J. Differential Equations 247 (2009) 2140–2162

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

A uniqueness result for a semilinear elliptic problem: A computer-assisted proof ☆

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ARTICLE INFO

Article history:
Received 4 March 2009
Revised 29 June 2009
Available online 30 July 2009

MSC:
35J25
35J60
65N15

Keywords:
Semilinear elliptic boundary value problem
Uniqueness
Computer-assisted proof

ABSTRACT

Starting with the famous article [A. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 209–243], many papers have been devoted to the uniqueness question for positive solutions of $-\Delta u = \lambda u + u^p$ in Ω, $u = 0$ on $\partial\Omega$, where $p > 1$ and λ ranges between 0 and the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of $-\Delta$. For the case when Ω is a ball, uniqueness could be proved, mainly by ODE techniques. But very little is known when Ω is not a ball, and then only for $\lambda = 0$. In this article, we prove uniqueness, for all $\lambda \in [0, \lambda_1(\Omega))$, in the case $\Omega = (0,1)^2$ and $p = 2$. This constitutes the first positive answer to the uniqueness question in a domain different from a ball. Our proof makes heavy use of computer assistance: we compute a branch of approximate solutions and prove existence of a true solution branch close to it, using fixed point techniques. By eigenvalue enclosure methods, and an additional analytical argument for λ close to $\lambda_1(\Omega)$, we deduce the non-degeneracy of all solutions along this branch, whence uniqueness follows from the known bifurcation structure of the problem.

☆ Research supported by MIUR, project "Variational methods and nonlinear differential equations".
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0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.06.023

1. Introduction

In the last decades considerable attention has been devoted to the study of semilinear elliptic equations of the type

$$-\Delta u = f(u) \quad \text{in } \Omega \tag{1.1}$$

where Ω is a bounded or unbounded domain and f is a C^1-function. In particular much progress has been made in the study of positive solutions of (1.1), under various boundary conditions. A much investigated boundary value problem is the Dirichlet problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

which naturally arises as a model problem in many applications.

Whenever a solution of (1.2) exists it is important to know whether it is unique or not. It is not difficult to provide cases when (1.2) admits only one solution as well as others when many solutions exist. Indeed, when f is a concave function in $(0, \infty)$, such as $f(u) = u^p$, $0 < p < 1$, uniqueness holds in any smooth bounded domain [7]. On the other hand, if $f(u) = u^p$, $p > 1$, there are examples of nonconvex domains and exponents p, in any dimension $N \geqslant 2$, for which more solutions exist. This is the case of dumb-bell shaped domains or annuli [11]. Moreover there are examples of convex nonlinearities, such as $f(u) = e^u$ or $f(u) = (1+u)^p$, $p > 1$, for which uniqueness fails even if Ω is a ball [18,29]. All the different results show that both the nonlinearity $f(u)$ and the shape of the bounded domain Ω play an important role for the uniqueness of the positive solution of (1.2). As a consequence, a conjecture has been formulated, with its roots in the paper [12] of Gidas, Ni, and Nirenberg.

Conjecture. *If Ω is bounded and convex and $f(u) = u^p + \lambda u$, $p > 1$, $p \leqslant \frac{N+2}{N-2}$ if $N \geqslant 3$, $\lambda \in \mathbb{R}$, then uniqueness holds as long as a solution of* (1.2) *exists.*

Let us point out that if $N \geqslant 3$ solutions of (1.2) do not exist if Ω is starshaped and $f(u) = u^p + \lambda u$, $p \geqslant \frac{N+2}{N-2}$, $\lambda \leqslant 0$, as a consequence of Pohozaev's identity [24]. More generally positive solutions do not exist in any bounded domain, for every $N \geqslant 2$, if $\lambda \geqslant \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplace operator $-\Delta$ with homogeneous Dirichlet boundary conditions in Ω. This can be easily seen multiplying the equation by the first eigenfunction of $-\Delta$ and integrating.

When Ω is a ball the conjecture has been proved for the full range of the values of λ and p for which existence holds, mainly exploiting ODE techniques. Indeed, in the case of a ball a well-known result of Gidas, Ni, and Nirenberg [12] asserts that every positive solution of (1.2) is radial and radially decreasing so that Eq. (1.1) can be rewritten as an ordinary differential equation. When $\lambda = 0$, i.e. $f(u) = u^p$ the proof of the uniqueness in the ball is not very difficult and is contained in [12]. Instead, when $\lambda \neq 0$ the uniqueness in the ball is much more difficult to obtain and the complete result is spread in several papers [18], [30], [28], [1], [2].

When Ω is not a ball very few results are available, and then only for the case $\lambda = 0$, i.e. $f(u) = u^p$. Some are of perturbative type like that of [31] where domains close to a ball are considered, or that of [13] where the exponent p is close to the critical Sobolev exponent in dimension $N \geqslant 3$ and the domain Ω is assumed to be symmetric and convex in N orthogonal directions.

As regards general results the only ones to our knowledge are those contained in the papers [17, 11], and [10], again for the case $\lambda = 0$. In [17] a partial result is obtained in the sense that the uniqueness is proved only for the so-called "least-energy" positive solution, and the result holds in any bounded convex set in the plane. In [11] it is proved that if Ω is a domain in $\mathbb{R}^2$, symmetric and convex in two orthogonal directions, then there exists only one positive solution. This proof is based on a continuation method, also introduced in [11], and on the already known uniqueness result for the ball.

In [10] the same result as in [11] is obtained, as well as some other qualitative properties of solutions of (1.2). However the proof of the uniqueness of the solution of (1.2) is completely different and is based on properties of the solutions of the associated linearized equations which are also investigated in [10]. This is a pure PDE approach, based on the maximum principle, which does not rely on the uniqueness of the positive solution in the ball but indeed provides an independent proof for the ball.

Additionally, the results of [10] enable one to establish properties of the solutions of (1.2) also when $\lambda \neq 0$, i.e. if $f(u) = u^p + \lambda u$, $\lambda < \lambda_1(\Omega)$, for which, to our knowledge, there are no uniqueness results in any bounded domain, other than the ball. However these properties are not sufficient in [10] to deduce the uniqueness of the positive solutions, mainly because of the difficulty in proving the nondegeneracy of solutions.

(Recall that a solution u of (1.2) is said to be nondegenerate if the linearized operator $L_u = -\Delta - f'(u)$ does not admit zero as an eigenvalue in Ω with zero Dirichlet boundary conditions.)

In this paper, by computer assistance, we provide the first rigorous proof of the conjecture in a domain different from a ball. More precisely we have

Theorem 1.1. *Let Ω be the unit square in $\mathbb{R}^2$, $\Omega = (0,1)^2$. Then the problem*

$$\begin{cases} -\Delta u = u^2 + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

admits only one solution for any $\lambda \in [0, \lambda_1(\Omega))$.

The choice of the exponent $p = 2$ is merely a matter of numerical convenience but other exponents could be tested in the same way. Also the domain has been chosen to be the unit square to simplify the numerical approach. Apart from the importance of the result in itself we believe that Theorem 1.1 is especially valuable because it shows how a combination of a theoretical approach and numerical methods can lead to the solution of important problems. Indeed without the preliminary results of [10] and [19] the uniqueness could not be proved solely by numerical methods. On the other hand the theoretical approach of [10] and [19] has not been sufficient to prove the conjecture.

Let us explain briefly the proof of Theorem 1.1 and, for simplicity let us denote $\lambda_1(\Omega)$ by λ_1. The starting point will be Theorem 1.3 of [19] which is a consequence of the results of [10] and asserts that all solutions of (1.3) lie on a simple continuous curve, in the space $\mathbb{R} \times C^{1,\alpha}(\bar{\Omega})$, joining the point $(\lambda_1, 0)$ with the point $(0, u_0)$, where u_0 is the unique positive solution of (1.3) for $\lambda = 0$. This implies that if we are able to construct (by whatever means) a branch of solutions connecting these two points and can show that along the branch solutions are nondegenerate, then uniqueness follows. Indeed, the nondegeneracy of the solutions ensures, by the Implicit Function Theorem, that there are neither turning points nor secondary bifurcations along the branch, so, for every λ there is only one solution on the curve.

To achieve this goal by computer-assisted methods we proceed in the following way:

(i) First we construct a continuous branch (ω_λ) of approximate solutions of (1.3) for λ in an interval $[0, \bar{\lambda}]$, with $\bar{\lambda} < \lambda_1(\Omega)$ suitably chosen.
(ii) Then we prove by the method described e.g. in [20,6,23,22] that a true solution u_λ of (1.3) exists near each ω_λ and get a precise estimate of the distance between u_λ and ω_λ both in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$. This allows us to obtain a smooth solution branch of true solutions in the interval $[0, \bar{\lambda}]$.
(iii) Using the linearized operator at the approximate solutions and a perturbation argument we prove that the true solutions u_λ are nondegenerate, for $\lambda \in [0, \bar{\lambda}]$.
(iv) Using an L^∞-estimate we prove that for $\lambda \in [\bar{\lambda}, \lambda_1)$ there is only one solution of (1.3) which is also nondegenerate.

The step (iv) will be proved in Section 2 as a consequence of a result of [10] and allows us to conclude the proof completing the construction of the branch in the interval $[\bar{\lambda}, \lambda_1)$. Indeed the construction of the steps (i)–(iii) cannot be carried out in the whole interval $[0, \lambda_1)$, since at λ_1 the only solution which is identically zero is obviously degenerate because the corresponding linearized operator is just $-\Delta - \lambda_1$ which has zero as first eigenvalue.

The outline of the paper is the following: In Section 2 we recall some known results and prove a preliminary result in general domains which allows us to achieve step (iv). In Sections 3 to 5 we describe the analytical background of our computer-assisted proof, and in Sections 6 and 7 the numerical tools used. Section 8 contains numerical results.

2. Preliminary results

We start recalling some results from [10] and [19] that are important to prove Theorem 1.1. In the whole paper we consider bounded domains $\Omega \subseteq \mathbb{R}^2$ which are symmetric with respect to two orthogonal axes intersecting at a point $x_M \in \Omega$ and convex in the directions orthogonal to these axes. For simplicity in this section we will choose x_M as the origin and the symmetry axes as the coordinate axes so that they are $T_i = \{x = (x_1, x_2) \in \mathbb{R}^2, x_i = 0\}$, $i = 1, 2$. In such domains we will give the following definition:

Definition 2.1. A function $u \in C^1(\Omega)$ is called symmetric and monotone if it is even in both variables and $\frac{\partial u}{\partial x_i} > 0$ in $\Omega_i^- = \{x \in \Omega\colon\ x_i < 0\}$, $i = 1, 2$.

Now we consider the semilinear elliptic equation:

$$-\Delta u = f(u) \quad \text{in } \Omega \tag{2.1}$$

where f is a C^1-function.

Let us remark that by a result of [12] all positive solutions of (2.1) which are zero on the boundary of Ω are symmetric and monotone.

The following result was proved in [10].

