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Morphic and principal-ideal group rings

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Abstract

For R an artinian ring and G a group, we characterize when RG is a principal ideal ring. In the case when G is finite (and R artinian), this yields a characterization of when RG is a left and right morphic ring. This extends work done by Passman, Sehgal and Fisher on principal ideal group rings when the coefficient ring is a field, and work of Chen, Li, and Zhou on morphic group rings.

Keywords: Principal ideal ring; Morphic ring; Group ring; Annihilator; Skew polynomial ring; Artinian ring

1. Introduction

Throughout this article, the term artinian ring will refer to a left and right artinian ring, and the term principal ideal ring will refer to a ring all of whose one-sided ideals are principal. Consider the following question:

Question 1. Given a ring R and a group G, when is the group ring RG a principal ideal ring?

The classical group algebra case of this question, when R is a (commutative) field, and G is an arbitrary group, was answered by Sehgal and Fisher in [4] in case G is nilpotent, and later completed (for arbitrary groups) by Passman in [14, Theorem 4.1]. With only minor changes (detailed in Appendix A, below), the proof given in [14] works for division rings as well.

On another, perhaps seemingly unrelated, topic, in [13], Nicholson and Sánchez Campos investigated the "morphic" rings. These are rings which satisfy a certain dual of the first iso-

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morphism theorem. Specifically, an element $a \in R$ is said to be left morphic if $R/Ra \cong \mathrm{ann}_\ell(a)$ (which is dual to the theorem that $R/\mathrm{ann}_\ell(a) \cong Ra$). A ring is said to be left morphic if each of its elements is left morphic; right morphic elements and rings are defined similarly. We say that R is morphic if it is left and right morphic. In [13, Example 36], an example is given of a group ring RG, for which R is artinian and left and right morphic, G is a finite group, but for which RG is not a morphic ring. Motivated by this example, in [1], Chen, Li, and Zhou investigated the question of when a group ring is morphic. In [1, Section 2], Chen, Li, and Zhou prove some general theorems about morphicity of group rings. For instance, if RG is left morphic then R is left morphic and G is locally finite [1, Theorem 2.1]; on the other hand, if RH is left morphic for each finite subgroup H of G, then RG is left morphic [1, Theorem 2.4]. After these general theorems, they classify when RG is left morphic in a few special cases: specifically, when R is either semisimple or $\mathbb{Z}_n$ for some n, and G is a finite abelian group. In addition, they complete the case when $G = D_n$ is a dihedral group and $R = \mathbb{Z}_{p^r}$ where p is prime and $r \geqslant 1$. The general problem of determination of when RG is morphic is left open, even in the case when R is a left and right artinian left and right morphic ring, and G is a finite group.

Nicholson and Sánchez Campos also investigated the interplay between (left, right, or left and right) morphic rings and (left, right, or left and right) principal ideal rings in [12]; following [12], a ring is said to be left (right) P-morphic if it is left (right) morphic and is a left (right) principal ideal ring. In particular, [12, Corollary 16] contains a structure theorem describing rings which are left artinian and left and right morphic, and it is shown that this class agrees with the class of rings which are left and right P-morphic. It seems to have been overlooked, however, that the structure theorem of [12, Corollary 16] (adding to the earlier [13, Theorem 35]) contains a condition equivalent to the classical structure theorem describing the artinian principal ideal rings found, for instance, in [5, Section 15] (stated with more modern terminology, for instance, in [3, Corollary 2.2]). Namely, a ring is an artinian principal ideal ring if and only if it is a finite direct product of matrix rings over local artinian principal ideal rings. In [12, Corollary 16], it is shown that the left artinian left and right morphic rings are precisely the finite direct products of matrix rings over left and right "special" rings (in the terminology of [13]). A left special ring is a local ring R for which the Jacobson radical of R is a left principal ideal, generated by a nilpotent element. It is easy to see that a ring is left special if and only if it is a local left artinian left principal ideal ring (using conditions (2) and (3) of [13, Theorem 9] and the fact that a left artinian ring has a nilpotent Jacobson radical), and hence a left and right special ring is precisely a local artinian principal ideal ring.

In view of this, there is another even more surprising equivalent condition that can be added to [12, Corollary 16]. Namely, the class described there is, in fact, the artinian principal ideal rings, allowing us to add the first condition below.

Theorem 2. *(Cf. [12, Corollary 16].) For any ring R, the following are equivalent*:

(1) *R is an artinian principal ideal ring.*
(2) *R is left and right P-morphic.*
(3) *R is left artinian and left and right morphic.*
(4) *R is semiprimary and left and right morphic.*
(5) *R is left perfect and left and right morphic.*
(6) *R is a semiperfect, left and right morphic ring in which J is nil and $S_r \subseteq^{ess} R_R$.*
(7) *R is a semiperfect, left and right morphic ring with ACC on principal left ideals in which $S_r \subseteq^{ess} R_R$.*

(8) *R is a finite direct product of matrix rings over local artinian principal ideal rings* (*i.e. left and right special rings*).

Consequently, whenever R is a left and right artinian principal ideal ring and G is a finite group, the group ring RG is morphic if and only if RG is a principal ideal ring. Thus, there is overlap in the study of morphic groups rings found in [1] with the earlier study of principal ideal group rings found in [4] and [14, Section 4]. In particular, one of the special cases handled in [1], when R is semisimple, and G is a finite abelian group, is essentially already contained in the results of [4] and [14, Section 4] (removing the hypothesis that G is a finite abelian group). As we mentioned above, Passman, Sehgal and Fisher only deal with the classical case with coefficients in a field, but their proofs essentially work in the case of a division ring, and easily imply a classification in the case of semisimple coefficient rings (see Theorem 20, below).

In this article, our main result is the characterization of when RG is a principal ideal ring, in the case when R is artinian and G is arbitrary, answering Question 1 in this case. Since the class of principal ideal rings is closed under quotients, it is clear that we are actually restricting to the class of artinian principal ideal rings. To obtain the characterization, we will first answer Question 1 in the case when R is a local artinian principal ideal ring and G is an arbitrary group. This extends [14, Theorem 4.1]; simultaneously it includes as special cases observations made in [1] on morphicity when the coefficient ring is $\mathbb{Z}_{p^n}$ (removing any hypothesis on the group G). We will then use the local case to answer Question 1 in the case when R is an artinian principal ideal ring and G is an arbitrary group. In particular, when R satisfies the hypotheses of Theorem 2 (i.e. is left artinian and left and right morphic), and G is finite, we completely characterize when RG is morphic. In particular, our results contain each of the special cases dealt with in [1, Sections 3, 4], and answer many of the question contained therein.

We will, of course, rely heavily on [14, Theorem 4.1] (for division rings), and this result will give us an extremely good start on our way. We will freely use the fact that all instances of "field" in [14, Section 4] can be replaced by "division ring" (we detail this in Appendix A, below). Also, Lemma 5, below, is motivated by [1, Theorem 2.8]; aside from this motivation, we will not rely upon any of the results found in [1].

Our ring-theoretic terminology will generally follow [7]. In particular, for a ring R, we denote by $J = J(R)$ the Jacobson radical of R, and for a group ring RG, we denote by ϵ the augmentation map $\epsilon : RG \to R$, whose kernel is the augmentation ideal $\Delta(RG)$. For an element x in a ring R, we denote by $\mathrm{ann}_\ell^R(x)$ and $\mathrm{ann}_r^R(x)$ the left and right annihilators of x in R, respectively. When the ring is clear from the context, we shall omit the superscript R. Also, a local ring is a (not necessarily noetherian) ring with a unique maximal left (equiv. right) ideal, which agrees with its Jacobson radical. We shall also need some group-theoretic terminology. Specifically, if $\mathcal{A}$ and $\mathcal{B}$ are two classes of groups, we say that a group G is $\mathcal{A}$-by-$\mathcal{B}$ if there exists $N \lhd G$ such that $N \in \mathcal{A}$ and $G/N \in \mathcal{B}$. Recall also that for finite groups G, we say that G is a p-group if $|G|$ is a power of p, and we say that G is a p'-group if $|G|$ is relatively prime to p. We will also allow ourselves the natural generalizations of this when π is a finite set of primes. In particular, if $\pi = \emptyset$, a finite π'-group is synonymous with a finite group, and the only finite π-group is the trivial group. We shall also freely use the fact that if R is a local artinian principal ideal ring, then $J(R) = cR = Rc$ for any $c \in J \setminus J^2$ (e.g. [13, Corollary 10] or [5, Theorem 38]).

2. The local case

As we mentioned in the introduction, Question 1 has a complete characterization in the case when R is a division ring. Completing work of Fisher and Sehgal [4] for nilpotent groups, in [14, Section 4], Passman showed that

Theorem 3. *(See [14, Theorem 4.1].) Let KG be the group ring of G over the division ring K. Then, the following are equivalent.*

(1) *KG is a right principal ideal ring.*
(2) *KG is right Noetherian and the augmentation ideal $\Delta(KG)$ is principal as a right ideal.*
(3)
- $\operatorname{char} K = 0$: *$G$ is finite or finite-by-infinite cyclic.*
- $\operatorname{char} K = p > 0$: *$G$ is finite p'-by-cyclic p, or finite p'-by-infinite cyclic.*

As we mentioned above, the statement found in [14, Theorem 4.1] requires that K is a field, but with only minor changes (detailed in Appendix A, below), its proof is valid when K is a division ring as well. For simplicity, we will refer to [14, Theorem 4.1], even in the case of division rings, as Passman's theorem.

