Note
On covering intersecting set-systems by digraphs

László Szegő

Department of Operations Research, Eötvös University, Kecskeméti utca 10–12., Budapest, H-1053, Hungary

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Abstract

We establish a common generalization of a theorem of Edmonds on the number of disjoint branchings and a theorem of Frank on kernel systems. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let \( \tilde{G} = (V, \tilde{E}) \) be a directed graph. For any \( R, \emptyset \neq R \subseteq V \), a branching \( B \) of \( \tilde{G} \), rooted at \( R \), is a subgraph of \( \tilde{G} \) such that for every node \( v \in V(B) \) there is exactly one directed path in \( B \) from a node in \( R \) to \( v \). A component of a branching is called an arborescence, if it is rooted at the node \( s \), we call it an \( s \)-arborescence. Edmonds in [1] proved the following theorem. \( \delta(X) \) denotes the number of edges that leave \( X \), \( \varrho(X) \) denotes the number of edges that enter \( X \).

Theorem 1.1. For any graph \( \tilde{G} \) and any sets \( R_i, \emptyset \neq R_i \subseteq V, i = 1, 2, \ldots, k \), there exist mutually edge-disjoint branchings \( B_i, i = 1, 2, \ldots, k \), of \( \tilde{G} \) rooted, respectively, at \( R_i \) if and only if

\[
\delta(X) \geq |\{i \in \{1, 2, \ldots, k\}: R_i \subseteq X\}| \quad \text{for all } X \subseteq V.
\]

Frank remarked [3] that the above theorem is equivalent to the following:
Theorem 1.2. Let $\tilde{G}=(V+s,\tilde{E})$ be a directed graph and $F_1,F_2,\ldots,F_k$ be $k$ edge-disjoint $s$-arborescences. They can be completed to $k$ edge-disjoint spanning $s$-arborescences if and only if

$$q'(X) \geq p(X) \text{ for all } X \subseteq V,$$

where $q'(X)$ denotes the number of edges entering $X$ not in any of the arborescences, and $p(X)$ denotes the number of the arborescences disjoint from $X$.

An interesting special case is the following:

Theorem 1.3. Let $\tilde{G}=(V+s,\tilde{E})$ be a directed graph. It has $k$ edge-disjoint spanning $s$-arborescences if and only if

$$q(X) \geq k \text{ for all } X \subseteq V.$$ 

Frank in [2] introduced the notion of kernel system. The family $\mathcal{F}$ of subsets of $V$ is called a kernel system with respect to $\tilde{G}$ if

- $q(F)>0$ for all $F \in \mathcal{F}$,
- if $F_1,F_2 \in \mathcal{F}$ and $F_1 \cap F_2 \neq \emptyset$ then $F_1 \cap F_2$ and $F_1 \cup F_2 \in \mathcal{F}$.

We say that $R \subseteq \tilde{E}$ covers $\mathcal{F}$ if $R$ contains at least one edge of $q(F)$ for every $F \in \mathcal{F}$. Frank proved the following theorem:

Theorem 1.4. Let $\tilde{G}=(V+s,\tilde{E})$ be a directed graph and $\mathcal{F}$ a kernel system. There exists a partition $R_1,R_2,\ldots,R_k$ of $\tilde{E}$ such that $R_i$ covers $\mathcal{F}$ for all $i=1,2,\ldots,k$ if and only if

$$q(X) \geq k \text{ for all } X \in \mathcal{F}.$$ 

If we set $\mathcal{F}=2^V$, then we get Theorem 1.3. In this note we give a common generalization of Theorems 1.2 and 1.4. Our proof is similar to Lovász’s in [4].

2. Covering intersecting set-systems

Let $\tilde{G}=(V+s,\tilde{E})$ be a directed graph and $\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_k$ be set-systems on the ground-set $V$ with the following two properties:

- $F_1,F_2 \in \mathcal{F}_i$, $F_1 \cap F_2 \neq \emptyset \Rightarrow F_1 \cap F_2$ and $F_1 \cup F_2 \in \mathcal{F}_i$,
- $F_1 \in \mathcal{F}_i$, $F_2 \in \mathcal{F}_j$ and $F_1 \cap F_2 \neq \emptyset \Rightarrow F_1 \cap F_2 \in \mathcal{F}_i \cap \mathcal{F}_j$.

If the first property is true for a set-system we call it intersecting. The second property will be referred to as the linking property. Let $X \subseteq V$, then $p(X)$ denotes
the number of the above set-systems which contain $X$. The following lemma is an immediate corollary of the linking property.

**Lemma 2.1.** If $X \in \mathcal{F}_{i_1}$ and $Y \in \mathcal{F}_{i_2}$ for some $i_1$ and $i_2$ and $X \cap Y \neq \emptyset$, then $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$. Moreover, equality holds if and only if $X \cap Y \in \mathcal{F}_{i}$ implies that $X$ or $Y \in \mathcal{F}_{i}$.

**Theorem 2.2.** There exists a partition $B_1, B_2, \ldots, B_k$ of $\tilde{E}$ such that $B_i$ covers $\mathcal{F}_i$ for all $i = 1, 2, \ldots, k$ if and only if

\[ \phi(X) \geq p(X) \quad \text{for all } X \subseteq V. \tag{1} \]

**Proof.** The necessity of (1) is immediate. The sufficiency is proved by induction on $\sum_i |\mathcal{F}_i|$. We can assume that $\mathcal{F}_1$ is not empty. Let us consider a maximal member $F_1$ of $\mathcal{F}_1$. By (1) there exists an edge $e$ entering $F_1$.

Let $\mathcal{F}_1' = \{ X \in \mathcal{F}_1 : e \text{ does not cover } X \}$. Clearly, $\mathcal{F}_1'$ is intersecting and $\mathcal{F}_1', \mathcal{F}_2, \ldots, \mathcal{F}_k$ satisfy the linking property. If not, then there exist $A \in \mathcal{F}_1'$, $B \in \mathcal{F}_j$ $(j \neq 1)$ such that $A \cap B \in \mathcal{F}_1 - \mathcal{F}_1'$, so $F_1 \cup A \in \mathcal{F}_1$ contradicts the maximality of $F_1$.

We call a subset $X \subseteq V$ tight if $\phi(X) = p(X) > 0$ and $X \notin \mathcal{F}_1$. If after deleting $e$ from $\tilde{G}$ the condition (1) holds, then we are done by the induction. If not, then $e$ enters a tight set.

Let us consider a minimal tight set $X$ which intersects $F_1$. (Such a set exists because the head of edge $e$ is in $F_1$, and $X - F_1$ is not empty because of the linking property and the fact that $X \notin \mathcal{F}_1$. There exists an edge $f$ from $X - F_1$ to $F_1 \cap X$ because of the linking property and (1). We claim that $f$ does not enter any tight set and so we are done. Suppose to the contrary that $f$ enters a tight set $Y$, then by (1) and the submodularity of the in-degree function $\phi$:

\[ p(X) + p(Y) = \phi(X) + \phi(Y) \geq \phi(X \cap Y) + \phi(X \cup Y) \geq p(X \cap Y) + p(X \cup Y). \]

So, equality holds everywhere and, by the lemma, $X \cap Y$ is a tight set and is smaller than $X$, a contradiction.

The above theorem implies the theorems of the introduction. In the case of Theorem 1.2 let $\mathcal{F}_i$ be the family of all the subsets which are disjoint from the nodes of the $s$-arborescence $F_i$. In the case of Theorem 1.4 $\mathcal{F}_1 = \mathcal{F}_2 = \cdots = \mathcal{F}_k = \mathcal{F}$.

**References**