Theorem 2.1. *Assume that u_1 and u_2 are positive, symmetric and monotone solutions of* (2.1)*, with f convex. If $\max_{\bar{\Omega}} u_1 = \max_{\bar{\Omega}} u_2$ and $u_1 \leqslant u_2$ on $\partial\Omega$, then $u_1 \equiv u_2$ in Ω.*

Note that no boundary conditions are assigned in the previous theorem, but the functions u_1 and u_2 are only required to be comparable on the boundary. Hence the previous result is similar to the uniqueness theorem for solutions of ordinary differential equations in an interval, in which case only prescribing the maximum value of the solution implies uniqueness.

Now let us consider the problem

$$\begin{cases} -\Delta u = u^p + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

with $p > 1$ and $\lambda \in [0, \lambda_1(\Omega))$, where $\lambda_1(\Omega)$ is the first eigenvalue of the operator $-\Delta$ with homogeneous Dirichlet boundary conditions.

Note that problem (2.2) can equivalently be reformulated as finding a *non-trivial* solution of

$$\begin{cases} -\Delta u = |u|^p + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

since by the strong maximum principle (for $-\Delta - \lambda$) every non-trivial solution of (2.3) is positive in Ω. In fact, this formulation is better suited for our computer-assisted approach than (2.2).

As a consequence of Theorem 2.1 and of a bifurcation theorem of [25], the following result was obtained in [19].

Theorem 2.2. *All solutions u_λ of* (2.2) *lie on a simple continuous curve Γ in* $[0, \lambda_1(\Omega)) \times C^{1,\alpha}(\bar{\Omega})$ *joining* $(\lambda_1(\Omega), 0)$ *with* $(0, u_0)$*, where u_0 is the unique solution of* (2.2) *for* $\lambda = 0$.

We recall that the uniqueness of the solution of (2.2) for $\lambda = 0$ was proved in [11] and [10]. As a consequence of the previous theorem we have

Corollary 2.1. *If all solutions on the curve Γ are nondegenerate then problem* (2.2) *admits only one solution for every* $\lambda \in [0, \lambda_1(\Omega))$.

Proof. The nondegeneracy of the solutions implies, by the Implicit Function Theorem, that neither turning points nor secondary bifurcations can exist along Γ. Then, for every $\lambda \in [0, \lambda_1(\Omega))$ there exists only one solution of (2.2) on Γ. By Theorem 2.2 all solutions are on Γ, hence uniqueness follows. □

Theorem 2.2 and Corollary 2.1 indicate that to prove the uniqueness of the solution of problem (2.2) for every $\lambda \in [0, \lambda_1(\Omega))$ it is enough to construct a branch of nondegenerate solutions which connects $(0, u_0)$ to $(\lambda_1(\Omega), 0)$. This is what we will do numerically in the next sections with a rigorous computer-assisted proof, obtaining so Theorem 1.1.

However, establishing the nondegeneracy of solutions u_λ for λ close to $\lambda_1(\Omega)$ numerically can be difficult, due to the fact that the only solution at $\lambda = \lambda_1(\Omega)$, which is the identically zero solution, is obviously degenerate because its linearized operator is $L_0 = -\Delta - \lambda_1$ which has the first eigenvalue equal to zero. In the next propositions we will show in a simple way that, using some L^∞-estimates and Theorem 2.1, it is possible to prove that for every domain Ω there exists a computable number $\bar{\lambda}(\Omega) \in (0, \lambda_1(\Omega))$ such that for any $\lambda \in (\bar{\lambda}(\Omega), \lambda_1(\Omega))$ problem (2.2) has only one solution which is also nondegenerate. Of course, from the well-known results of Crandall and Rabinowitz [8,9], one can establish that for λ "close to" λ_1, all solutions u_λ are nondegenerate. However, in order to complete our program, we need to calculate a precise and *explicit* estimate of how close they need to be. This allows us to carry out the numerical computation only in the interval $[0, \bar{\lambda}(\Omega)]$ as we will do later.

Let us denote by $\lambda_1 = \lambda_1(\Omega)$ and $\lambda_2 = \lambda_2(\Omega)$ the first and second eigenvalue of the operator $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions. We have

Proposition 2.1. *Assume that for some* $\lambda \in (0, \lambda_1)$ *and for a solution u_λ of* (2.2) *we have*

$$\|u_\lambda\|_\infty \leqslant \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}}. \tag{2.4}$$

Then u_λ is nondegenerate. Moreover if two solutions u_1 and u_2, corresponding to the same value of λ, satisfy (2.4) *then $u_1 \equiv u_2$ in Ω.*

Proof. Arguing by contradiction let u_λ be a degenerate solution of (2.2) satisfying (2.4). This implies that there exists a non-trivial solution v of the linearized equation at u_λ:

$$-\Delta v - p u_\lambda^{p-1} v - \lambda v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

Then v would change sign because zero cannot be the first eigenvalue of the linearized equation for any solution of (2.2). Indeed this first eigenvalue is negative for all solutions of (2.2), as can be deduced by comparison with the linear operator $-\Delta - u_\lambda^{p-1} - \lambda$, which has 0 as first eigenvalue (with eigenfunction u_λ; see [17]).

Setting $\beta(x) = pu_\lambda^{p-1}$, this means that v is an eigenfunction of the linear operator $L_\beta = (-\Delta - \beta(x))$ corresponding to the eigenvalue λ and by (2.4) we have

$$\beta(x) \leqslant \lambda_2 - \lambda_1. \tag{2.5}$$

Since v changes sign λ must be greater than or equal to the second eigenvalue of L_β that we denote by $\mu_2(\beta)$. Then, using also (2.5) we have

$$\lambda \geqslant \mu_2(\beta) \geqslant \lambda_2 - (\lambda_2 - \lambda_1) = \lambda_1$$

contradicting the fact that $\lambda \in (0, \lambda_1)$.

This proves that u_λ must be nondegenerate. In the same way we can also prove the second assertion. Indeed if u_1, u_2 are two different solutions of (2.2) satisfying (2.4), it is easy to show that the difference $w = u_1 - u_2$ must change sign (see [10, proof of Theorem 4.3]) and satisfy the equation

$$-\Delta w = \alpha(x) w + \lambda w \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega$$

with

$$\alpha(x) = \int_0^1 p\big(tu_1 + (1-t)u_2\big)^{p-1} \, dt \leqslant \lambda_2 - \lambda_1 \tag{2.6}$$

by (2.4).

Hence w is an eigenfunction of the linear operator $L_\alpha = -\Delta - \alpha(x)$ corresponding to the eigenvalue λ which must be greater than or equal to the second eigenvalue of L_α, because w changes sign. Therefore the same contradiction arises as in the proof of the nondegeneracy of the solution. □

The previous proposition asserts that we can deduce nondegeneracy and local uniqueness from the L^∞-estimate (2.4). Next we show how to verify this estimate for λ in an interval $(\bar{\lambda}, \lambda_1)$.

Lemma 2.1. *Let Γ be the unique continuous branch of solutions of* (2.2) *given by Theorem* 2.2 *and consider the function g defined on $\Gamma^* := \Gamma \setminus \{(0, u_0)\}$ by*

$$g(\lambda, u_\lambda) = \|u_\lambda\|_\infty \cdot \frac{1}{\lambda^{\frac{1}{p-1}}}, \quad (\lambda, u_\lambda) \in \Gamma^*.$$

Then $g(\Gamma^*) = (0, +\infty)$, g is injective and hence monotone decreasing along Γ^*.

Proof. Recall that because Ω is a doubly symmetric domain and u_λ is a symmetric and monotone function (in the sense of Definition 2.1), it achieves its maximum at the origin 0 which is the intersection of the symmetry axes of Ω, i.e.

$$\|u_\lambda\|_\infty = u_\lambda(0).$$

Define

$$h_\lambda(x) = u_\lambda\left(\frac{x}{\sqrt{\lambda}}\right) \frac{1}{\lambda^{\frac{1}{p-1}}}, \quad x = (x_1, x_2) \in \Omega_{\sqrt{\lambda}}$$

where $\Omega_{\sqrt{\lambda}} = \{x = \sqrt{\lambda} y, \ y \in \Omega\}$.

Then h satisfies

$$(P_\lambda)\quad \begin{cases} -\Delta h = h^p + h & \text{in } \Omega_{\sqrt{\lambda}},\\ h > 0 & \text{in } \Omega_{\sqrt{\lambda}},\\ h = 0 & \text{on } \partial\Omega_{\sqrt{\lambda}}.\end{cases}$$

Clearly $g(\lambda, u_\lambda) = h_\lambda(0) \to 0$ as $\lambda \to \lambda_1$ and $g(\lambda, u_\lambda) = h_\lambda(0) \to +\infty$ as $\lambda \to 0$. If there exist λ' and λ'', $\lambda' < \lambda''$, such that $g(\lambda', u_{\lambda'}) = g(\lambda'', u_{\lambda''})$ then the corresponding functions $h_{\lambda'}$ and $h_{\lambda''}$ will be two solutions of $(P_{\lambda'})$ and $(P_{\lambda''})$ with the same maximum and comparable on the boundary of $\Omega_{\sqrt{\lambda'}}$. This is a contradiction with the statement of Theorem 2.1 and hence the assertion is proved. □

Proposition 2.2. *If there exist $\bar\lambda \in (0, \lambda_1)$ and a solution $u_{\bar\lambda}$ of* (2.2) *with $\lambda = \bar\lambda$ such that*

$$\|u_{\bar\lambda}\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar\lambda}{\lambda_1}\right)^{\frac{1}{p-1}}, \tag{2.7}$$

then

$$\|u_\lambda\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \tag{2.8}$$

for all solutions u_λ of (2.2) *belonging to the branch $\Gamma_2 \subset \Gamma$ which connects $(\bar\lambda, u_{\bar\lambda})$ to $(\lambda_1, 0)$.*

(Recall that Γ is the unique continuous branch of solutions given by Theorem 2.2.)

Proof. We set $\Gamma = \Gamma_1 \cup \Gamma_2$, with Γ_1 connecting $(0, u_0)$ to $(\bar\lambda, u_{\bar\lambda})$.