Before stating our extension of Passman's theorem to local artinian principal ideal rings, we shall need a definition, which will require a bit of preliminary set up. This work, and some of the work done when we discuss associated graded rings (in the beginning of Section 3) is similar to that found in [6, Chapter 2, Section 6], specifically, [6, Chapter 2, Lemma 6.2]. We are working in a more restricted case when compared with that studied in [6], and a more elementary exposition is therefore possible. In the interests of keeping the exposition elementary and relatively self-contained, we will deal explicitly with our special case, instead of extracting it from the results from [6].

Suppose that R is a local artinian principal ideal ring with $J^2 = 0$, and for which $J \neq 0$. In this case, the only nontrivial ideal of R is $J = cR = Rc$ (see, for instance, [13, Theorem 9, ff.]). We associate to a ring with these properties a ring automorphism φ of R/J as follows. Note that, for each $r \in R$, $cr = sc$ for some $s \in R$. Since $\operatorname{ann}_\ell(c) = J$ (see [13, Theorem 9]), it is clear that s is determined uniquely modulo J, and so it determines a unique element of R/J. Observe that $c1 = 1c$, and if $cr = sc$ and $cr' = s'c$, then $c(r + r') = (s + s')c$ and $c(rr') = scr' = (ss')c$. Thus, we have a well-defined ring homomorphism $\varphi : R \to R/J$ defined by setting $\varphi(r) = s + J$, such that $cr = sc$. Furthermore, σ is surjective, since if $s \in R$, $sc \in Rc = cR$, so $sc = cr$ for some $r \in R$, and hence $\varphi(r) = s$. Observe that $cr = 0 = 0c$ if and only if $r \in \operatorname{ann}_r(c) = J$, so $\ker(\varphi) = J$. We conclude that σ induces a ring automorphism, which we will refer to as σ, of R/J. Note that if $J = Rc' = c'R$, then $c' = uc$ for some $u \in U(R)$. Then, $c'r = ucr = usc = usu^{-1}uc = usu^{-1}c'$. Thus, the map σ is only determined up to conjugation by a nonzero element of R/J. For future reference, let us observe that the skew polynomial rings $(R/J)[t;\sigma]$ and $(R/J)[x;\rho_u \circ \sigma]$, where ρ_u denotes conjugation by u, are isomorphic, by sending t to ux. For our uses later, this conjugation of σ will not be relevant, since we will primarily be dealing with properties of the associated skew polynomial rings which are preserved by isomorphism, so we will, in general, refer imprecisely to a single map σ.

Given a local artinian principal ideal ring R, we associate to R the (conjugacy class of) $\sigma \in \operatorname{Aut}(R/J)$ corresponding to the above construction for the quotient ring R/J^2. Now, if G is a finite group G with $|G| \cdot 1 \in U(R/J)$, by Maschke's theorem, $(R/J)G$ is semisimple. Let $\{e_1, \dots, e_n\}$ be the set of centrally primitive idempotents of $(R/J)G$. The automorphism σ extends to an automorphism of $(R/J)G$, acting on G trivially, and must permute the set $\{e_1, \ldots, e_n\}$ of centrally primitive idempotents. Note that this is reminiscent of Lemmas 5 and 6 of [4].

We shall say that a finite group G with $|G| \cdot 1 \in U(R/J)$ is R-admissible if σ induces the identity permutation on the set of centrally primitive idempotents of $(R/J)G$. Note that this does not depend on the choice of σ, since conjugation by a nonzero element of R/J certainly must fix any central element of $(R/J)G$. The condition that G is R-admissible is equivalent to saying that if f_i is any lift of e_i to $(R/J^2)G$, then $f_i c = c f_i$. It is easy to see from this that R-admissibility is equivalent to requiring that the centrally primitive idempotents of $(R/J)G$ lift to centrally primitive idempotents of $(R/J^2)G$. In particular, R-admissibility is actually the statement that the block decomposition of the artinian ring $(R/J)G$ lifts to a block decomposition of the artinian ring RG (see [7, Section 22]).

In treating the group ring case specifically, we prefer to view R-admissibility as a property of the automorphism σ. For instance, in the case when $k = R/J$ is an algebraically closed field, this is equivalent to the condition that σ fixes $\chi(g)$ for each irreducible k-character χ of G, and each $g \in G$. This is certainly ensured if σ fixes all $|G|$th roots of unity. Generalizing our results to other classes of rings may well be possible, however, using the lifting of block decompositions (or centrally primitive idempotents) as one's starting point. Our main theorem for local artinian principal ideal rings is the following, which extends Passman's theorem. We do not know, however, if there is a valid analogue of condition (b) of Passman's theorem.

Theorem 4. *Suppose R is a local artinian principal ideal ring and G is a group. Then, the following are equivalent*:

(1) *RG is a principal ideal ring*
(2) • $\mathrm{char}(R/J) = 0$: *G is a finite or finite-by-infinite cyclic. If R is not a division ring, then G is an R-admissible finite group.*
  • $\mathrm{char}(R/J) = p > 0$: *$G$ is finite p'-by-cyclic p, or a finite p'-by-infinite cyclic. If R is not a division ring, then G is a finite R-admissible p'-group.*

Much of the forward implication follows immediately from Theorem 3, since if RG is a principal ideal ring, then $(R/J)G$ is a principal ideal ring as well, and R/J is a division ring, so we may apply Theorem 3 to obtain information about the group G. This gets us off to a very good start, however, there is still much to be done. In the forward implication, it remains only to show that, if R is not a division ring, then G is finite with $|G| \cdot 1 \in U(R)$, and that G is R-admissible. Our next lemma will complete everything in the forward implication except for the R-admissibility. For the reverse implication, the entire case when R is not a division ring remains. We shall break the proof of Theorem 4 into a few steps, over the course of the next few sections.

Our first step is to prove two lemmas. The first, in characteristic p, is motivated by [1, Theorem 2.8]. The argument in [1, Theorem 2.8] is specific to $\mathbb{Z}_{p^r}$ (possibly able to be extended to local artinian principal ideal rings for which J has a central generator). Our argument is completely different, obtaining a slightly weaker conclusion than [1, Theorem 2.8], but for general local artinian principal ideal rings. When restricting to groups which admit surjections onto nontrivial p-groups whenever p divides $|G|$ (e.g. finite nilpotent groups), the results contained herein yield the same type of conclusion found in [1, Theorem 2.8].

Lemma 5. *Suppose R is a local artinian principal ideal ring whose residue division ring, R/J, has characteristic p, and suppose G is a finite p-group. If RG is a principal ideal ring, then R is a division ring.*

Proof. Suppose that $\operatorname{char}(R/J) = p > 0$, G is a finite p-group, and that RG is a principal ideal ring. By [11], RG is an (artinian) local ring, which, by assumption is a principal ideal ring. It is easy to see that $\epsilon^{-1}(J(R))$ is the maximal left ideal of RG, since the left ideals of RG form a chain (see [13, Theorem 9]) and the only left ideals of R are powers of J, so $I = \epsilon^{-1}(J)$ is a maximal left ideal of RG, and hence the unique maximal left ideal because RG is local. Also, it is apparent that $I^2 = \epsilon^{-1}(J^2)$. In particular, if R is not a division ring, then $J \setminus J^2$ is nonempty, and if $c \in J \setminus J^2$, the element $c \cdot 1$ is clearly an element of $I \setminus I^2$, hence it generates I as a right ideal (e.g. [5, Theorem 38] or [13, Claim 1, p. 395]). But, since G is nontrivial, we may find $1 \neq g \in G$, and the element $1 - g$ is an element of I. But $I = c(RG)$, so there must exist $x \in RG$ such that $(c \cdot 1)x = (1 - g)$. Comparing constant coefficients (noting that $c = c \cdot 1$ is a scalar), we find that c is right invertible, which is clearly impossible, since $c \in J(R)$. We conclude that $J(R) = J(R)^2$ so $J(R) = 0$ (since R is artinian), and hence R is a division ring. □

Our next lemma is in the same vein, for the group $\mathbb{Z}$, and applies to all characteristics. The previous lemma was stated only in the local case, since this is the only case we shall use, and since it is simpler to state due to the restriction on the characteristic. The following lemma will be just as easy to state without the condition that R is local. The crux of the proof, however, is the local case, as in the last lemma.

