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Koszul differential graded algebras and BGG correspondence

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Abstract

The concept of Koszul differential graded algebra (Koszul DG algebra) is introduced. Koszul DG algebras exist extensively, and have nice properties similar to the classic Koszul algebras. A DG version of the Koszul duality is proved. When the Koszul DG algebra A is AS-regular, the Ext-algebra E of A is Frobenius. In this case, similar to the classical BGG correspondence, there is an equivalence between the stable category of finitely generated left E-modules, and the quotient triangulated category of the full triangulated subcategory of the derived category of right DG A-modules consisting of all compact DG modules modulo the full triangulated subcategory consisting of all the right DG modules with finite dimensional cohomology. The classical BGG correspondence can be derived from the DG version.

Keywords: Differential graded algebra; Derived category; Koszul algebra; BGG correspondence

Introduction

In his book [Ma] Manin presented an open question: How to generalize the Koszulity to differential graded (DG for short) algebras? Attempts have been made by several authors as in [PP] and [Be]. In their terminology, a DG algebra is said to be Koszul if the underlying graded algebra is Koszul. Koszul DG algebras in their sense are applied to discuss configuration spaces.

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In this paper, we take a different point of view. Let k be a field. A connected DG algebra over k is a positively graded k-algebra $A = \bigoplus_{n \geqslant 0} A^n$ with $A^0 = k$ such that there is a differential $d : A \to A$ of degree 1 which is also a graded derivation. A connected DG algebra A is said to be a *Koszul DG algebra* if the minimal semifree resolution of the trivial DG module ${}_A k$ has a semifree basis consisting of homogeneous elements of degree zero (Definition 2.1). Our definition of Koszul DG algebra is a natural generalization of the usual Koszul algebra. As we will see in Section 2, a connected graded algebra regarded as a DG algebra with zero differential is a Koszul DG algebra if and only if it is a Koszul algebra in the usual sense. Examples of Koszul DG algebras can be found in various fields. For example, let M be a connected n-dimensional C^∞ manifold, and let $(\mathcal{A}^*(M) = \bigoplus_{i=0}^n \mathcal{A}^i(M), d)$ be the de Rham complex of M, then $(\mathcal{A}^*(M), d)$ is a commutative DG algebra and by de Rham theorem [M] the 0th cohomology group $H^0(\mathcal{A}^*(M)) \cong \mathbb{R}$. Hence the DG algebra $\mathcal{A}^*(M)$ has a minimal model A [KM] or Sullivan model [FHT2], which is certainly a connected DG algebra. If the manifold M has some further properties (e.g., $M = T^n$ the n-dimensional torus), then the de Rham cohomology algebra $H(\mathcal{A}^*(M))$ is a Koszul algebra. Hence the cohomology algebra of its minimal model (or Sullivan model) A is Koszul as A is quasi-isomorphic to $\mathcal{A}^*(M)$. Then A is a Koszul DG algebra by Proposition 2.3. More examples of Koszul DG algebra will be given in Section 2. In fact, we will see that any Koszul algebra can be viewed as the cohomology algebra of some Koszul DG algebra.

Bernstein–Gelfand–Gelfand in [BGG] established an equivalence between the stable category of finitely generated graded modules over the exterior algebra $\bigwedge V$ with $V = kx_0 \oplus kx_1 \oplus \cdots \oplus kx_n$, and the bounded derived category of coherent sheaves on the projective space $\mathbb{P}^n$. This equivalence is now called the *BGG correspondence*. BGG correspondence has been generalized to noncommutative projective geometry by several authors. Let R be a (noncommutative) Koszul algebra. If R is AS-regular, Jørgensen proved in [Jo] that there is an equivalence between the stable category over the graded Frobenius algebra $E(R) = \mathrm{Ext}^*_R(k, k)$ and the derived category of the noncommutative analogue $\mathrm{QGr}(R)$ of the quasi-coherent sheaves over R; Martínez Villa and Saorín proved in [MS] that the stable category of the finite dimensional modules over $E(R)$ is equivalent to the bounded derived category of the noncommutative analogue $\mathrm{qgr}\, R$ of the coherent sheaves over R. Mori in [Mo] proved a similar version under a more general condition. One of our purposes in this paper is to establish a DG version of the BGG correspondence. In some special case, the DG version of the BGG correspondence coincides with the classical one as established in [BGG] and [MS].

The paper is organized as follows.

In Section 1, we give some preliminaries and fix some notations for the paper.

In Section 2, we first propose a definition for Koszul DG algebras (Definition 2.1), then give some examples and discuss some basic properties of Koszul DG algebras. For any connected DG algebra A, we prove that if the cohomology algebra $H(A)$ is Koszul in the usual sense, then A is a Koszul DG algebra (Proposition 2.3). The converse is not true in general.

In Section 3, we discuss the structure of the Ext-algebras of Koszul DG algebras. For any Koszul DG algebra A, we prove that the Ext-algebra $E = \mathrm{Ext}^*_A({}_A k, {}_A k)$ of A is an augmented, filtered algebra. Moreover, if $H(A)$ is a Koszul algebra, then the associated graded algebra $gr(E)$ is isomorphic to the dual Koszul algebra $(H(A))^!$ (Theorem 3.3). If further, ${}_A k$ is compact, then E is a finite dimensional local algebra; when $H(A)$ is Koszul, the filtration on E is exactly the Jacobson radical filtration (Theorem 3.5). Using bar and cobar constructions, we prove the following version of the Koszul duality on the Ext-algebras (Theorem 3.8):

Theorem *(Koszul duality on Ext-algebra). Let A be a Koszul DG algebra and E be its Ext-algebra. If ${}_Ak$ is compact, then* $\mathrm{Ext}_E^*({}_Ek, {}_Ek) \cong H(A)$.

As a corollary, we show that the Ext-algebra of a Koszul DG algebra A with ${}_Ak$ compact is strongly quasi-Koszul [GM] if and only if its cohomology algebra $H(A)$ is a Koszul algebra.

In Section 4, by using Lefèvre-Hasegawa's theorem in [Le, Ch. 2] (see Theorem 4.1), we establish a DG version of Koszul equivalence and duality (Theorems 4.4 and 4.7).

Theorem *(Koszul equivalence and duality). Let A be a Koszul DG algebra and E be its Ext-algebra. Suppose ${}_Ak$ is compact. Then there is an equivalence of triangulated categories between $\mathcal{D}^+(E)$ and $\mathcal{D}^+_{dg}(A^{op})$; and there is a duality of triangulated categories between $\mathcal{D}^b(\text{mod-}E^{op})$ and $\mathcal{D}^c(A^{op})$.*

Here $\mathcal{D}^+(E)$ is the derived category of bounded below cochain complexes of left E-modules; $\mathcal{D}^b(\text{mod-}E)$ (resp. $\mathcal{D}^b(\text{mod-}E^{op})$) is the bounded derived category of finitely generated left (resp. right) E-modules; $\mathcal{D}_{dg}(A^{op})$ (resp. $\mathcal{D}^+_{dg}(A^{op})$) is the derived category of right DG A-modules (resp. bounded below right DG A-modules), and $\mathcal{D}^c(A^{op})$ is the full triangulated subcategory of $\mathcal{D}^+_{dg}(A^{op})$ consisting of all the compact objects.

As a corollary, we show that each finite dimensional local algebra with residue field k can be viewed as the Ext-algebra of some Koszul DG algebra. As a result, we see that the cohomology algebra of a Koszul DG algebra may not be Koszul.

In Section 5, we introduce the concept of AS-regular DG algebra. Based on the result obtained in Section 4, we show that the Ext-algebra of an AS-regular Koszul DG algebra is Frobenius (Proposition 5.4 and Corollary 5.5). We then prove a correspondence between some quotient category of the derived category of a Koszul AS-regular DG algebra and the stable category of its Ext-algebra, which is similar to the classical BGG correspondence (Theorems 5.7 and 5.8).

