), \mathbf{Y}' = (Y,\sigma,\alpha',\mu')$ *are computable* T_0*-spaces with open-open multi-representation* δ_{oo} *of* $\mathrm{C}_p(\mathbf{X},\mathbf{Y})$ *and* δ'_{oo} *of* $\mathrm{C}_p(\mathbf{X}',\mathbf{Y}')$ *respectively.*

- (i) $\delta_{\mathrm{oo}} \equiv \delta'_{\mathrm{oo}}$*, if* $\mathbf{X}$ *and* $\mathbf{X}'$ *are recursively related and* $\mathbf{Y}$ *and* $\mathbf{Y}'$ *are recursively related.*
- (ii) $\delta_{\mathrm{oo}} \equiv_t \delta'_{\mathrm{oo}}$

Proof:

(i) We show $\delta_{\mathrm{oo}} \leq \delta'_{\mathrm{oo}}$. Suppose $f \in \delta_{\mathrm{oo}}(r)$, then

$$
\begin{aligned}
f^{-1}[\mu'(v')] &= f^{-1}[\bigcup_{(v,v')\in B'} \mu(v)] \\
&= \bigcup_{(v,v')\in B'} f^{-1}[\mu(v)] \\
&= \bigcup_{(v,v')\in B'} \textstyle\bigcup_{\iota\langle u,v\rangle\lhd r} \nu(u) \cap \mathrm{dom}(f) \\
&= \bigcup_{(v,v')\in B'} \bigcup_{\iota\langle u,v\rangle\lhd r} \bigcup_{(u,u')\in A} \nu'(u') \cap \mathrm{dom}(f)
\end{aligned}
$$

holds for all $v' \in \mathrm{dom}(\mu')$ and $u' \in \mathrm{dom}(\nu')$. There is a Type-2 machine that on input r computes a sequence s such that $\iota\langle u', v'\rangle \lhd s$ iff

$$(\exists u, v)(\iota\langle u, v\rangle \lhd r \wedge (u, u') \in A' \wedge (v, v') \in B').$$

For $\delta'_{\rm oo} \leq \delta_{\rm oo}$ a similar argument holds.

(ii) For $\delta_{\rm oo} \leq_t \delta'_{\rm oo}$ use A and B' of the previous proof as oracles.

□

For a continuous function the preimage of an open set is open. We prove a fully computable version for partial functions.

Lemma 4.3 (pre-image of open sets) *Let* $\mathbf{X_1}$ *and* $\mathbf{X_2}$ *be computable* T_0*-spaces. Then* $\mathrm{PI} : \mathrm{C}_p(\mathbf{X_1}, \mathbf{X_2}) \times \tau_2 \rightrightarrows \tau_1$ *defined by*

$$U \in \mathrm{PI}(f, V) :\iff U \cap \mathrm{dom}(f) = f^{-1}(V)$$

is $(\delta_{\rm oo}, \theta_2, \theta_1)$*-computable.*

Proof: Suppose $f \in \delta_{\rm oo}(p)$ and $V = \theta_2(q)$, then

$$\begin{aligned} f^{-1}[\theta_2(q)] &= f^{-1}[\bigcup_{\iota(v)\lhd q} \nu_2(v)] \\ &= \bigcup_{\iota(v)\lhd q} f^{-1}[\nu_2(v)] \\ &= \bigcup_{\iota(v)\lhd q} \bigcup_{\iota\langle u,v\rangle\lhd p} \nu_1(u) \cap \mathrm{dom}(f). \end{aligned}$$

There is a Type-2 machine that on input p and q computes a sequence r such that $\iota(u) \lhd r$ iff

$$(\exists v)(\iota(v) \lhd q \wedge \iota\langle u, v\rangle \lhd p).$$

□

The next lemma is a generalization of [14, Lemma 6.2.4.4]. Notice that we consider only total continuous functions $f : X_1 \to X_2$.