Lemma 6. *Let R be an artinian principal ideal ring. If the ring $R\mathbb{Z}$ (which is isomorphic to the Laurent polynomial ring $R\langle x\rangle$) is a principal ideal ring, then R is semisimple.*

Proof. Using the structure theorem for artinian principal ideal rings, we write $R \cong \prod_{i=1}^{n} \mathbb{M}_{k_i}(S_i)$, where S_i is a local artinian principal ideal ring, and $k_i > 1$. If R is not semisimple, there is some S_i for which S_i is not a division ring. Since R is a principal ideal ring, so is its quotient $\mathbb{M}_{k_i}(S_i/J(S_i)^2)$. Thus, it suffices to consider the case when $R = \mathbb{M}_n(S)$, where $n \geqslant 1$ and S is a local artinian principal ideal ring for which $J(S)^2 = 0$.

Let $T = R\langle x\rangle$, $J(R) = Rc = cR$ and let $K = J(R)\langle x\rangle = cR\langle x\rangle = R\langle x\rangle c$, which is an ideal of T. We shall write $\overline{T} = S/K \cong (R/J(R))\langle x\rangle$ and for $t \in T$, we will denote by $\bar{t}$ the image of t in $\overline{T}$. Since R is artinian, $K \subseteq J(T)$ by [2, Proposition 9]. Consider the right ideal $I = (1 + x)T + cT$. We will show that I is not principal. Note first that $\overline{I} = (1 + x)\overline{T}$, which is a proper right ideal of $\overline{T}$. We conclude that $I < T$. Suppose that $I = fT$. We have $f = \sum_{i\in\mathbb{Z}} a_i x^i$. Separate those coefficients which are in $J(R)$ from those which are not, and write $f = f_0 + f_1$, where $f_1 \in J(R)\langle x\rangle$, and each coefficient of f_0 is in $R \setminus J$. Observe that $\overline{f} = \overline{f_0}$.

Since $fT = I$, we have $fg = 1 + x$ for some $g \in T$. Note that $\overline{f_0}\,\overline{g_0} = 1 + x$, and that $1 + x$ is not a zero divisor in $\overline{T}$, so $\operatorname{ann}_\ell(\overline{f_0}) = 0$. Also, $\overline{T} \cong (R/J(R))\mathbb{Z} \cong \mathbb{M}_n(S/J(S))\mathbb{Z}$ is a left and right Noetherian ring, and is a prime ring by [7, Theorem 10.20] and [7, Connell's Theorem, p. 161]. We conclude that $\overline{T}$ is left and right nonsingular (e.g. [8, Corollary 7.19]) and hence we conclude that $\operatorname{ann}_r(\overline{f_0}) = 0$, by [15, Lemma 10.4.9].

On the other hand, since $c \in I$, there must exist $h \in T$, written as $h = h_0 + h_1$, such that $fh = c$. Then, $c = fh = f_0h_0 + f_1h_0 + f_0h_1$, since $J(R)^2 = 0$. Reducing modulo K, we see that $0 = \bar{c} = \overline{f_0}\,\overline{h_0}$. Since $\operatorname{ann}_r(f_0) = 0$, we conclude that $\overline{h_0} = 0$. By our choice of h_0 and h_1, we see that $h_0 = 0$. Now, $K = R\langle x\rangle c$, so we write $h_1 = g_1 c$ for some $g_1 \in R\langle x\rangle$. Thus, we have $c = f_0 g_1 c$, so $(1 - f_0 g_1)c = 0$. Using [13, Theorem 9], we see that every coefficient of $1 - f_0 g_1$ is in $J(R)$ (looking first at coefficients with respect to x, and then at the entries of the matrices). We conclude that $f_0 g_1 \in 1 + J(R)\langle x\rangle$. In particular, since $J(R)\langle x\rangle \subseteq J(T)$, we conclude that $f_0 g_1$ is a unit. We conclude that f_0 is right invertible. It is easy to see that $T = R\mathbb{Z}$ is noetherian, hence Dedekind-finite, so we conclude that f_0 is a unit. Therefore, $I = f f_0^{-1} T$, where $f f_0^{-1} = 1 + f_1 f_0^{-1}$. Since each coefficient of $f_1 f_0^{-1}$ is in $J(R)$, and $J(R)^2 = 0$, we see that $f_1 f_0^{-1}$ is nilpotent. We conclude that $f f_0^{-1} = 1 + f_1 f_0^{-1}$ is unipotent, hence a unit. We conclude that $I = T$, a contradiction. □

The next step is to reduce to the case when $J(R)^2 = 0$. This is motivated by [7, Chapter 22]; the reduction to the case of square zero radical is a standard technique (see [7, p. 332]) in studying artinian rings.

Lemma 7. *Suppose R is a local artinian ring. Then, R is a principal ideal ring if and only if R/J^2 is a principal ideal ring.*

Proof. We need only prove the reverse implication, since the class of principal ideal rings is closed under homomorphic images. Assume R/J^2 is a principal ideal ring. Thus, $J/J^2 = c(R/J^2) = (R/J^2)c$ for some $c \in R/J^2$. Lift c to an element of $J \setminus J^2$ in R. Thus, $J^2 + cR = J$ and $J^2 + Rc = J$. Since R is artinian, J is nilpotent. Thus, by [7, Theorem 23.16], we conclude that $Rc = J$ and $cR = J$. By [13, Theorem 9], R is a principal ideal ring. □

Using the structure theorem for artinian principal ideal rings, we can easily remove the assumption that R is local from Lemma 7.

Corollary 8. *Suppose R is an artinian ring. Then, R is a principal ideal ring if and only if R/J^2 is a principal ideal ring.*

Proof. Only the reverse implication needs proof. Assume R/J^2 is a principal ideal ring, so that $R/J^2 = \prod_{i=1}^{n} \mathbb{M}_{k_i}(S_i)$. By [7, Theorem 22.9], we may lift the centrally primitive idempotents $\{e_1, \ldots, e_n\}$ corresponding to the previous product to a full set $\{f_1, \ldots, f_n\}$ of centrally primitive idempotents of R. Note that $f_i R f_i / \mathrm{rad}(f_i R f_i) \cong e_i R e_i / \mathrm{rad}(e_i R e_i) \cong \mathbb{M}_{k_i}(S_i/\mathrm{rad}(S_i))$. Since $f_i R f_i$ is artinian, we conclude by [7, Theorem 23.10] that $f_i R f_i \cong \mathbb{M}_{l_i}(K_i)$ for some local ring K_i. But then, $\mathbb{M}_{l_i}(K_i/J(K_i)^2) \cong \mathbb{M}_{k_i}(S_i)$. The uniqueness asserted in [7, Theorem 23.10] implies that $l_i = k_i$ and $K_i/J(K_i)^2 \cong S_i$. Thus, K_i is a local artinian ring for which $K_i/J(K_i)^2$ is a principal ideal ring. By Lemma 7, we conclude that K_i is a principal ideal ring. We conclude that $R \cong \prod_{i=1}^{n} \mathbb{M}_{k_i}(K_i)$ is a principal ideal ring. □

At this point, we obtain our desired reduction.

Corollary 9. *If R is a local artinian principal ideal ring and G is a finite group with $|G| \cdot 1 \in U(R)$, then RG is a principal ideal ring if and only if $(R/J^2)G$ is a principal ideal ring.*

Proof. By [2, Proposition 9], $J(R)G \subseteq J(RG)$ in case R is artinian or G is locally finite (both of those conditions are true in our situation). In this case, however, we obtain equality. To see this, note that

$$\frac{RG}{J(R)G} \cong \left(\frac{R}{J(R)}\right)G,$$

which is semisimple, since $R/J(R)$ is semisimple (J-semisimple and artinian) and G is a finite group with $|G| \cdot 1 \in U(R/J(R))$. We conclude from [7, Ex. 4.11], that $J(RG) \subseteq J(R)G$, and hence $J(RG) = J(R)G$.

Now, only the reverse implication needs proof, as usual. We have

$$\left(R/J^2\right)G \cong RG/\left(J(R)^2G\right) \cong RG/\left(J(RG)^2\right).$$

Since RG is artinian, the result now follows from Corollary 8. □

3. Associated graded rings and the case $J^2 = 0$

In this section, we will handle completely the case when R is a local artinian principal ideal ring with $J(R)^2 = 0$, and G is a finite group with $|G| \cdot 1 \in U(R)$, and we shall use this to complete the proof of Theorem 4. We will first look at the simpler case when R is an associated graded ring with respect to its Jacobson radical. As we shall see the prototype for this type of ring is a skew polynomial ring of the form $D[t;\varphi]/(t^2)$, where D is a division ring and φ is a ring automorphism of D (cf. [6, Chapter 2, Section 6]).