Theorem *(BGG correspondence). Let A be a Koszul DG AS-regular algebra with Ext-algebra $E = \mathrm{Ext}_A^*(k,k)$. Then there is a duality of triangulated categories between $\overline{\text{mod}}\text{-}E^{op}$ and $\mathcal{D}^c(A^{op})/\mathcal{D}_{fd}(A^{op})$ and an equivalence of triangulated categories between $\overline{\text{mod}}\text{-}E$ and $\mathcal{D}^c(A^{op})/\mathcal{D}_{fd}(A^{op})$.*

Here $\overline{\text{mod}}\text{-}E^{op}$ (resp. $\overline{\text{mod}}\text{-}E$) is the stable category of finitely generated right (resp. left) E-modules. $\mathcal{D}_{fd}(A^{op})$ is the full triangulated subcategory of the derived category of right DG A-modules consisting of all the DG modules with finite dimensional cohomology.

The results above are generalized to Adams connected DG algebras in Section 6. We show that the noncommutative BGG correspondence between the triangulated categories established in [Jo] and [MS] can be deduced from the BGG correspondence on Adams connected DG algebras (Theorem 6.8).

1. Preliminaries

Throughout, k is a field and all algebras are k-algebras; unadorned $\otimes$ means $\otimes_k$ and Hom means Hom_k.

By a *graded algebra* we mean a $\mathbb{Z}$-graded algebra. An *augmented* graded algebra is a graded algebra A with an augmentation map $\varepsilon : A \to k$ which is a graded algebra morphism. A positively graded algebra $A = \bigoplus_{n \geqslant 0} A_n$ with $A_0 = k$ is called a *connected* graded algebra. Let M and N be graded A-modules. $\underline{\mathrm{Hom}}_A(M, N)$ is the set of all graded A-module morphisms. If L is a graded vector space, $L^\# = \underline{\mathrm{Hom}}(L, k)$ is the graded vector space dual.

By a (cochain) *DG algebra* we mean a graded algebra $A = \bigoplus_{n \in \mathbb{Z}} A^n$ with a differential $d : A \to A$ of degree 1, which is also a graded derivation. An *augmented DG algebra* is a DG algebra A such that the underlying graded algebra is augmented with augmentation map $\varepsilon : A \to k$ satisfying $\varepsilon \circ d = 0$. $\ker \varepsilon$ is called the augmented ideal of A. A *connected* DG algebra is a DG algebra such that the underlying graded algebra is connected. Any graded algebra can be viewed as a DG algebra with differential $d = 0$; in this case it is called a DG algebra with *trivial* differential.

Let (A, d_A) be a DG algebra. A *left differential graded module over* A (DG A-module for short) is a left graded A-module M with a differential $d_M : M \to M$ of degree 1 such that d_M satisfies the graded Leibniz rule

$$d_M(am) = d_A(a)m + (-1)^{|a|} a d_M(m)$$

for all graded elements $a \in A$, $m \in M$.

A right DG module over A is defined similarly. We denote A^{op} as the opposite DG algebra of A, whose product is defined as $a \cdot b = (-1)^{|a| \cdot |b|} ba$ for all graded elements $a, b \in A$. Right DG modules over A can be identified with DG A^{op}-modules.

Dually, by a (cochain) *DG coalgebra* we mean a graded coalgebra $C = \bigoplus_{n \in \mathbb{Z}} C^n$ with a differential $d : C \to C$ of degree 1, which is also a graded coderivation. A *coaugmented DG coalgebra* is a DG coalgebra C with a graded coalgebra map $\eta : k \longrightarrow C$, called coaugmentation map, such that $d \circ \eta = 0$. If C is a coaugmented DG coalgebra, then C has a decomposition $C = k \oplus \bar{C}$, where $\bar{C}$ is the kernel of the counit ε_C, which is isomorphic to the cokernel $\widetilde{C}$ of η. There is a coproduct $\bar{\Delta} : \bar{C} \to \bar{C} \otimes \bar{C}$ defined by $\bar{\Delta}(c) = \Delta(c) - 1 \otimes c - c \otimes 1$, such that $(\bar{C}, \bar{\Delta})$ is a coalgebra without counit. Δ induces a coproduct $\widetilde{\Delta}$ over $\widetilde{C}$. $(\bar{C}, \bar{\Delta})$ and $(\widetilde{C}, \widetilde{\Delta})$ are isomorphic as coalgebras. A coaugmented DG coalgebra C is *cocomplete* if, for any homogeneous element $x \in \bar{C}$, there is an integer n such that $\bar{\Delta}^n(x) = (\bar{\Delta} \otimes 1^{\otimes n-1}) \circ \cdots \circ (\bar{\Delta} \otimes 1) \circ \bar{\Delta}(x) = 0$. A right DG C-comodule N is a graded right C-comodule with a graded coderivation d_N (i.e. $\rho_N d_N = (d_N \otimes 1 + 1 \otimes d_C)\rho_N$) of degree 1. A cocomplete right DG C-comodule is defined similarly [Le].

For the standard facts about DG modules, semifree modules and semifree resolutions of DG modules, etc., refer to [AFH] and [FHT2]. A DG A-module M is said to be *bounded below* if $M^n = 0$ for $n \ll 0$. Let A be a DG algebra, M and N be left DG A-modules, W be a right DG A-module. Following [KM] and [We], the differential Ext and Tor are defined as

$$\mathrm{Ext}^n_A(M, N) = H^n\big(\mathrm{RHom}_A(M, N)\big) \quad \text{and} \quad \mathrm{Tor}^n_A(W, M) = H^n\big(W \otimes^L_A M\big)$$

for all $n \in \mathbb{Z}$.

Let s be the suspension map (shifting map) with $(sX)^n = X^{n-1}$ for any cochain complex X. Thus $s^i : X \to s^i X$ is of degree i for any $i \in \mathbb{Z}$.

1.1. Bar constructions

Let A be an augmented DG algebra with differential d. Let $I(A) = \cdots \oplus A^{-1} \oplus \bar{A}^0 \oplus A^1 \oplus \cdots$ be its augmented ideal. Let

$$\begin{aligned} B(A) &= T\big(s^{-1}\big(I(A)\big)\big) \\ &= k \oplus s^{-1}\big(I(A)\big) \oplus s^{-1}\big(I(A)\big) \otimes s^{-1}\big(I(A)\big) \oplus \big[s^{-1}\big(I(A)\big)\big]^{\otimes 3} \oplus \cdots. \end{aligned}$$

The homogeneous element $s^{-1}a_1 \otimes s^{-1}a_2 \otimes \cdots \otimes s^{-1}a_n$ of $B(A)$ is written as $[a_1|a_2|\cdots|a_n]$ for homogeneous elements $a_1, \ldots, a_n \in I(A)$. The coproduct

$$\Delta : B(A) \to B(A) \otimes B(A)$$

is defined by

$$\begin{aligned} \Delta\big([a_1|a_2|\cdots|a_n]\big) = 1 \otimes [a_1|a_2|\cdots|a_n] + [a_1|a_2|\cdots|a_n] \otimes 1 \\ + \sum_{1 \leqslant i \leqslant n-1} [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n], \end{aligned}$$

and define a counit $\varepsilon : B(A) \to k$ by $\varepsilon|_k = 1_k$ and $\varepsilon([a_1|\cdots|a_n]) = 0$ for $n \geqslant 1$. It is easy to check that $(B(A), \Delta, \varepsilon)$ is a coaugmented graded coalgebra.

Define $\delta_0 : B(A) \to B(A)$ by

$$\delta_0\big([a_1|\cdots|a_n]\big) = -\sum_{i=1}^{n} (-1)^{\omega_i} \big[a_1|\cdots|d(a_i)|\cdots|a_n\big],$$

and define $\delta_1 : B(A) \to B(A)$ by

$$\delta_1\big([a_1]\big) = 0 \quad \text{and} \quad \delta_1\big([a_1|\cdots|a_n]\big) = \sum_{i=2}^{n} (-1)^{\omega_i} [a_1|\cdots|a_{i-1}a_i|\cdots|a_n],$$

where $\omega_i = \sum_{j<i}(|a_j| - 1)$.

It is easy to see that $\delta_0^2 = \delta_1\delta_0 + \delta_0\delta_1 = \delta_1^2 = 0$. Set $\delta = \delta_0 + \delta_1$. Then δ is a differential and $(B(A), \delta)$ is a coaugmented DG coalgebra, which is called the *bar construction* of A.