Lemma 4.4 (image of closed sets) *Let* $\mathbf{X_1}$ *and* $\mathbf{X_2}$ *be computable* T_0*-spaces. Then* $\mathrm{IM} : \mathrm{C}(\mathbf{X_1}, \mathbf{X_2}) \times \tau_1^c \to \tau_2^c$ *defined by* $\mathrm{IM}(f, A) := \overline{f[A]}$ *is* $(\delta_{\rm oo}, \psi_1^<, \psi_2^<)$*-computable.*

Proof: Let $f = \delta_{\rm oo}(p)$ and $A = \psi_1^<(q)$. Then

$$\begin{aligned} \nu_2(v) \cap \overline{f[A]} \neq \emptyset &\iff \nu_2(v) \cap f[A] \neq \emptyset \\ &\iff f^{-1}[\nu_2(v)] \cap A \neq \emptyset \\ &\iff \bigcup_{\iota\langle u,v\rangle\lhd p} \nu_1(u) \cap A \neq \emptyset. \end{aligned}$$

There is a Type-2 machine that on input p and q computes a sequence r such that

$\iota(v) \lhd r$ iff

$$(\exists u)(\iota(u) \lhd q \wedge \iota\langle u, v\rangle \lhd p).$$

□

Since the standard representation δ of a computable T_0-space is admissible, by the *main theorem* ([14], Thm. 3.2.11) a partial function $f :\subseteq X \to X'$ between computable T_0-spaces is topological continuous iff it has a continuous realization $f \in \mathbf{F}^{\omega\omega}$. In the following any η-name of a realization of f is a name of f.

Definition 4.5 [multi-representation via realization] Let $\mathbf{X}$ and $\mathbf{X}'$ be computable T_0-spaces with standard representations δ and δ' respectively. Define the multi-representation $\delta_{\to} :\subseteq \Sigma^\omega \rightrightarrows \mathrm{C}_p(\mathbf{X}, \mathbf{X}')$ by

$$f \in \delta_{\to}(r) :\Longleftrightarrow \Big(f \circ \delta(p) = \delta' \circ \eta_r(p) \text{ whenever } \delta(p) \in \mathrm{dom}(f)\Big)$$

where η is the standard representation of $\mathrm{F}^{\omega\omega}$ (Def. 2.3.10 in [14]).

$$\begin{tikzcd} \Sigma^\omega \arrow[r, "\eta_r"] \arrow[d, "\delta"'] & \Sigma^\omega \arrow[d, "\delta'"'] & \begin{array}{c} f\circ\delta(p)\\ =\\ \delta'\circ\eta_r(p)\end{array} & p \arrow[r, mapsto, "\eta_r"] \arrow[d, mapsto, "\delta"] & \eta_r(p) \arrow[d, mapsto, "\delta'"'] \\ X \arrow[r, "f"] & X' & & x \arrow[r, mapsto, "f"] & f(x) \end{tikzcd}$$

Fig. 2. $f \in \delta_{\to}(r)$

The representation η can be replaced by any equivalent one. Notice that the restrictions of $\delta_{\to}$ and δ_{oo} to the set $\mathrm{C}(\mathbf{X}, \mathbf{X}')$ of the total continuous functions are single-valued representations.

Theorem 4.6 (equivalence) *$\delta_{\to} \equiv \delta_{\mathrm{oo}}$, if $\mathbf{X}, \mathbf{X}'$ are computable T_0-spaces.*

Proof: $\delta_{\to} \leq \delta_{\mathrm{oo}}$:

Let $h :\subseteq \Sigma^* \to \Sigma^\omega$ be a computable translation from $\bigcap \nu^*$ to θ (4), let P be the r.e. set of all prefixes of elements in $\mathrm{dom}(\delta)$ (5) and let U be a universal machine of $\eta^{\omega\omega}$.

Suppose η_p realizes f and $\delta(p) \in \mathrm{dom}(f)$. If the machine U on input (p, q) after some t steps has read the prefix y of q and has written the word z, then $f[\bigcap \nu^*(y)] \subseteq \nu'(v)$ whenever $\iota(v) \lhd z$. Since the set P is r.e. and $\bigcap \nu^* \leq \theta$ for every v we can list words $u_0, u_1, \ldots$ such that $f^{-1}[\nu'(v)] = \mathrm{dom}(f) \cap \bigcup_i \nu(u_i)$. More precisely, there is a Type-2 machine M that on input p computes a sequence r such that $\iota\langle u, v\rangle \lhd r$ iff for some $t \in \mathbb{N}$ and some $y \in P$

– on input $(p, y0^\omega)$ in t steps U reads exactly y
  from the 2nd input tape and writes z on the output tape and

– $\iota(u) \lhd h(y)$ and $\iota(v) \lhd z$.