Lemma 10. *Suppose D is a division ring, and φ is a ring automorphism of D. Then, $D[t;\varphi]$ is a principal ideal ring.*

Proof. Apply [10, Theorem 1.2.9], noting that since φ is an automorphism, $D[t;\varphi]$ can be viewed as both a right and left skew polynomial ring. □

Note that $D[t;\varphi]/(t^2)$ is a local artinian principal ideal ring with radical (t). Now, suppose instead that we start with any local artinian principal ideal ring R for which $J(R)^2 = 0$. We may form the associated graded ring of R with respect to the ideal $J(R)$, which in this case is $(R/J(R)) \oplus J(R)$, since $J(R)^2 = 0$. Fix a $c \in R$ such that $J(R) = cR = Rc$, and an associated ring automorphism $\sigma : R/J \to R/J$ (as in Section 2). From the definition of σ, it is easy to see that $\mathrm{gr}_J R \cong (R/J)[t;\sigma]/(t^2)$, with t corresponding to c (recall that the choice of σ does not affect the isomorphism type of $(R/J)[t;\sigma]$). Thus, as we eluded to earlier, rings of the form $R = D[t;\varphi]/(t^2)$ are the general form of local artinian principal ideal rings with $J(R)^2 = 0$ for which $\mathrm{gr}_J R \cong R$. Note that the study of group rings over such rings are much easier to study than general local artinian principal ideal rings with $J^2 = 0$, since $(D[t;\varphi]/(t^2))G \cong DG[t;\varphi]/(t^2)$, where φ is the automorphism of DG obtained by extending the automorphism φ linearly, acting trivially on G. The group ring DG is semisimple, but, the difficulty lies in the fact that φ need not respect the blocks (centrally primitive idempotents) of DG. It is clear that the blocks are preserved precisely when G is $D[t;\varphi]$-admissible, and we shall see that this is precisely the case when $DG[t;\varphi]$ is a principal ideal ring.

Therefore, we shall now work to characterize when RG is a principal ideal ring, in the case that $R = \mathrm{gr}_J R$ is a local artinian principal ideal ring with $J(R)^2 = 0$. First, we shall need to describe the automorphisms of $\mathbb{M}_n(D)$, where D is a division ring, so that we may characterize the skew polynomial rings with coefficients in the simple artinian ring $\mathbb{M}_n(D)$.

Lemma 11. *Let D be a division ring, and $n > 0$. Let φ be any ring automorphism of $\mathbb{M}_n(D)$. Then, $\mathbb{M}_n(D)[t;\varphi] \cong \mathbb{M}_n(D[x;\sigma])$ for some ring automorphism σ of D; moreover, $\mathbb{M}_n(D)[t;\varphi]/(t^k) \cong \mathbb{M}_n(D[x;\sigma]/(x^k))$ for each $k \geqslant 1$.*

Proof. Our first goal is to describe the automorphisms of $\mathbb{M}_n(D)$. Let $\{e_{ij}\}$ denote the usual matrix units of $\mathbb{M}_n(D)$, and set $f_{ii} = \varphi(e_{ii})$. Since $\{e_{11}, \dots, e_{nn}\}$ is a collection of orthogonal local idempotents which sum to 1, it is easy to see that the same is true for $\{f_{11}, \dots, f_{nn}\}$. By [7, Exercise 21.17], there is a unit $u \in \mathbb{M}_n(D)$ and a permutation $\pi \in S_n$ such that $f_{i,i} = u^{-1}e_{\pi(i),\pi(i)}u$. There is a unit $v \in \mathbb{M}_n(D)$ (a permutation matrix) such that $v^{-1}e_{ii}v = e_{\pi(i),\pi(i)}$. Then, $f_{ii} = (vu)^{-1}e_{ii}(vu)$.

Now, let us consider the automorphism $\psi = \rho_{(vu)^{-1}} \circ \varphi$, where, throughout this proof, ρ denotes conjugation. By our choice of u and v, $\psi(e_i) = e_i$. Consider $f_{ij} = \psi(e_{ij})$. Note that $e_r f_{ij} e_s = \psi(e_r e_{ij} e_s)$. If $i \neq r$ and $j \neq s$, then $e_r f_{ij} e_s = 0$. Thus, if we express $f_{ij} = \sum_{ij} a_{ij}e_{ij}$, for $a_{ij} \in D$, we see that $f_{ij} = a_{ij}e_{ij}$. Since $\psi(e_i) = e_i$, we see that $a_{ii} = 1$. Note that $a_{ij}a_{jk} = a_{ik}$ for all i, j, k. In particular, $a_{ij}a_{ji} = 1$ so each a_{ij} is a unit. Consider the diagonal matrix $w = \operatorname{diag}(a_{11}, a_{21}, \dots, a_{n1})$, which is an invertible matrix with inverse $\operatorname{diag}(a_{11}, a_{12}, \dots, a_{1n})$. Consider $\rho_w \circ \psi$. Note that $\rho_w \circ \psi(e_{ij}) = e_{ij}$ for all i, j.

Now, let $d \in D$, and consider $d' = \rho_w \circ \psi(dI_n)$. Note that $e_{ii}d'e_{jj} = (\rho_w \circ \psi)(e_{ii}dI_ne_{jj})$, which is zero if $i \neq j$. Thus, d' is a diagonal matrix. Consider the permutation matrix $x = e_{12} + e_{23} + \dots + e_{n-1,n} + e_{n,1}$, whose inverse is $x^{-1} = e_{21} + e_{32} + \dots + e_{n,n-1} + e_{1n}$. For any diagonal matrix z, conjugation by x applies a cyclic shift on the entries of z. In particular, $x^{-1}(dI)x = dI$, and hence $x^{-1}d'x = d'$. It follows that d' is a diagonal matrix of the form cI_n for some $c \in D$. It is easy to see that $(\rho_w \circ \psi)|_D I_n$ is a ring automorphism of DI_n. We shall refer to this automorphism of $D \cong DI_n$ as σ, and we shall also use σ to denote the automorphism of $\mathbb{M}_n(D)$ obtained by applying σ componentwise.

It is easy now to see that $\sigma^{-1} \circ \rho_w \circ \psi$ is the identity map on $\mathbb{M}_n(D)$. Indeed, $\sigma^{-1} \circ \rho_w \circ \psi$ fixes each e_{ij} and fixes DI_n elementwise, from which it follows that it fixes $\sum_{ij} a_{ij}e_{ij} = \sum_{ij}(a_{ij}I_n)e_{ij}$. We conclude that $\varphi = \rho_{uvw^{-1}} \circ \sigma$.

Let $z = (uvw^{-1})^{-1}$. Now, consider the map $g : \mathbb{M}_n(D)[t;\varphi] \to \mathbb{M}_n(D[x;\sigma])$ defined by embedding $\mathbb{M}_n(D)$ in $\mathbb{M}_n(D[x;\psi])$ and sending t to zx. Note that $\mathbb{M}_n(D[x;\sigma]) \cong \mathbb{M}_n(D)[x;\sigma]$, where the first σ refers to the ring automorphism of D, and the second refers to the ring automorphism of $\mathbb{M}_n(D)$ it induces componentwise.

Note that $tA = \varphi(A)t$. Note that $g(tA) = zxA$, and $g(\varphi(A)t) = \varphi(A)zx$, but $z^{-1}\varphi(A)z = \sigma(A)$, so $g(\varphi(A)t) = z\sigma(A)x = zxA = g(tA)$. Since $\mathbb{M}_n(D)[t;\varphi] = \mathbb{M}_n(D)[t]/\langle\{tA - \varphi(A)t\colon A \in \mathbb{M}_n(D)\}\rangle$, we conclude that g is a well-defined ring homomorphism.

Suppose that $p(t) = A_0 + A_1t + \dots + A_nt^n$, then

$$\begin{aligned} g\big(p(t)\big) &= A_0 + A_1zx + A_2(zx)^2 + \dots + A_n(zx)^n \\ &= A_0 + A_1zx + A_2z\sigma(z)x^2 + \dots + A_nz\sigma(z)\cdots\sigma^{n-1}(z)x^n. \end{aligned}$$

In particular, we see that g is bijective, since z is a unit (as are its images under powers of σ). We conclude that $\mathbb{M}_n(D)[t;\varphi] \cong \mathbb{M}_n(D[x;\sigma])$. It is also clear that for each $k \geqslant 1$, $g((t^k)) = (x^k)$, from which it follows that $\mathbb{M}_n(D)[t;\varphi]/(t^k) \cong \mathbb{M}_n(D[x;\sigma]/(x^k))$. $\square$

The following proposition is now essentially obvious.

Proposition 12. *Suppose $R = \mathrm{gr}_J R$ is a local artinian principal ideal ring with $J^2 = 0$ and G is a finite group with $|G| \cdot 1 \in U(R)$. If G is R-admissible, then RG is a principal ideal ring.*

Proof. We have $R \cong \overline{R}[t;\sigma]/(t^2)$ as above. By Maschke's theorem with the Artin–Wedderburn theorem, $\overline{R}G \cong \prod_{i=1}^k \mathbb{M}_{n_i}(D_i)$ for division rings D_i and $n_i > 0$. The automorphism σ of $\overline{R}$ extends to an automorphism of $\overline{R}G$, which, by assumption, fixes the centrally primitive idempotents (which correspond to the direct product decomposition above). In particular, σ acts as the direct product of automorphisms σ_i of $\mathbb{M}_{n_i}(D_i)$. Using Lemma 11 to obtain automorphisms ψ_i of D_i, it is straightforward to see that

$$RG \cong \left(\overline{R}[t;\sigma]/\left(t^2\right)\right)G \cong (\overline{R}G)[t,\sigma]/\left(t^2\right) \cong \prod_{i=1}^{k}\left(\mathbb{M}_{n_i}(D_i)[t_i;\sigma_i]/(t_i)^2\right)$$

$$\cong \prod_{i=1}^{k}\left(\mathbb{M}_{n_i}\left(D_i[x_i;\psi_i]/(x_i)^2\right)\right).$$

By Lemma 10 and the structure theorem for artinian principal ideal rings, we conclude that RG is a principal ideal ring. □

The reverse implication is easier, and will not require the assumption that the ring is an associated graded ring. We will therefore, delay this until the complete characterization for the local case (Proposition 14).