Let (M, d_M) be a right DG A-module. The *bar construction of* M is the complex $B(M; A) = M \otimes B(A)$ with differential $\delta = \delta_0 + \delta_1$, where

$$\begin{aligned} \delta_0\big(m[a_1|\cdots|a_n]\big) &= d_M(m)[a_1|\cdots|a_n] \\ &\quad - \sum_{i=1}^{n} (-1)^{\omega_i + |m|} m\big[a_1|\cdots|d_A(a_i)|\cdots|a_n\big], \end{aligned}$$

and

$$\begin{aligned} \delta_1(m) &= 0, \\ \delta_1\big(m[a_1|\cdots|a_n]\big) &= (-1)^{|m|} m a_1 [a_2|\cdots|a_n] \\ &\quad + \sum_{i=2}^{n} (-1)^{\omega_i + |m|} m[a_1|\cdots|a_{i-1}a_i|\cdots|a_n]. \end{aligned}$$

$B(M; A)$ is a right DG $B(A)$-comodule.

1.2. Cobar constructions

Let C be a coaugmented DG coalgebra with differential d, and let $\bar{C} = \cdots \oplus C^{-1} \oplus \bar{C}^0 \oplus C^1 \oplus \cdots$ be the cokernel of the coaugmentation map. Let

$$\Omega(C) = T(s\bar{C}) = k \oplus s\bar{C} \oplus s\bar{C} \otimes s\bar{C} \oplus [s\bar{C}]^{\otimes 3} \oplus \cdots$$

be the tensor algebra, which is augmented. Define $\partial_0 : \Omega(C) \to \Omega(C)$ by

$$\partial_0\big([x_1|\cdots|x_n]\big) = -\sum_{i=1}^{n}(-1)^{\kappa_i}\big[x_1|\cdots|d(x_i)|\cdots|x_n\big],$$

and $\partial_1 : \Omega(C) \to \Omega(C)$ by

$$\partial_1\big([x_1|\cdots|x_n]\big) = \sum_{i=1}^{n}\sum_{(x_i)}(-1)^{\kappa_i+|x_{i(1)}|+1}[x_1|\cdots|x_{i(1)}|x_{i(2)}|\cdots|x_n],$$

where $\kappa_i = \sum_{j<i}(|x_j|+1)$ and $\sum_{(x_i)} x_{i(1)} \otimes x_{i(2)} = \bar{\Delta}(x_i)$. Set $\partial = \partial_0 + \partial_1$. Then $(\Omega(C), \partial)$ is an augmented DG algebra, called the *cobar construction* of C.

Let (M, ρ, d_M) be a right DG C-comodule. Then we have a composition

$$\bar{\rho} : M \xrightarrow{\rho} M \otimes C \longrightarrow M \otimes \bar{C}.$$

The *cobar construction of M* is the complex $\Omega(M; C) = M \otimes \Omega(C)$ with differential $\partial = \partial_0 + \partial_1$, where

$$\partial_0\big(m[x_1|\cdots|x_n]\big) = d_M(m)[x_1|\cdots|x_n] - \sum_{i=1}^{n}(-1)^{\kappa_i+|m|}m\big[x_1|\cdots|d_C(x_i)|\cdots|x_n\big],$$

and

$$\partial_1\big(m[x_1|\cdots|x_n]\big) = \sum_{(m)}(-1)^{|m_{(0)}|}m_{(0)}[m_{(1)}|x_1|\cdots|x_n] + \sum_{i=1}^{n}\sum_{(x_i)}(-1)^{\kappa_i+|m|+|x_{i(1)}|+1}m[x_1|\cdots|x_{i(1)}|x_{i(2)}|\cdots|x_n],$$

where $\sum_{(m)} m_{(0)} \otimes m_{(1)} = \bar{\rho}(m)$.

$\Omega(M; C)$ is a right DG $\Omega(C)$-module.

Lemma 1.1. *(See [FHT2, Ex. 2, P. 272].) Let A be an augmented DG algebra. Then there is a quasi-isomorphism of DG algebras $\zeta : \Omega B(A) \longrightarrow A$.*

Lemma 1.2. *(See [FHT2, Proposition 19.2].) The augmentation map*

$$B(A;A) = A \otimes B(A) \xrightarrow{\epsilon\otimes\epsilon} {}_Ak$$

is a quasi-isomorphism, and $A \otimes B(A)$ is a semifree resolution of ${}_Ak$.

Dually, we have

Lemma 1.3. *The coaugmentation map $\eta : k \longrightarrow \Omega(C;C) = C \otimes \Omega(C)$ is a quasi-isomorphism of left DG C-comodules.*

Since C is coaugmented, $C = k \oplus \bar{C}$. Let $\phi : \Omega(C;C) \longrightarrow k$ be the natural linear projection map. Then it is a right DG $\Omega(C)$-module morphism. Since $\phi \circ \eta = id$ and η is a quasi-isomorphism, it follows that ϕ is a quasi-isomorphism, that is, $k_{\Omega(C)}$ and $\Omega(C;C) = C \otimes \Omega(C)$ are quasi-isomorphic as DG $\Omega(C)$-modules.

1.3. Some notations

Let A be an augmented DG algebra. $\mathcal{D}_{dg}(A)$ stands for the derived category of left DG A-modules and $\mathcal{D}_{dg}(A^{op})$ for the derived category of right DG A-modules; $\mathcal{D}^c(A)$ (resp. $\mathcal{D}^c(A^{op})$) stands for the full triangulated subcategory of $\mathcal{D}_{dg}(A)$ (resp. $\mathcal{D}_{dg}(A^{op})$) consisting of all the compact objects [Ke1, Sect. 5]. If A is a connected DG algebra, then $\mathcal{D}^c(A)$ (resp. $\mathcal{D}^c(A^{op})$) is equivalent to the full triangulated subcategory $\langle {}_AA\rangle$ (resp. $\langle A_A\rangle$) generated by the object ${}_AA$ (resp. A_A), that is, the smallest full triangulated subcategory containing ${}_AA$ (resp. A_A) as an object and closed under isomorphisms.

Let E be an algebra. The notation $\mathcal{D}^*(E)$ ($* = +, -, b$) stands for the derived category of bounded below (resp. bounded above, bounded) cochain complexes of left E-modules. $\mathcal{D}^*(E^{op})$ stands for the right version of $\mathcal{D}^*(E)$.

2. Koszul DG algebras

In this section, we give a definition of Koszul DG algebras, and discuss some basic properties of Koszul DG algebras.

First of all we recall some classical definitions and well-known results. Let V be a finite dimensional vector space, and $T(V) = k \oplus V \oplus V^{\otimes 2} \oplus \cdots$ be the tensor algebra over V. With the usual grading, $T(V)$ is a graded algebra. A *quadratic algebra* is a quotient algebra $R = T(V)/(U)$ for some finite dimensional vector space V and some subspace $U \subseteq V \otimes V$; the *quadratic dual* $R^!$ of R is defined as $T(V^*)/(U^\perp)$, where V^* is the dual vector space of V and $U^\perp \subseteq (V \otimes V)^* \cong V^* \otimes V^*$ is the orthogonal complement of U. A quadratic algebra R is *Koszul* [Pr,BGS,Sm] if the trivial R-module ${}_Rk$ admits a free resolution

$$\cdots \to Q_n \to \cdots \to Q_1 \to Q_0 \to {}_Rk \to 0$$

with Q_n generated in degree n for all $n \geqslant 0$. If R is a Koszul algebra, then its Yoneda Ext-algebra $\mathrm{Ext}^*_R({}_Rk, {}_Rk) \cong R^!$ [Sm,BGS]. For more properties about Koszul algebras, we refer to the references [BGS,Pr,Sm].

