Containment regions for zeros of polynomials from numerical ranges of companion matrices

Hansjörg Linden

Fachbereich Mathematik, Fernuniversität-Gesamthochschule, Postfach 940, Lützowstr. 125, D-58084 Hagen, Germany

Received 24 March 1999; accepted 22 December 2001

Abstract

Containment regions for the zeros of a monic polynomial are given with the aid of results for containment regions for the numerical range of certain bordered diagonal matrices which are applied to different types of companion matrices of the polynomial. © 2002 Elsevier Science Inc. All rights reserved.

AMS classification: 15A60; 65J05; 26C10; 30C15; 65H05

Keywords: Numerical range; Bordered diagonal matrix; Companion matrix; Zeros of polynomials

1. Introduction

Let

\[ P_n(x) := x^n - \alpha_1 x^{n-1} - \cdots - \alpha_n, \]
\[ \alpha_j \in \mathbb{C}, \quad j = 1, 2, \ldots, n, \quad \alpha_n \neq 0, \]

be a monic polynomial of degree \( n \geq 3 \). In this paper we give some containment regions for the zeros of \( P_n \).

In several numerical methods it is of importance to have regions for zeros of polynomials. For example, initial regions for zeros of polynomials are needed before
improving, simultaneously, the zeros of a polynomial by applying a local algorithm with rapid convergence (cf. for example [16]). In many cases the approach to this problem uses matrix analysis (cf. [1,3,4,8,9], [10, pp. 139–146] and [15, pp. 125–155]). In particular, there were used estimates of the numerical radius of companion matrices of the given polynomial to give bounds for the zeros of the polynomial (cf. [1,4,9] and [5, p. 122]).

In this paper we also use matrix analysis to give regions for the zeros of polynomials. We apply results on containment regions for the numerical range of certain companion matrices of the given polynomial to give containment regions for the zeros of that polynomial. The regions which usually are given are circular disks with center at the origin. The regions which we derive are circular disks with center not necessarily at the origin (only in special cases), the convex hull of two circular disks with the same radius, the intersection of two such sets, and the sum of a certain elliptical disk with a circular disk with center at the origin. From these regions there can be easily derived rectangular containment regions for the zeros. The localization of the zeros of a polynomial is a classical problem which has been considered by many authors, see [10–14].

To be more specific: In Section 2 we consider bordered diagonal matrices of the following type: Let $a := (\alpha_1, \ldots, \alpha_{n-1})^T$, $b := (\beta_1, \ldots, \beta_{n-1})^T \in \mathbb{C}^{n-1}$, $D := \text{diag} (\delta_1, \ldots, \delta_{n-1})$, $\delta_1, \ldots, \delta_{n-1} \in \mathbb{C}$, be given, and let $\gamma \in \mathbb{C}$. Furthermore, let $\overline{a} := (\overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1})^T$ and $\overline{b} := (\overline{\beta}_1, \ldots, \overline{\beta}_{n-1})^T$, and let $A \neq 0$ be the square matrix of order $n$ given by

$$A := \begin{pmatrix} D & b \\ a^T & \gamma \end{pmatrix}. \tag{2}$$

For the special case that $\delta_k = \delta, k = 1, \ldots, n-1$, the numerical range of $A$ is either a circular disk or a (noncircular) elliptical disk (possibly degenerated to a line segment) as follows from results in [7]. With the aid of these descriptions we give regions which contain the numerical range of the general matrix $A$ given by (2).

In Section 3—the main part of this paper—we consider monic polynomials of type given by (1), and two different types of generalized Frobenius companion matrices of $P_n$ which we have already used in [8,9]. Each of these companion matrices can be decomposed in different ways in the sum of a shift matrix and of a bordered diagonal matrix. To the special shift matrices we can apply a result of Chien (cf. Theorem 3 in [2]) to describe their numerical range. To the special bordered diagonal matrices we apply the results from Section 2 on numerical ranges of bordered diagonal matrices. From this we get containment regions for the zeros of $P_n$, since these are contained in the sum of the numerical ranges of the matrices of the decompositions. In some special cases these containment regions coincide with known ones, but mainly these are new ones, and in many cases they are better as known ones as examples show.
2. The numerical range of bordered diagonal matrices

We denote by $(\cdot, \cdot)$ the usual inner product in $\mathbb{C}^n$ and by $\| \cdot \| := \sqrt{(\cdot, \cdot)}$ the corresponding norm. Let $M$ be a square matrix of order $n$. Then the numerical range (or the field of values) $W(M)$ of $M$ is defined by $W(M) := \{(Mx, x) : x \in \mathbb{C}^n, \|x\| = 1\}$, and the numerical radius $r(M)$ of $M$ is defined by $r(M) := \max\{|z| : z \in W(M)\}$. $W(M)$ is a compact convex set which contains the spectrum (eigenvalues) of $M$. See [6, Chapter 1] for further details on the numerical range and the numerical radius.

We consider bordered diagonal matrices of the type given by (2). In the following proposition we describe the numerical range $W(A)$ in the special case that $\delta_1 = \cdots = \delta_{n-1} =: \delta$. From this we derive regions in the complex plane which contain the numerical range of $A$ in the general case.

**Proposition 1.** Let $A$ be as given by (2), where $\delta_1 = \cdots = \delta_{n-1} =: \delta$, and let $\eta := \frac{1}{2} (\gamma - \delta)^2 + (a, \overline{b})$.

(a) Suppose that $\eta = 0$. Then $W(A)$ is a circular disk with center at $\frac{1}{2} (\gamma + \delta)$ and radius $\frac{1}{2} (\frac{1}{2} |\gamma - \delta|^2 + \|a\|^2 + \|b\|^2)^{1/2}$.

(b) (i) Suppose that $\eta \neq 0$ and $|\eta| \neq \frac{1}{4} |\gamma - \delta|^2 + \frac{1}{2} \|a\|^2 + \frac{1}{2} \|b\|^2$. Then $W(A)$ is an elliptical (noncircular) disk with center at $\frac{1}{2} (\gamma + \delta)$, major axis of length $2\alpha$, minor axis of length $2\beta$ and the inclination of the major axis with the real axis is equal to $\varphi$, where

$$\varphi := \frac{1}{2} \arg(\eta), \quad \alpha := \frac{1}{2} (2|\eta| + \frac{1}{2} |\gamma - \delta|^2 + \|a\|^2 + \|b\|^2)^{1/2}, \quad \beta := \frac{1}{2} (-2|\eta| + \frac{1}{2} |\gamma - \delta|^2 + \|a\|^2 + \|b\|^2)^{1/2}.$$

That is, $W(A)$ is the closed interior of the ellipse $$\{ z \in \mathbb{C} : z = \frac{1}{2} (\gamma + \delta) + \frac{1}{2} e^{i\varphi}((\alpha + \beta)e^{i\Theta} + (\alpha - \beta)e^{-i\Theta}), 0 \leq \Theta < 2\pi \}.$$

(ii) Suppose that $\eta \neq 0$ and $|\eta| = \frac{1}{4} |\gamma - \delta|^2 + \frac{1}{2} \|a\|^2 + \frac{1}{2} \|b\|^2$. Then $A$ is normal, and $W(A)$ is the line segment from $\frac{1}{2} (\gamma + \delta) - \sqrt{\eta} i$ to $\frac{1}{2} (\gamma + \delta) + \sqrt{\eta}$.

**Proof.** Let $E_n$ denote the identity matrix of order $n$. The matrix $A$ is unitarily similar to the matrix $\delta E_{n-3} \oplus \tilde{A}$, where $\tilde{A}$ is given by

$$\tilde{A} := \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & \|b\| \\ \tilde{\alpha}_1 & \tilde{\alpha}_2 & \gamma \end{pmatrix},$$

and $\tilde{\alpha}_1, \tilde{\alpha}_2$ are such that $|\tilde{\alpha}_1|^2 + |\tilde{\alpha}_2|^2 = \|a\|^2$. The construction of the unitary transformation matrix $U$ is as follows: Let $U$ be given by
Let $A$ be as given by Corollary 1. The numerical range of the matrix $A$ is contained in the intersection of the two sets

$$W(A) = \{ a, b : a, b \} = \{ a, b : a, b \}$$

where $a_1, a_2, \ldots, a_{n-1} \in \mathbb{C}^{n-1}$, and $\omega \in \mathbb{C}^{n-1}$ is the null vector in $\mathbb{C}^{n-1}$.