At this point, we know that if G is R-admissible, then $\mathrm{gr}_{JG}(RG) = (\mathrm{gr}_J R)G$ is a principal ideal ring. What is not clear is the nature of the relationship between RG and $\mathrm{gr}_{JG}(RG)$. Our goal is a theorem of the type sought in [6, Chapter 2, Section 7]. We seek to conclude that a ring is a principal ideal ring, knowing that its associated graded ring (with respect to its Jacobson radical) is a principal ideal ring (note that $P(R) = J(R)$ when R is artinian). In general, this type of question is difficult, but we have imposed strong chain conditions which help us. Moreover, the main trick that we need is that the R-admissibility of G allows us to lift the centrally primitive idempotents of $(R/J)G$ to centrally primitive idempotents of RG.

One important special case of this type of result (lifting through the associated graded ring) is found in [6, Proposition 7.7]; the following is a special case of that result.

Lemma 13. *Suppose that R is a local artinian ring. Then, the following are equivalent*:

(1) *R is a principal ideal ring,*
(2) *$\mathrm{gr}_J R$ is a principal ideal ring.*

Proof. Any (one-sided) artinian local ring is completely primary, and its prime radical agrees with its Jacobson radical (e.g. [7, Theorem 10.30] and the fact that the Jacobson radical of an artinian ring is nilpotent). The result is then a special case of [6, Proposition 7.7] applied on the left and the right.

The special case when $J^2 = 0$ lends itself to a simpler proof, since $\mathrm{gr}_J R$ takes on a particularly simple form. In particular, the ideal J in R is also an ideal (i.e. $0 \oplus J$) of $\mathrm{gr}_J R$, and it is easy to check that the ideal J is (left, respectively right) principal in R if and only if $0 \oplus J$ is (left, respectively right) principal in S. □

Proposition 14. *If R is a local artinian principal ideal ring with $J^2 = 0$, and G is a finite group with $|G| \cdot 1 \in U(R)$, then RG is a principal ideal ring if and only if G is R-admissible.*

Proof. Let $S = RG$. Since G is finite and $|G| \cdot 1 \in U(R)$, we see that $J(RG) = JG$. First, lift the set $\{e_1, \ldots, e_n\}$ of centrally primitive idempotents of RG/JG to orthogonal primitive idempotents $\{f_1, \ldots, f_n\}$ (for instance, by [7, Corollary 21.32]). Note that $f_1 + \cdots + f_n$ reduces to $e_1 + \cdots + e_n = 1$, modulo $J(RG)$, so $f_1 + \cdots + f_n$ is an idempotent unit, hence equals 1. If $i \neq j$, note that $f_i S f_j \subseteq J(S)$. To see this, note that reducing modulo $J(RG) = JG$, we obtain $e_i((R/J)G)e_j = 0$, since e_i, e_j are orthogonal and central.

For the forward implication, suppose S is an artinian principal ideal ring. By Theorem 2, S is morphic. By [13, Corollary 19], $f_i S f_j = 0$ whenever $i \neq j$. Thus, each f_i is a central idempotent of S, since $1 - f_i = \sum_{j \neq i} f_j$, so $f_i S(1 - f_i) = 0 = (1 - f_i) S f_i$, from which we conclude that f_i is central (by [7, Lemma 21.5]). In particular, $f_i c = c f_i$. It follows that $\sigma(e_i) = \sigma(\overline{f_i}) = \overline{f_i} = e_i$, so G is R-admissible.

For the converse, suppose that G is R-admissible. We claim that each f_i is central in S. First, note that $f_i c = c f_i$ for each i, by hypothesis, since G is R-admissible. Also, note that, even without R-admissibility, for any $r \in R$, $c(f_i r) = c(r f_i)$ and $(f_i r)c = (r f_i)c$, since $f_i r - r f_i$ is in $J = \operatorname{ann}(c)$, since $\overline{f_i} = e_i$ is central in $\overline{S}$. Let $r \in R$, and let $i \neq j$. Note that $f_i r f_j \in f_i S f_j \subseteq J(S)$. We may write $f_i r f_j = ch = h'c$, for some $h, h' \in S$, since $J(S) = J(R)G = c(RG) = (RG)c$. Note that $f_i r f_j = f_i(f_i r f_j) = f_i ch = c f_i h = ch f_i = f_i r f_j f_i = 0$. We conclude that $f_i S f_j = 0$ if $i \neq j$, and, as before, that each f_i is central. We conclude that the f_i are a complete set of centrally primitive idempotents of S. The ring $f_i S f_i$ is an artinian ring which is a simple artinian ring modulo its radical. By [7, Theorem 23.10], $f_i S f_i \cong \mathbb{M}_{k_i}(S_i)$ for some $k_i > 0$ and some local ring S_i. At this point, we know that $RG \cong \prod_{i=1}^n \mathbb{M}_{k_i}(S_i)$. Next, we will show that each S_i is a principal ideal ring.

Note that $(\operatorname{gr}_J R)G \cong \operatorname{gr}_J(RG) \cong \prod_{i=1}^n \mathbb{M}_{k_i}(\operatorname{gr}_J S_i)$. By hypothesis, G is R-admissible, so it is also $\operatorname{gr}_J R$ admissible (the induced automorphism of R/J is the same); by Proposition 12 $(\operatorname{gr}_J R)G$ is an artinian principal ideal ring. Since the class of artinian principal ideal rings is Morita invariant (e.g. [12, Corollary 17]), $\operatorname{gr}_J S_i$ is a local artinian principal ideal ring. By Lemma 13, we have that S_i is a local artinian principal ideal ring. It follows that $RG \cong \prod_{i=1}^n \mathbb{M}_{k_i}(S_i)$ is a principal ideal ring. □

Remark 15. In light of the Artin–Wedderburn theorem and the structure theorem for artinian principal ideal rings, we view Proposition 14 as an analogue of Maschke's theorem (e.g. as stated in [7, Theorem 6.1]).

We can now put all of this together to prove Theorem 4.

Proof of Theorem 4. For the implication (1) ⇒ (2), suppose RG is a principal ideal ring. Then, its quotient $(R/J)G$ is a principal ideal ring, to which we may apply Theorem 3. If $\operatorname{char}(R/J) = 0$, we conclude that G is finite or finite-by-infinite cyclic. In the latter case, RG surjects onto $R\mathbb{Z}$, which must therefore be a principal ideal ring. By Lemma 6, we conclude that G must be finite if R is not a division ring.

If $\mathrm{char}(R/J) = p > 0$, we conclude from Theorem 3 that G is finite p'-by-cyclic p or finite p'-by-infinite cyclic. In either case, if the cyclic group in question is nontrivial, RG surjects onto RH for some nontrivial cyclic p-group H, and RH is a principal ideal ring. By Lemma 5, we conclude that, if R is not a division ring, then the cyclic group must be trivial, and hence G must be a p'-group.

We have shown that if R is not a division ring, then G is a finite group with $|G| \cdot 1 \in U(R)$. Since RG is a principal ideal ring, its homomorphic image $(R/J^2)G$ is a principal ideal ring, and by Proposition 14, G is R-admissible.

For the implication (2) $\Rightarrow$ (1), Theorem 3 handles the case when R is a division ring. In the remaining case, R is not a division ring, G is an R-admissible finite group with $|G| \cdot 1 \in U(R)$. By Proposition 14, $(R/J^2)G$ is a principal ideal ring, and by Corollary 9, we conclude that RG is a principal ideal ring. □

Before moving on to general artinian rings, let us obtain a result which does not directly follow from our results on group rings, but does follow from our arguments. In our study of when a group ring is a principal ideal ring when the coefficient ring is an associated graded ring, we have seen that rings of the form $D[x;\sigma]/(x^2)$ play an important role, and our arguments show the following.

Corollary 16. *Let R be a semisimple ring, and σ an automorphism of R. Then, the following are equivalent*:

(1) *All centrally primitive idempotents of R are fixed by σ.*
(2) *All central idempotents of R are fixed by σ.*
(3) *$R[x;\sigma]/(x^n)$ is morphic for all $n \geqslant 1$.*
(4) *$R[x;\sigma]/(x^2)$ is morphic.*

Proof. Since R is semisimple, by the Artin–Wedderburn theorem there are division rings $D_1, \ldots, D_k$, and integers $n_i > 0$ for $1 \leqslant i \leqslant k$ such that $R \cong \prod_{i=1}^{k} \mathbb{M}_{n_i}(D_i)$. Let $e_1, \ldots, e_k$ be the complete set of centrally primitive idempotents which corresponds to the above direct product decomposition (see [7, Chapter 22]). It is clear that the automorphism σ induces a permutation on the set of centrally primitive idempotents.