Ungraded Koszul algebra was defined by Green–Martínez Villa [GM]. Let E be a noetherian semiperfect algebra with Jacobson radical J. E is called a *quasi-Koszul algebra* if the quotient module E/J has a minimal projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} E/J \longrightarrow 0$$

such that

$$\ker \delta_n \cap J^2 P_n = J \ker \delta_n \quad \text{for all } n \geqslant 0.$$

E is called a *strongly quasi-Koszul algebra* if

$$\ker \delta_n \cap J^i P_n = J^{i-1} \ker \delta_n \quad \text{for all } i \geqslant 2 \text{ and } n \geqslant 0.$$

More properties and applications of (strongly) quasi-Koszul algebras may be found in [GM] and [Mar]. We point out here that if E (with $E/J \cong k$) is a strongly quasi-Koszul algebra then $gr(E)$, the associated graded algebra, is Koszul [GM].

Now let A be a connected DG algebra, and let $I = \bigoplus_{n\geqslant 1} A^n$. A DG A-module M with differential d is said to be *minimal* if $d(M) \subseteq IM$. If M is a bounded below DG A-module, then M has a minimal semifree resolution [KM,MW]. Recall that a DG A-module P is called *semifree* if there is a filtration of DG submodules

$$0 \subseteq P(0) \subseteq P(1) \subseteq \cdots \subseteq P(n) \subseteq \cdots$$

such that $P = \bigcup_{n\geqslant 0} P(n)$ and each $P(n)/P(n-1)$ is free on a basis of cocycles.

A graded subset E of a DG A-module P is called a *semibasis* if it is a basis of the graded module P over the graded algebra A and has a decomposition $E = \bigsqcup_{n\geqslant 0} E^n$ as a union of disjoint graded subsets E^n such that

$$d(E^0) = 0 \quad \text{and} \quad d(E^n) \subseteq \bigoplus_{e\in(\bigsqcup_{i<n} E^i)} Ae \quad \text{for all } n > 0.$$

A DG A-module is semifree if and only if it has a semibasis [AFH, Proposition 2.5].

We now give a definition of the Koszulity for DG algebras.

Definition 2.1. A connected DG algebra A is called a *left Koszul* DG algebra if the trivial DG module ${}_Ak$ has a minimal semifree resolution $\varepsilon : P \to {}_Ak$ such that the semibasis of P consists of elements of degree zero.

Right Koszul DG algebra is defined similarly. The next proposition tells us that a connected DG algebra is left Koszul if and only it is right Koszul.

Proposition 2.2. *Let A be a connected DG algebra. The following statements are equivalent.*

(i) *A is a left Koszul DG algebra*;
(ii) $\mathrm{Ext}^n_A({}_Ak, {}_Ak) = 0$ *for all* $n \neq 0$;
(iii) $\mathrm{Tor}^n_A(k_A, {}_Ak) = 0$ *for all* $n \neq 0$;
(iv) *A is a right Koszul DG algebra.*

Proof. Using the minimal semifree resolution of the trivial module. □

Let R be a connected graded algebra. Suppose that

$$\cdots \to Q_n \to \cdots \to Q_1 \to Q_0 \to {}_Rk \to 0$$

is a minimal free resolution of the trivial module ${}_Rk$. If we consider R as a DG algebra with trivial differential, and view $\cdots \to Q_n \to Q_{n-1} \to \cdots$ as a double complex by using the sign trick, then the associated total complex (that is, $Q_0 \oplus Q_1[-1] \oplus \cdots \oplus Q_n[-n] \oplus \cdots$) is a minimal semifree resolution of the trivial DG module ${}_Rk$. Therefore R is a Koszul algebra in the usual sense if and only if it is a Koszul DG algebra with trivial differential.

Proposition 2.3. *Let A be a connected DG algebra. If the cohomology algebra $H(A)$ is a Koszul algebra, then A is a Koszul DG algebra.*

Proof. We use the Eilenberg–Moore spectral sequence [FHT2,KM]

$$E_2^{p,q} = \mathrm{Tor}_{H(A)}^{p,q}(k,k) = \mathrm{Tor}_{-p}^{H(A)}(k,k)^q \implies \mathrm{Tor}_A^{p+q}(k,k),$$

where q is the grading induced by the gradings on $H(A)$ and ${}_{H(A)}k$. This is a convergent bounded below cohomology spectral sequence. Since $H(A)$ is a Koszul algebra, $E_2^{p,q} = 0$ for $p+q \neq 0$. Thus $\mathrm{Tor}_A^n(k,k) = 0$ for all $n \neq 0$. □

Before proceeding to discuss further properties of Koszul DG algebras, we give some examples here.

Example 2.4. Let A be the graded algebra $k\langle x,y\rangle/(y^2, yx)$, where $|x| = |y| = 1$. Let $d(x) = xy$ and $d(y) = 0$. Then d induces a differential d over A and A is a DG algebra. It is not hard to check that $H(A) = k \oplus ky$, which is a Koszul algebra. Hence by Proposition 2.3, A is a Koszul DG algebra.

The following example shows that Koszul DG algebras with nontrivial differentials exist extensively.

Example 2.5. Each Koszul algebra R is the cohomology algebra of a certain Koszul DG algebra with nontrivial differential. In fact, R can be viewed as a connected DG algebra with a trivial differential. Then by Lemma 1.2, $\Omega B(R)$ is quasi-isomorphic to R as DG algebras. Hence $H(\Omega B(R)) \cong H(R) \cong R$. Clearly $\Omega B(R)$ is a connected DG algebra with a nontrivial differential. By Proposition 2.3, $\Omega B(R)$ is a Koszul DG algebra.

The converse of Proposition 2.3 is not true, as we will see at the end of Section 4. However we have the following proposition.

Proposition 2.6. *Let A be a Koszul DG algebra. If the global dimension* $\operatorname{gldim} H(A) \leqslant 2$*, then $H(A)$ is a Koszul algebra.*

Proof. Let

$$Q_\bullet:\quad 0 \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow H(A) \longrightarrow k \longrightarrow 0$$

be a minimal free resolution of the trivial module ${}_{H(A)}k$. It is direct to check that in this case the Eilenberg–Moore resolution [FHT2,KM] of the trivial DG module ${}_Ak$ arising from $Q_\bullet$ can be chosen to be minimal. If A is Koszul, then the minimal free resolution $Q_\bullet$ must be linear and hence $H(A)$ is Koszul. $\square$

The Koszulity of DG algebras is preserved under taking quasi-isomorphisms.

Lemma 2.7. *(See [KM, Proposition 4.2].) Let A and B be DG algebras. If there is a quasi-isomorphism of DG algebras $f: A \longrightarrow B$, then the restriction of f induces an equivalence of triangulated categories $f^*: \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ with the inverse functor $B \otimes_A^L -$. The same is true for $\mathcal{D}(B^{op})$ and $\mathcal{D}(A^{op})$.*

Proposition 2.8. *Let A and B be connected DG algebras. Suppose that there is a quasi-isomorphism of DG algebras $f: A \longrightarrow B$. If A (resp. B) is a Koszul DG algebra, then so is B (resp. A).*

Proof. If A is a Koszul DG algebra, then $\mathrm{Ext}_A^n({}_Ak, {}_Ak) = 0$ for all $n \neq 0$, that is, $\mathrm{Hom}_{\mathcal{D}(A)}({}_Ak, {}_Ak[n]) = 0$ for all $n \neq 0$. Hence

$$\mathrm{Hom}_{\mathcal{D}(B)}\big({}_Bk, {}_Bk[n]\big) \cong \mathrm{Hom}_{\mathcal{D}(A)}\big(f^*({}_Bk), f^*({}_Bk)[n]\big) = \mathrm{Hom}_{\mathcal{D}(A)}\big({}_Ak, {}_Ak[n]\big) = 0$$

for all $n \neq 0$. Hence $\mathrm{Ext}_B^n({}_Bk, {}_Bk) = 0$ for all $n \neq 0$, and B is Koszul. $\square$

3. The Ext-algebra of a Koszul DG algebra

In this section, we study the structure of the Ext-algebra of a Koszul DG algebra. We prove a version of the Koszul duality on Ext-algebra for Koszul DG algebras.

Let P be a semifree DG A-module with a semifree filtration

$$0 \subseteq P(0) \subseteq P(1) \subseteq \cdots \subseteq P(n) \subseteq \cdots.$$

We may adjust the semifree filtration of P to get a *standard filtration* of P as in the following.

