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Note

On covering intersecting set-systems by digraphs ☆

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Abstract

We establish a common generalization of a theorem of Edmonds on the number of disjoint branchings and a theorem of Frank on kernel systems. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\vec{G}=(V,\vec{E})$ be a directed graph. For any R, $\emptyset \neq R \subseteq V$, a *branching* B of $\vec{G}$, *rooted* at R, is a subgraph of $\vec{G}$ such that for every node $v \in V(B)$ there is exactly one directed path in B from a node in R to v. A component of a branching is called an *arborescence*, if it is rooted at the node s, we call it an s-arborescence. Edmonds in [1] proved the following theorem. $\delta(X)$ denotes the number of edges that leave X, $\varrho(X)$ denotes the number of edges that enter X.

Theorem 1.1. *For any graph $\vec{G}$ and any sets R_i, $\emptyset \neq R_i \subseteq V$, $i=1,2,\ldots,k$, there exist mutually edge-disjoint branchings B_i, $i=1,2,\ldots,k$, of $\vec{G}$ rooted, respectively, at R_i if and only if*

$$\delta(X) \geqslant |\{i \in \{1,2,\ldots,k\}\colon R_i \subseteq X\}| \quad \text{for all } X \subset V.$$

Frank remarked [3] that the above theorem is equivalent to the following:

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Theorem 1.2. *Let $\vec{G}=(V+s,\vec{E})$ be a directed graph and $F_1,F_2,\ldots,F_k$ be k edge-disjoint s-arborescences. They can be completed to k edge-disjoint spanning s-arborescences if and only if*

$$\varrho'(X)\geqslant p(X) \quad \textit{for all } X\subseteq V,$$

where $\varrho'(X)$ denotes the number of edges entering X not in any of the arborescences, and $p(X)$ denotes the number of the arborescences disjoint from X.

An interesting special case is the following:

Theorem 1.3. *Let $\vec{G}=(V+s,\vec{E})$ be a directed graph. It has k edge-disjoint spanning s-arborescences if and only if*

$$\varrho(X)\geqslant k \quad \textit{for all } X\subseteq V.$$

Frank in [2] introduced the notion of *kernel system*. The family $\mathscr{F}$ of subsets of V is called a kernel system with respect to $\vec{G}$ if

- $\varrho(F)>0$ for all $F\in\mathscr{F}$,
- if $F_1,F_2\in\mathscr{F}$ and $F_1\cap F_2\neq\emptyset$ then $F_1\cap F_2$ and $F_1\cup F_2\in\mathscr{F}$.

We say that $R\subseteq\vec{E}$ covers $\mathscr{F}$ if R contains at least one edge of $\varrho(F)$ for every $F\in\mathscr{F}$. Frank proved the following theorem:

Theorem 1.4. *Let $\vec{G}=(V+s,\vec{E})$ be a directed graph and $\mathscr{F}$ a kernel system. There exists a partition $R_1,R_2,\ldots,R_k$ of $\vec{E}$ such that R_i covers $\mathscr{F}$ for all $i=1,2,\ldots,k$ if and only if*

$$\varrho(X)\geqslant k \quad \textit{for all } X\in\mathscr{F}.$$

If we set $\mathscr{F}=2^V$, then we get Theorem 1.3. In this note we give a common generalization of Theorems 1.2 and 1.4. Our proof is similar to Lovász's in [4].

2. Covering intersecting set-systems

Let $\vec{G}=(V+s,\vec{E})$ be a directed graph and $\mathscr{F}_1,\mathscr{F}_2,\ldots,\mathscr{F}_k$ be set-systems on the ground-set V with the following two properties:

- $F_1,F_2\in\mathscr{F}_i,\ F_1\cap F_2\neq\emptyset\Rightarrow F_1\cap F_2$ and $F_1\cup F_2\in\mathscr{F}_i$,
- $F_1\in\mathscr{F}_i,\ F_2\in\mathscr{F}_j$ and $F_1\cap F_2\neq\emptyset\Rightarrow F_1\cap F_2\in\mathscr{F}_i\cap\mathscr{F}_j$.

If the first property is true for a set-system we call it *intersecting*. The second property will be referred to as the *linking property*. Let $X\subseteq V$, then $p(X)$ denotes the number of the above set-systems which contain X. The following lemma is an immediate corollary of the linking property.

Lemma 2.1. *If $X \in \mathscr{F}_{i_1}$ and $Y \in \mathscr{F}_{i_2}$ for some i_1 and i_2 and $X \cap Y \neq \emptyset$, then $p(X) + p(Y) \leqslant p(X \cap Y) + p(X \cup Y)$. Moreover, equality holds if and only if $X \cap Y \in \mathscr{F}_i$ implies that X or $Y \in \mathscr{F}_i$.* □

Theorem 2.2. *There exists a partition $B_1, B_2, \ldots, B_k$ of $\vec{E}$ such that B_i covers $\mathscr{F}_i$ for all $i = 1, 2, \ldots, k$ if and only if*

$$\varrho(X) \geqslant p(X) \quad \text{for all } X \subseteq V. \tag{1}$$

Proof. The necessity of (1) is immediate. The sufficiency is proved by induction on $\sum_i |\mathscr{F}_i|$. We can assume that $\mathscr{F}_1$ is not empty. Let us consider a maximal member F_1 of $\mathscr{F}_1$. By (1) there exists an edge e entering F_1.

Let $\mathscr{F}'_1 = \{X \in \mathscr{F}_1$: e does not cover $X\}$. Clearly, $\mathscr{F}'_1$ is intersecting and $\mathscr{F}'_1, \mathscr{F}_2, \ldots, \mathscr{F}_k$ satisfy the linking property. If not, then there exist $A \in \mathscr{F}'_1$, $B \in \mathscr{F}_j$ $(j \neq 1)$ such that $A \cap B \in \mathscr{F}_1 - \mathscr{F}'_1$, so $F_1 \cup A \in \mathscr{F}_1$ contradicts the maximality of F_1.

We call a subset $X \subseteq V$ *tight* if $\varrho(X) = p(X) > 0$ and $X \notin \mathscr{F}_1$. If after deleting e from $\vec{G}$ the condition (1) holds, then we are done by the induction. If not, then e enters a tight set.

Let us consider a minimal tight set X which intersects F_1. (Such a set exists because the head of edge e is in F_1.) $X - F_1$ is not empty because of the linking property and the fact that $X \notin \mathscr{F}_1$. There exists an edge f from $X - F_1$ to $F_1 \cap X$ because of the linking property and (1). We claim that f does not enter any tight set and so we are done. Suppose to the contrary that f enters a tight set Y, then by (1) and the submodularity of the in-degree function ϱ:

$$p(X) + p(Y) = \varrho(X) + \varrho(Y) \geqslant \varrho(X \cap Y) + \varrho(X \cup Y) \geqslant p(X \cap Y) + p(X \cup Y).$$

So, equality holds everywhere and, by the lemma, $X \cap Y$ is a tight set and is smaller than X, a contradiction. □

The above theorem implies the theorems of the introduction. In the case of Theorem 1.2 let $\mathscr{F}_i$ be the family of all the subsets which are disjoint from the nodes of the s-arborescence F_i. In the case of Theorem 1.4 $\mathscr{F}_1 = \mathscr{F}_2 = \cdots = \mathscr{F}_k = \mathscr{F}$.

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