Now let $f \in \delta(p)$ and $r = f_M(p)$. Suppose $\iota\langle u, v\rangle \lhd r$. Then for some t, y the above conditions hold true. We can conclude $f[\nu(u)]\subseteq\nu'(v)$, hence

$$\mathrm{dom}(f) \cap \bigcup_{\iota\langle u,v\rangle\lhd r} \nu(u)\subseteq f^{-1}[\nu'(v)].$$

On the other hand suppose $x \in f^{-1}[\nu'(v)]$. Then $x \in \mathrm{dom}(f)$ and $\delta(q) = x$ for some $q \in \mathrm{dom}(\delta)$. There are some t, y such that the machine U on input (p, q) in t steps has written $\iota(v)$ (somewhere on its output tape) and reads exactly the prefix y form the 2nd input tape. Then the machine U also on input $(p, y0^\omega)$ in t steps writes $\iota(v)$ and reads exactly the prefix y form the 2nd input tape. Since

$$x = \delta(q) \in \bigcap\{\nu(w) \mid \iota(w) \lhd y\} = \bigcup\{\nu(w) \mid \iota(w) \lhd h(y)\}\,,$$

there is some u such that $x \in \nu(u)$ and $\iota(u) \lhd h(y)$, hence some u such that $\iota\langle u, v\rangle$ is listed by M on input p. Therefore,

$$f^{-1}[\nu'(v)]\subseteq\mathrm{dom}(f) \cap \bigcup\{\nu(u) \mid \iota\langle u, v\rangle \lhd r\,.$$

This shows that the machine M translates $\delta_\to$ δ_{oo}.

$\delta_{\mathrm{oo}} \le \delta_\to$: There is a Type-2 machine M, which on input $(p, q) \in \Sigma^\omega \times \Sigma^\omega$ computes a list of all $\iota(v)$ such that for some u, $\iota\langle u, v\rangle \lhd p$ and $\iota(u) \lhd q$.

If $f \in \delta_{\mathrm{oo}}(p)$ and $x = \delta(q) \in \mathrm{dom}(f)$ then $f(x) = \delta'(f_M(p, q))$. By the smn-theorem for η there is a computable function $h : \Sigma^\omega \to \Sigma^\omega$ such that $f_M(p, q) = \eta_{h(p)}(q)$. The function h translates δ_{oo} to $\delta_\to$. □

For all $f \in \mathrm{C}_p(\mathbf{X}, \mathbf{X}')$ the following holds:

$$(\forall x \in \mathrm{dom}(f))(\forall V \in \beta', f(x) \in V)(\exists U \in \beta)\,(x \in U \wedge f[U] \subseteq V).$$

The open set U does not depend continuously on x and V, there is, however, a continuous multi-function.

Lemma 4.7 *In general, for $f \in \mathrm{C}_p(\mathbf{X}, \mathbf{X}')$ there is no continuous (single-valued) function $\hat{f} : \subseteq X \times \beta' \to \beta$ with*

(12) $\mathrm{dom}(\hat{f}) := \{(x, V) \in X \times \beta' \mid x \in \mathrm{dom}(f) \wedge f(x) \in V\}$

(13) $(\forall(x, V) \in \mathrm{dom}(\hat{f}))[x \in \hat{f}(x, V) \wedge f[\hat{f}(x, V)] \subseteq V]$

Proof: Let $\mathbf{X} = \mathbf{X}' := (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ where ν is a standard notation of all rational intervals. Let $f(x) := x^3$ and suppose $\hat{f} : \subseteq \mathbb{R} \times \mathrm{Cb}^{(1)} \to \mathrm{Cb}^{(1)}$ such that (12,13) is (δ, ν, ν)-continuous. Then $g : \subseteq \mathbb{R} \to \beta$ defined by $g(x) := \hat{f}(x, (0; 2))$ is (ρ, ν)-continuous and therefore constant on its domain $(0; \sqrt[3]{2})$ (Lemma 4.3.15 in [14]). Therefore, there are rational numbers a, b such that $\hat{f}(x, (0; 2)) = (a; b)$ for all $x \in (0; \sqrt[3]{2})$. Then $(0; \sqrt[3]{2})\subseteq(a; b)$ and $(a^3; b^3)\subseteq(0; 2)$. This cannot be true for ratinal numbers a, b. □