Let E be a semifree basis of P. Then as a graded A-module, $P = A \otimes kE$, where $kE = \bigoplus_{e\in E} ke$ is a graded k-vector space. Set inductively,

$$V_{\leqslant 0} = V(0) = \big\{v \in kE \;\big|\; d(v) = 0\big\} \quad \text{and} \quad F(0) = A \otimes V(0) \subseteq P,$$
$$V_{\leqslant 1} = \big\{v \in kE \;\big|\; d(v) \in F(0)\big\} \quad \text{and} \quad F(1) = A \otimes V_{\leqslant 1} \subseteq P,$$
$$V_{\leqslant n} = \big\{v \in kE \;\big|\; d(v) \in F(n-1)\big\} \quad \text{and} \quad F(n) = A \otimes V_{\leqslant n} \subseteq P.$$

Let $V(n)$ be a subspace of $V_{\leqslant n}$ such that $V_{\leqslant n} = V_{\leqslant n-1} \oplus V(n)$. Then for any $0 \neq v \in V(n)$, $d(v) \in F(n-1) \backslash F(n-2)$.

Obviously, $\bigcup_{n\geqslant 0} F(n) = P$ and $F(n)/F(n-1) \cong A \otimes V(n)$ is a free DG module over a basis of cocycles. Hence

$$0 \subseteq F(0) \subseteq F(1) \subseteq \cdots \subseteq F(n) \subseteq \cdots$$

is a new semifree filtration on P, which is called the *standard semifree filtration* of P associated to the semibasis E.

As we will see in next example, the standard semifree filtration depends on the choice of the semibasis.

Example 3.1. Let A be a connected DG algebra such that there is an element $a \in A^1$ with $d_A(a) \neq 0$. Let $P = Ae_0 \oplus Ae_1$ as a graded free A-module with $\deg(e_i) = i$ for $i = 0, 1$. Define $d(e_0) = 0$ and $d(e_1) = d_A(a)e_0$. Then P is a semifree DG A-module with a semifree filtration

$$\mathbf{P}\colon\quad 0 \subseteq P(0) \subseteq P(1) = P$$

where $P(0) = Ae_0$ and $P(1) = Ae_0 \oplus Ae_1 = Ae_0 \oplus A(e_1 - ae_0) = P$.

Then $E = \{e_0, e_1\}$ and $E' = \{e_0, e_1 - ae_0\}$ are two semibasis of the semifree DG module P. Associated to the semibasis E, the standard filtration is the original one

$$\mathbf{P}\colon\quad 0 \subseteq P(0) \subseteq P(1) = P.$$

Associated to the semibasis E', the standard filtration is

$$\mathbf{F}\colon\quad 0 \subseteq F(0) = Ae_0 \oplus A(e_1 - ae_0) = P.$$

The main reason to introduce the standard filtration is that DG morphism preserves the standard filtration as in the following lemma, which is needed in the proof of Theorem 3.3.

Lemma 3.2. *Let A be a connected DG algebra, M and N be minimal semifree DG A-modules with the standard filtration $0 \subseteq M(0) \subseteq M(1) \subseteq \cdots$ and $0 \subseteq N(0) \subseteq N(1) \subseteq \cdots$ respectively. If the semibasis of M and N consist of elements of degree 0, then any DG module morphism $f : M \to N$ preserves the filtration.*

Proof. Assume that there are graded vector spaces $U(i)$ and $W(i)$ for $i \geqslant 0$ such that $M(i)/M(i-1) = A \otimes U(i)$ and $N(i)/N(i-1) = A \otimes W(i)$. For any $u \in U(0)$, $f(u) \in \bigoplus_{i\geqslant 0} W(i)$ and $d(f(u)) = 0$ since f is a cochain map. Let $f(u) = v_{i_0} + \cdots + v_{i_t}$ with $0 \neq v_{i_j} \in W(i_j)$ for $0 \leqslant j \leqslant t$ and $i_0 < i_1 < \cdots < i_t$. Suppose that $t \geqslant 1$. By the definition of standard filtration of N, $d(v_{i_j}) \in N(i_j - 1)$ and $d(v_{i_j}) \notin N(i_j - 2)$. However, $0 = d(f(u)) = d(v_{i_0} + \cdots + v_{i_t}) = d(v_{i_0} + \cdots + v_{i_{t-1}}) + d(v_{i_t})$. It follows that $d(v_{i_t}) = -d(v_{i_0} + \cdots + v_{i_{t-1}}) \in N(i_{t-1} - 1) \subseteq N(i_t - 2)$, a contradiction. Hence $t = 0$ and $f(u) \in W(0)$, which implies $f(M(0)) \subseteq N(0)$.

Now suppose $f(M(n)) \subseteq N(n)$. Let $\bar{M} = M/M(n)$ and $\bar{N} = N/N(n)$. Then f induces a DG morphism $\bar{f} : \bar{M} \to \bar{N}$. $\bar{M}$ and $\bar{N}$ are minimal semifree modules with standard semifree filtration

$$\bar{M}(0) = M(n+1)/M(n) \subseteq \bar{M}(1) = M(n+2)/M(n) \subseteq \cdots \quad \text{and}$$
$$\bar{N}(0) = N(n+1)/N(n) \subseteq \bar{N}(1) = N(n+2)/N(n) \subseteq \cdots$$

respectively. By the previous narratives, we have $\bar{f}(\bar{M}(0)) \subseteq \bar{N}(0)$, which in turn implies $f(M(n+1)) \subseteq N(n+1)$. $\square$

Theorem 3.3. *Let A be a Koszul DG algebra. Then*

(i) *the Ext-algebra $E = \mathrm{Ext}_A^*({}_Ak, {}_Ak)$ of A is an augmented algebra*;
(ii) *there is a filtration*

$$\mathbf{F}\colon \quad E = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

on E such that E is a filtered algebra. Moreover, if $H(A)$ is a Koszul algebra, then the associated graded algebra $gr_{\mathbf{F}}(E)$ is isomorphic to the dual Koszul algebra $(H(A))^!$.

Proof. (i) and the first part of (ii) may be proved by using the bar construction of A. We give a direct proof here for later use.

Let $\varepsilon : P \longrightarrow {}_Ak$ be a minimal semifree resolution of the trivial DG module ${}_Ak$. Suppose that

$$0 \subseteq P(0) \subseteq P(1) \subseteq \cdots \subseteq P(n) \subseteq \cdots$$

is a standard semifree filtration of P associated to some semibasis. We have graded vector spaces $V(0), V(1), \ldots, V(n), \ldots$ such that $P(0) = A \otimes V(0)$ and $P(n)/P(n-1) = A \otimes V(n)$ for all $n \geqslant 1$. By the minimality of P, it is easy to see that $V(0) = k$. Since A is Koszul,

$$E = \mathrm{Ext}_A^*({}_Ak, {}_Ak) = \mathrm{Ext}_A^0({}_Ak, {}_Ak) = \prod_{i \geqslant 0} V(i)^* = k \oplus \prod_{i \geqslant 1} V(i)^*.$$

Define a decreasing filtration $\mathbf{F}$ on E by

$$\mathbf{F}\colon \quad F_0 = E \quad \text{and} \quad F_n = \prod_{i \geqslant n} V(i)^* \quad \text{for } n \geqslant 1.$$

We claim that E is a filtered algebra with this filtration. For any $x \in F_n = \prod_{i \geqslant n} V(i)^*$ and $y \in F_m = \prod_{i \geqslant m} V(i)^*$, we still use x to denote the corresponding DG module morphism $x : P/P(n-1) \longrightarrow {}_Ak$, and y the corresponding DG module morphism $y : P/P(m-1) \longrightarrow {}_Ak$. Since $P/P(n-1)$ is semifree, there is a DG module morphism $f_x : P/P(n-1) \longrightarrow P$ such that $\varepsilon \circ f_x = x$ [AFH, Lemma 6.5.3]. Let g be the composition

$$P \xrightarrow{\pi} P/P(n-1) \xrightarrow{f_x} P,$$

where π is the natural projection map. By Lemma 3.2, f_x preserves the filtration, hence $g(P(n-1)) = 0$ and $g(P(n+i)) \subseteq P(i)$ for all $i \geqslant 0$. Let h be the composition

$$P \xrightarrow{\pi} P/P(m-1) \xrightarrow{y} k.$$

By definition, the product $y \cdot x$ in the algebra E is the restriction of $h \circ g$ to $\bigoplus_{i \geqslant 0} V(i)$. Since $h \circ g(P(n+m-1)) \subseteq h(P(m-1)) = 0$, it follows that $y \cdot x \in \prod_{i \geqslant n+m} V(i)^*$. Hence E is a filtered algebra with filtration $\{F_n\}$.