By (2.7) we have

$$g(\bar\lambda, u_{\bar\lambda}) = \|u_{\bar\lambda}\|_\infty \cdot \frac{1}{\bar\lambda^{\frac{1}{p-1}}} < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \cdot \frac{1}{\lambda_1^{\frac{1}{p-1}}} =: \beta.$$

By continuity we have that the function g takes on Γ_1 all values in the interval $[\beta, +\infty)$. If on Γ_2 there was a solution $u_{\lambda'}$, $\lambda' \in (0, \lambda_1)$, such that $\|u_{\lambda'}\|_\infty \geqslant (\frac{\lambda_2-\lambda_1}{p})^{\frac{1}{p-1}}$ then we would have

$$g(\lambda', u_{\lambda'}) = \|u_{\lambda'}\|_\infty \cdot \frac{1}{(\lambda')^{\frac{1}{p-1}}} > \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \cdot \frac{1}{\lambda_1^{\frac{1}{p-1}}} = \beta.$$

As a consequence on Γ_2 the function g would take again some values in the interval $[\beta, +\infty)$, contradicting Lemma 2.1 which asserts the injectivity of the function g. □

Corollary 2.2. *If on the branch Γ there exists a solution $u_{\bar\lambda}$, $\bar\lambda \in (0, \lambda_1)$, such that*:

(i) *on the sub-branch Γ_1 connecting $(0, u_0)$ with $(\bar\lambda, u_{\bar\lambda})$ all solutions are nondegenerate, and*

(ii)
$$\|u_{\bar\lambda}\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar\lambda}{\lambda_1}\right)^{\frac{1}{p-1}}, \tag{2.9}$$

then all solutions of (2.2) *are nondegenerate, for all $\lambda \in (0, \lambda_1)$, and therefore problem* (2.2) *admits only one solution for every $\lambda \in [0, \lambda_1(\Omega))$.*

Proof. Let us split as before the branch Γ into the two sub-branches Γ_1 and Γ_2. On Γ_1 we have that all solutions are nondegenerate by (i). On the other hand the hypothesis (ii) allows us to apply Proposition 2.2 and obtain (2.8) for all solutions u_λ belonging to Γ_2. Then Proposition 2.1 implies that all solutions on Γ_2 are nondegenerate. Hence there is nondegeneracy all along Γ so the assertion follows from Corollary 2.1. □

The last corollary suggests the method of proving the uniqueness through computer assistance: first we construct a branch of nondegenerate "true" solutions near approximate ones in a certain interval $[0, \bar{\lambda}]$ and then verify (ii) for the solution $u_{\bar{\lambda}}$. Note that the estimate (2.9) depends only on p and on the eigenvalues λ_1 and λ_2 of the operator $-\Delta$ in the domain Ω. So the constant on the right-hand side is easily computable. When Ω is the unit square which is the case considered in Theorem 1.1 and analyzed in the next sections, the estimate (2.9) becomes:

$$\|u_{\bar{\lambda}}\|_\infty < \left(\frac{3\pi^2}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar{\lambda}}{2\pi^2}\right)^{\frac{1}{p-1}} = \left(\frac{3\bar{\lambda}}{2p}\right)^{\frac{1}{p-1}}$$

because $\lambda_1 = 2\pi^2$ and $\lambda_2 = 5\pi^2$.

Fixing $p = 2$ or 3 we finally get the conditions

$$\|u_{\bar{\lambda}}\|_\infty < \frac{3}{4}\bar{\lambda} \quad \text{if } p = 2, \tag{2.10}$$

$$\|u_{\bar{\lambda}}\|_\infty < \sqrt{\frac{\bar{\lambda}}{2}} \quad \text{if } p = 3. \tag{2.11}$$

3. The basic existence and enclosure theorem

We start the computer-assisted part of our proof with a basic theorem on existence, local uniqueness, and non-degeneracy of solutions to problem (2.3), assuming $p \geqslant 2$ now. In this section, the parameter $\lambda \in [0, \lambda_1(\Omega))$ is fixed.

Let $H_0^1(\Omega)$ be endowed with the inner product $\langle u, v\rangle_{H_0^1} := \langle \nabla u, \nabla v\rangle_{L^2} + \sigma\langle u, v\rangle_{L^2}$; the choice of some *positive* σ will turn out to be advantageous. Let $H^{-1}(\Omega)$ denote the (topological) dual of $H_0^1(\Omega)$, endowed with the usual operator sup-norm.

Suppose that an *approximate* solution $\omega_\lambda \in H_0^1(\Omega)$ of problem (2.3) has been computed by numerical means, and that a bound $\delta_\lambda > 0$ for its *defect* is known, i.e.

$$\left\| -\Delta\omega_\lambda - \lambda\omega_\lambda - |\omega_\lambda|^p \right\|_{H^{-1}} \leqslant \delta_\lambda, \tag{3.1}$$

as well as a constant K_λ such that

$$\|v\|_{H_0^1} \leqslant K_\lambda \left\| L_{(\lambda,\omega_\lambda)}[v] \right\|_{H^{-1}} \quad \text{for all } v \in H_0^1(\Omega). \tag{3.2}$$

Here, $L_{(\lambda,\omega_\lambda)}$ denotes the operator *linearizing* problem (2.3) at ω_λ; more generally, for $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$, let the linear operator $L_{(\lambda,u)} : H_0^1(\Omega) \to H^{-1}(\Omega)$ be defined by

$$L_{(\lambda,u)}[v] := -\Delta v - \lambda v - p|u|^{p-2}uv \quad \big(v \in H_0^1(\Omega)\big). \tag{3.3}$$

The practical computation of bounds δ_λ and K_λ will be addressed in Sections 6 and 7.

Let C_{p+1} denote a norm bound (embedding constant) for the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, which is bounded since $\Omega \subset \mathbb{R}^2$. C_{p+1} can be calculated e.g. according to the explicit formula given in [22, Lemma 2]. Finally, let

$$\gamma := \frac{1}{2} p(p-1) C_{p+1}^3.$$

In our example case where $\Omega = (0,1)^2$, and e.g. for $p = 2$, the above-mentioned explicit formula gives (with the choice $\sigma := 1$)

$$\gamma = \frac{1}{\sqrt{2}(2\pi^2 + \frac{3}{2})} \left(< \frac{1}{30} \right).$$

Theorem 3.1. *Suppose that some $\alpha_\lambda > 0$ exists such that*

$$\delta_\lambda \leqslant \frac{\alpha_\lambda}{K_\lambda} - \gamma \alpha_\lambda^2 \big(\|\omega_\lambda\|_{L^{p+1}} + C_{p+1}\alpha_\lambda \big)^{p-2} \tag{3.4}$$

and

$$2K_\lambda \gamma \alpha_\lambda \big(\|\omega_\lambda\|_{L^{p+1}} + C_{p+1}\alpha_\lambda \big)^{p-2} < 1. \tag{3.5}$$

Then, the following statements hold true:

(a) (*Existence*) *There exists a solution $u_\lambda \in H_0^1(\Omega)$ of problem* (2.3) *such that*

$$\|u_\lambda - \omega_\lambda\|_{H_0^1} \leqslant \alpha_\lambda. \tag{3.6}$$

(b) (*Local uniqueness*) *Let $\eta > 0$ be chosen such that* (3.5) *holds with $\alpha_\lambda + \eta$ instead of α_λ. Then,*

$$\left. \begin{array}{l} u \in H_0^1(\Omega) \text{ solution of (2.3)} \\ \|u - \omega_\lambda\|_{H_0^1} \leqslant \alpha_\lambda + \eta \end{array} \right\} \implies u = u_\lambda. \tag{3.7}$$

(c) (*Nondegeneracy*)

$$\left. \begin{array}{l} u \in H_0^1(\Omega) \\ \|u - \omega_\lambda\|_{H_0^1} \leqslant \alpha_\lambda \end{array} \right\} \implies L_{(\lambda,u)} : H_0^1(\Omega) \to H^{-1}(\Omega) \text{ is bijective}, \tag{3.8}$$

whence in particular $L_{(\lambda,u_\lambda)}$ is bijective (*by* (3.6)).

Corollary 3.1. *Suppose that* (3.4) *and* (3.5) *hold, and in addition that $\|\omega_\lambda\|_{H_0^1} > \alpha_\lambda$. Then, the solution u_λ given by Theorem* 3.1 *is non-trivial* (*and hence positive*).

Remark 3.1. (a) The function $\psi(\alpha) := \frac{\alpha}{K_\lambda} - \gamma\alpha^2(\|\omega_\lambda\|_{L^{p+1}} + C_{p+1}\alpha)^{p-2}$ has obviously a positive maximum on $[0,\infty)$, and the crucial condition (3.4) requires that

$$\delta_\lambda \leqslant \max_{\alpha \in [0,\infty)} \psi(\alpha), \tag{3.9}$$

i.e. δ_λ has to be sufficiently small. According to (3.1), this means that ω_λ must be computed with sufficient accuracy, which leaves the "hard work" to the computer!

Furthermore, a "small" defect bound δ_λ allows (via (3.4)) a "small" error bound α_λ, if K_λ is not too large.

(b) If we require, slightly stronger than (3.9) (but without much "practical" difference), that

$$\delta_\lambda < \max_{\alpha\in[0,\infty)} \psi(\alpha), \tag{3.10}$$

and if moreover we choose the *minimal* α_λ satisfying (3.4), then the additional condition (3.5) follows automatically, which can be seen as follows: There is a unique $\bar{\alpha} > 0$ such that $\psi(\bar{\alpha}) = \max\psi(\alpha)$, and (3.10) and the minimal choice of α_λ show that $\alpha_\lambda < \bar{\alpha}$. $\bar{\alpha}$ is determined by $\psi'(\bar{\alpha}) = 0$, which implies $2K_\lambda\gamma\bar{\alpha}(\|\omega_\lambda\|_{L^{p+1}} + C_{p+1}\bar{\alpha})^{p-2} \leqslant 1$ (with equality holding if $p = 2$), and therefore (3.5).

Since we will anyway try to find α_λ (satisfying (3.4)) close to the minimal one, condition (3.5) is "practically" always satisfied if (3.4) holds. (Nevertheless, it must of course be checked.)

(c) In the case $p = 2$, condition (3.10) reads

$$\delta_\lambda < \frac{1}{4\gamma K_\lambda^2}, \tag{3.11}$$

and if it holds, (3.4) and (3.5) are satisfied for

$$\alpha_\lambda := \frac{2K_\lambda\delta_\lambda}{1+\sqrt{1-4\gamma\delta_\lambda K_\lambda^2}}. \tag{3.12}$$

For the proof of Theorem 3.1, and also for later purposes, we will need two lemmata.