Lemma 4.8 *Let $\mathbf{X}, \mathbf{X}'$ be computable T_0-spaces. For $f \in \mathrm{C}_p(\mathbf{X}, \mathbf{X}')$ define $\hat{f} : \subseteq X \times \beta' \rightrightarrows \beta$ by*

(14) $\mathrm{dom}(\hat{f}) := \{(x, V) \in X \times \beta' \mid x \in \mathrm{dom}(f) \wedge f(x) \in V\}$

(15) $U \in \hat{f}(x, V) :\iff x \in U \wedge f[U] \subseteq V.$

Then $\hat{f}$ *is* (δ, ν', ν)*-continuous.*

Proof: Consider the oracle-machine M that on input $\langle p, w\rangle \in \Sigma^\omega$ and oracle $o \in \Sigma^\omega$, where $\iota\langle u, v\rangle \lhd o \iff f[\nu(u)] \subseteq \nu'(v)$ in stage k works as follows:
if $\iota\langle u, w\rangle \lhd o_{<k}$ and $\iota(u) \lhd p_{<k}$ then M writes u. Then f_M is a continuous realization of $\hat{f}$. □

Definition 4.9 [multi-representation via pointwise continuity] Let $\mathbf{X}$ and $\mathbf{X}'$ be computable T_0-spaces and let δ be the standard representation of X. Define the pointwise multi-representation $\hat{\delta}_\to : \subseteq\Sigma^\omega \rightrightarrows \mathrm{C}_p(\mathbf{X}, \mathbf{X}')$ by $f \in \hat{\delta}_\to(r)$, iff

$$\nu \circ \eta_r\langle p, w\rangle \in \hat{f} \circ [\delta, \nu']\langle p, w\rangle, \quad \text{whenever } (\delta(p), \nu'(w)) \in \mathrm{dom}(\hat{f})$$

or equivalently,

$$(\delta(p) \in \nu \circ \eta_r\langle p, w\rangle \quad \text{and} \quad f[\nu \circ \eta_r\langle p, w\rangle]\subseteq\nu'(w) \text{ for } \delta(p) \in \nu'(w)\,.$$

Theorem 4.10 (equivalence) $\hat{\delta}_\to \equiv \delta_{\mathrm{oo}}$.

Proof. $\delta_{\mathrm{oo}} \leq \hat{\delta}_\to$: Let $f \in \delta_{\mathrm{oo}}(r)$ and let t be a codeword for the oracle-machine M described in the proof of Lemma 4.8, thus $f_M = \xi_t$. Then

$$r \to \langle t, r\rangle$$

is a computable translation from δ_{oo} to $\hat{\delta}_\to$:

$$f_M\langle r, \langle p, w\rangle\rangle = \xi_t\langle r, \langle p, w\rangle\rangle = \eta_{\langle t,r\rangle}\langle p, w\rangle.$$

$\hat{\delta}_\to \leq \delta_{\mathrm{oo}}$: Let $f \in \hat{\delta}_\to(r)$, where r lists the graph of some monoton function $h : \Sigma^* \to \Sigma^*$. The machine M works on stage k as follows: if $\iota\langle x, y\rangle \lhd r_{<k}$ and there exist subwords $\iota(v) \lhd x \wedge \iota(u) \lhd y$ such that $\nu(v) \in \mathrm{dom}(\nu') \wedge \nu(u) \in \mathrm{dom}(\nu)$ then M writes $\iota\langle u, v\rangle$. □

Next we define a compact-open representation of the set of total continuous functions.

Definition 4.11 [compact-open representation] Let $\mathbf{X}$ be a computably locally compact space and $\mathbf{X}'$ a computable T_0-space. Define the compact-open representation $\delta_{\mathrm{co}} : \subseteq\Sigma^\omega \to \mathrm{C}(\mathbf{X}, \mathbf{X}')$ by

$$\delta_{\mathrm{co}}(p) := f \iff \{(u, v)|\iota\langle u, v\rangle \lhd p\} = \{(u, v)|f[\overline{\nu(u)}] \subseteq \nu'(v)\}$$

Theorem 4.12 (equivalence) *Let* $\mathbf{X}$ *be a computably locally compact space and* $\mathbf{X}'$ *a computable* T_0*-space. Restricted to* $\mathrm{C}(\mathbf{X}, \mathbf{X}')$ *the following equivalences hold true*

$$\delta_{\mathrm{oo}} \equiv \delta_{\mathrm{co}} \equiv \delta_\to.$$

Proof.