Define a map $\epsilon : E \longrightarrow k$ by $\epsilon|_k = id_k$ and $\epsilon|_{F_1} = 0$. Since F_1 is an ideal, ϵ is an algebra morphism, hence an augmentation map. (i) is proved.

Now we prove the second part of (ii). Suppose that $H(A)$ is a Koszul algebra. The trivial $H(A)$-module ${}_{H(A)}k$ has a linear projective resolution

$$\cdots \longrightarrow H(A) \otimes V'(n) \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} H(A) \otimes V'(1) \xrightarrow{\delta_1} H(A) \otimes V'(0) \xrightarrow{\delta_0} {}_{H(A)}k \longrightarrow 0.$$

The Eilenberg–Moore resolution [FHT2, Proposition 20.11] P' of the DG module ${}_Ak$ arising from the previous resolution of ${}_{H(A)}k$ is minimal. Hence $P \cong P'$ as DG A-modules since A is connected and then $V(i) \cong V'(i)$ as vector spaces for all $i \geqslant 0$. For convenience, we identify $V(i)$ with $V'(i)$ for all $i \geqslant 0$ and P with P'. By the construction of the filtration $\mathbf{F}$ on E, we get $F_n/F_{n-1} \cong V(n)^*$ for all $n \geqslant 0$. Hence we have

$$gr_{\mathbf{F}}(E) \cong \bigoplus_{n \geqslant 0} V(n)^* \cong \operatorname{Ext}^*_{H(A)}(k, k) \tag{1}$$

as graded vector spaces. Pick elements $x \in V(n)^*$ and $y \in V(m)^*$. As we know, x and y can be extended to be DG module maps (also denoted by x and y respectively) $P/P(n-1) \xrightarrow{x} {}_Ak$ and $P/P(m-1) \xrightarrow{y} {}_Ak$. As before, there are filtration-preserving DG module morphisms $f_x : P/P(n-1) \longrightarrow P$ and $f_y : P/P(m-1) \longrightarrow P$ such that $\varepsilon \circ f_x = x$ and $\varepsilon \circ f_y = y$. Let g be the composition of the DG module morphisms

$$g : P \xrightarrow{\pi} P/P(n-1) \xrightarrow{f_x} P \xrightarrow{\pi} P/P(m-1) \xrightarrow{y} k.$$

Then the product $y \cdot x \in V(n+m)^*$ of x and y in $gr_{\mathbf{F}}(E)$ is the restriction of g to $V(n+m)$. Since it is filtration-preserving, f_x induces a morphism of spectral sequences

$$E^{p,q}_*(f_x) : E^{p,q}_*\big(P/P(n-1)\big) \longrightarrow E^{p,q}_*(P).$$

In particular, $E^{p,q}_1(P/P(n-1)) = H^{p+q}(A) \otimes V(n-p)$ and $E^{p,q}_1(P) = H^{p+q}(A) \otimes V(-p)$ for all $p \leqslant 0$ and $p+q \geqslant 0$. Now we regard $x \in V(n)^*$ and $y \in V(m)^*$ as elements in $\operatorname{Ext}^*_{H(A)}(k, k)$. Let $h_{-p} = \bigoplus_{p \geqslant -q} E^{p,q}_1(f_x)$. Then we get a commutative diagram

$$\begin{array}{ccccccccccc}
\longrightarrow & H(A) \otimes V(n+m) & \xrightarrow{\delta_{n+m}} & \cdots & \xrightarrow{\delta_{n+2}} & H(A) \otimes V(n+1) & \xrightarrow{\delta_{n+1}} & H(A) \otimes V(n) & \longrightarrow & 0 & \\
 & \downarrow{\scriptstyle h_m} & & & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} & \searrow{\scriptstyle \eta_x} & & \\
\longrightarrow & H(A) \otimes V(m) & \xrightarrow{\delta_m} & \cdots & \xrightarrow{\delta_2} & H(A) \otimes V(1) & \xrightarrow{\delta_1} & H(A) \otimes V(0) & \xrightarrow{\delta_0} & k & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \eta_y} & & & & & & & & & \\
 & k & & & & & & & & &
\end{array}$$

where η_x and η_y are graded $H(A)$-module morphisms induced by x and y. To avoid the possible confusion, we temporarily denote the Yoneda product on $\operatorname{Ext}^*_{H(A)}(k, k)$ by $y * x$. By the definition of Yoneda product, $y * x$ is equal to the restriction of $\eta_y \circ h_m$ to $V(n+m)$. Let $\tau_m : \bigoplus_{i=0}^m V(i) \to V(m)$ be the projection map. For any $v \in V(n+m)$,

$$\eta_y \circ h_m(v) = \eta_y\big(E_1^{-(n+m),n+m}(f_x)(v)\big) = \eta_y \circ \tau_m \circ f_x(v) = g(v).$$

Hence $y * x = y \cdot x$, that is, the products on $gr_{\mathbf{F}}(E)$ and $\mathrm{Ext}^*_{H(A)}(k,k)$ coincide under the isomorphism in (1). Since $H(A)$ is Koszul, $\mathrm{Ext}^*_{H(A)}(k,k) \cong (H(A))^!$. Hence $gr_{\mathbf{F}}(E) \cong (H(A))^!$. □

Let A be a connected DG algebra. When the trivial module ${}_Ak$ lies in $\mathcal{D}^c(A)$, the DG algebra A usually has good properties. The following proposition is clear.

Proposition 3.4. *Let A be a connected DG algebra. If ${}_Ak \in \mathcal{D}^c(A)$ and $H(A)$ is a Koszul algebra, then* gldim $H(A) < \infty$.

Theorem 3.5. *Let A be a connected DG algebra. Suppose ${}_Ak \in \mathcal{D}^c(A)$.*

(i) *If A is a Koszul DG algebra, then the Ext-algebra $E = \mathrm{Ext}^0_A({}_Ak, {}_Ak)$ is a finite dimensional local algebra with $E/J = k$, where J is the Jacobson radical of E.*
(ii) *If $H(A)$ is a Koszul algebra, then $gr(E) = (H(A))^!$, where $gr(E)$ is the graded algebra associated with the radical filtration of the local algebra E.*

Proof. We use the notations in the proof of Theorem 3.3.

(i) Since ${}_Ak \in \mathcal{D}^c(A)$, there is an integer m such that $P(m)/P(m-1) \neq 0$ and $P(i)/P(i-1) = 0$ for all $i > m$. Hence the filtration $\mathbf{F}$: $E = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$ stops at the mth step. By Theorem 3.3, E is a filtered algebra, hence for $x \in F_1$, $x^{m+1} = 0$. Thus E is a local algebra with Jacobson radical $J = F_1$ and $E/J \cong k$.

(ii) If $H(A)$ is a Koszul algebra, then by Proposition 3.4, gldim $H(A) < \infty$. Assume that gldim $H(A) = n$. Then the filtration $\mathbf{F}$ stops at the nth step, and $J^{n+1} = 0$. By Theorem 3.3, $gr_{\mathbf{F}}(E) \cong (H(A))^!$. If we can show $J^i = F_i$ for all $1 \leqslant i \leqslant n$, then we are done. Since $V(j) = 0$ for $j \geqslant n+1$, $E = k \oplus V(1)^* \oplus \cdots \oplus V(n)^*$ and $F_i = \bigoplus_{j=i}^n V(j)^*$. By Theorem 3.3, $gr_{\mathbf{F}}(E)$ is generated in degree 1, so $(F_1)^n = F_n = V(n)^*$, that is, $J^n = F_n$. Similarly, since $V(n)^* = J^n \subseteq J^{n-1}$, $V(n-1)^* \subseteq (F_1)^{n-1} + V(n)^* = J^{n-1}$ and $F_{n-1} = V(n-1)^* \oplus V(n)^* \subseteq J^{n-1}$. On the other hand, $J^{n-1} \subseteq V(n-1)^* \oplus V(n)^*$. Hence $F_{n-1} = J^{n-1}$. An easy induction shows that $J^i = F_i$ for all $1 \leqslant i \leqslant n$. □

We next prove a theorem similar to the Koszul duality for Koszul algebras [BGS].