**Case 1:** $\|a\| = 0$, $\|b\| = 0$. Then $U = E_n$.

**Case 2:** $\|a\| \neq 0$, $\|b\| = 0$. Let $a_{n-2} = a/\|a\|$, and let $\{a_1, \ldots, a_{n-3}, a_{n-1}\}$ be an orthonormal system, where each vector $a_j$ is orthogonal to $a_{n-2}$. With $\tilde{a}_1 = \|a\|, \tilde{a}_2 = 0$ we have the desired result.

**Case 3:** $\|a\| = 0$, $\|b\| \neq 0$. Let $a_{n-1} = b/\|b\|$, and let $\{a_1, \ldots, a_{n-2}\}$ be an orthonormal system, where each vector $a_j$ is orthogonal to $a_{n-1}$. With $\tilde{a}_1 = 0, \tilde{a}_2 = 0$, then $U^*AU$ will have the desired form.

**Case 4:** $\|a\| \neq 0$, $\|b\| \neq 0$, and $\alpha$ and $b$ are linearly dependent. Let $a_{n-1} = b/\|b\|$, and let $\{a_1, \ldots, a_{n-2}\}$ be an orthonormal system, where each vector $a_j$ is orthogonal to $a_{n-1}$. With $\tilde{a}_2 = (a, b)/\|b\|$ and $\tilde{a}_1 = 0$ we have $\delta E_{n-3} \oplus \tilde{A} = U^*AU$.

**Case 5:** $\|a\| \neq 0$, $\|b\| \neq 0$, and $a$ and $b$ are linearly independent. Let $a_{n-1} = b/\|b\|$, and let $\{a_1, \ldots, a_{n-3}\}$ be an orthonormal system, where each vector $a_j$ is orthogonal to $a_{n-1}$ and $b$. Furthermore, let $a_{n-2} = (\alpha - (\tilde{a}, b) b/\|b\|^2)/\|\alpha - (\tilde{a}, b)b/\|b\|^2\|$. With $\tilde{a}_1 = (a, b)/\|b\|$ and $\tilde{a}_1 = (\|a\|^2 - |(\alpha, b)|^2/\|b\|^2)/\|\alpha - (\tilde{a}, b)b/\|b\|^2\|$ again we have $\delta E_{n-3} \oplus \tilde{A} = U^*AU$.

In all cases the eigenvalues of $\tilde{A}$ are $\frac{1}{2}(\gamma - \delta) + \sqrt{\eta}, \frac{1}{2}(\gamma + \delta) + \sqrt{\eta}, \delta$. Therefore, $W(A) = W(\tilde{A})$. The assertion of (a) now follows from Corollary 2.5 in [7]. The assertions of (b) follow from Theorem 2.4 in [7].

In the next corollary we describe regions in the complex plane which contain the numerical range of the matrix $A$ of the general form given by (2).

**Corollary 1.** Let $A$ be as given by (2).

(a) Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ denote the two square roots of $-(a, b)$, Then $W(A)$ is contained in the intersection of the two sets

$$\bigoplus_{\delta_1 + \tilde{\gamma}_j, \ldots, \delta_{n-1} + \tilde{\gamma}_j, \gamma - \tilde{\gamma}_j} \{ z \in \mathbb{C} : |z| \leq \frac{1}{2} (2(a, b) + \|a\|^2 + \|b\|^2)^{1/2} \} \quad \text{for} \quad j = 1, 2.$$  

(b) Suppose that $(a, b) \neq 0$. Let

$$\phi^{(1)} := \frac{1}{2} \arg(a, b),$$

$$\alpha^{(1)} := \frac{1}{2} \big( 2 \|a\|^2 + \|b\|^2 \big)^{1/2},$$

$$\beta^{(1)} := \frac{1}{2} \big( -2 \|a\|^2 + \|b\|^2 \big)^{1/2},$$

$$E^{(1)} := \bigg\{ z \in \mathbb{C} : z = \frac{1}{2} e^{\phi^{(1)}} ((\alpha^{(1)} + \beta^{(1)}) e^{i\theta} + (\alpha^{(1)} - \beta^{(1)}) e^{-i\theta}) \bigg\}$$

$$0 \leq \theta < 2\pi.$$
Then
\[ W(A) \subset \text{co} \{ \delta_1, \ldots, \delta_{n-1}, \gamma \} \text{ + the closed interior of the ellipse } E^{(1)}. \]

**Proof.** (a) We decompose \( A \) in the form \( A := \text{diag} \{ \delta_1 + \gamma_j, \ldots, \delta_{n-1} + \gamma_j, \gamma - \gamma_j \} + \hat{A}_j \), where \( \hat{A}_j \) is given by
\[
\hat{A}_j := \text{diag} \{ -\gamma_j, \ldots, -\gamma_j, \gamma_j \} + \begin{pmatrix} 0 & b \\ a^T & 0 \end{pmatrix} \quad \text{for } j = 1, 2.
\]
Then
\[ W(A) \subset \text{co} \{ \delta_1 + \hat{\gamma}_j, \ldots, \delta_{n-1} + \hat{\gamma}_j, \gamma - \hat{\gamma}_j \} + W(\hat{A}_j). \] (3)
Furthermore, \( \hat{A}_j \) satisfies the assumptions of Proposition 1(a). Thus, from Proposition 1(a) it follows that \( W(\hat{A}_j) \) is a circular disk with center at the origin and radius \( \frac{1}{2}(2|\gamma_j|^2 + \|a\|^2 + \|b\|^2)^{1/2} \). According to the definition of \( \hat{\gamma}_j \) from (3) the assertion follows.

(b) We decompose \( A \) in the form \( A = \text{diag} \{ \delta_1, \ldots, \delta_{n-1}, \gamma \} + A^{(1)} \), where
\[
A^{(1)} := \begin{pmatrix} 0 & \beta \\ a^T & 0 \end{pmatrix}.
\]
Then \( W(A) \subset \text{co} \{ \delta_1 + \gamma_j, \ldots, \delta_{n-1} + \gamma_j, \gamma - \gamma_j \} + W(A^{(1)}) \), and from Proposition 1(b) it follows that \( W(A^{(1)}) \) is equal to the closed interior of the ellipse \( E^{(1)} \). The assertion follows. \( \square \)

3. Regions for the zeros of polynomials

In this section we apply the results of Section 2 to certain different types of companion matrices of the monic polynomial \( P_n \) given by (1). From this containment regions for the zeros of \( P_n \) are derived. In the following theorem we use companion matrices of \( P_n \) which come from a diagonal similarity of the usual Frobenius companion matrix. Let \( P_n \) be as given by (1). We suppose that there exist complex numbers \( \gamma_1, \ldots, \gamma_n \in \mathbb{C}, 0 \neq \beta_1, \ldots, \beta_{n-1} \in \mathbb{C} \) such that
\[
\alpha_1 := \gamma_1, \quad \alpha_2 := \gamma_2 \beta_1, \quad \vdots \quad \alpha_n := \gamma_n \beta_{n-1} \cdots \beta_1. \]

Furthermore, let
\[
\delta_1 := \min \left\{ \cos \frac{\pi}{n+1}, \frac{1}{2} \max_{k=1, \ldots, n-1} |\beta_k|, \frac{1}{2} \max_{k=1, \ldots, n-2} (|\beta_k| + |\beta_{k+1}|) \right\},
\]
\[
\delta_2 := \min \left\{ \cos \frac{\pi}{n}, \frac{1}{2} \max_{k=2, \ldots, n-1} |\beta_k|, \frac{1}{2} \max_{k=2, \ldots, n-2} (|\beta_k| + |\beta_{k+1}|) \right\},
\]
and for $0 \neq \hat{\alpha} \in \mathbb{C}$ let
\[
\hat{\delta}_2 := \min \left\{ \cos \frac{\pi}{n+1} \max \{ |\hat{\alpha}|, |\beta_k|, k = 2, \ldots, n-1 \}, \right. \\
\left. \frac{1}{2} \max \{ |\hat{\alpha}| + |\beta_{n-1}|, |\beta_k| + |\beta_{k+1}|, k = 2, \ldots, n-2 \} \right\}.
\]

Decompositions of type (4) of the coefficients of $P_n$ are always possible. The simplest one is $\gamma_k := \alpha_k$, $k = 1, \ldots, n$, $\beta_1 = \cdots = \beta_{n-1} := 1$. We propose three different special choices for the decompositions (4) (cf. also [8]). Special regions which can be obtained by using some of these special choices are given in Corollary 2.