(i) $\delta_{\mathrm{oo}} \leq \delta_{\mathrm{co}}$:

Let $f = \delta_{\mathrm{oo}}(p)$. It is

$$f[\overline{\nu(u)}] \subseteq \nu'(v) \iff \overline{\nu(u)} \subseteq f^{-1}[\nu'(v)] = \bigcup_{\iota\langle u_v, v\rangle \lhd p} \nu(u_v)$$

There is a Type-2 machine that on input p computes a sequence r such that $\iota\langle u, v\rangle \lhd r$ iff

$$(\exists w)(\forall \iota(u_v) \lhd w)(\iota\langle u_v, v\rangle \lhd p \wedge \iota(w) \lhd q)$$

where q is a κ-name of $\overline{\nu(u)}$.

(ii) $\delta_{\mathrm{co}} \leq \delta_{\rightarrow}$:

Let $f = \delta_{\mathrm{co}}(p)$ and $\delta(q) = x$. It is

$$f(x) \in \nu'(v) \iff (\exists u)(x \in \nu(u) \text{ and } f[\overline{\nu(u)}] \subseteq \nu'(v))$$

as X is regular.

There is a Type-2 machine M that on input p and q computes a sequence r such that $\iota(v) \lhd r$ iff

$$(\exists u)(\iota(u) \lhd q \wedge \iota\langle u, v\rangle \lhd p)\,.$$

By smn-Theorem there is a function r such that $f_M(p, q) = \eta_{r(p)}(q)$, that is r translates δ_{co} to $\delta_{\rightarrow}$.

(iii) $\delta_{\rightarrow} \leq \delta_{\mathrm{oo}}$:

Since $\delta_{\rightarrow} \equiv \delta_{\mathrm{oo}}$ holds for computable T_0-spaces, $\delta_{\rightarrow} \leq \delta_{\mathrm{oo}}$ holds for computably locally compact spaces obviously.

□

5 Stone-Weierstrass Representation

By the Weierstrass approximation theorem the set of polynomial functions $f : [0;1] \to \mathbb{R}$ is dense in the space $C[0;1]$ of real valued continuous functions on the unit interval with the metric $d(f, g) := \max_{x\in[0;1]} |f(x) - g(x)|$. Obviously, also the countable set Pn of polynomial functions with rational coefficients is dense in the space $C[0;1]$.

For a natural notation ν_{Pn} of the set Pn, $\mathbf{X} := (C[0;1], d, \mathrm{Pn}, \nu_{\mathrm{Pn}})$ is a computable metric space [14, Section 8.1]. By [14, Section 6.1] the Cauchy representation $\delta_C^{[0;1]}$ of $\mathbf{X}$ is equivalent to the representation $[\rho \to \rho]_{[0;1]}$ (where ρ is the stadard representation of the real numbers). This is a computable version of the Weierstrass approximation theorem.

In this section we prove a computable version of the more general Stone-Weierstrass approximation theorem for compact Hausdorff spaces. A set A of real-valued functions on a set X is an *algebra* if $f \cdot g \in A$ and $af + bg \in A$ for all $a, b \in \mathbb{R}$, whenever $f, g \in A$. A separates the points of X, if for all $x, y \in X$ such that $x \neq y$ there is some function $f \in A$ such that $f(x) \neq g(x)$

Theorem 5.1 (Stone-Weierstrass [3]) *Let X be a compact Hausdorff space. If A is an algebra of continuous real-valued functions on X that contains the constant functions and separates the points of X, then A is dense in the space $C(X)$ of continuous real-valued functions on X with metric $d(f, g) = \max_{x\in X} |f(x) - g(x)|$.*

In the following let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space that is computably Hausdorff and computably locally compact with standard representations δ, θ, ψ and κ of the points, the open sets, the closed sets and the compact sets, respectively. Furthermore, let κ_{mc} be the minimal-cover representation of the compact sets. The support $\mathrm{supp}(f)$ of a function $f : X \to \mathbb{R}$ is the closure of $\{x | f(x) \neq 0\}$.