Let A be an augmented DG algebra, and let $R = B(A)$ be its bar construction.

Lemma 3.6. *(See [FHT2, P. 272].) The map $\varphi : R^\# \longrightarrow \mathrm{End}_A(A \otimes R)$ defined by*

$$\varphi(f)\big(1[a_1|\cdots|a_n]\big) = \sum_{i=0}^n (-1)^{|f|\omega_i} 1[a_1|\cdots|a_i] f\big([a_{i+1}|\cdots|a_n]\big)$$

is a quasi-isomorphism of DG algebras, where $\omega_i = |a_1| + \cdots + |a_i| - i$.

Lemma 3.7. *Let A be a Koszul DG algebra and E be its Ext-algebra. If ${}_Ak \in \mathcal{D}^c(A)$, then $E^\#$ is a coaugmented coalgebra and there is a quasi-isomorphism of DG algebras*

$$\psi : \Omega\big(E^\#\big) \longrightarrow A.$$

Proof. Let $R = B(A)$ be the bar construction of A. Then R is a coaugmented DG coalgebra, and is concentrated in non-negative degrees. The graded vector space dual $R^{\#}$ is an augmented DG algebra. It follows from Lemmas 1.2 and 3.6 that $E \cong H(\mathrm{End}_A(A \otimes R)) \cong H(R^{\#})$. Since A is Koszul, E is concentrated in degree zero. The last isomorphism implies that $H^i(R) = 0$ for all $i > 0$. Then there is naturally a quasi-isomorphism of coaugmented DG coalgebras

$$Z^0(R) \longrightarrow R,$$

which induces a quasi-isomorphism of augmented DG algebras

$$R^{\#} \longrightarrow \left(Z^0(R)\right)^{\#}.$$

Therefore $E \cong H(R^{\#}) \cong (Z^0(R))^{\#}$ as augmented algebras. Since ${}_Ak \in \mathcal{D}^c(A)$, E is a finite dimensional algebra. Hence $E^{\#} \cong Z^0(R)$ as coaugmented coalgebras, and there is a quasi-isomorphism of coaugmented DG coalgebras

$$E^{\#} \longrightarrow R.$$

This induces a quasi-isomorphism of DG algebras

$$\xi : \Omega\left(E^{\#}\right) \longrightarrow \Omega(R) = \Omega B(A).$$

There is also a quasi-isomorphism of DG algebras $\zeta : \Omega B(A) \longrightarrow A$ by Lemma 1.1. Hence the composition

$$\Omega\left(E^{\#}\right) \xrightarrow{\xi} \Omega B(A) \xrightarrow{\zeta} A \tag{2}$$

gives a quasi-isomorphism of DG algebras $\psi = \zeta \circ \xi : \Omega(E^{\#}) \longrightarrow A$. The proof is completed. □

Theorem 3.8 *(Koszul duality on Ext-algebra). Let A be a Koszul DG algebra and E be its Ext-algebra. If ${}_Ak \in \mathcal{D}^c(A)$, then $\mathrm{Ext}^*_E({}_Ek, {}_Ek) \cong H(A)$.*

Proof. By Lemma 3.6, $\Omega(E^{\#}) = B(E)^{\#}$ is quasi-isomorphic to $\mathrm{End}_E(E \otimes B(E))$. Hence

$$\mathrm{Ext}^*_E({}_Ek, {}_Ek) \cong H\left(\mathrm{End}_E\left(E \otimes B(E)\right)\right) \cong H\left(\Omega\left(E^{\#}\right)\right).$$

It follows from Lemma 3.7 that $\mathrm{Ext}^*_E({}_Ek, {}_Ek) \cong H(A)$. □

As an application of Theorem 3.8, we have the following two corollaries, which establish relations between Koszul DG algebras and (strongly) quasi-Koszul algebras.

Corollary 3.9. *Let A be a Koszul DG algebra. If ${}_Ak \in \mathcal{D}^c(A)$, then the following are equivalent*:

(i) *The Ext-algebra E of A is a quasi-Koszul algebra*;
(ii) *$H(A)$ is generated in degree* 1.

Proof. By Theorem 3.5, E is a finite dimensional local algebra with the residue field k. The equivalence of (i) and (ii) follows from Theorem 3.8 and [GM, Theorem 4.4]. □

Corollary 3.10. *Let A be a Koszul DG algebra. If ${}_Ak \in \mathcal{D}^c(A)$, then the following are equivalent*:

(i) *The Ext-algebra E of A is a strongly quasi-Koszul algebra*;
(ii) *$H(A)$ is a Koszul algebra.*

Proof. (i) ⇒ (ii). By [GM, Theorem 6.1] and its proof, $\mathrm{Ext}_E^*({}_Ek, {}_Ek)$ is a Koszul algebra. Theorem 3.8 implies that $H(A) \cong \mathrm{Ext}_E^*({}_Ek, {}_Ek)$ is a Koszul algebra.

(ii) ⇒ (i). Applying Theorem 3.8 again, $\mathrm{Ext}_E^*({}_Ek, {}_Ek) \cong H(A)$ is a Koszul algebra. By [GM, Theorem 9.1], E is a strongly quasi-Koszul algebra. □

4. Koszul duality

Let B be an augmented DG algebra and C be a coaugmented DG coalgebra. Lefèvre-Hasegawa [Le, Proposition 2.2.4.1] established an equivalence between the derived category $\mathcal{D}(B)$ and the so-called coderived category $\mathcal{D}(C)$ when B and C satisfy certain conditions. Thanks for the result of Lefèvre-Hasegawa we can prove a version of Koszul duality [BGS] for Koszul DG algebras.

Let (B, m_B, d_B) be an augmented DG algebra with an augmentation map $\varepsilon_B : B \to k$, and (C, Δ, d_C) be a coaugmented DG coalgebra with a coaugmentation map $\eta_C : k \to C$. A graded linear map $\tau : C \to B$ of degree 1 is called a *twisting cochain* from C to B [HMS,Le] if

$$\varepsilon_B \circ \tau \circ \eta_C = 0, \quad \text{and}$$
$$m_B \circ (\tau \otimes \tau) \circ \Delta + d_B \circ \tau + \tau \circ d_C = 0.$$

Let $\Omega(C)$ be the cobar construction of C. The twisting cochains from C to B are one-to-one corresponding to the DG algebra morphisms from $\Omega(C)$ to B. There is a *canonical twisting cochain* $\tau_0 : C \to \Omega(C)$ given by $\tau_0(c) = [c]$ for any $c \in \bar{C}$ and $\tau_0(k) = 0$.

Let $\tau : C \to B$ be a twisting cochain. For any right DG C-comodule N, the *twisted tensor product* $N \otimes_\tau B$ [Le,Ke2] is the right DG B-module defined by

(i) $N \otimes_\tau B = N \otimes B$ as a right graded B-module;
(ii) the differential $\delta = d_N \otimes 1 + 1 \otimes d_B + (1 \otimes m_B)(1 \otimes \tau \otimes 1)(\rho_N \otimes 1)$, i.e.

$$\delta(n \otimes a) = d(n) \otimes a + (-1)^{|n|} n \otimes d(a) + \sum_{(n)} (-1)^{|n_{(0)}|} n_{(0)} \otimes \tau(n_{(1)}) a,$$

for any homogeneous elements $n \in N$ and $a \in B$.

Dually, for any DG B-module M, the *twisted tensor product* $M \otimes_\tau C$ is the right DG C-comodule defined by

(i) $M \otimes_\tau C = M \otimes C$ as a vector space;

(ii) the differential $\delta = d_M \otimes 1 + 1 \otimes d_C - (m_M \otimes 1)(1 \otimes \tau \otimes 1)(1 \otimes \Delta)$, i.e.

$$\delta(m \otimes c) = d(m) \otimes c + (-1)^{|m|} m \otimes d(c) - \sum_{(c)} (-1)^{|m|} m\tau(c_{(1)}) \otimes c_{(2)},$$

for any homogeneous elements $m \in M$ and $c \in C$.