**Special Choice I.**

\[
\gamma_1 := \alpha_1, \quad \beta_1 := \max_{k=1, \ldots, n} \|\alpha_k\|^{1/k}, \\
\beta_k := \max_{j=k+1, \ldots, n} \left| \frac{\alpha_j}{\beta_1 \beta_2 \cdots \beta_{k-1}} \right|^{1/(j-k+1)}, \quad k = 2, \ldots, n-1, \\
\gamma_{k+1} := \frac{\alpha_{k+1}}{\beta_1 \beta_2 \cdots \beta_k}, \quad k = 1, \ldots, n-1.
\]

**Special Choice II.**

\[
\beta_1 := \cdots := \beta_{n-1} =: \beta := \max_{k=1, \ldots, n} |\alpha_k|^{1/k}, \\
\gamma_k := \frac{\alpha_k}{\beta^{k-1}}, \quad k = 1, 2, \ldots, n.
\]

**Special Choice III.** Let $\alpha_k \neq 0$, $k = 1, \ldots, n$. Then

\[
\beta_1 := \gamma_1 := \alpha_1, \\
\beta_k := \gamma_k := \frac{\alpha_k}{\alpha_{k-1}}, \quad k = 2, \ldots, n-1, \\
\gamma_n := \frac{\alpha_n}{\alpha_{n-1}}.
\]

**Theorem 1.** Let $P_n$ be as given by (1), and let its coefficients satisfy (4).

(a) Let $\hat{\alpha}_{2,1}, \hat{\alpha}_{2,2}$ denote the two square roots of $-\alpha_2$. Then all zeros of $P_n$ lie in the intersection of the two sets given by

\[
\text{co} \left\{ \hat{\alpha}_{2,j}, \alpha_1 - \hat{\alpha}_{2,j} \right\} + \left\{ z \in \mathbb{C} : |z| \leq \delta_2 + \frac{1}{2} \left( 2|\alpha_2| + |\beta_1|^2 + \sum_{k=2}^n |\gamma_k|^2 \right)^{1/2} \right\}
\]

for $j = 1, 2$. 
(b) All zeros of $P_n$ lie in the set

$$\text{coistine} \{ z \in \mathbb{C} : |z| \leq \delta_1 + \frac{1}{2} \left( \sum_{k=2}^{n} |\gamma_k|^2 \right)^{1/2} \}.$$

(c) Let $0 \neq \hat{\alpha} \in \mathbb{C}$, and let $\kappa_1, \kappa_2$ denote the two square roots of $\hat{\alpha} \alpha_1 - \alpha_2$. Then all zeros of $P_n$ lie in the intersection of the two sets given for $j = 1, 2, \ldots$

$$\text{co} \left\{ \kappa_j, \alpha_1 + \hat{\alpha} - \kappa_j \right\} \cup \left\{ z \in \mathbb{C} : |z| \leq \delta_2 + \frac{1}{2} \left( 2|\hat{\alpha} \alpha_1 - \alpha_2| + |\beta_1|^2 + |\gamma_n|^2 + \sum_{k=1}^{n-1} |\gamma_{k+1} - \hat{\alpha} \frac{\gamma_k}{\beta_k}|^{1/2} \right) \right\}. \quad (5)$$

(d) Suppose that $\alpha_1 \neq 0$. Let

$$\varphi_1 := \arg(\alpha_1), \quad a_1 := \frac{1}{2} \left( \sum_{k=1}^{n} |\gamma_k|^2 \right)^{1/2}, \quad b_1 := \frac{1}{2} \left( \sum_{k=2}^{n} |\gamma_k|^2 \right)^{1/2}. \quad (6)$$

Let $E_1$ denote the closed interior of the ellipse given by

$$\left\{ z \in \mathbb{C} : z = \frac{1}{2} \alpha_1 + \frac{1}{2} e^{i\varphi_1} ((a_1 + b_1)e^{i\theta} + (a_1 - b_1)e^{-i\theta}), 0 \leq \Theta < 2\pi \right\}.$$

Then all zeros of $P_n$ lie in the region given by $E_1 \cup \{ z \in \mathbb{C} : |z| \leq \delta_1 \}$.

**Proof.** Let $A_c := (\tau_{i,j})_{i,j=1}^{n}$ be the square matrix of order $n$ given by

$$\tau_{i,j} := \begin{cases} \gamma_{n-j+1}, & i = n, \quad j = 1, \ldots, n, \\
\beta_{n-i}, & i = 1, \ldots, n-1, \quad j = i+1, \\
0, & \text{otherwise}. \end{cases} \quad (7)$$

Then $\det(x E_n - A_c) = P_n(x)$. That is, $A_c$ is a companion matrix of $P_n$. We set $A_c := A_1 + A_2 + A_3 + A_4$, where $A_k := (\tau_{i,j}^{(k)})_{i,j=1}^{n}$ for $k = 1, \ldots, 4$, and

$$A_2 := \text{diag}(0, \ldots, 0, \gamma_1).$$

$$\tau_{i,j}^{(2)} := \begin{cases} \beta_1, & i = n - 1, \quad j = n, \\
0, & \text{otherwise}, \end{cases}$$

$$\tau_{i,j}^{(3)} := \begin{cases} \beta_{n-i}, & i = 1, \ldots, n-2, \quad j = i+1, \\
0, & \text{otherwise}. \end{cases}$$

(a) We have $W(A_c) \subset W(A_1 + A_2 + A_3) + W(A_4)$. Using the block form of the shift matrix $A_4$ it follows from a result of Chien (cf. Theorem 3 in [2]; we use this
theorem subsequently several times without further citing) that $W(A_4)$ is a circular disk centered at the origin with

$$r(A_4) \leq \min \left\{ \cos \pi n \sum_{k=2}^{n} |\beta_k|, \frac{1}{2} \sum_{k=2}^{n-2} (|\beta_k| + |\beta_{k+1}|) \right\}.$$  

We apply Corollary 1(a) to the matrix $A_1 + A_2 + A_3$. From this the assertion follows.

(b) We have $W(A_c) \subset W(A_1) + W(A_2) + W(A_3 + A_4)$. From Proposition 1(a) it follows that $W(A_1)$ is a circular disk with center at the origin and radius

$$\frac{1}{2} \left( \sum_{k=2}^{n} |\gamma_k|^2 \right)^{1/2}.$$  

Furthermore $W(A_2) = \text{co}\{0, \alpha_1\}$, and $W(A_3 + A_4)$ is a circular disk with center at the origin with

$$r(A_3 + A_4) \leq \min \left\{ \cos \pi n \sum_{k=2}^{n} |\beta_k|, \frac{1}{2} \sum_{k=2}^{n-2} (|\beta_k| + |\beta_{k+1}|) \right\}.$$  

The assertion follows.

(c) We consider the square matrix $A_{\hat{\alpha}} := \left( \tau(\hat{\alpha}) \right)_{i,j=1}^{n+1}$ of order $n + 1$ for $0 \neq \hat{\alpha} \in \mathbb{C}$ given by

$$\tau(\hat{\alpha}) := \begin{cases} \hat{\alpha}, & i = 1, \ j = 2, \\ \beta_{n-i+1}, & i = 2, \ldots, n, \ j = i + 1, \\ \gamma_{n-j+2} - \hat{\alpha} \frac{\gamma_{n-j+1}}{\beta_{n-j+1}}, & i = n + 1, \ j = 2, \ldots, n, \\ \gamma_1 + \hat{\alpha}, & i = n + 1, \ j = n + 1, \\ 0, & \text{otherwise}. \end{cases}$$

$A_{\hat{\alpha}}$ is a companion matrix of the polynomial $Q_{n+1}(x) := (x - \hat{\alpha})P_n(x)$. We decompose $A_{\hat{\alpha}} := B_{\hat{\alpha}} + C_{\hat{\alpha}}$ (analogous as in the proof of (a)), where

$$B_{\hat{\alpha}} := \left( \rho_{i,j} \right)_{i,j=1}^{n+1}, \quad C_{\hat{\alpha}} := \left( \sigma_{i,j} \right)_{i,j=1}^{n+1},$$

and

$$\rho_{i,j}(\hat{\alpha}) := \begin{cases} \hat{\alpha}, & i = 1, \ j = 2, \\ \beta_{n-i+1}, & i = 2, \ldots, n - 1, \ j = i + 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\sigma_{i,j}(\hat{\alpha}) := \begin{cases} \beta_1, & i = n, \ j = n + 1, \\ -\gamma_n, & i = n + 1, \ j = 1, \\ \gamma_{n-j+2} - \hat{\alpha} \frac{\gamma_{n-j+1}}{\beta_{n-j+1}}, & i = n + 1, \ j = 2, \ldots, n, \\ \gamma_1 + \hat{\alpha}, & i = n + 1, \ j = n + 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $W(A_{\hat{\alpha}}) \subset W(B_{\hat{\alpha}}) + W(C_{\hat{\alpha}})$. The block form of $B_{\hat{\alpha}}$ implies that $W(B_{\hat{\alpha}})$ is a circular disk centered at the origin with
Again we apply Corollary 1(a) to the matrix $C_{\hat{\alpha}}$. From this the assertion follows.