First, we show that there is a computable sequence of continuous functions with compact support, which separates the points of X.

Lemma 5.2 *There is a sequence $(e_i)_i$ of continuous functions $e_i : X \to \mathbb{R}$ such that*

(i) *$\{e_i \mid i \in \mathbb{N}\}$ separates the points of X,*

(ii) *there is a $(\nu_{\mathbb{N}}, \kappa)$-computable function h such that* $\mathrm{supp}(e_i) \subseteq \kappa \circ h(i)$,

(iii) *the function $(i, x) \mapsto e_i(x)$ is $(\nu_{\mathbb{N}}, \delta, \rho)$-computable.*

Proof. In [9] it is shown that the space X is computably regular. This means that there is a computable function $\mathrm{t}_3 :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$ such that $R := \mathrm{dom}(\mathrm{t}_3)$ is recursively enumerable,

(16) $(\forall v \in \mathrm{dom}(\nu)),\ \nu(v) = \bigcup_{(u,v)\in R} \nu(u),$ and

(17) $(\forall (u, v) \in R)\, \nu(u) \subseteq \psi(\mathrm{t}_3\,(u, v)) \subseteq \nu(v)\,.$

In [6] for for a computably regular space from a computable enumeration $(u_i, v_i)_i$ of R a sequence of continuous functions $(f_i)_i$ is constructed such that

(18) $(\forall\ i, x)\, 0 \leq f_i(x) \leq 1\,,$

(19) $(\forall i)\ f_i(x) = \begin{cases} 0 \text{ if } x \in \nu(u_i) \\ 1 \text{ if } x \notin \nu(v_i)\,. \end{cases}$

(20) $(i, x) \mapsto f_i(x)$ is $(\nu_{\mathbb{N}}, \delta, \rho)$-computable

Define $e_i(x) := 1 - f_i(x)$.

If $x \neq y$ then by (16) and the Hausdorff property there is some i such that $x \in \nu(u_i)$ and $y \notin \nu(v_i)$. By (19), $e_i(x) = 1$ and $e_i(y) = 0$. Therefore, $\{e_i \mid i \in \mathbb{N}\}$ separates the points of X.

By (18) $e_i(x) = 0$ for $x \notin \nu(v_i)$, hence the support of e_i is in the compact set $h(i) := \overline{\nu(v_i)}$. The function h is $(\nu_{\mathbb{N}}, \kappa)$-computable as the space is computably locally compact.

Finally (iii) follows from (20). □

For $K \subseteq X$ let $\delta_K :\subseteq \Sigma^\omega \to K$ be the restriction of δ to K and let $C(K)$ be the set continuous real functions $f : K \to \mathbb{R}$. Since δ is admissible also δ_K is admissible and so $[\delta_K \to \rho]$ is a representation of $C(K)$ [14, Sections 3.2, 3.3]. For the set $C_p(X, \mathbb{R})$ of partial continuous functions we use the multi-representation $[\delta \to_p \rho]$.

Define the "restriction" operator $\mathrm{res}_K :\subseteq C_p(X, \mathbb{R}) \to C(K)$ by $\mathrm{dom}(\mathrm{res}_K) = \{f \in C_p(X, \mathbb{R}) \mid K \subseteq \mathrm{dom}(f)\}$ and $\mathrm{graph}(\mathrm{res}_K(f)) := \mathrm{graph}(f) \cap K \times \mathbb{R}$ (abbreviation: $f_K := \mathrm{res}_K(f)$). Define the "embedding" operator $\mathrm{emb}_K : C(K) \to C_p(X, \mathbb{R})$

by $\mathrm{graph}(\mathrm{emb}_K(g)) := \mathrm{graph}(g)$.