Lemma 3.1. *For all $u, \tilde{u}, v \in H_0^1(\Omega)$,*

$$\big\| p\big[|u|^{p-2}u - |\tilde{u}|^{p-2}\tilde{u}\big]v\big\|_{H^{-1}} \leqslant 2\gamma \max\big\{\|u\|_{L^{p+1}}, \|\tilde{u}\|_{L^{p+1}}\big\}^{p-2} \cdot \|u-\tilde{u}\|_{H_0^1}\|v\|_{H_0^1}.$$

Proof. The Mean Value Theorem gives

$$|u|^{p-2}u - |\tilde{u}|^{p-2}\tilde{u} = \int_0^1 (p-1)\big|tu + (1-t)\tilde{u}\big|^{p-2}\,dt \cdot (u-\tilde{u}) \quad \text{on } \Omega,$$

whence, for all $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned}
\left|\int_\Omega p\big[|u|^{p-2}u - |\tilde{u}|^{p-2}\tilde{u}\big]v\varphi\,dx\right| &= p(p-1)\left|\int_0^1\int_\Omega \big|tu+(1-t)\tilde{u}\big|^{p-2}(u-\tilde{u})v\varphi\,dx\,dt\right| \\
&\leqslant p(p-1)\int_0^1 \big\|tu+(1-t)\tilde{u}\big\|_{L^{p+1}}^{p-2}\|u-\tilde{u}\|_{L^{p+1}}\|v\|_{L^{p+1}}\|\varphi\|_{L^{p+1}}\,dt \\
&\leqslant p(p-1)C_{p+1}^3\int_0^1 \big\|tu+(1-t)\tilde{u}\big\|_{L^{p+1}}^{p-2}\,dt \cdot \|u-\tilde{u}\|_{H_0^1}\|v\|_{H_0^1}\|\varphi\|_{H_0^1}
\end{aligned}$$

implying the assertion. □

Lemma 3.2. *Let $(\lambda, u), (\tilde{\lambda}, \tilde{u}) \in \mathbb{R} \times H_0^1(\Omega)$, and suppose that, for some $K > 0$,*

$$\|v\|_{H_0^1} \leqslant K \left\| L_{(\tilde{\lambda},\tilde{u})}[v] \right\|_{H^{-1}} \quad \textit{for all } v \in H_0^1(\Omega)$$

(*with $L_{(\lambda,u)}$ defined in* (3.3)), *and*

$$\kappa := K\left[\frac{1}{\lambda_1(\Omega)+\sigma}|\lambda-\tilde{\lambda}| + 2\gamma \max\left\{\|u\|_{L^{p+1}}, \|\tilde{u}\|_{L^{p+1}}\right\}^{p-2}\|u-\tilde{u}\|_{H_0^1}\right] < 1. \tag{3.13}$$

Then,

$$\|v\|_{H_0^1} \leqslant \frac{K}{1-\kappa}\left\| L_{(\lambda,u)}[v] \right\|_{H^{-1}} \quad \textit{for all } v \in H_0^1(\Omega).$$

Proof. Using (3.3), Lemma 3.1, and $\|v\|_{H^{-1}} \leqslant \frac{1}{\lambda_1(\Omega)+\sigma}\|v\|_{H_0^1}$, we obtain

$$\begin{aligned}\|v\|_{H_0^1} &\leqslant K\left\|L_{(\tilde{\lambda},\tilde{u})}[v]\right\|_{H^{-1}} \leqslant K\left[\left\|L_{(\lambda,u)}[v]\right\|_{H^{-1}} + \left\|(\lambda-\tilde{\lambda})v + p\left[|u|^{p-2}u - |\tilde{u}|^{p-2}\tilde{u}\right]v\right\|_{H^{-1}}\right] \\ &\leqslant K\left\|L_{(\lambda,u)}[v]\right\|_{H^{-1}} + \kappa\|v\|_{H_0^1} \quad \text{for all } v \in H_0^1(\Omega),\end{aligned}$$

whence (3.13) implies the assertion. □

Proof of Theorem 3.1. By (3.2), $L_{(\lambda,\omega_\lambda)}: H_0^1(\Omega) \to H^{-1}(\Omega)$ is one-to-one, and hence bijective due to Fredholm's Alternative for linear boundary value problems. Problem (2.3) is therefore equivalent to the following fixed-point problem for the error $v = u - \omega_\lambda$:

$$v = T(v) := L^{-1}_{(\lambda,\omega_\lambda)}\left[\left(\Delta\omega_\lambda + \lambda\omega_\lambda + |\omega_\lambda|^p\right) + \left(|\omega_\lambda + v|^p - |\omega_\lambda|^p - p|\omega_\lambda|^{p-2}\omega_\lambda v\right)\right]. \tag{3.14}$$

Below we will prove that the fixed-point operator $T: H_0^1(\Omega) \to H_0^1(\Omega)$

(i) maps $D := \{v \in H_0^1(\Omega)\colon \|v\|_{H_0^1} \leqslant \alpha_\lambda\}$ into itself,
(ii) is contractive on $D_\eta := \{v \in H_0^1(\Omega)\colon \|v\|_{H_0^1} \leqslant \alpha_\lambda + \eta\}$.

Then, Banach's Fixed-Point Theorem yields a fixed-point $v_\lambda \in D$ of T, whence by construction $u_\lambda := \omega_\lambda + v_\lambda$ is a solution of (2.3) satisfying (3.6), which proves part (a) of the theorem. Furthermore, by (ii), v_λ is the *only* fixed-point of T within D_η, whence also part (b) follows. For u as in the premise of (3.8), and $(\tilde{\lambda}, \tilde{u}) := (\lambda, \omega_\lambda)$, the assumptions of Lemma 3.2 hold for $K = K_\lambda$ and

$$\kappa = 2K_\lambda\gamma \max\left\{\|u\|_{L^{p+1}}, \|\omega_\lambda\|_{L^{p+1}}\right\}^{p-2}\|u-\omega_\lambda\|_{H_0^1} \leqslant 2K_\lambda\gamma\left(\|\omega_\lambda\|_{L^{p+1}} + C_{p+1}\alpha_\lambda\right)^{p-2}\alpha_\lambda$$

which is indeed less than 1 by (3.5). Thus, $L_{(\lambda,u)}$ is one-to-one by Lemma 3.2, and hence bijective due to Fredholm's Alternative. This proves the final part (c) of the theorem.

To prove (i) and (ii), we first note that, for $v, \tilde{v} \in H_0^1(\Omega)$,

$$\begin{aligned}&|\omega_\lambda + v|^p - |\omega_\lambda + \tilde{v}|^p - p|\omega_\lambda|^{p-2}\omega_\lambda(v-\tilde{v}) \\ &\quad = \int_0^1 \frac{d}{dt}\left[\left|\omega_\lambda + tv + (1-t)\tilde{v}\right|^p - tp|\omega_\lambda|^{p-2}\omega_\lambda(v-\tilde{v})\right]dt \\ &\quad = \int_0^1 p\left[\left|\omega_\lambda + tv + (1-t)\tilde{v}\right|^{p-2}\left(\omega_\lambda + tv + (1-t)\tilde{v}\right) - |\omega_\lambda|^{p-2}\omega_\lambda\right](v-\tilde{v})\,dt\end{aligned}$$

on Ω. Multiplying by a test function, integrating over Ω, exchanging the order of integration on the right-hand side, and applying Lemma 3.1, we obtain

$$\begin{aligned}
&\big\| |\omega_\lambda + v|^p - |\omega_\lambda + \tilde{v}|^p - p|\omega_\lambda|^{p-2}\omega_\lambda(v-\tilde{v})\big\|_{H^{-1}} \\
&\quad \leqslant 2\gamma \int_0^1 \max\big\{\big\|\omega_\lambda + tv + (1-t)\tilde{v}\big\|_{L^{p+1}}, \|\omega_\lambda\|_{L^{p+1}}\big\}^{p-2} \big\|tv + (1-t)\tilde{v}\big\|_{H_0^1}\,dt \cdot \|v-\tilde{v}\|_{H_0^1} \\
&\quad \leqslant 2\gamma\big[\|\omega_\lambda\|_{L^{p+1}} + C_{p+1}\max\big\{\|v\|_{H_0^1}, \|\tilde{v}\|_{H_0^1}\big\}\big]^{p-2} \cdot \frac{1}{2}\big(\|v\|_{H_0^1} + \|\tilde{v}\|_{H_0^1}\big) \cdot \|v-\tilde{v}\|_{H_0^1}. \qquad (3.15)
\end{aligned}$$

By (3.14), (3.2), (3.1), (3.15) (with $\tilde{v} := 0$), and (3.4), we obtain for $v \in D$:

$$\begin{aligned}
\big\|T(v)\big\|_{H_0^1} &\leqslant K_\lambda \big\|\big(\Delta\omega_\lambda + \lambda\omega_\lambda + |\omega_\lambda|^p\big) + \big(|\omega_\lambda + v|^p - |\omega_\lambda|^p - p|\omega_\lambda|^{p-2}\omega_\lambda v\big)\big\|_{H^{-1}} \\
&\leqslant K_\lambda\big[\delta_\lambda + \gamma\big(\|\omega_\lambda\|_{L^{p+1}} + C_{p+1}\alpha_\lambda\big)^{p-2}\alpha_\lambda^2\big] \leqslant \alpha_\lambda,
\end{aligned}$$

i.e. $T(v) \in D$, which proves (i). Furthermore, (3.14), (3.2), and (3.15) imply, for $v, \tilde{v} \in D_\eta$:

$$\begin{aligned}
\big\|T(v) - T(\tilde{v})\big\|_{H_0^1} &\leqslant K_\lambda \big\||\omega_\lambda + v|^p - |\omega_\lambda + \tilde{v}|^p - p|\omega_\lambda|^{p-2}\omega_\lambda(v-\tilde{v})\big\|_{H^{-1}} \\
&\leqslant 2K_\lambda\gamma\big(\|\omega_\lambda\|_{L^{p+1}} + C_{p+1}(\alpha_\lambda + \eta)\big)^{p-2}(\alpha_\lambda + \eta)\|v - \tilde{v}\|_{H_0^1},
\end{aligned}$$

whence (3.5), respectively the choice of η, proves (ii). □

4. The branch (u_λ)

Fixing some $\bar{\lambda} \in (0, \lambda_1(\Omega))$ (the actual choice of which is made on the basis of Proposition 2.2; see also Section 5), we assume now that for *every* $\lambda \in [0, \bar{\lambda}]$ an approximate solution $\omega_\lambda \in H_0^1(\Omega)$ is at hand, as well as a defect bound δ_λ satisfying (3.1), and a bound K_λ satisfying (3.2). Furthermore, we assume now that, for every $\lambda \in [0, \bar{\lambda}]$, some $\alpha_\lambda > 0$ satisfies (3.4) and (3.5), and the additional non-triviality condition $\|\omega_\lambda\|_{H_0^1} > \alpha_\lambda$ (see Corollary 3.1). We suppose that some *uniform* (λ-independent) $\eta > 0$ can be chosen such that (3.5) holds with $\alpha_\lambda + \eta$ instead of α_λ (compare Theorem 3.1(b)). Hence Theorem 3.1 gives a positive solution $u_\lambda \in H_0^1(\Omega)$ of problem (2.3) with the properties (3.6)–(3.8), for every $\lambda \in [0, \bar{\lambda}]$.

Finally, we assume that the approximate solution branch $([0, \bar{\lambda}] \to H_0^1(\Omega), \lambda \mapsto \omega_\lambda)$ is continuous, and that $([0, \bar{\lambda}] \to \mathbb{R}, \lambda \mapsto \alpha_\lambda)$ is lower semi-continuous.

In Sections 6 and 7, we will address the actual computation of such branches $(\omega_\lambda), (\delta_\lambda), (K_\lambda), (\alpha_\lambda)$.

So far we know nothing about continuity or smoothness of $([0, \bar{\lambda}] \to H_0^1(\Omega), \lambda \mapsto u_\lambda)$, which however we will need to conclude that $(u_\lambda)_{\lambda \in [0, \bar{\lambda}]}$ coincides with the sub-branch Γ_1 introduced in Corollary 2.2.