(d) We have $W(A_1 + A_2) \subset W(A_3 + A_4)$. $W(A_3 + A_4)$ is a circular disk with center at the origin with

$$r(A_3 + A_4) \leq \min \left\{ \cos \frac{\pi}{n+1} \max \{|\hat{\alpha}|, |\beta_k|, k = 2, \ldots, n-1\}, \right.$$

$$\left. \frac{1}{2} \max \{|\hat{\alpha}| + |\beta_{n-1}|, |\beta_k| + |\beta_{k-1}|, k = 2, \ldots, n-2\} \right\}.$$

From Proposition 1(b) it follows that $W(A_1 + A_2)$ is an elliptical disk as described by the ellipse $E_1$. The assertion follows. \[\Box\]

**Remark 1.**

(a) If $\frac{1}{4} \alpha_1^2 + \alpha_2 = 0$, then the application of Theorem 1(a) gives that all zeros of $P_n$ lie in the circular disk given by

$$|z - \frac{1}{2} \alpha_1| \leq \delta_2 + \frac{1}{2} \left( \frac{1}{2} |\alpha_1|^2 + |\beta_1|^2 + \sum_{k=2}^{n} |\gamma_k|^2 \right)^{1/2}.$$

(b) If $\alpha_1 = 0$, then the application of Theorem 1(b) gives that all zeros of $P_n$ lie in the circular disk given by

$$|z| \leq \delta_1 + \frac{1}{2} \left( \sum_{k=2}^{n} |\gamma_k|^2 \right)^{1/2}.$$

(c) If in Theorem 1(c) we choose $\hat{\alpha}$ to be equal to $\alpha_1 + 2\hat{\alpha}_{2,j}$, $j = 1, 2$, then all zeros of $P_n$ lie in the intersection of the two circular disks given by

$$|z - (\alpha_1 + \hat{\alpha}_{2,j})|$$

$$\leq \hat{\delta}_{2,j} + \frac{1}{2} \left( 2|\alpha_1 + \hat{\alpha}_{2,j}|^2 + |\beta_1|^2 + |\gamma_n|^2 + \sum_{k=1}^{n-1} \left| \gamma_{k+1} - \hat{\alpha}_j \cdot \frac{\gamma_k}{\beta_k} \right|^2 \right)^{1/2}$$

for $j = 1, 2$, where $\hat{\alpha}_j = \alpha_1 + 2\hat{\alpha}_{2,j}$, and

$$\hat{\delta}_{2,j} := \min \left\{ \cos \frac{\pi}{n+1} \max \{|\hat{\alpha}_j|, |\beta_k|, k = 2, \ldots, n-1\}, \right.$$

$$\left. \frac{1}{2} \max \{|(\hat{\alpha}_j) + |\beta_{n-1}|), |\beta_k| + |\beta_{k-1}|, k = 2, \ldots, n-2\} \right\}.$$

(d) The application of the techniques of the proof of Theorem 1 to the inverse matrix $A_1^{-1}$ of the matrix $A_1$ given by (7) yields further regions for the zeros of $P_n$ by the transformation $z \mapsto 1/z$. But except for some special cases the resulting
regions are complicated to describe in closed form. Therefore we do not consider the
details.
(e) For \( \alpha_1 \neq 0 \) the region from (d) is properly contained in the region from (b);
only the latter is to handle somewhat easier. The same is valid for the region from
Proposition 2, Formula (6), in our paper [9] (cf. also the region of Fujii and Kubo
from Theorem 2 in [4] and the region from Theorem 1 in our paper [9]), because in
these cases the numerical range of the matrix \( A_1 + A_2 \) is in Theorem 1(d) exactly
determined as an elliptical disk, whereas at the cited places only the determination of
the numerical radius of \( A_1 + A_2 \) gives a circular disk which contains \( W(A_1 + A_2) \)
properly. In this context also confer the comparison of the bounds from Fujii and
Kubo [4], from Parodi [15, p. 131] and of Carmichael-Mason (cf. [10, p. 125]) in
[9].

We consider some special cases in the following corollary to Theorem 1 by choos-
ing \( \beta_1 = \cdots = \beta_{n-1} =: \beta \).

Corollary 2. Let \( P_n \) satisfy the assumptions of Theorem 1 with \( \beta_1 = \cdots = \beta_{n-1} =: \beta \).
(a) All zeros of \( P_n \) lie in the intersection of the two sets given by
\[
\text{co} \left\{ \hat{\alpha}_{2,j}, \alpha_1 - \hat{\alpha}_{2,j} \right\} + \left\{ z \in \mathbb{C} : |z| \leq |\beta| \cos \frac{\pi}{n} + \frac{1}{2} \left( |\beta|^2 + 2|\alpha_2| + \sum_{k=2}^{n} \frac{|\alpha_k|^2}{|\beta|^{2(k-1)}} \right)^{1/2} \right\}
\]
for \( j = 1, 2 \).
(b) All zeros of \( P_n \) lie in the set
\[
\text{co} \{ 0, \alpha_1 \} + \left\{ z \in \mathbb{C} : |z| \leq |\beta| \cos \frac{\pi}{n+1} + \frac{1}{2} \left( \sum_{k=2}^{n} \frac{|\alpha_k|^2}{|\beta|^{2(k-1)}} \right)^{1/2} \right\}.
\]
(c) All zeros of \( P_n \) lie in the intersection of the two circular disks given by
\[
|z - (\alpha_1 + \hat{\alpha}_{2,j})| \leq \hat{\delta}_{2,j} + \frac{1}{2} \left( 2|\alpha_1 + \hat{\alpha}_{2,j}|^2 + |\beta|^2 + \frac{|\alpha_n|^2}{|\beta|^{2(n-1)}} \right) \sum_{k=1}^{n-1} \frac{1}{|\beta|^{2k}} (|\alpha_k + 1 - \beta \hat{\alpha}_j |^2 + \beta \hat{\alpha}_j |\alpha_k|^2)^{1/2}
\]
for \( j = 1, 2 \), where \( \hat{\alpha}_j = \alpha_1 + 2 \hat{\alpha}_{2,j}, j = 1, 2 \), and
\[
\hat{\delta}_{2,j} := \min \left\{ \cos \frac{\pi}{n+1} \max \{|\alpha_j|, |\beta|\}, \max \left\{ \frac{1}{2} (|\alpha_j| + |\beta|), |\beta| \right\} \right\}.
\]
(d) Suppose that $\alpha_1 \neq 0$. Let
\[
a_{1\beta} := \frac{1}{2} \left( \sum_{k=1}^{n} \frac{|\alpha_k|^2}{|\beta|^{2(k-1)}} \right)^{1/2}, \quad b_{1\beta} := \frac{1}{2} \left( \sum_{k=2}^{n} \frac{|\alpha_k|^2}{|\beta|^{2(k-1)}} \right)^{1/2}.
\]

Let $E_{1\beta}$ denote the closed interior of the ellipse given by
\[
\{ z \in \mathbb{C} : z = \frac{1}{2} \alpha_1 + \frac{1}{2} e^{\Theta i} \left( (a_{1\beta} + b_{1\beta}) e^{i\Theta} + (a_{1\beta} - b_{1\beta}) e^{-i\Theta} \right), 0 \leq \Theta < 2\pi \}.
\]

Then all zeros of $P_n$ lie in the region given by
\[
E_{1\beta} + \left\{ z \in \mathbb{C} : |z| \leq |\beta| \cos \frac{\pi}{n+1} \right\}.
\]

Since some of the containment regions are not easily comparable, we consider an example.