Lemma 5.3 (i) *The restriction* res_K *is* $([\delta \to_p \rho], [\delta_K \to \rho])$*-computable.*
(ii) *The embedding* emb_K *is* $([\delta_K \to \rho], [\delta \to_p \rho])$*-computable.*
In both cases the identity on Σ^ω *is a realizaton.*

Proof: Suppose $K \subseteq \mathrm{dom}(f)$. Then

$$\begin{aligned} & f \in [\delta \to_p \rho](p) \\ \Longrightarrow\ & (\forall\, q,\ \delta(q) \in \mathrm{dom}(f))\, f\delta(q) = \rho\eta_p(q) \\ \Longrightarrow\ & (\forall\, q,\ q \in \mathrm{dom}(\delta_K))\, f\delta_K(q) = \rho\eta_p(q) \\ \Longrightarrow\ & \mathrm{res}_K(f) = [\delta_K \to \rho](p)\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} & g = [\delta_K \to \rho](p) \\ \Longrightarrow\ & (\forall\, q \in \mathrm{dom}(\delta_K))\, g\delta_K(q) = \rho\eta_p(q) \\ \Longrightarrow\ & (\forall\, q,\ \delta(q) \in K)\, g\delta(q) = \rho\eta_p(q) \\ \Longrightarrow\ & \mathrm{emb}_K(g) \in [\delta \to_p \rho](p)\,. \end{aligned}$$

□

Let $\mathbf{M} = (M, d, D, \alpha)$ such that (M, d) is a metric space and α is a notation with recursive domain of the dense set D. According to [14, Section 8.1], $\mathbf{M}$ is a *semi-computable* metric space if the distance d is $(\alpha, \alpha, \rho_>)$-computable on D and a *computable* metric space if d is (α, α, ρ)-computable on D. The Cauchy representation δ_C of M is defined by

$$\delta_C(p) = y \iff p = \iota(u_0)\iota(u_1)\ldots \quad \text{such that} \quad (\forall\, i)\, d(\alpha(u_i), y) \le 2^{-i}\,.$$

For compact $K \subseteq X$ define a metric $d_K : C(K) \times C(K) \to \mathbb{R}$ on $C(K)$ by

$$d_K(f, g) := \max_{x \in K} |f(x) - g(x)|\,.$$

Lemma 5.4 *The function* d_K *is* $([\delta_K \to \rho], [\delta_K \to \rho], \rho_>)$*-computable for* κ*-computable* K *and* $([\delta_K \to \rho], [\delta_K \to \rho], \rho)$*-computable for* κ_{mc}*-computable* K*.*

Proof: For the compact subsets of the real numbers let $\kappa^{\mathbb{R}}$ be the cover representation and let let $\kappa^{\mathbb{R}}_{\mathrm{mc}}$ be the minimal-cover representation (called κ_c and κ_{mc}, respectively, in [14, Section 5.2]). By [14, Lemma 5.2.6], $L \mapsto \max L$ for compact $L \in \mathbb{R}$ is $(\kappa^{\mathbb{R}}, \rho_>)$-computable and $(\kappa^{\mathbb{R}}_{mc}, \rho)$-computable.

Since evaluation $(f, x) \mapsto f(x)$ is $([\delta_K \to \rho], \delta_K, \rho)$-computable, the function $(f, g, x) \mapsto |f(x) - g(x)|$ is $([\delta_K \to \rho], [\delta_K \to \rho], \delta_K, \rho)$-computable. Therefore, the function $(f, g) \mapsto h$, $h(x) := |f(x) - g(x)|$, is $([\delta_K \to \rho], [\delta_K \to \rho], [\delta_K \to \rho])$-computable. By lemma 5.3 the function $(f, g) \mapsto \mathrm{emb}_K(h)$ is $([\delta_K \to \rho], [\delta_K \to \rho], [\delta \to_p \rho])$-computable and hence $([\delta_K \to \rho], [\delta_K \to \rho], \delta_{oo})$-computable by Theorem 4.6.

For $r :\subseteq X \to \mathbb{R}$ and compact $K' \subseteq \mathrm{dom}(r)$, the function $(r, K') \mapsto r[K']$ is $(\delta_{oo}, \kappa, \kappa^{\mathbb{R}})$-computable by [7, Lemma 12]. From Lemma 4.4 we can conclude that the function is also $(\delta_{oo}, \kappa_{mc}, \kappa^{\mathbb{R}}_{mc})$-computable. Therefore, $r \mapsto \max r[K]$

is $(\delta_{oo}, \rho_>)$-computable for κ-computable K and (δ_{oo}, ρ)-computable for κ_{mc}-computable K.