The equivalence of (1) and (2) is a consequence of [7, Proposition 22.1], while the implication (3) $\Rightarrow$ (4) is a tautology. The argument for (2) $\Rightarrow$ (3) is essentially contained in the proof of Proposition 12. Since σ fixes the centrally primitive idempotents, it fixes the associated Peirce corner rings, which are the factors $\mathbb{M}_{n_i}(D_i)$; in particular, σ induces a ring automorphism σ_i on $\mathbb{M}_{n_i}(D_i)$, and is the direct product of the automorphisms σ_i. By Lemma 11, there exist automorphisms ψ_i of D_i such that $\mathbb{M}_{n_i}(D_i)[x_i;\sigma_i]/(x_i^k) \cong \mathbb{M}_{n_i}(D_i[t_i;\psi_i]/(t_i^k))$. Combining this,

$$R[x;\sigma]/\left(x^n\right) \cong \prod_{i=1}^{k} \mathbb{M}_{n_i}(D_i)[x_i;\sigma_i]/\left(x_i^n\right) \cong \prod_{i=1}^{k} \mathbb{M}_{n_i}\left(D_i[t_i;\psi_i]/\left(t_i^k\right)\right).$$

As in the proof of Proposition 12, we conclude from Lemma 10 and the structure theorem for artinian principal ideal rings that $R[x;\sigma]/(x^n)$ is an artinian principal ideal ring, which is equivalent to saying that it is an artinian morphic ring.

Finally, we shall show that (4) $\Rightarrow$ (1), by an argument similar to a portion of the proof of Proposition 14. Let $S = R[x;\sigma]/(x^2)$. Let $e_1, \ldots, e_k$ be the (orthogonal) set of centrally primitive idempotents of R. Observe that if $i \neq j$, then $e_i S e_j = e_i R\sigma(e_j)x$. In particular, $e_i S e_j$ is contained in Sx which is a nilpotent ideal of S, so $e_i S e_j \subseteq J(R)$. Since $R[x;\sigma]/(x^2)$ is morphic, we conclude from [13, Corollary 19] that $e_i S e_j = e_j S e_i = 0$. In particular, $e_i\sigma(e_j)R = e_i R\sigma(e_j) = 0$. Since σ induces a permutation on the finite set of centrally primitive idempotents of R, we conclude that $e_i = \sigma(e_i)$, since $e_i\sigma(e_j) = 0$ (in particular, $\sigma(e_j) \neq e_i$) whenever $i \neq j$. □

In [9], Lee and Zhou examine morphicity of rings of the form $R[x;\sigma]/(x^n)$. In particular, in [9, Theorem 2] Lee and Zhou showed that for unit regular rings if σ is an endomorphism and fixes *all* idempotents of R, then $R[x;\sigma]/(x^n)$ is morphic. For the smaller class of semisimple rings, and assuming that σ is an automorphism, Corollary 16 provides the necessary and sufficient condition for $R[x;\sigma]/(x^n)$ to be morphic: σ need only fix all centrally primitive idempotents of R. The coefficient ring in [9, Example 3] is semisimple, and in light of Corollary 16, it does not actually show that the assumption that σ must fix *all* idempotents cannot be removed, it demonstrates the result of Corollary 16, that σ must fix all central idempotents. It is, however, easy to find examples where R is semisimple, σ is an automorphism, and σ does not fix all idempotents, but $R[x;\sigma]/(x^n)$ is morphic. For instance, let D be a division ring, let $R = \mathbb{M}_2(D)$, and let σ be conjugation by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since R has no nontrivial central idempotents, σ fixes them (so $R[x;\sigma]/(x^n)$ is morphic), but, it does not fix all idempotents of R. It remains an open question for unit regular rings, however, to determine the necessary and sufficient conditions on σ which guarantee that $R[x;\sigma]/(x^n)$ is morphic. In particular, it is unknown whether fixing the centrally primitive idempotents ensures morphicity.

4. General artinian principal ideal rings

Given the structure theorem for artinian principal ideal rings, we are now in position to easily study RG when R is an artinian principal ideal ring. We shall first need a few completely elementary group theoretic lemmas.

Lemma 17. *Suppose that G is a finite group, and that $H, K \lhd G$ such that G/H and G/K are cyclic groups with relatively prime order. Then, $G/(H \cap K)$ is a cyclic group of order $|G/H| \cdot |G/K|$.*

Suppose that $k \geqslant 1$ and $H_1, \ldots, H_k$ are normal subgroups of a finite group G, such that G/H_i is cyclic for each i, and that $|G/H_i|$ is relatively prime to $|G/H_j|$ if $i \neq j$. Then, $G/(H_1 \cap \cdots \cap H_k)$ is a cyclic group of order $|G/H_1| \cdot |G/H_2| \cdots |G/H_k|$.

Proof. This follows immediately from the structure theorem for finitely generated abelian groups. □

Corollary 18. *Let π be a nonempty finite set of primes, and let G be a group. Then, G is finite π'-by-cyclic π if and only if G is finite p'-by-cyclic p for each $p \in \pi$.*

Proof. We are given that for each $p \in \pi$, there exists a normal p'-subgroup $H_p \lhd G$ such that $|G/H_p|$ is a cyclic p-group. Applying Lemma 17, we find that the group $G/(\bigcap_{p\in\pi} H_p)$ is cyclic and its order is $\prod_{p\in\pi} |G/H_p|$. In particular, $G/(\bigcap_{p\in\pi} H_p)$ is a cyclic π-group. On the other hand, $\bigcap_{p\in\pi} H_p$ is a subgroup of each H_p, which is a p'-group. We conclude that $\bigcap_{p\in\pi} H_p$ is a finite normal π'-group, and we conclude that G is finite π'-by-cyclic p. □

Similarly, we have the following lemma in the infinite case.

Lemma 19. *Let π be a nonempty finite set of primes, and let G be a group. Then, G is finite π'-by-infinite cyclic if and only if G is finite p'-by-infinite cyclic for each $p \in \pi$.*

Proof. Suppose that G is finite π'-by-infinite cyclic, so there is a finite π'-group $H \lhd G$ such that G/H is infinite cyclic. For each $p \in \pi$, H is a p'-group. We conclude that G is finite p-by-infinite cyclic.

On the other hand, suppose that G is finite p'-by-infinite cyclic for each $p \in \pi$. We claim that G is finite π'-by-infinite cyclic. We induct on the size of π, the result being trivial if $|\pi| = 1$. Thus, suppose that $\pi = \pi_1 \cup \{p\}$, where $|\pi_1| < |\pi|$. By the inductive hypothesis, G is finite π_1'-by-infinite cyclic. Thus, we have a finite normal π_1'-subgroup $H_1 \lhd G$ such that G/H_1 is infinite cyclic. We also have a finite normal p'-subgroup $H \lhd G$ such that G/H is infinite cyclic. Note that the subgroup $H_1 H$ is a finite normal subgroup, and its image in G/H and G/H_1 is thus trivial, since an infinite cyclic group has no nontrivial finite subgroups. We conclude that $H = H_1$. In particular, $H = H_1$ is a normal π-subgroup for which G/H is infinite cyclic. We conclude that G is finite π'-by-infinite cyclic. □

We reformulate Passman's theorem for semisimple rings as follows; note that the statement is somewhat simpler than that of Passman's theorem, not distinguishing the characteristic.

Theorem 20. *Let R be a semisimple ring, and G a group. Let π be the set of primes which are not invertible in R. Then, the following are equivalent*:

(1) *RG is a principal ideal ring.*
(2) *G is finite π'-by-cyclic π, or finite π'-by-infinite cyclic.*

The statement for artinian principal ideal rings is similarly the following.

Theorem 21. *Suppose R is an artinian principal ideal ring, and that G is a group. Write $R \cong \prod_{i=1}^{n} \mathbb{M}_{k_i}(S_i)$ where each S_i is a local artinian principal ideal ring. Let π denote the set of primes which are not invertible in R. Then, the following are equivalent*:

(1) *RG is a principal ideal ring.*
(2) *G is finite π'-by-cyclic π or finite π'-by-infinite cyclic. If R is not semisimple, then G is finite, and for each $i \in \{1, \dots, n\}$ for which S_i is not a division ring, $|G| \cdot 1 \in U(S_i)$ and G is S_i-admissible.*

We shall prove Theorems 20 and 21 together. In case G is finite, these theorems are simply repackagings of their analogues Theorems 3 and 4, using the relevant structure theorems and the fact that the class of *artinian* principal ideal rings is Morita invariant (passing to and from matrix rings). We shall need to do some work, however, even in the semisimple case, to deal with arbitrary (not necessarily finite) groups. The arguments in that case model essentially those found in [14, Section 4], but need to be adapted slightly. In particular, we do not know whether, even for semisimple rings, an analogue of condition (b) of Passman's theorem holds.