**Example 1.** We take the polynomial $P_3(x) := x^3 - 3x^2 + 4x - 2$ from the book of Gustafson and Rao [5, p. 122], which has the zeros $x_1 := 1 - i$, $x_2 := 1$, $x_3 := 1 + i$. Then from Corollary 2 we have (Corollary 2(a) with $\beta := \frac{3}{2}$)
\[
x_k \in \text{co}\{1, 2\} + \left\{ z \in \mathbb{C} : |z| \leq 2.8803 \right\},
\]
and thus
\[
x_k \in \{ z \in \mathbb{C} : -1.8803 \leq \Re z \leq 4.8803, -2.8803 \leq \Im z \leq 2.8803 \},
\]
and (Corollary 2(b) with $\beta := 2$)
\[
x_k \in \text{co}\{0, 3\} + \left\{ z \in \mathbb{C} : |z| \leq 2.4450 \right\},
\]
and thus
\[
x_k \in \{ z \in \mathbb{C} : -2.4450 \leq \Re z \leq 5.4450, -2.4450 \leq \Im z \leq 2.4450 \},
\]
and (Corollary 2(c) with $\beta := 2^{1/3}$)
\[
x_k \in \{ z \in \mathbb{C} : |z - 1| \leq 2.3803 \},
\]
and (Corollary 2(d) with $\beta := 2$)
\[
x_k \in \{ z \in \mathbb{C} : -1.7342 \leq \Re z \leq 4.7344, -2.4449 \leq \Im z \leq 2.4450 \},
\]
whereas for example the bound from Abdurakhmanov [1] gives $|x_k| \leq 4.7581$ (see [5, p. 122], cf. also the containment regions in [5, p. 117] coming from Gershgorin-type sets). For this example the containment region from Corollary 2(c) gives the best result.

Using a different type of generalized companion matrix not coming from a diagonal similarity of the Frobenius companion matrix we get some more regions for the zeros of $P_n$. Let $P_n$ be as given by (1). We suppose now that there exist complex numbers $\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_2^{(2)}, \ldots, \alpha_n^{(1)} - \alpha_n^{(2)}, \ldots, \alpha_n^{(n)} \in \mathbb{C}$ such that
\[ \alpha_1 := \alpha_1^{(1)}, \]
\[ \alpha_2 := \alpha_2^{(1)} \alpha_2^{(2)}, \]
\[ \vdots \]
\[ \alpha_n := \alpha_n^{(1)} \alpha_n^{(2)} \cdots \alpha_n^{(n)}. \]

Furthermore, let
\[ \widehat{\beta}_1 := \max \left\{ \frac{1}{2} |\alpha_2^{(2)}|, \frac{1}{2} |\alpha_3^{(2)}|, \max_{k=4, \ldots, n} \cos \frac{\pi}{k} \max_{j=2, \ldots, k-1} |\alpha_k^{(j)}|, \frac{1}{2} \max_{j=2, \ldots, k-2} \left( |\alpha_k^{(j)}| + |\alpha_k^{(j+1)}| \right) \right\}, \]
\[ \widehat{\beta}_2 := \max \left\{ \frac{1}{2} |\alpha_3^{(2)}|, \max_{k=4, \ldots, n} \cos \frac{\pi}{k} \max_{j=2, \ldots, k-1} |\alpha_k^{(j)}|, \frac{1}{2} \max_{j=2, \ldots, k-2} \left( |\alpha_k^{(j)}| + |\alpha_k^{(j+1)}| \right) \right\}, \]
and for \( 0 \neq \alpha \in \mathbb{C} \) let
\[ \widehat{\beta}_3 := \max \left\{ \frac{1}{2} |\alpha_3^{(2)}|, \max_{k=4, \ldots, n} \cos \frac{\pi}{k} \max_{j=2, \ldots, k-1} |\alpha_k^{(j)}|, \frac{1}{2} \max_{j=2, \ldots, k-2} \left( |\alpha_k^{(j)}| + |\alpha_k^{(j+1)}| \right) \right\}, \]
\[ \min \left\{ \cos \frac{\pi}{n+1} \max \left\{ |\alpha_1|, |\alpha_n^{(2)}|, \ldots, |\alpha_n^{(n-1)}| \right\}, \frac{1}{2} \max_{j=2, \ldots, n-2} \left( |\alpha_1| + |\alpha_n^{(2)}|, |\alpha_n^{(j)}| + |\alpha_n^{(j+1)}| \right) \right\}. \]

Decompositions of type (8) of the coefficients of \( P_n \) are always possible. The simplest one is if we choose \( \alpha_k^{(1)} := \alpha_k, \ k = 1, 2, \ldots, n, \ \alpha_k^{(j)} := 1, \ j = 2, \ldots, k, k = 2, \ldots, n. \) But also the decomposition (4) is a special case of decomposition (8). In this case a decomposition of type (4) generates many different decompositions (8). Furthermore the Special Choices I–III are applicable here. We propose some further special choices of the decomposition coefficients in (8), which give special estimates for the zeros of \( P_n \) (cf. also Corollaries 3 and 4). Many other useful decompositions are possible (cf. [8] for some further discussion):

Special Choice IV. Let \( \alpha \) be a positive real number. Then
\[ \alpha_1 := \alpha_1, \]
\[ \alpha_k^{(1)} := \left( \frac{|\alpha_k|}{\alpha_k^{k-2}} \right)^{1/2} \exp(i \arg(\alpha_k)), \ k = 2, \ldots, n, \]
\[ \alpha^{(k)}_k := \left( \frac{\vert \alpha_k \vert}{\alpha^{k-2}} \right)^{1/2}, \quad k = 2, \ldots, n, \]
\[ \alpha^{(j)}_k := \alpha, \quad j = 2, \ldots, k-1, \quad k = 3, \ldots, n. \]

The additional special choices \( \alpha := 1 \) (cf. Corollary 3) and \( \alpha := \max_{k=1,\ldots,n} |\alpha_k|^{1/k} \) are useful.

**Special Choice V.**
\[ \alpha^{(1)}_1 := \alpha_1, \]
\[ \alpha^{(1)}_k := |\alpha_k|^{1/k} \exp(i \arg(\alpha_k)), \quad k = 2, \ldots, n, \]
\[ \alpha^{(j)}_k := |\alpha_k|^{1/k}, \quad j = 2, \ldots, k, \quad k = 2, \ldots, n. \]

**Theorem 2.** Let \( P_n \) be as given by (1), and let its coefficients satisfy (8).

(a) All zeros of \( P_n \) lie in the intersection of the two sets given by
\[
\text{co}\{\hat{\alpha}_2, j, \alpha_1 - \hat{\alpha}_2, j\} + \left\{ z \in \mathbb{C} : \vert z \vert \leq \beta_2 + \frac{1}{2} \left( 2 |\alpha_2| + \sum_{k=2}^{n} \left( |\alpha^{(1)}_k| + |\alpha^{(k)}_k| \right) \right)^{1/2} \right\}
\]
for \( j = 1, 2 \).

(b) All zeros of \( P_n \) lie in the set
\[
\text{co}\{0, \alpha_1\} + \left\{ z \in \mathbb{C} : \vert z \vert \leq \beta_1 + \frac{1}{2} \left( |\alpha^{(1)}_2| + \sum_{k=3}^{n} \left( |\alpha^{(1)}_k| + |\alpha^{(k)}_k| \right) \right)^{1/2} \right\}.
\]

(c) Let \( 0 \neq \hat{\alpha} \in \mathbb{C} \), let \( \alpha_2, \ldots, \alpha_n \neq 0 \), and let \( \kappa_1, \kappa_2 \) denote the two square roots of \( \hat{\alpha}_1 \alpha_1 - \hat{\alpha}_2 \). Then all zeros of \( P_n \) lie in the intersection of the two sets given for \( j = 1, 2 \), by
\[
\text{co}\{\kappa_j, \alpha_1 + \hat{\alpha} - \kappa_j\} + \left\{ z \in \mathbb{C} : \vert z \vert \leq \beta_2 + \frac{1}{2} \left( 2|\alpha_1 - \alpha_2| + |\alpha^{(1)}_n| + |\alpha^{(n)}_n| + \sum_{k=2}^{n} |\alpha^{(1)}_k| \right.ight.
\]
\[
+ \sum_{k=2}^{n} |\alpha^{(k)}_k| \left( 1 - \hat{\alpha} \frac{\alpha^{k-1}}{\alpha_k} \right)^{1/2} \right\}.
\]

(d) Suppose that \( \alpha_1 \neq 0 \). Let
\[
\hat{\alpha}_1 := \frac{1}{2} \left( |\alpha_1|^2 + \sum_{k=2}^{n} \left( |\alpha^{(1)}_k|^2 + |\alpha^{(k)}_k|^2 \right) \right)^{1/2},
\]
\[ \hat{b}_1 := \frac{1}{2} \left( \sum_{k=2}^{n} \left( |a_k^{(1)}|^2 + |a_k^{(k)}|^2 \right) \right)^{1/2}. \]

Let \( \hat{E}_1 \) denote the closed interior of the ellipse given by
\[ \{ z \in \mathbb{C} : z = \frac{1}{2} \alpha_1 + \frac{1}{2} e^{\Theta i} \left( (\hat{a}_1 + \hat{b}_1)e^{i\Theta} + (\hat{a}_1 - \hat{b}_1)e^{-i\Theta} \right), 0 \leq \Theta < 2\pi \}. \]

Then all zeros of \( P_n \) lie in the region given by \( \hat{E}_1 + \{ z \in \mathbb{C} : |z| \leq \hat{b}_1 \} \).