Since $d_K(f, g) = \max(|f - g|[K])$, the lemma follows by composition of the above functions. □

Theorem 5.5 (computable Stone-Weierstrass) *For every κ-computable set K there is a notation $\nu_K :\subseteq \Sigma^* \to D_K$ such that $\mathbf{M}_K = (C(K), d_K, D_K, \nu_K)$ is a semi-computable metric space, the Cauchy representation δ^C_K of which is equivalent to $[\delta_K \to \rho]$. Furthermore, if K is κ_{mc}-computable then $\mathbf{M}_K$ is a computable metric space.*

Proof. Let $(e_i)_i$ be the sequence of functions from Lemma 5.2. Define a canonical notation ν_1 of the finite products of the e_i with rational coefficients by

$$\nu_1(\iota(w))(x) := \nu_{\mathbb{Q}}(w)$$
$$\nu_1(\iota(w)0^{n_1}1 \ldots 0^{n_k}1 := \nu_{\mathbb{Q}}(w) \cdot e_{n_1} \cdot \ldots \cdot e_{n_k}$$

and the notation $\alpha :\subseteq \Sigma^* \to D$ of the rational linear span D of the $\{e_i\}$ and the constant 1 function on X by

$$\alpha(\iota(v_1) \ldots \iota(v_m)) := \nu_1(v_1) + \ldots + \nu_1(v_m)\,.$$

From Lemma 5.2(iii) we obtain

(21) $\alpha \le [\delta \to \rho]\,.$

Define a notation $\nu_K :\subseteq \Sigma^* \to D_K$ (with recursive domain) by

(22) $D_K := \{f_K \mid f \in D\}, \quad \nu_K(u) := \alpha(u)_K (= \mathrm{res}_K(\alpha(u)))\,.$

By Lemma 5.2(i), the set D_K separates the points of K. The set D_K is dense (with respect to d_K) in the linear span A of the e_{iK} and the constant 1 function on K. By the (classical) Stone-Weierstrass theorem, A is dense in $C(K)$. Therefore, D_K is dense in $C(K)$.

By (21) there is a computale function h such that $\alpha(u) = [\delta \to \rho]h(u)$, hence $\alpha(u) \in [\delta \to_p \rho]h(u)$. By Lemma 5.3, $\nu_K(u) = \mathrm{res}_K(\alpha(u)) = [\delta_K \to \rho]h(u)$, hence

(23) $\nu_K \le [\delta_K \to \rho]$.

It follows from Lemma 5.4 that d_K is $(\nu_K, \nu_K, \rho_>)$-computable for κ-computable K and (ν_K, ν_K, ρ)-computable for κ_{mc}-computable K. Therefore, the metric space $\mathbf{M}_K$ is semi-computable for κ-computable $K$and computable for κ_{mc}-computable K.

Finally, we have to show $\delta^C_K \equiv [\delta_K \to \rho]$.

"$[\delta_K \to \rho] \le \delta^C_K$": By Lemma 5.4 and since $\nu_K \le [\delta_K \to \rho]$, the relation $\{(f, g, i) \in C(K) \times D_K \times \mathbb{N} \mid d_K(f, g) < 2^{-i}\}$ is $([\delta_K \to \rho], \nu_K, \nu_{\mathbb{N}})$-recursively enumerable. Therefore, for every $f \in C(K)$ and every i some u can be computed such that $d_K(f, \nu_K(u)) < 2^{-i}$. We conclude $[\delta_K \to \rho] \le \delta^C_K$.

"$\delta^C_K \le [\delta_K \to \rho]$": By the definition of the Cauchy representation, the multi-function $f \rightrightarrows (g_i)_i$ such that $|f(x) - g_i(x)|$ for all x is $(\delta^C_K, [\nu_K]^\omega)$-computable. Since $\nu_K \le [\delta_K \to \rho]$, the function $((g_i)_i, x) \mapsto (g_i(x))_i$ is $([\delta_K \to \rho], \delta_K, \rho)$-computable. Since the limit of real Cauchy sequences is computable by [14, Theorem 4.3.7], the function $(f, x) \mapsto f(x)$ is $(\delta_K^C, \delta_K, \rho)$-computable, hence $f \mapsto f$ is $(\delta_K^C, [\delta_K \to \rho])$-computable. This means $\delta_K^C \leq [\delta_K \to \rho]$.

□

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