Theorem 4.1. *The solution branch*

$$\left\{ \begin{array}{rcl} [0, \bar{\lambda}] & \to & H_0^1(\Omega) \\ \lambda & \mapsto & u_\lambda \end{array} \right\}$$

is continuously differentiable.

Proof. The mapping

$$\mathcal{F}: \left\{ \begin{array}{rcl} \mathbb{R} \times H_0^1(\Omega) & \to & H^{-1}(\Omega) \\ (\lambda, u) & \mapsto & -\Delta u - \lambda u - |u|^p \end{array} \right\}$$

is continuously differentiable, with $(\partial \mathcal{F}/\partial u)(\lambda, u) = L_{(\lambda,u)}$ (see (3.3)), and $\mathcal{F}(\lambda, u_\lambda) = 0$ for all $\lambda \in [0, \bar{\lambda}]$. (Note that $L_{(\lambda,u)}$ depends indeed continuously on (λ, u) by Lemma 3.1.) It suffices to prove the asserted smoothness locally. Thus, fix $\lambda_0 \in [0, \bar{\lambda}]$. Since $L_{(\lambda_0, u_{\lambda_0})}$ is bijective by Theorem 3.1(c), the Implicit Function Theorem gives a C^1-smooth solution branch

$$\left\{ \begin{array}{rcl} (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) & \to & H_0^1(\Omega) \\ \lambda & \mapsto & \hat{u}_\lambda \end{array} \right\}$$

to problem (2.3), with $\hat{u}_{\lambda_0} = u_{\lambda_0}$. By (3.6),

$$\|\hat{u}_{\lambda_0} - \omega_{\lambda_0}\|_{H_0^1} \leqslant \alpha_{\lambda_0}. \tag{4.1}$$

Since $\hat{u}_\lambda$ and ω_λ depend continuously on λ, and α_λ lower semi-continuously, (4.1) implies

$$\|\hat{u}_\lambda - \omega_\lambda\|_{H_0^1} \leqslant \alpha_\lambda + \eta \quad \big(\lambda \in [0, \bar{\lambda}] \cap (\lambda_0 - \tilde{\varepsilon}, \lambda_0 + \tilde{\varepsilon})\big)$$

for some $\tilde{\varepsilon} \in (0, \varepsilon)$. Hence Theorem 3.1(b) provides

$$\hat{u}_\lambda = u_\lambda \quad \big(\lambda \in [0, \bar{\lambda}] \cap (\lambda_0 - \tilde{\varepsilon}, \lambda_0 + \tilde{\varepsilon})\big),$$

implying the desired smoothness in some neighborhood of λ_0 (which of course is one-sided if $\lambda_0 = 0$ or $\lambda_0 = \bar{\lambda}$). □

As a consequence of Theorem 4.1, $(u_\lambda)_{\lambda \in [0, \bar{\lambda}]}$ is a continuous solution curve connecting the point $(0, u_0)$ with $(\bar{\lambda}, u_{\bar{\lambda}})$, and thus must coincide with the sub-branch Γ_1, connecting these two points, of the unique simple continuous curve Γ given by Theorem 2.2. Using Theorem 3.1(c), we obtain

Corollary 4.1. *On the sub-branch Γ_1 of Γ which connects $(0, u_0)$ with $(\bar{\lambda}, u_{\bar{\lambda}})$, all solutions are nondegenerate.*

Thus, if we can choose $\bar{\lambda}$ such that condition (2.9) holds true, Corollary 2.2 will give the desired uniqueness result.

5. Choice of $\bar{\lambda}$

We have to choose $\bar{\lambda}$ such that condition (2.9) is satisfied. For this purpose, we use computer assistance again. With x_M denoting the intersection of the symmetry axes of the (doubly symmetric) domain Ω, i.e. $x_M = (\frac{1}{2}, \frac{1}{2})$ for $\Omega = (0,1)^2$, we choose $\bar{\lambda} \in (0, \lambda_1(\Omega))$, not too close to $\lambda_1(\Omega)$, such that our *approximate* solution $\omega_{\bar{\lambda}}$ satisfies

$$\omega_{\bar{\lambda}}(x_M) < \left(\frac{\lambda_2(\Omega) - \lambda_1(\Omega)}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar{\lambda}}{\lambda_1(\Omega)}\right)^{\frac{1}{p-1}}, \tag{5.1}$$

with "not too small" difference between right- and left-hand side. Such a $\bar{\lambda}$ can be found within a few numerical trials.

Here, we impose the additional requirement

$$\omega_{\bar{\lambda}} \in H^2(\Omega) \cap H_0^1(\Omega), \tag{5.2}$$

which is in fact a condition on the numerical method used to compute $\omega_{\bar{\lambda}}$. (Actually, condition (5.2) could be avoided if we were willing to accept additional technical effort.) Moreover, exceeding (3.1), we will now need an L^2-bound $\hat{\delta}_{\bar{\lambda}}$ for the defect:

$$\left\| -\Delta\omega_{\bar{\lambda}} - \bar{\lambda}\omega_{\bar{\lambda}} - |\omega_{\bar{\lambda}}|^p \right\|_{L^2} \leqslant \hat{\delta}_{\bar{\lambda}}. \tag{5.3}$$

Finally, we make the assumption that Ω is *convex*, and hence in particular H^2-regular, whence every solution $u \in H_0^1(\Omega)$ of problem (2.3) is in $H^2(\Omega)$. Again, this additional assumption could be avoided with a lot of technical effort, but this is not worth doing here since anyway we are aiming at the convex domain $\Omega = (0,1)^2$.

Using the method described in Section 3, we obtain, by Theorem 3.1(a), a positive solution $u_{\bar{\lambda}} \in H^2(\Omega) \cap H_0^1(\Omega)$ of problem (2.3) satisfying

$$\|u_{\bar{\lambda}} - \omega_{\bar{\lambda}}\|_{H_0^1} \leqslant \alpha_{\bar{\lambda}}, \tag{5.4}$$

provided that (3.4) and (3.5) hold, and that $\|\omega_{\bar{\lambda}}\|_{H_0^1} > \alpha_{\bar{\lambda}}$.

Now we make use of the explicit version of the Sobolev embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ given in [20]. There, explicit constants $\hat{C}_0$, $\hat{C}_1$, $\hat{C}_2$ are computed such that

$$\|u\|_\infty \leqslant \hat{C}_0\|u\|_{L^2} + \hat{C}_1\|\nabla u\|_{L^2} + \hat{C}_2\|u_{xx}\|_{L^2} \quad \text{for all } u \in H^2(\Omega),$$

with $\|u_{xx}\|_{L^2}$ denoting the L^2-Frobenius norm of the Hessian matrix u_{xx}. E.g. for $\Omega = (0,1)^2$, [20] gives

$$\hat{C}_0 = 1, \qquad \hat{C}_1 = 1.1548 \cdot \sqrt{\frac{2}{3}} \leqslant 0.9429, \qquad \hat{C}_2 = 0.22361 \cdot \sqrt{\frac{28}{45}} \leqslant 0.1764.$$

Moreover, $\|u_{xx}\|_{L^2} \leqslant \|\Delta u\|_{L^2}$ for $u \in H^2(\Omega) \cap H_0^1(\Omega)$ since Ω is convex (see e.g. [15]). Consequently,

$$\|u_{\bar{\lambda}} - \omega_{\bar{\lambda}}\|_\infty \leqslant \hat{C}_0\|u_{\bar{\lambda}} - \omega_{\bar{\lambda}}\|_{L^2} + \hat{C}_1\|u_{\bar{\lambda}} - \omega_{\bar{\lambda}}\|_{H_0^1} + \hat{C}_2\|\Delta u_{\bar{\lambda}} - \Delta\omega_{\bar{\lambda}}\|_{L^2}. \tag{5.5}$$

To bound the last term on the right-hand side, we first note that

$$\begin{aligned}
\left\| |u_{\bar{\lambda}}|^p - |\omega_{\bar{\lambda}}|^p \right\|_{L^2} &= \left\| p\int_0^1 \left|\omega_{\bar{\lambda}} + t(u_{\bar{\lambda}} - \omega_{\bar{\lambda}})\right|^{p-2}\left(\omega_{\bar{\lambda}} + t(u_{\bar{\lambda}} - \omega_{\bar{\lambda}})\right) dt \cdot (u_{\bar{\lambda}} - \omega_{\bar{\lambda}}) \right\|_{L^2} \\
&\leqslant p\int_0^1 \left\| \left|\omega_{\bar{\lambda}} + t(u_{\bar{\lambda}} - \omega_{\bar{\lambda}})\right|^{p-1} \cdot |u_{\bar{\lambda}} - \omega_{\bar{\lambda}}| \right\|_{L^2} dt \\
&\leqslant p\int_0^1 \left\|\omega_{\bar{\lambda}} + t(u_{\bar{\lambda}} - \omega_{\bar{\lambda}})\right\|_{L^{2p}}^{p-1} \|u_{\bar{\lambda}} - \omega_{\bar{\lambda}}\|_{L^{2p}}\, dt \\
&\leqslant p\int_0^1 \left(\|\omega_{\bar{\lambda}}\|_{L^{2p}} + tC_{2p}\alpha_{\bar{\lambda}}\right)^{p-1} dt \cdot C_{2p}\alpha_{\bar{\lambda}},
\end{aligned} \tag{5.6}$$

using (5.4) and an embedding constant C_{2p} for the embedding $H_0^1(\Omega) \hookrightarrow L^{2p}(\Omega)$ in the last line; see e.g. [22, Lemma 2] for its computation. Moreover, by (2.3) and (5.3),

$$\|\Delta u_{\bar\lambda} - \Delta\omega_{\bar\lambda}\|_{L^2} \leqslant \hat\delta_{\bar\lambda} + \bar\lambda\|u_{\bar\lambda} - \omega_{\bar\lambda}\|_{L^2} + \big\||u_{\bar\lambda}|^p - |\omega_{\bar\lambda}|^p\big\|_{L^2}. \tag{5.7}$$

Using (5.4)–(5.7), and the Poincaré inequality

$$\|u\|_{L^2} \leqslant \frac{1}{\sqrt{\lambda_1(\Omega)+\sigma}}\|u\|_{H_0^1} \quad \big(u \in H_0^1(\Omega)\big), \tag{5.8}$$

we finally obtain

$$\|u_{\bar\lambda} - \omega_{\bar\lambda}\|_\infty \leqslant \left[\frac{\hat C_0 + \bar\lambda \hat C_2}{\sqrt{\lambda_1(\Omega)+\sigma}} + \hat C_1 + pC_{2p}\hat C_2\int_0^1 \big(\|\omega_{\bar\lambda}\|_{L^{2p}} + tC_{2p}\alpha_{\bar\lambda}\big)^{p-1}\,dt\right]\cdot\alpha_{\bar\lambda} + \hat C_2\hat\delta_{\bar\lambda}, \tag{5.9}$$

and the right-hand side is “small” if $\alpha_{\bar\lambda}$ and $\hat\delta_{\bar\lambda}$ are “small”, which can (again) be achieved by sufficiently accurate numerical computations.