**Proof.** Let \( \hat{n} := 1 + \frac{1}{2} n(n-1) \), and let \( \hat{A}_c := (\hat{\tau}_{i,j})_{i,j=1}^{\hat{n}} \) be the square matrix of order \( \hat{n} \) with
\[
\hat{\tau}_{i,j} := \begin{cases} 
\alpha_{\mu}^{(1)}, & i = \hat{n}, \ j = \hat{n} - \frac{1}{2} \mu (\mu - 1), \ 1 \leq \mu \leq n, \\
\alpha_{\mu}^{(\mu)}, & i = \hat{n} - 1 - \frac{1}{2} (\mu - 1) (\mu - 2), \ j = \hat{n}, \ 2 \leq \mu \leq n, \\
\alpha_{\nu}^{(\mu)}, & i = 3 - \mu + \frac{1}{2} \nu (\nu - 1), \ j = i - 1, \ 2 \leq \mu < \nu \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]

Then
\[
\det \left( x E_{\hat{n}} - \hat{A}_c \right) = x^{(n-1)(n-2)/2} P_n(x),
\]
where \( E_{\hat{n}} \) is the identity matrix of order \( \hat{n} \). Thus \( \hat{A}_c \) is a generalized companion matrix of \( P_n \). We set \( \hat{A}_2 := \hat{A}_1 + \hat{A}_2 + \hat{A}_3 + \hat{A}_4 \), where \( \hat{A}_k := (\hat{\tau}_{i,j}^{(k)})_{i,j=1}^{\hat{n}} \) for \( k = 1, \ldots, 4 \), and
\[
\hat{\tau}_{i,j}^{(1)} := \begin{cases} 
\alpha_{\mu}^{(1)}, & i = \hat{n}, \ j = \hat{n} - \frac{1}{2} \mu (\mu - 1), \ 2 \leq \mu \leq n, \\
\alpha_{\mu}^{(\mu)}, & i = \hat{n} - 1 - \frac{1}{2} (\mu - 1) (\mu - 2), \ j = \hat{n}, \ 3 \leq \mu \leq n, \\
0, & \text{otherwise},
\end{cases}
\]
\[
\hat{A}_2 := \text{diag}(0, \ldots, 0, \alpha_{1}^{(1)}),
\]
\[
\hat{\tau}_{i,j}^{(2)} := \begin{cases} 
\alpha_{2}^{(2)}, & i = \hat{n} - 1, \ j = \hat{n}, \\
0, & \text{otherwise},
\end{cases}
\]
\[
\hat{\tau}_{i,j}^{(3)} := \begin{cases} 
\alpha_{\nu}^{(\mu)}, & i = 3 - \mu + \frac{1}{2} \nu (\nu - 1), \ j = i - 1, \ 2 \leq \mu < \nu \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]

(a) We have \( W(\hat{A}_c) \subset W(\hat{A}_1 + \hat{A}_2 + \hat{A}_3) + W(\hat{A}_4) \). Using the block form of the shift matrix \( \hat{A}_4 \) it follows that \( W(\hat{A}_4) \) is a circular disk centered at the origin with \( r(\hat{A}_4) \leq \hat{b}_2 \). We apply Corollary 1(a) to the matrix \( \hat{A}_1 + \hat{A}_2 + \hat{A}_3 \). The assertion follows.
(b) We have $W(\hat{A}_c) \subset W(\hat{A}_1) + W(\hat{A}_2) + W(\hat{A}_3 + \hat{A}_4)$. From Proposition 1(a) it follows that $W(\hat{A}_1)$ is a circular disk with center at the origin and radius 

$$\frac{1}{2} \left( |\alpha_2^{(1)}|^2 + \sum_{k=3}^{n} (|\alpha_k^{(1)}|^2 + |\alpha_k^{(k)}|^2) \right)^{1/2}.$$ 

Furthermore, $W(\hat{A}_2) = \text{co}\{0, \alpha_1\}$, and the block form of the shift matrix $\hat{A}_3 + \hat{A}_4$ implies that $W(\hat{A}_3 + \hat{A}_4)$ is a circular disk with center at the origin with $r(\hat{A}_3 + \hat{A}_4) \leq \hat{\beta}_1$. From this the assertion follows.

(c) We consider the square matrix $\hat{A}_{\hat{\alpha}} := \left( \hat{\tau}_{i,j}^{(\hat{\alpha})} \right)_{i,j=1}^{\hat{n}+n}$ of order $\hat{n} + n$ for $0 \neq \hat{\alpha} \in \mathbb{C}$ given by

$$\hat{A}_{\hat{\alpha}} := \left\{ \begin{array}{ll}
\alpha_i^{(1)} + \hat{\alpha}, & i = \hat{n} + n, \quad j = \hat{n} + n, \\
\alpha_i^{(1)} \hat{\alpha}, & i = \hat{n} + n, \quad j = \hat{n} + n - \frac{1}{2} \mu (\mu - 1), \\
\alpha_\mu^{(\mu)} (1 - \hat{\alpha} \alpha_{\mu-1}/\alpha_{\mu}), & i = \hat{n} + n - 1 - \frac{1}{2} (\mu - 1)(\mu - 2), \\
\alpha_i^{(\mu)}, & i = \hat{n} + n, \quad 2 \leq \mu \leq n, \\
\alpha_i^{(1)} \hat{\alpha}, & i = 3 - \mu + \frac{1}{2} \nu (\nu - 1), \\
\alpha_i^{(1)} \hat{\alpha}, & j = i - 1, \quad 2 \leq \mu < \nu \leq n, \\
\alpha_i^{(n)}, & i = \hat{n} + n, \quad j = 1, \\
\alpha_i^{(n)}, & i = 2, \ldots, n - 1, \quad j = 3, \ldots, n, \\
\alpha_i^{(n)}, & i = n, \quad j = \hat{n} + n, \\
\hat{\alpha}, & i = 1, \quad j = 2, \\
0, & \text{otherwise.}
\end{array} \right\}$$

$\hat{A}_{\hat{\alpha}}$ is also a generalized companion matrix of the polynomial $Q_{n+1}(x) := (x - \hat{\alpha})P_n(x)$. We decompose $\hat{A}_{\hat{\alpha}} = \hat{B}_{\hat{\alpha}} + \hat{C}_{\hat{\alpha}}$, where

$\hat{B}_{\hat{\alpha}} := \left( \hat{\rho}_{i,j}^{(\hat{\alpha})} \right)_{i,j=1}^{\hat{n}+n}$, \quad $\hat{C}_{\hat{\alpha}} := \left( \hat{\sigma}_{i,j}^{(\hat{\alpha})} \right)_{i,j=1}^{\hat{n}+n}$,

and

$$\hat{\rho}_{i,j}^{(\hat{\alpha})} := \left\{ \begin{array}{ll}
\alpha_i^{(\mu)}, & i = 3 - \mu + \frac{1}{2} \nu (\nu - 1), \quad j = i - 1, \quad 2 \leq \mu < \nu \leq n, \\
\alpha_i^{(1)} \hat{\alpha}, & i = 2, \ldots, n - 1, \quad j = 3, \ldots, n, \\
\hat{\alpha}, & i = 1, \quad j = 2, \\
0, & \text{otherwise.}
\end{array} \right\}$$
zeros of $P_n$


(a) Remark 2.