Proof of Theorems 20 and 21. The reverse implication is straightforward. Indeed, either S_i is a division ring and G is finite π'-by-cyclic π or finite π'-by-infinite cyclic; or else, $|G|$ is finite with $|G| \cdot 1 \in U(S_i)$, and G is S_i-admissible. By Theorems 4 and 3, S_iG is a principal ideal ring. By [5, Theorem 40], $\mathbb{M}_{k_i}(S_iG)$ is a principal ideal ring, and by [5, Lemma on p. 70], $R \cong \prod_{i=1}^{n} \mathbb{M}_{k_i}(S_i)$ is a principal ideal ring.

The forward implication requires a slight amount of work. As we mentioned before starting the proof, in case G is finite, this work evaporates, since the class of artinian principal ideal rings is Morita invariant; in particular, if RG is an artinian principal ideal ring, so is S_iG for each i, from which we can easily complete the argument. For principal ideal rings we do not know, in general, whether the (full) Peirce corner rings of a principal ideal ring need to be principal ideal rings. We shall sidestep this problem, however.

First, let us deal with the semisimple case. We will suppose first that $R \cong \mathbb{M}_n(K)$ is simple artinian, where $n \geqslant 1$ and K is a division ring. Essentially, we will argue as in the proof of the implication (b) $\Rightarrow$ (c) in Passman's theorem (to obtain information about G), however, we will need to adapt those arguments slightly to our situation. We will, however, only prove the implication (a) $\Rightarrow$ (c) in this context, which allows us more flexibility.

Proceeding as in the implication (b) $\Rightarrow$ (c) (though we are doing the analogue of the implication (a) $\Rightarrow$ (c)) of the proof of Passman's theorem, we conclude that RG is noetherian, and hence all subgroups of G are finitely generated, and in particular, we have $\Delta^+(G)$ finite. As in Passman's theorem, setting $\overline{G} = G/\Delta^+(G)$, we see that $K\overline{G}$ is a prime ring (e.g. [7, Connell's Theorem, p. 161]), and hence $RG \cong \mathbb{M}_n(K)\overline{G} \cong \mathbb{M}_n(K\overline{G})$ is a prime ring (by [7, Theorem 10.20]).

At this point, we seek to apply [14, Lemma 4.4], which does not apply directly to our situation. Fortunately, we are only attempting to prove an analogue of the implication (a) $\Rightarrow$ (c) as opposed to the more restrictive implication (b) $\Rightarrow$ (c). We will show that the conclusion of [14, Lemma 4.4] is valid if we assume RG is a principal ideal ring, instead of only assuming that $\Delta(RG)$ is a principal right ideal and RG is noetherian.

Indeed, using the argument found in [14, Lemma 4.4], with R simple artinian,[1] we conclude as before that when $\mathrm{char}(K) = 0$,[2] G/G' is infinite cyclic. Similarly, if $\mathrm{char}(K) = p > 0$ we conclude that $|G/H|$ is infinite, where $H = \bigcap_{n=1}^{\infty} \mathcal{D}_n(RG)$. As before, we conclude that $G_1 = G/\mathcal{D}_n(RG)$ is a finite p-group. Now, $RG_1 \cong \mathbb{M}_n(KG_1)$ is an *artinian* principal ideal ring, and, by Morita invariance, we conclude that KG_1 is a principal ideal ring. Now, we are in position to apply [14, Lemma 4.3] (for division rings, see Appendix A), to conclude that $G/\mathcal{D}_n(RG)$ is cyclic, and hence $G' \subseteq H$, so G/G' is infinite. The rest of the proof of [14, Lemma 4.4] carries through routinely. Indeed, G/G' is an infinite finitely generated abelian group, so there is a normal subgroup W of G for which G/W is infinite cyclic. If we set $B = \Delta(RW)RG$, then B is a prime ideal of RG since $RG/B \cong R(G/W) \cong \mathbb{M}_n(K(G/W))$ is a prime ring by [7, Connell's Theorem, p. 161] and [7, Theorem 10.20]. Applying [14, Lemma 4.2(ii)], we conclude that $B = 0$, and hence $W = 1$.

[1] In Appendix A, below, we observe that the basic properties of the dimension subgroups needed apply in this situation, since $\mathbb{Q}$ or $\mathbb{Z}_p$ embeds in R.

[2] Note that in this case $\mathbb{Q}$ embeds in R, so we obtain the usual basic properties of the dimension subgroups; see Appendix A, below.

Returning to the main proof, we therefore conclude that $\overline{G}$ is infinite cyclic or else $\overline{G} = 1$. Thus, G is finite or finite-by-infinite cyclic, and if $\mathrm{char}(K) = 0$, we are done. Now, suppose $\mathrm{char}(K) = p > 0$. If G is finite, then, since $RG \cong \mathbb{M}_n(KG)$ is an artinian principal ideal ring, we conclude that KG is a principal ideal ring, and hence we conclude from Theorem 3 that G is finite p'-by-cyclic p. Finally, we argue as in the last paragraph of the implication (b) ⇒ (c) of the proof of Passman's theorem, and, instead of applying [14, Lemma 4.3] to $R\tilde{G}$, we note first that $R\tilde{G} \cong \mathbb{M}_n(K\tilde{G})$ is an artinian principal ideal ring, so $K\tilde{G}$ is a principal ideal ring, to which we may apply [14, Lemma 4.3], and we conclude as in the original proof, that either G is finite p'-by-cyclic p, or else G is finite p'-by-infinite cyclic.

Putting all of the information we have together, suppose now that R is semisimple, so that $R \cong \prod_{i=1}^{n} \mathbb{M}_{k_i}(K_i)$, and that G is a group, for which RG is a principal ideal ring. Looking at quotients, we find that $\mathbb{M}_{k_i}(K_i G)$ is a principal ideal ring. Let π be the set of primes which are not invertible in R; equivalently, $p \in \pi$ if and only if $p = \mathrm{char}(K_i) > 0$ for some i. First suppose that G is finite. If π is empty, then considering any i, we find that G is finite; equivalently, G is finite π'-by-cyclic π. If π is nonempty, then, we find that G is finite p'-by-cyclic p for each $p \in \pi$ (considering any i for which K_i has characteristic p). By Lemma 18, we conclude that G is finite π'-by-cyclic π. We conclude in each case that if G is finite, then G is finite π'-by-cyclic π.

Next, suppose that G is infinite. If $\pi = \emptyset$, then each K_i has characteristic 0; we conclude that G is finite-by-infinite cyclic (we conclude this for each i); equivalently, G is finite π'-by-infinite cyclic. If π is nonempty, then, we conclude for each $p \in \pi$ that G is finite p'-by-infinite cyclic (considering any i for which K_i has characteristic p). By Lemma 19, we conclude that G is finite π'-by-infinite cyclic. We therefore conclude that, in any event, G is finite π'-by-infinite cyclic, and the forward implication has been proved when R is semisimple.

With the semisimple case completed, we will now tackle the general case. Suppose that RG is a principal ideal ring, where R is an artinian principal ideal ring. Thus, $(R/J)G$ is a principal ideal ring, but R/J is semisimple (since it is J-semisimple and artinian) and applying the semisimple case, we find that G is finite π'-by-cyclic π or finite π'-by-infinite cyclic. In the latter case, $R\mathbb{Z}$ is a quotient of RG, so $R\mathbb{Z}$ is a principal ideal ring. By Lemma 6, we conclude that R is semisimple. In particular, if any S_i is not a division ring, then G is finite π'-by-cyclic π. Thus, suppose that S_i is not a division ring. Clearly, since RG is a principal ideal ring, its quotient $\mathbb{M}_{n_i}(S_i G)$ is a principal ideal ring as well. Since G is finite π'-by-cyclic π, G is finite, so $\mathbb{M}_{n_i}(S_i G)$ is an artinian principal ideal ring, so $S_i G$ is a principal ideal ring. By Theorem 4, we conclude that $|G| \cdot 1 \in U(S_i)$ and G is S_i-admissible, which completes the forward implication.

For the implication (1) ⇒ (2), Theorem 4 implies that G is a finite p'-by-cyclic p group for each $p \in \pi$; if S_i is not a division ring, then $|G| \cdot 1 \in U(R)$ and G is S_i-admissible. By Corollary 18, G is a finite π'-by-cyclic π group, and the proof is complete. □

Using Theorem 2 we can now obtain a characterization of when RG is morphic (which is merely a restatement of Theorem 21), in the case when G is finite and R is artinian.