Finally, since

$$u_{\bar\lambda}(x_M) \leqslant \omega_{\bar\lambda}(x_M) + \|u_{\bar\lambda} - \omega_{\bar\lambda}\|_\infty,$$

(5.9) yields an upper bound for $u_{\bar\lambda}(x_M)$ which is “not too much” larger than $\omega_{\bar\lambda}(x_M)$. Hence condition (2.9) can easily be checked, and (5.1) (with “not too small” difference between right- and left-hand side) implies a good chance that this check will be successful; otherwise, $\bar\lambda$ has to be chosen a bit larger.

6. Computation of ω_λ, δ_λ, K_λ for fixed λ

In this section we report on the computation of an approximate solution $\omega_\lambda \in H^2(\Omega) \cap H_0^1(\Omega)$ to problem (2.3), and of bounds δ_λ and K_λ satisfying (3.1) and (3.2), where $\lambda \in [0, \lambda_1(\Omega))$ is *fixed* (or one of *finitely* many values). We will restrict ourselves to the unit square $\Omega = (0,1)^2$ now.

An approximation ω_λ is computed by a *Newton iteration* applied to problem (2.3), where the linear boundary value problems

$$L_{(\lambda,\omega_\lambda^{(n)})}[v_n] = \Delta\omega_\lambda^{(n)} + \lambda\omega_\lambda^{(n)} + \big|\omega_\lambda^{(n)}\big|^p \tag{6.1}$$

occurring in the single iteration steps are solved approximately by an ansatz

$$v_n(x_1,x_2) = \sum_{i,j=1}^N \alpha_{ij}^{(n)} \sin(i\pi x_1)\sin(j\pi x_2) \tag{6.2}$$

and a Ritz–Galerkin method (with the basis functions in (6.2)) applied to problem (6.1). The update $\omega_\lambda^{(n+1)} := \omega_\lambda^{(n)} + v_n$ concludes the iteration step.

The Newton iteration is terminated when the coefficients $\alpha_{ij}^{(n)}$ in (6.2) are “small enough”, i.e. their modulus is below some pre-assigned tolerance.

To *start* the Newton iteration, i.e. to find an appropriate $\omega_\lambda^{(0)}$ of the form (6.2), we first consider some λ close to $\lambda_1(\Omega)$, and choose $\omega_\lambda^{(0)} = \alpha\sin(\pi x_1)\sin(\pi x_2)$; with an appropriate choice of $\alpha > 0$ (to be determined in a few numerical trials), the Newton iteration will “converge” to a non-trivial approximation $\omega^{(\lambda)}$. Then, starting at this value, we diminish λ in small steps until we arrive at

$\lambda = 0$, while in each of these steps the approximation $\omega^{(\lambda)}$ computed in the *previous* step is taken as a start of the Newton iteration. In this way, we find approximations ω_λ to problem (2.3) for "many" values of λ. Note that all approximations ω_λ obtained in this way are of the form (6.2).

The computation of an L^2-defect bound $\hat{\delta}_\lambda$ satisfying

$$\left\| -\Delta\omega_\lambda - \lambda\omega_\lambda - |\omega_\lambda|^p \right\|_{L^2} \leqslant \hat{\delta}_\lambda \tag{6.3}$$

amounts to the computation of an integral over Ω. In the case $p = 2$, which is the only one which we treated completely rigorously, this integral can easily and quickly be computed in closed form, since ω_λ is of the form (6.2) and hence only products of trigonometric functions occur in the integrand. After calculating them, various sums $\sum_{i=1}^N$ remain to be evaluated. In order to obtain a *rigorous* bound $\hat{\delta}_\lambda$, these computations (in contrast to those for obtaining ω_λ as described above) need to be carried out in *interval* arithmetic [14,27], to take rounding errors into account.

For the case $p = 3$ we just approximated the needed integral by a quadrature formula, which of course spoils mathematical rigor in this case; note however that this lack of rigor occurs on a rather "technical" level only, whence we are convinced that also this case can be treated completely rigorously when we use refined numerics.

Once an L^2-defect bound $\hat{\delta}_\lambda$ (satisfying (6.3)) has been computed, an H^{-1}-defect bound δ_λ (satisfying (3.1)) is easily obtained via the embedding

$$\|u\|_{H^{-1}} \leqslant \frac{1}{\sqrt{\lambda_1(\Omega) + \sigma}} \|u\|_{L^2} \quad \bigl(u \in L^2(\Omega)\bigr) \tag{6.4}$$

which is a result of the corresponding dual embedding (5.8). Indeed, (6.3) and (6.4) imply that

$$\delta_\lambda := \frac{1}{\sqrt{\lambda_1(\Omega) + \sigma}} \hat{\delta}_\lambda$$

satisfies (3.1).

For computing a constant K_λ satisfying (3.2), we use the isometric isomorphism

$$\Phi : \left\{ \begin{array}{rcl} H_0^1(\Omega) & \to & H^{-1}(\Omega) \\ u & \mapsto & -\Delta u + \sigma u \end{array} \right\}, \tag{6.5}$$

and note that $\Phi^{-1} L_{(\lambda,\omega_\lambda)} : H_0^1(\Omega) \to H_0^1(\Omega)$ is $\langle\cdot,\cdot\rangle_{H_0^1}$-symmetric since

$$\bigl\langle \Phi^{-1} L_{(\lambda,\omega_\lambda)}[u], v \bigr\rangle_{H_0^1} = \int_\Omega \bigl[\nabla u \cdot \nabla v - \lambda u v - p|\omega_\lambda|^{p-2}\omega_\lambda u v\bigr]\, dx, \tag{6.6}$$

and hence selfadjoint. Since $\|L_{(\lambda,\omega_\lambda)}[u]\|_{H^{-1}} = \|\Phi^{-1} L_{(\lambda,\omega_\lambda)}[u]\|_{H_0^1}$, (3.2) thus holds for any

$$K_\lambda \geqslant \bigl[\min\bigl\{|\mu|\colon\ \mu \text{ is in the spectrum of } \Phi^{-1} L_{(\lambda,\omega_\lambda)}\bigr\}\bigr]^{-1}, \tag{6.7}$$

provided the min is positive.

A particular consequence of (6.6) is that

$$\bigl\langle \bigl(I - \Phi^{-1} L_{(\lambda,\omega_\lambda)}\bigr)[u], u \bigr\rangle_{H_0^1} = \int_\Omega W_\lambda u^2\, dx \quad \bigl(u \in H_0^1(\Omega)\bigr) \tag{6.8}$$

where

$$W_\lambda(x) := \sigma + \lambda + p\left|\omega_\lambda(x)\right|^{p-2}\omega_\lambda(x) \quad (x \in \Omega). \tag{6.9}$$

Choosing the parameter $\sigma > 0$ in the H_0^1-product large enough (where $\sigma := 1$ turned out to be sufficient in the actual computations), we obtain $W_\lambda > 0$ on $\bar{\Omega}$. Thus, (6.8) shows that all eigenvalues μ of $\Phi^{-1}L_{(\lambda,\omega_\lambda)}$ are less than 1, and that its essential spectrum consists of the single point 1. Therefore, (6.7) requires the computation of *eigenvalue bounds* for the eigenvalue(s) μ neighboring 0.

Using the transformation $\kappa = 1/(1-\mu)$, the eigenvalue problem $\Phi^{-1}L_{(\lambda,\omega_\lambda)}[u] = \mu u$ is easily seen to be equivalent to

$$-\Delta u + \sigma u = \kappa W_\lambda u,$$

or, in weak formulation,

$$\langle u, v\rangle_{H_0^1} = \kappa \int_\Omega W_\lambda u v\, dx \quad \bigl(v \in H_0^1(\Omega)\bigr), \tag{6.10}$$

and we are interested in bounds to the eigenvalue(s) κ neighboring 1. It is therefore sufficient to compute two-sided bounds to the first N eigenvalues $\kappa_1 \leqslant \cdots \leqslant \kappa_N$ of problem (6.10), where N is (at least) such that $\kappa_N > 1$. In all our practical examples, the computed enclosures $\kappa_i \in [\underline{\kappa}_i, \bar{\kappa}_i]$ are such that $\bar{\kappa}_1 < 1 < \underline{\kappa}_2$, whence by (6.7) and $\kappa = 1/(1-\mu)$ we can choose

$$K_\lambda := \max\left\{\frac{\bar{\kappa}_1}{1-\bar{\kappa}_1}, \frac{\underline{\kappa}_2}{\underline{\kappa}_2 - 1}\right\}. \tag{6.11}$$

The desired *eigenvalue bounds* for problem (6.10) can be obtained by computer-assisted means of their own. For example, *upper* bounds to $\kappa_1, \ldots, \kappa_N$ (with $N \in \mathbb{N}$ given) are easily and efficiently computed by the *Rayleigh–Ritz* method [26]:

Let $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_N \in H_0^1(\Omega)$ denote linearly independent trial functions, for example approximate eigenfunctions obtained by numerical means, and form the matrices

$$A_1 := \bigl(\langle\tilde{\varphi}_i, \tilde{\varphi}_j\rangle_{H_0^1}\bigr)_{i,j=1,\ldots,N}, \qquad A_0 := \left(\int_\Omega W_\lambda \tilde{\varphi}_i \tilde{\varphi}_j\, dx\right)_{i,j=1,\ldots,N}.$$

Then, with $\Lambda_1 \leqslant \cdots \leqslant \Lambda_N$ denoting the eigenvalues of the matrix eigenvalue problem

$$A_1 x = \Lambda A_0 x$$

(which can be enclosed by means of verifying numerical linear algebra; see [3]), the Rayleigh–Ritz method gives

$$\kappa_i \leqslant \Lambda_i \quad \text{for } i = 1, \ldots, N.$$

However, also *lower* eigenvalue bounds are needed, which constitute a more complicated task than upper bounds. The most accurate method for this purpose has been proposed by Lehmann [16], and improved by Goerisch concerning its range of applicability [4]. Its numerical core is again (as in the Rayleigh–Ritz method) a matrix eigenvalue problem, but the accompanying analysis is more involved. In particular, in order to compute lower bounds to the first N eigenvalues, a *rough* lower bound to the $(N+1)$-st eigenvalue must be known already. This a priori information can usually be obtained via a *homotopy method* connecting a simple “base problem” with known eigenvalues to the given eigenvalue problem, such that all eigenvalues increase (index-wise) along the homotopy; see [21] or [5] for details on this method, a detailed description of which would be beyond the scope of this article. In fact, [5] contains the newest version of the homotopy method, where only very small (2×2 or even 1×1) matrix eigenvalue problems need to be treated rigorously in the course of the homotopy.