From Proposition 1(b) it follows that $W(\hat{\mathcal{A}})$ with

\[
\hat{\sigma}_{i,j}(\hat{\alpha}) := \begin{cases} 
\alpha_1^{(1)} + \hat{\alpha}, & i = \hat{n} + n, \ j = \hat{n} + n, \\
\alpha_\mu^{(1)}, & i = \hat{n} + n, \ j = \hat{n} + n - \frac{1}{2}\mu(\mu - 1), \\
\alpha_\mu^{(\mu)}(1 - \hat{\alpha}\alpha_{\mu-1}/\alpha_\mu), & i = \hat{n} + n - 1 - \frac{1}{2}(\mu - 1)(\mu - 2), \\
\alpha_\mu^{(i)}, & j = \hat{n} + n, 2 \leq \mu \leq n, \\
\alpha_\mu^{(n)}, & i = n, \ j = \hat{n} + n, \\
0, & \text{otherwise.}
\end{cases}
\]

Then $W(\hat{\mathcal{A}}) \subset W(\hat{\mathcal{B}}) + W(\hat{\mathcal{C}})$. Using the block form of the shift matrix $\hat{\mathcal{B}}$ it follows that $W(\hat{\mathcal{B}})$ is a circular disk centered at the origin with

\[
r(\hat{\mathcal{B}}) \leq \max \left\{ \frac{1}{2} |\alpha_3^{(2)}|, \ \max_{k=4,\ldots,n} \left\{ \cos \frac{\pi}{k} \max_{j=2,\ldots,k-1} |\alpha_k^{(j)}|, \right. \right. \\
\left. \left. \frac{1}{2} \max_{j=2,\ldots,k-2} \left( |\alpha_k^{(j)}| + |\alpha_k^{(j+1)}| \right) \right\}, \right. \\
\left. \min \left\{ \cos \frac{\pi}{n+1} \max \left\{ |\hat{\alpha}|, |\alpha_n^{(2)}|, \ldots, |\alpha_n^{(n-1)}| \right\}, \right. \\
\left. \left. \frac{1}{2} \max_{j=2,\ldots,n-2} \left\{ |\hat{\alpha}| + |\alpha_n^{(2)}|, |\alpha_n^{(j)}| + |\alpha_n^{(j+1)}| \right\} \right\} \right\},
\]

and we apply Corollary 1(a) to the matrix $\hat{\mathcal{C}}$. The assertion follows.

(d) We have $W(\hat{\mathcal{A}}_c) \subset W(\hat{\mathcal{A}}_1 + \hat{\mathcal{A}}_2) + W(\hat{\mathcal{A}}_3 + \hat{\mathcal{A}}_4)$. The block form of the shift matrix $\hat{\mathcal{A}}_3 + \hat{\mathcal{A}}_4$ implies that $W(\hat{\mathcal{A}}_3 + \hat{\mathcal{A}}_4)$ is a circular disk with center at the origin with

\[
r(\hat{\mathcal{A}}_3 + \hat{\mathcal{A}}_4) \leq \hat{\beta}_1.
\]

From Proposition 1(b) it follows that $W(\hat{\mathcal{A}}_1 + \hat{\mathcal{A}}_2)$ is an elliptical disk as described by the ellipse $\hat{E}_1$. From this the assertion follows. 

\[\square\]

Remark 2. (a) If $\frac{1}{4}\alpha_1^2 + \alpha_2 = 0$, then the application of Theorem 2(a) gives that all zeros of $P_n$ lie in the circular disk given by

\[
|z - \frac{1}{2}\alpha_1| \leq \hat{\beta}_2 + \frac{1}{2} \left( 2|\alpha_2| + \sum_{k=2}^n \left( |\alpha_k^{(1)}|^2 + |\alpha_k^{(k)}|^2 \right) \right)^{1/2}.
\]

(b) If $\alpha_1 = 0$, then the application of Theorem 2(b) gives that all zeros of $P_n$ lie in the circular disk given by

\[
|z| \leq \hat{\beta}_1 + \frac{1}{2} \left( \sum_{k=2}^n \left( |\alpha_k^{(1)}|^2 + |\alpha_k^{(k)}|^2 \right) \right)^{1/2}.
\]
(c) If in Theorem 1(c) we choose \( \hat{\alpha} \) to be equal to \( \alpha_1 + 2\hat{\alpha}_{2,j}, j = 1, 2 \), then all zeros of \( P_n \) lie in the intersection of the two circular disks given by

\[
|z - (\alpha_1 + \hat{\alpha}_{2,j})| \leq \beta_{2,j} + \frac{1}{2} \left( 2|\alpha_1 + \hat{\alpha}_{2,j}|^2 + |\alpha_n^{(1)}|^2 + |\alpha_n^{(n)}|^2 + \sum_{k=2}^{n} |\alpha_k^{(k)}|^2 \right)^{1/2}
\]

for \( j = 1, 2 \), where \( \hat{\alpha}_j = \alpha_1 + 2\hat{\alpha}_{2,j}, j = 1, 2 \), and

\[
\beta_{2,j} := \max \left\{ \frac{1}{2}|\alpha_2^{(2)}|, \max_{k=4,\ldots,n} \min \left\{ \cos \frac{\pi}{k}, \max_{j=2,\ldots,k-1} |\alpha_k^{(j)}|, \frac{1}{2} \max_{j=2,\ldots,k-2} \left( |\alpha_k^{(j)}| + |\alpha_k^{(j+1)}| \right) \right\} \right. 
\]

\[
\left. \min \left\{ \cos \frac{\pi}{n+1} \max \left\{ |\alpha_j|, |\alpha_2^{(2)}|, \ldots, |\alpha_n^{(n-1)}| \right\}, \frac{1}{2} \max_{j=2,\ldots,n-2} \left\{ |\alpha_j| + |\alpha_2^{(2)}|, |\alpha_n^{(n)}| + |\alpha_n^{(n+1)}| \right\} \right\} \right\}
\]

See also Remark 2(c).

(d) For \( \alpha_1 \neq 0 \) the region from Theorem 2(d) is properly contained in the region from (b); only the latter is to handle somewhat easier.

We consider two special cases of the decompositions of the coefficients of \( P_n \) in the following two corollaries to Theorem 2.

**Corollary 3.** Let \( P_n \) be as given by (1).

(a) All zeros of \( P_n \) lie in the intersection of the two sets given by

\[
\text{co}\{\hat{\alpha}_{2,j}, \alpha_1 - \hat{\alpha}_{2,j}\} + \left\{ z \in \mathbb{C} : |z| \leq \cos \frac{\pi}{n} + \frac{1}{2} \sqrt{2} \left( |\alpha_2| + \sum_{k=2}^{n} |\alpha_k| \right)^{1/2} \right\}
\]

for \( j = 1, 2 \).

(b) All zeros of \( P_n \) lie in the set

\[
\text{co}\{0, \alpha_1\} + \left\{ z \in \mathbb{C} : |z| \leq \max \left\{ \frac{1}{2} |\alpha_2|^{1/2}, \cos \frac{\pi}{n} \right\}, \right. 
\]

\[
\left. + \frac{1}{2} \sqrt{2} \left( \frac{1}{2} |\alpha_2| + \sum_{k=3}^{n} |\alpha_k| \right)^{1/2} \right\}.
\]
Let \( \alpha_2, \ldots, \alpha_n \neq 0 \). Then all zeros of \( P_n \) lie in the intersection of the two circular disks given by

\[
|z - (\alpha_1 + \hat{\alpha}_{2,j})| \leq \hat{\beta}_{2,j} + \frac{1}{2} \left( 2|\alpha_1 + \hat{\alpha}_{2,j}|^2 + 2|\alpha_n| + \sum_{k=2}^{n} |\alpha_k| \left( 1 + \left| 1 - \frac{\alpha_{k-1}}{\alpha_k} \right|^2 \right) \right)^{1/2}
\]

for \( j = 1, 2 \), where \( \hat{\alpha}_j = \alpha_1 + 2\hat{\alpha}_{2,j}, j = 1, 2 \), and

\[
\hat{\beta}_{2,j} := \max \left\{ \cos \frac{\pi}{n}, \min \left\{ \cos \frac{\pi}{n+1} \max \{ |\hat{\alpha}_j|, 1 \}, \max \left\{ \frac{1}{2} (|\hat{\alpha}_j| + 1), 1 \right\} \right\} \right\}.
\]