Corollary 22. *Suppose R is an artinian principal ideal ring, and that G is a finite group. Write $R \cong \prod_{i=1}^{n} \mathbb{M}_{k_i}(S_i)$ where each S_i is a local artinian principal ideal ring. Let π denote the set of primes which are not invertible in R. Then, the following are equivalent*:

(1) *RG is a principal ideal ring.*
(2) *RG is a morphic ring.*

(3) *G is finite π'-by-cyclic π. If R is not semisimple, then for each $i \in \{1, \ldots, n\}$ for which S_i is not a division ring, $|G| \cdot 1 \in U(S_i)$ and G is S_i-admissible.*

Remark 23. [1, Theorem 3.7] characterizes, for R semisimple and G a finite abelian group, when RG is (strongly) left morphic, which, by Theorem 2 is equivalent to requiring that RG is a principal ideal ring (strongly morphic and morphic are equivalent for artinian rings). The second condition found there, that for each $p \in \pi$, each Sylow p-subgroup of G is cyclic, is equivalent to condition (3) of Corollary 22 above in the case when G is a finite nilpotent group.

Also, Corollary 22, in the case when R is a commutative artinian principal ideal ring (for which any group is R-admissible), reduces to the statement that RG is a principal ideal ring if and only if G is finite π'-by-cyclic π and, for any $p \in \pi$, if $p \in J(S_i)$, then S_i is a division ring. The characterization of when $\mathbb{Z}_n G$ is morphic (i.e. a principal ideal ring) appears in [1, Theorem 3.15], but its equivalence to this condition is somewhat obscured, since the statement and proof find the number theoretic condition that p^2 does not divide n, which happens to be equivalent to the aforementioned ring-theoretic condition for the ring $\mathbb{Z}_n$.

5. Examples

Our work settles a number of questions raised in [1]. In particular, Theorem 21 answers in the affirmative [1, Conjecture 4.14] and [1, Question 4.15]. Next, let us resolve [1, Question 2.6] in the negative. First, we will need the following useful example, due to the author and A. Diesl.

Example 24. Let $R = \mathbb{C}[t; \sigma]/(t^2)$, where σ is complex conjugation, and $G = C_3$. Note that G is not R-admissible, since the central idempotents of $(R/J)G \cong \mathbb{C}C_3$ are $\frac{1+\alpha g+\alpha^2 g^2}{3}$, where $\alpha^3 = 1$, but σ does not fix the cube roots of unity. By Theorem 4, RG is not a principal ideal ring (equivalently, it is not morphic, since RG is artinian).

Now, let us use Example 24 to answer [1, Question 2.6] in the negative. Consider the ring R from Example 24, let $H = C_3$, viewed as a subgroup of $G = S_3$. We have seen that RC_3 is not a principal ideal ring. The ring RS_3 is, however, a principal ideal ring, since S_3 is R-admissible (since each entry of the character table of S_3 is in $\mathbb{Z}$, the coefficients of the centrally primitive idempotents of $\mathbb{C}S_3$ are all in $\mathbb{Q}$, and hence are fixed by σ).

We note, in passing, that if R is a local artinian principal ideal ring for which $J(R)$ has a central generator, then every finite group G, for which $|G| \cdot 1 \in U(R)$, is R-admissible, since the automorphism of R/J is the identity map. In particular, if G is finite, R is such a ring, then if RG is morphic, the same is true for any subgroup H of G (since $|H|$ divides $|G|$).

It should also be noted that the likely motivation for the authors of [1] to ask [1, Question 2.6] lies in the statement and proof of [1, Theorem 2.4]. The full strength of the hypotheses of [1, Theorem 2.4] are not needed in the proof and can be weakened. Namely, if G is a locally finite group with the property that every element $x \in RG$ is left morphic as an element of RH for some finite subgroup H of G, then RG is left morphic (instead of letting H be the subgroup generated by the support, simply take H to be the finite subgroup for which x is left morphic in RH; the rest of the proof is unchanged). In fact, this gives a local condition for morphicity in the group ring RG.

Proposition 25. *Let G be a locally finite group. Then, RG is left morphic if and only if for each $x \in RG$, there is a finite subgroup H of G such that x is left morphic in RH.*

Proof. The reverse implication is proved as in the proof of [1, Theorem 2.4] (replacing the subgroup generated by the support of the element by the larger subgroup guaranteed by hypothesis). For the converse, if $x \in RG$ is left morphic, then there is some y such that $\mathrm{ann}_\ell^{RG}(x) = RGy$ and $\mathrm{ann}_\ell^{RG}(y) = RGx$. Consider the subgroup H generated by the supports of x and y; H is finite, since G is locally finite. Clearly, $RHx \subseteq \mathrm{ann}_\ell^{RH}(y)$ and $RHy \subseteq \mathrm{ann}_\ell^{RH}(x)$. Conversely, if $z \in \mathrm{ann}_\ell^{RH}(y)$, then $z \in \mathrm{ann}_\ell^{RG}(y) = RGx$. We write $z = wx$, for some $w \in RG$, and write $w = \sum g_i b_i$, where $\{g_i\}$ is a left transversal for H in G, and $b_i \in RH$. We have $z = \sum g_i(b_i x)$. Comparing coefficients of elements of H, we see that $z = g_0 b_0 x$, where $g_0 \in H$. Since $g_0 b_0 \in RH$, we conclude that $z \in RHx$, and hence $RHx \supseteq \mathrm{ann}_\ell^{RH}(y)$. It follows that $RHx = \mathrm{ann}_\ell^{RH}(y)$. Similarly, $RHy = \mathrm{ann}_\ell^{RH}(x)$, and we conclude that x is left morphic in H. □

An interesting question, however, is whether RG left morphic implies that for each $x \in RG$, there is a finite subgroup H of G for which $x \in RH$ and RH is left morphic (as opposed to simply x being left morphic in RH); nor do we know whether each finite subgroup K of G is contained in a finite subgroup H of G for which RH is left morphic. Certainly, both of these statements are trivially true if the group G is a finite group (take $H = G$). We conclude with a few more examples.

Example 26. Let R be an artinian principal ideal ring, and let G be an infinite locally finite group for which RH is a principal ideal ring for each nontrivial (finite) subgroup of G. For instance, we may take R to be a division ring, and p a prime number which is invertible in R, we may take $G = \{x \in \mathbb{C}\colon x^{p^r} = 1 \text{ for some } r \geqslant 0\}$. Then, RG is not a principal ideal ring by Passman's theorem (it is infinite, but has no elements of infinite order), however, by [1, Theorem 2.4], RG is a left and right morphic ring.

Example 27. Let $R = \mathbb{C}[t;\sigma]/(t^2)$, where σ is an automorphism of $\mathbb{C}$ which fixes the algebraic numbers (there are such maps which are nontrivial, see, for instance, [16]). Then, any finite group G is R-admissible (see the discussion preceding Theorem 4). In particular, RG is a principal ideal ring for each finite group G. Note that nontrivial σ give rise to noncommutative rings R with this property.

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Appendix A

In this appendix, we will detail slight changes to the arguments found in [14, Section 4] which allow one to replace the hypothesis that K is a field with the hypothesis that K is a division ring in each of the results found in [14, Section 4]. For this section, we recommend that the reader have a copy of [14] to follow along with.

First, suppose R is a ring which has a subring, with the same unity as R, which is isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}_p$ for some prime p; we then view R as having characteristic 0 or characteristic p, accordingly. We may define the dimension subgroups as in [15, Section 3.3], and [15, Lemma 3.3.1] and [15, Lemma 3.3.2] remain valid (we need $(x-1)^p = x^p - 1$, and we need to divide by positive integers, in the characteristic p and 0 cases, respectively). Also, observe that $\mathcal{D}_n(\mathbb{M}_n(R)) = \mathcal{D}_n(R)$, since $\Delta(\mathbb{M}_n(R))^i = \mathbb{M}_n(\Delta(R)^i)$.

We will next detail why [14, Lemma 4.3] remains valid for division rings. In the proof of [14, Lemma 4.3], the first paragraph remains valid for any division ring K, with dimension interpreted as left K-vector space dimension. The next paragraph (finding a subgroup H for which $|H| \neq 0$ in the division ring) requires no changes. By the properties cited for the dimension subgroups in this context, G/H is a p-group, and if it is not cyclic, it has a homomorphic image which is elementary abelian of order p^2. We need to make a slight change in the last paragraph, since the ring KW need not be commutative. Arguing as in the first paragraph of the proof, if $\Delta(KW)$ is principal as a right ideal, say $\Delta(KW) = \alpha KW$, then $\Delta(KW) = KW\alpha$. It follows that $\Delta(KW)^p = KW\alpha^p$. But, if $\alpha = \sum_{g\in W} a_g g$, then $\alpha^p = \sum_{g\in W} a_g^p = \epsilon(\alpha)^p = 0$. We conclude that if $\Delta(KW)$ is principal, it must be nilpotent of degree p. It is, however, easy to see that the subring $\mathbb{Z}_p W$ of KW is nilpotent of degree $2p-1 > p$, from which it follows that $\Delta(KW)$ is not principal, and hence G/H is a cyclic p-group.

Finally, we observe that [14, Lemma 4.4] remains valid for K a division ring, with no changes needed. The proof of Passman's theorem proceeds as before for the implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c), using [14, Lemma 4.3] and [14, Lemma 4.4] which hold for division rings. The implication (c) $\Rightarrow$ (a) is essentially unchanged, using Maschke's theorem and [4, Lemma 6] (more details of this type of argument are found in the proof of the main theorem in [4]).

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