Finding a base problem for problem (6.10), and a suitable homotopy connecting them, is rather simple here since Ω is a bounded rectangle, whence the eigenvalues of $-\Delta$ on $H_0^1(\Omega)$ are known: We choose a constant upper bound c_0 for $|\omega_\lambda|^{p-2}\omega_\lambda$ on Ω, and the coefficient homotopy

$$W_\lambda^{(s)}(x) := \sigma + \lambda + p\big[(1-s)c_0 + s\big|\omega_\lambda(x)\big|^{p-2}\omega_\lambda(x)\big] \quad (x \in \Omega,\ 0 \leqslant s \leqslant 1).$$

Then, the family of eigenvalue problems

$$-\Delta u + \sigma u = \kappa^{(s)} W_\lambda^{(s)} u$$

connects the explicitly solvable constant-coefficient base problem ($s = 0$) to problem (6.10) ($s = 1$), and the eigenvalues increase in s, since the Rayleigh quotient does, by Poincaré's min–max principle.

7. Computation of branches (ω_λ), (δ_λ), (K_λ), (α_λ)

In the previous section we described how to compute approximations ω_λ for a grid of finitely many values of λ within $[0, \lambda_1(\Omega))$. After selecting $\bar{\lambda}$ (among these) according to Section 5, we are left with a grid

$$0 = \lambda^0 < \lambda^1 < \cdots < \lambda^M = \bar{\lambda}$$

and approximate solutions $\omega^i = \omega_{\lambda^i} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ $(i = 0, \ldots, M)$. Furthermore, according to the methods described in the previous section, we can compute bounds $\delta^i = \delta_{\lambda^i}$ and $K^i = K_{\lambda^i}$ such that (3.1) and (3.2) hold at $\lambda = \lambda^i$.

Now we define a piecewise linear (and continuous) approximate solution branch $([0, \bar{\lambda}] \to H_0^1(\Omega), \lambda \mapsto \omega_\lambda)$ by

$$\omega_\lambda := \frac{\lambda^i - \lambda}{\lambda^i - \lambda^{i-1}}\omega^{i-1} + \frac{\lambda - \lambda^{i-1}}{\lambda^i - \lambda^{i-1}}\omega^i \quad \big(\lambda^{i-1} < \lambda < \lambda^i,\ i = 1, \ldots, M\big). \tag{7.1}$$

To compute corresponding defect bounds δ_λ, we fix $i \in \{1, \ldots, M\}$ and $\lambda \in [\lambda^{i-1}, \lambda^i]$, and let $t := (\lambda - \lambda^{i-1})/(\lambda^i - \lambda^{i-1}) \in [0, 1]$, whence

$$\lambda = (1-t)\lambda^{i-1} + t\lambda^i, \qquad \omega_\lambda = (1-t)\omega^{i-1} + t\omega^i. \tag{7.2}$$

Using the classical linear interpolation error bound we obtain, for fixed $x \in \Omega$,

$$\begin{aligned}
&\big|\big|\omega_\lambda(x)\big|^p - \big[(1-t)\big|\omega^{i-1}(x)\big|^p + t\big|\omega^i(x)\big|^p\big]\big| \\
&\quad \leqslant \frac{1}{2}\max_{s\in[0,1]}\left|\frac{d^2}{ds^2}\big|(1-s)\omega^{i-1}(x) + s\omega^i(x)\big|^p\right| \cdot t(1-t) \\
&\quad \leqslant \frac{1}{8}p(p-1)\max_{s\in[0,1]}\big|(1-s)\omega^{i-1}(x) + s\omega^i(x)\big|^{p-2} \cdot \big(\omega^i(x) - \omega^{i-1}(x)\big)^2 \\
&\quad \leqslant \frac{1}{8}p(p-1)\max\big\{\big\|\omega^{i-1}\big\|_\infty, \big\|\omega^i\big\|_\infty\big\}^{p-2}\big\|\omega^i - \omega^{i-1}\big\|_\infty^2,
\end{aligned} \tag{7.3}$$

$$\begin{aligned}
&\left|\lambda\omega_\lambda(x) - \left[(1-t)\lambda^{i-1}\omega^{i-1}(x) + t\lambda^i\omega^i(x)\right]\right| \\
&\quad \leqslant \frac{1}{2}\max_{s\in[0,1]}\left|\frac{d^2}{ds^2}\left[\left((1-s)\lambda^{i-1}+s\lambda^i\right)\left((1-s)\omega^{i-1}(x)+s\omega^i(x)\right)\right]\right|\cdot t(1-t) \\
&\quad \leqslant \frac{1}{4}\left(\lambda^i-\lambda^{i-1}\right)\left\|\omega^i-\omega^{i-1}\right\|_\infty.
\end{aligned} \tag{7.4}$$

Since $\|u\|_{H^{-1}} \leqslant C_1\|u\|_\infty$ for all $u \in L^\infty(\Omega)$, with C_1 denoting an embedding constant for the embedding $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$ (e.g. $C_1 = \sqrt{|\Omega|}C_2$), (7.3) and (7.4) imply

$$\begin{aligned}
&\left\| |\omega_\lambda|^p - \left[(1-t)\left|\omega^{i-1}\right|^p + t\left|\omega^i\right|^p\right]\right\|_{H^{-1}} \\
&\quad \leqslant \frac{1}{8}p(p-1)C_1\max\left\{\left\|\omega^{i-1}\right\|_\infty, \left\|\omega^i\right\|_\infty\right\}^{p-2}\left\|\omega^i-\omega^{i-1}\right\|_\infty^2 =: \rho_i,
\end{aligned} \tag{7.5}$$

$$\left\|\lambda\omega_\lambda - \left[(1-t)\lambda^{i-1}\omega^{i-1} + t\lambda^i\omega^i\right]\right\|_{H^{-1}} \leqslant \frac{1}{4}C_1\left(\lambda^i-\lambda^{i-1}\right)\left\|\omega^i-\omega^{i-1}\right\|_\infty =: \tau_i. \tag{7.6}$$

Now (7.2), (7.5), (7.6) give

$$\begin{aligned}
&\left\|-\Delta\omega_\lambda - \lambda\omega_\lambda - |\omega_\lambda|^p\right\|_{H^{-1}} \\
&\quad \leqslant (1-t)\left\|-\Delta\omega^{i-1} - \lambda^{i-1}\omega^{i-1} - \left|\omega^{i-1}\right|^p\right\|_{H^{-1}} + t\left\|-\Delta\omega^i - \lambda^i\omega^i - \left|\omega^i\right|^p\right\|_{H^{-1}} + \tau_i + \rho_i \\
&\quad \leqslant \max\left\{\delta^{i-1}, \delta^i\right\} + \tau_i + \rho_i =: \delta_\lambda.
\end{aligned} \tag{7.7}$$

Thus, we obtain a branch $(\delta_\lambda)_{\lambda\in[0,\bar\lambda]}$ of defect bounds which is constant on each subinterval $[\lambda^{i-1}, \lambda^i]$. In the points $\lambda^1, \ldots, \lambda^{M-1}$, δ_λ is possibly doubly defined by (7.7), in which case we choose the smaller of the two values. Hence, $([0,\bar\lambda] \to \mathbb{R}, \lambda \mapsto \delta_\lambda)$ is lower semi-continuous.

Note that δ_λ given by (7.7) is "small" if δ^{i-1} and δ^i are small (i.e. if the approximations ω^{i-1} and ω^i have been computed with sufficient accuracy; see Remark 3.1(a)) *and* if ρ_i, τ_i are small (i.e. if the grid is chosen sufficiently fine; see (7.5), (7.6)).

In order to compute bounds K_λ satisfying (3.2) for $\lambda \in [0, \bar\lambda]$, with ω_λ given by (7.1), we fix $i \in \{1, \ldots, M-1\}$ and $\lambda \in [\frac{1}{2}(\lambda^{i-1}+\lambda^i), \frac{1}{2}(\lambda^i+\lambda^{i+1})]$. Then,

$$\begin{aligned}
\left|\lambda-\lambda^i\right| &\leqslant \frac{1}{2}\max\left\{\lambda^i-\lambda^{i-1}, \lambda^{i+1}-\lambda^i\right\} =: \mu_i, \\
\left\|\omega_\lambda-\omega^i\right\|_{H_0^1} &\leqslant \frac{1}{2}\max\left\{\left\|\omega^i-\omega^{i-1}\right\|_{H_0^1}, \left\|\omega^{i+1}-\omega^i\right\|_{H_0^1}\right\} =: \nu_i,
\end{aligned} \tag{7.8}$$

whence Lemma 3.2, applied for $(\tilde\lambda, \tilde u) := (\lambda^i, \omega^i)$ and $u := \omega_\lambda$, implies: If

$$\kappa_i := K^i\left[\frac{1}{\lambda_1(\Omega)+\sigma}\mu_i + 2\gamma\left(\left\|\omega^i\right\|_{L^{p+1}} + C_{p+1}\nu_i\right)^{p-2}\nu_i\right] < 1, \tag{7.9}$$

then (3.2) holds for

$$K_\lambda := \frac{K^i}{1-\kappa_i}. \tag{7.10}$$

Note that (7.9) is indeed satisfied if the grid is chosen sufficiently fine, since then μ_i and ν_i are "small" by (7.8).

Analogous estimates give K_λ also on the two remaining half-intervals $[0, \frac{1}{2}\lambda^1]$ and $[\frac{1}{2}(\lambda^{M-1} + \lambda^M), \lambda^M]$.

Choosing again the smaller of the two values at the points $\frac{1}{2}(\lambda^{i-1} + \lambda^i)$ $(i = 1, \dots, M)$ where K_λ is possibly doubly defined by (7.10), we obtain a lower semi-continuous, piecewise constant branch $([0, \bar{\lambda}] \to \mathbb{R}, \lambda \mapsto K_\lambda)$.

According to the above construction, both $\lambda \mapsto \delta_\lambda$ and $\lambda \mapsto K_\lambda$ are constant on the $2M$ half-intervals. Moreover, (7.1) implies that, for $i = 1, \dots, M$,

$$\|\omega_\lambda\|_{L^{p+1}} \leqslant \left\{ \begin{array}{l} \max\{\|\omega^{i-1}\|_{L^{p+1}}, \frac{1}{2}(\|\omega^{i-1}\|_{L^{p+1}} + \|\omega^i\|_{L^{p+1}})\} \text{ for } \lambda \in [\lambda^{i-1}, \frac{1}{2}(\lambda^{i-1} + \lambda^i)] \\ \max\{\frac{1}{2}(\|\omega^{i-1}\|_{L^{p+1}} + \|\omega^i\|_{L^{p+1}}), \|\omega^i\|_{L^{p+1}}\} \text{ for } \lambda \in [\frac{1}{2}(\lambda^{i-1} + \lambda^i), \lambda^i] \end{array} \right\}$$

and again we choose the smaller of the two values at the points of double definition.