(d) Suppose that \( \alpha_1 \neq 0 \). Let

\[
\hat{a}_{1Q} := \frac{1}{2} \left( |\alpha_1|^2 + 2 \sum_{k=2}^{n} |\alpha_k| \right)^{1/2}, \quad \hat{b}_{1Q} := \frac{1}{2} \left( 2 \sum_{k=2}^{n} |\alpha_k| \right)^{1/2}.
\]

Let \( \hat{E}_{1Q} \) denote the closed interior of the ellipse given by

\[
\{ z \in \mathbb{C} : z = \frac{1}{2} \alpha_1 + \frac{1}{2} e^{i\Theta} ((\hat{a}_{1Q} + \hat{b}_{1Q}) e^{i\Theta} + (\hat{a}_{1Q} - \hat{b}_{1Q}) e^{-i\Theta}) , 0 \leq \Theta < 2\pi \}.
\]

Then all zeros of \( P_n \) lie in the region given by

\[
\hat{E}_{1Q} + \left\{ z \in \mathbb{C} : |z| \leq \max \left\{ \frac{1}{2} |\alpha_2|^{1/2}, \cos \frac{\pi}{n} \right\} \right\}.
\]

**Proof.** In Theorem 2 we make the special choice \( \alpha_{k}^{(1)} = \alpha_{k}^{(k)} \) to be equal to the same square root of \( \alpha_k, k = 2, \ldots, n \), and all other \( \alpha_{k}^{(j)} \) to be equal to 1. Then from Theorem 2 the assertions follow. \( \square \)

**Corollary 4.** Let \( P_n \) be as given by (1).

(a) All zeros of \( P_n \) lie in the intersection of the two sets given by

\[
\text{co}\left[ \hat{\alpha}_{2,j}, \alpha_1 - \hat{\alpha}_{2,j} \right] + \left\{ z \in \mathbb{C} : |z| \leq \max_{k=3, \ldots, n} |\alpha_k|^{1/k} \cos \frac{\pi}{k} \right. \left. + \frac{1}{2} \sqrt{2} \left( |\alpha_2| + \sum_{k=2}^{n} |\alpha_k|^{2/k} \right)^{1/2} \right\}
\]

for \( j = 1, 2 \).
(b) All zeros of \( P_n \) lie in the set
\[
\text{co}\{0, \alpha_1\} + \left\{ z \in \mathbb{C} : |z| \leq \max \left\{ \frac{1}{2} |\alpha_2|^{1/2}, \max_{k=3,\ldots,n} |\alpha_k|^{1/k} \cos \frac{\pi}{k} \right\} \right. \\
\left. + \frac{1}{2} \sqrt{2} \left( \frac{1}{2} |\alpha_2| + \sum_{k=3}^{n} |\alpha_k|^{2/k} \right)^{1/2} \right\}.
\]

(c) Let \( \alpha_2, \ldots, \alpha_n \neq 0 \). Then all zeros of \( P_n \) lie in the intersection of the two circular disks given by
\[
|z - (\alpha_1 + \hat{\alpha}_{2,j})| \leq \hat{\beta}_{2,j} + \frac{1}{2} \left( 2|\alpha_1 + \hat{\alpha}_{2,j}|^2 + 2|\alpha_n|^{2/n} \right. \\
\left. + \sum_{k=2}^{n} |\alpha_k|^{2/k} \left( 1 + \left| 1 - \hat{\alpha}_j \frac{\alpha_{k-1}}{\alpha_k} \right|^2 \right) \right)^{1/2}
\]
for \( j = 1, 2 \), where \( \hat{\alpha}_j = \alpha_1 + 2\hat{\alpha}_{2,j}, j = 1, 2 \), and
\[
\hat{\beta}_{2,j} := \max \left\{ \max_{k=3,\ldots,n} |\alpha_k|^{1/k} \cos \frac{\pi}{k}, \min \left\{ \cos \frac{\pi}{n+1}, \max \left\{ |\hat{\alpha}_j|, |\alpha_n|^{1/n} \right\} \right\}, \right. \\
\left. \max \left\{ \frac{1}{2} \left( |\hat{\alpha}_j| + |\alpha_n|^{1/n} \right), |\alpha_n|^{1/n} \right\} \right\}.
\]

(d) Suppose that \( \alpha_1 \neq 0 \). Let
\[
\hat{\alpha}_{1R} := \frac{1}{2} \left( |\alpha_1|^2 + 2 \sum_{k=2}^{n} |\alpha_k|^{2/k} \right)^{1/2}, \quad \hat{\beta}_{1R} := \frac{1}{2} \left( 2 \sum_{k=2}^{n} |\alpha_k|^{2/k} \right)^{1/2}.
\]
Let \( \hat{E}_{1R} \) denote the closed interior of the ellipse given by
\[
\left\{ z \in \mathbb{C} : z = \frac{1}{2} \alpha_1 + \frac{1}{2} e^{\Theta i} (\hat{\alpha}_{1R} + \hat{\beta}_{1R}) e^{i\Theta} + (\hat{\alpha}_{1R} - \hat{\beta}_{1R}) e^{-i\Theta}, 0 \leq \Theta < 2\pi \right\}.
\]
Then all zeros of \( P_n \) lie in the region given by
\[
\hat{E}_{1R} + \left\{ z \in \mathbb{C} : |z| \leq \max \left\{ \frac{1}{2} |\alpha_2|^{1/2}, \max_{k=3,\ldots,n} |\alpha_k|^{1/k} \cos \frac{\pi}{k} \right\} \right\}.
\]

Proof. In Theorem 2 we make the special choice \( \alpha_k^{(j)}, j = 1, \ldots, k \), to be equal to the same \( k \)th root of \( \alpha_k \) for \( k = 2, \ldots, n \). Then from Theorem 2 the assertions follow. 

Example 2. We take the polynomial \( P_3 \) from Example 1. Then we have (Corollary 3(a))
\[ x_k \in \text{co}\{1, 2\} + \{z \in \mathbb{C} : |z| \leq 2.7361\}, \]

and thus
\[ x_k \in \left\{ z \in \mathbb{C} : -1.7360 \leq \Re z \leq 4.7361, -2.7360 \leq \Im z \leq 2.7361 \right\}, \]

and (Corollary 3(b))
\[ x_k \in \text{co}\{0, 3\} + \{z \in \mathbb{C} : |z| \leq 2.4142\}, \]

and thus
\[ x_k \in \left\{ z \in \mathbb{C} : -2.4141 \leq \Re z \leq 5.4143, -2.4141 \leq \Im z \leq 2.4141 \right\}, \]

and (Corollary 3(c))
\[ x_k \in \left\{ z \in \mathbb{C} : |z - 1| \leq 2.5946 \right\}, \]

and (Corollary 3(d))
\[ x_k \in \left\{ z \in \mathbb{C} : -2 \leq \Re z \leq 5, -3 \leq \Im z \leq 3 \right\}. \]

Furthermore, we have (Corollary 4(a))
\[ x_k \in \text{co}\{1, 2\} + \{z \in \mathbb{C} : |z| \leq 2.8195\}, \]

and thus
\[ x_k \in \left\{ z \in \mathbb{C} : -1.8194 \leq \Re z \leq 4.8195, -2.8194 \leq \Im z \leq 2.8195 \right\}, \]

and (Corollary 4(b))
\[ x_k \in \text{co}\{0, 3\} + \{z \in \mathbb{C} : |z| \leq 2.3393\}, \]

and thus
\[ x_k \in \left\{ z \in \mathbb{C} : -2.3392 \leq \Re z \leq 5.3393, -2.3392 \leq \Im z \leq 2.3393 \right\}, \]

and (Corollary 4(c))
\[ x_k \in \left\{ z \in \mathbb{C} : |z - 1| \leq 2.6657 \right\}, \]

and (Corollary 4(d))
\[ x_k \in \left\{ z \in \mathbb{C} : -1.7459 \leq \Re z \leq 4.7459, -2.6714 \leq \Im z \leq 2.6715 \right\}, \]

whereas for example a bound from [9, Theorem 5] gives \(|x_k| \leq 4.5\). This bound is based on a theorem from [1] applied to the matrix \( \hat{A}_c \).

Note. The author is indebted to anonymous referees for their remarks, in particular for the present formulation and proof of Proposition 1.
References