Using these bounds, the crucial inequalities (3.4) and (3.5) (which have to be satisfied for all $\lambda \in [0, \bar{\lambda}]$) result in *finitely* many inequalities which can be fulfilled with "small" and piecewise constant α_λ if δ_λ is sufficiently small, i.e. if $\omega^0, \dots, \omega^M$ have been computed with sufficient accuracy (see Remark 3.1(a)) and if the grid has been chosen sufficiently fine (see (7.5)–(7.7)). Moreover, since $\lambda \mapsto \delta_\lambda$, $\lambda \mapsto K_\lambda$ and the above piecewise constant upper bound for $\|\omega_\lambda\|_{L^{p+1}}$ are lower semi-continuous, the structure of the inequalities (3.4) and (3.5) clearly shows that also $\lambda \mapsto \alpha_\lambda$ can be chosen to be lower semi-continuous, as required in Section 4. Finally, since (3.5) now consists in fact of *finitely* many strict inequalities, a uniform (λ-independent) $\eta > 0$ can be chosen in Theorem 3.1(b), as needed for Theorem 4.1.

8. Numerical results

We carried out the computer-assisted proof explained in the previous sections for the case $p = 2$ with full mathematical rigor, and for the case $p = 3$ using quadrature approximations to various integrals needed. Thus, we give a mathematical proof for $p = 2$ only, but the lack of rigor in the case $p = 3$ occurs on a rather "technical" level only.

All computations have been performed on an Intel Core 2 Duo T 7300 (2 GHz) and on an Intel Pentium M (1.86 GHz) processor, using MATLAB (versions 7.1 and 7.4, resp.) and the interval toolbox INTLAB [27]. Our source code can be found on our webpage.[1]

In the following, we first report on some more detailed numerical results for the (completely verified) case $p = 2$.

Fig. 1 shows an approximate branch $[0, 2\pi^2) \to \mathbb{R}$, $\lambda \mapsto \|\omega_\lambda\|_\infty$. (The continuous plot has been created by interpolation of grid points.)

Using $\bar{\lambda} = 18.5$ (which is not the minimally possible choice 13.1) and $M + 1 = 94$ values $0 = \lambda^0 < \lambda^1 < \dots < \lambda^{93} = 18.5$ (with $\lambda^1 = 0.1, \lambda^2 = 0.3$ and the remaining grid points equally spaced with distance 0.2) we computed approximations $\omega^0, \dots, \omega^{93}$ with $N = 15$ in (6.2), as well as defect bounds $\delta^0, \dots, \delta^{93}$ and constants $K^0, \dots, K^{93}$, by the methods described in Section 6.

For some selected values of λ, Table 1 shows, with an obvious sub- and superscript notation for enclosing intervals, the eigenvalue bounds for problem (6.10) (giving K_λ by (6.11)), which were computed using the Rayleigh–Ritz and the Lehmann–Goerisch method, and the homotopy method briefly mentioned at the end of Section 6. The homotopy is illustrated, for the single value $\lambda = 0$, in Fig. 2.

Table 1
Eigenvalue enclosures for the first two eigenvalues.

	ω_0	$\omega_{2.7}$	$\omega_{6.7}$	$\omega_{10.7}$	$\omega_{14.7}$	$\omega_{18.5}$
κ_1	0.51204_{011}^{304}	0.54795_{075}^{375}	0.6123_{196}^{226}	0.6947_{581}^{607}	0.80347_{300}^{444}	0.943494_{808}^{991}
κ_2	1.5558_{173}^{925}	1.630_{6362}^{7122}	1.761_{0959}^{1681}	1.9203_{291}^{862}	2.1158_{316}^{624}	2.34331_{332}^{763}

[1] http://www.mathematik.uni-karlsruhe.de/mi1plum/~roth/page/publ/en.

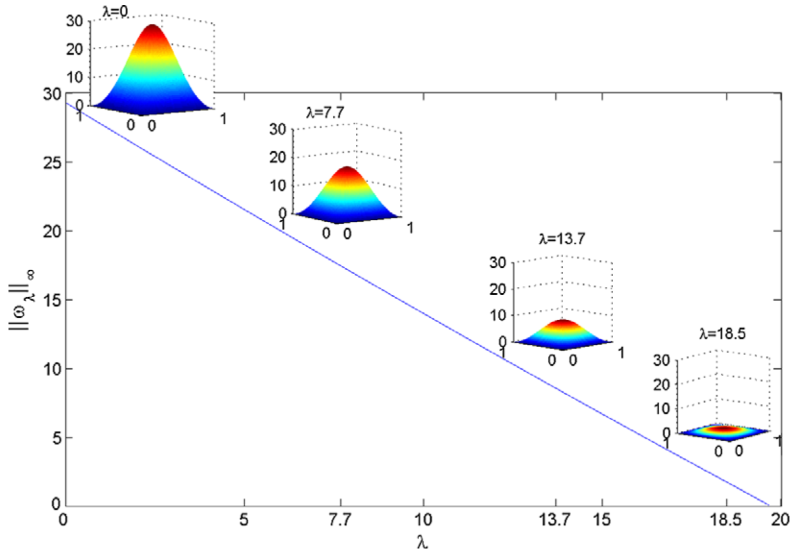

Fig. 1. Curve $(\lambda, \|\omega_\lambda\|_\infty)$ with samples of ω_λ in the case $p = 2$.

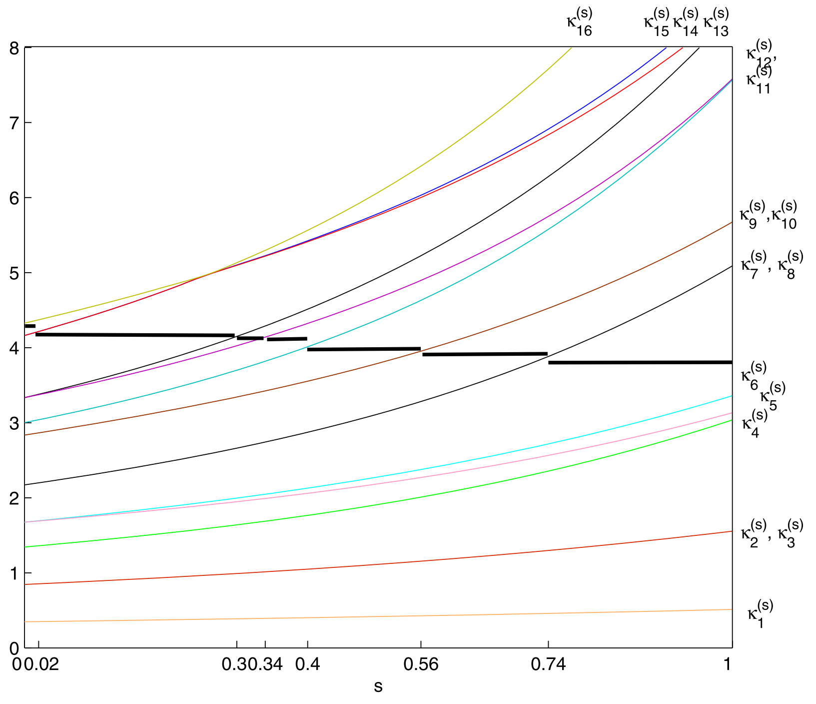

Fig. 2.

Table 2 contains, for some selected of the 186 λ-half-intervals,

(a) the defect bounds δ_λ obtained by (7.7) from the grid-point defect bounds δ^{i-1}, δ^i, and from the grid-width characteristics ρ_i, τ_i defined in (7.5), (7.6),

Table 2

λ-interval	δ_λ	K_λ	α_λ
[0, 0.05)	0.0515864	2.9006274	0.1518597
(2, 2.1)	0.0512280	2.8159871	0.1462628
(6, 6.1)	0.0367367	2.4988079	0.0925099
(10, 10.1)	0.0238838	2.2558125	0.0540970
(14, 14.1)	0.0138680	4.0259597	0.0562560
(16, 16.1)	0.0103536	6.6289293	0.0697054
(18.4, 18.5]	0.0077985	29.2929778	0.3435319

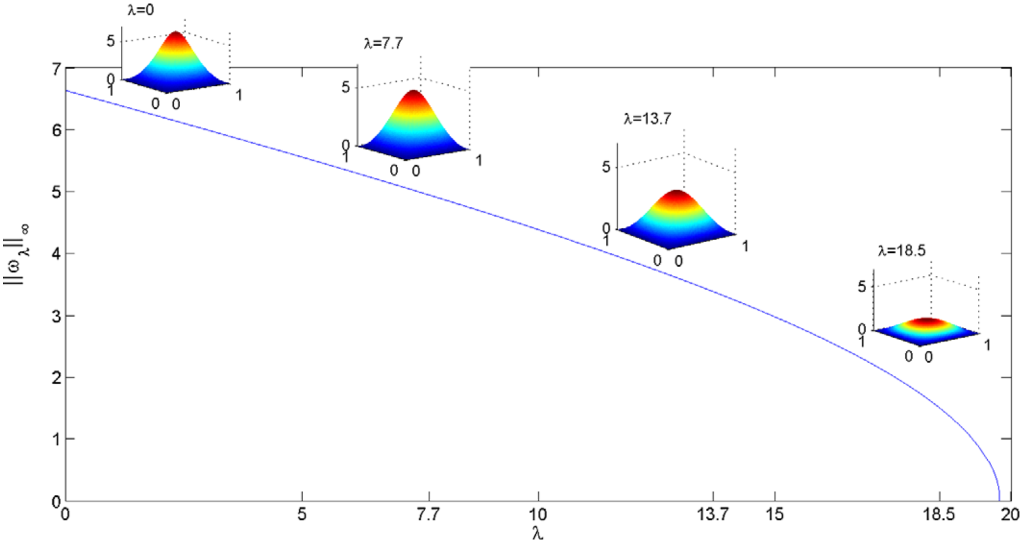

Fig. 3. Curve $(\lambda, \|\omega_\lambda\|_\infty)$ with samples of ω_λ in the case $p = 3$.

Table 3

λ-interval	δ_λ	K_λ	α_λ
[0, 0.05)	0.0142569	4.7672700	0.0949471
(2, 2.1)	0.0098265	4.2643990	0.0476681
(6, 6.1)	0.0039591	3.3147384	0.0134391
(10, 10.1)	0.0025861	2.6513609	0.0069136
(14, 14.1)	0.0027013	2.1806280	0.0059173
(16, 16.1)	0.0031378	2.8967269	0.0091579
(18.4, 18.5]	0.0049595	9.2818185	0.0499900

(b) the constants K_λ obtained by (7.10) from the grid-point constants K^i and the grid-width parameters ν_i defined in (7.8) (note that $\mu_i = 0.1$ for all i),
(c) the error bounds α_λ computed according to (3.4), (3.5).

For the case $p = 3$, an approximate solution branch is displayed in Fig. 3. We carried out the computations described before also here, but—as mentioned above—using quadrature approximations to the integrals needed to compute the bounds δ^i and the eigenvalue bounds (giving K^i). With these non-verified bounds (which nevertheless are very likely "almost" correct), Table 3 shows the terms described for Table 2 already.

Acknowledgment

The authors are grateful to the anonymous referee for his helpful remarks and suggestions.

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