Let DGmod-B be the category of right DG B-modules and DGcom-C be the category of right DG C-comodules. Then there is a pair of adjoint functors (L, R) [Ke2,Le]:

$$\text{DGcom-}C \underset{R=-\otimes_\tau C}{\overset{L=-\otimes_\tau B}{\rightleftarrows}} \text{DGmod-}B.$$

Let C be a cocomplete DG coalgebra, and DGcomc-C be the category of cocomplete right DG C-comodules. For any $M, N \in$ DGcomc-C, a DG comodule morphism $f : M \to N$ is called a *weak equivalence related to* τ [Ke2,Le] if $L(f) : LM \to LN$ is a quasi-isomorphism. Note that a weak equivalence related to τ_0 ($B = \Omega(C)$) is a quasi-isomorphism. But the converse is not true in general [Ke2]. Let $\mathcal{K}(C)$ be the homotopy category of DGcomc-C. Equipped with the natural exact triangles, $\mathcal{K}(C)$ is a triangulated category. Let $\mathcal{W}$ be the class of weak equivalences in the category $\mathcal{K}(C)$. Then $\mathcal{W}$ is a multiplicative system. The coderived category $\mathcal{D}_{dg}(C)$ of C is defined to be $\mathcal{K}(C)[\mathcal{W}^{-1}]$, the localization of $\mathcal{K}(C)$ at the class $\mathcal{W}$ of weak equivalences [Ke2,Le]. Let $\mathcal{D}_{dg}(B^{op})$ be the derived category of right DG B-modules. The following theorem is proved by Lefèvre-Hasegawa in [Le, Ch. 2], and also can be found in [Ke2].

Theorem 4.1. *Let C be a cocomplete DG coalgebra, B an augmented DG algebra and $\tau : C \to B$ be a twisting cochain. Then the following are equivalent*:

(i) *The map τ induces a quasi-isomorphism $\Omega(C) \to B$*;
(ii) *The adjunction map*

$$B \otimes_\tau C \otimes_\tau B \to B$$

is a quasi-isomorphism;
(iii) *The functors L and R induce an equivalence of triangulated categories* (*also denoted by L and R*)

$$\mathcal{D}_{dg}(C) \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}_{dg}\big(B^{op}\big).$$

Now let A be a Koszul DG algebra. Suppose ${}_Ak \in \mathcal{D}^c(A)$. By Theorem 3.5, its Ext-algebra E is a finite dimensional local algebra with the residue field k. Hence the vector space dual $E^* = E^\#$ is a coaugmented coalgebra which is of course cocomplete. Hence all the DG E^*-comodules are cocomplete. Let $C = E^*$ and $B = \Omega(C)$. Clearly, B is a connected DG algebra, and the canonical twisting cochain $\tau_0 : C \to \Omega(C)$ satisfies the condition (i) in the Theorem 4.1. Hence we have the following equivalence of triangulated categories

$$\mathcal{D}_{dg}(E^*) \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}_{dg}(\Omega(E^*)^{op}).$$

Let $\mathcal{D}^+_{dg}(\Omega(E^*)^{op})$ be the derived category of all bounded below right DG $\Omega(E^*)$-modules, that is, consisting of objects M with $M^n = 0$ for $n \ll 0$. Since $\Omega(E^*)$ is connected, it is not hard to see that $\mathcal{D}^+_{dg}(\Omega(E^*)^{op})$ is a full triangulated subcategory of $\mathcal{D}_{dg}(\Omega(E^*)^{op})$. Similarly, let $\mathcal{K}^+_{dg}(C)$ be the homotopy category of bounded below DG cocomplete comodules, and let $\mathcal{D}^+_{dg}(E^*)$ be the localization of $\mathcal{K}^+_{dg}(E^*)$ at the class of weak equivalences $\mathcal{W}^+$ in $\mathcal{K}^+_{dg}(E^*)$ ($\mathcal{W}^+$ is also a multiplicative system). One can check that $\mathcal{D}^+_{dg}(E^*)$ is a full triangulated subcategory of $\mathcal{D}_{dg}(E^*)$. Restricting L and R to the subcategories $\mathcal{D}^+_{dg}(E^*)$ and $\mathcal{D}^+_{dg}(\Omega(E^*)^{op})$ respectively, we get the following proposition.

Proposition 4.2. *Let A be a Koszul DG algebra and E be its Ext-algebra. If ${}_Ak \in \mathcal{D}^c(A)$, then the following is an equivalence of triangulated categories*

$$\mathcal{D}^+_{dg}(E^*) \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}^+_{dg}(\Omega(E^*)^{op}).$$

Since E^* is concentrated in degree zero, a DG E^*-comodule is exactly a cochain complex of E^*-comodules. Hence $\mathcal{K}^+_{dg}(E^*) = \mathcal{K}^+(E^*)$, the homotopy category of bounded below cochain complexes of right E^*-comodules. It is not hard to see that the class $\mathcal{W}^+$ of weak equivalences related to τ_0 is exactly the class of quasi-isomorphisms. Hence $\mathcal{D}^+_{dg}(E^*) = \mathcal{K}^+_{dg}(E^*)[(\mathcal{W}^+)^{-1}] = D^+(E^*)$, the derived category of bounded below cochain complexes of right E^*-comodules. By Proposition 4.2 we have the following proposition.

Proposition 4.3. *Let A be a Koszul DG algebra and E be its Ext-algebra. If ${}_Ak \in \mathcal{D}^c(A)$, then there is an equivalence of triangulated categories* (*we use the same notations of the equivalent functors as in Proposition* 4.2).

$$D^+(E^*) \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}^+_{dg}(\Omega(E^*)^{op}).$$

Since E is a finite dimensional algebra, the category of left E-modules is isomorphic to the category of right E^*-comodules [Mon, 1.6.4]. Hence there is an equivalence of triangulated categories

$$\mathcal{D}^+(E) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}^+(E^*),$$

where $\mathcal{D}^+(E)$ is the derived category of bounded below cochain complexes of left E-modules.

By Lemma 3.7, there is a quasi-isomorphism of DG algebras $\varphi:\Omega(E^*)\longrightarrow A$. Hence by Lemma 2.7, the following gives an equivalence of triangulated categories

$$\mathcal{D}^+\big(A^{op}\big)\underset{-\otimes^L_{\Omega(E^*)}A}{\overset{\varphi^*}{\rightleftarrows}}\mathcal{D}^+\big(\Omega\big(E^*\big)^{op}\big).$$

Let $\Phi=(-\otimes^L_{\Omega(E^*)}A)\circ L\circ F$ and $\Psi=G\circ R\circ\varphi^*$. We have the following theorem.

Theorem 4.4 *(Koszul equivalence). Let A be a Koszul DG algebra and E be its Ext-algebra. If ${}_Ak\in\mathcal{D}^c(A)$, then we have an equivalence of triangulated categories*

$$\mathcal{D}^+(E)\underset{\Psi}{\overset{\Phi}{\rightleftarrows}}\mathcal{D}^+_{dg}\big(A^{op}\big).$$

It is easy to see that $\Phi({}_Ek)=L(k^{E^*})\otimes^L_{\Omega(E^*)}A=\Omega(E^*)\otimes^L_{\Omega(E^*)}A=A_A$. Temporarily write $\langle {}_Ek\rangle$ the full triangulated subcategory of $\mathcal{D}^+(E)$ generated by ${}_Ek$. By restricting Φ and Ψ, we get an equivalence of triangulated categories

$$\langle {}_Ek\rangle\underset{\Psi_{\mathrm{res}}}{\overset{\Phi_{\mathrm{res}}}{\rightleftarrows}}\mathcal{D}^c(A).$$

Lemma 4.5. $\langle {}_Ek\rangle=\mathcal{D}^b(\text{mod-}E)$*, where* mod-$E$ *is the category of finitely generated left E-modules.*

Proof. It suffices to show that all the finitely generated E-modules are in $\langle {}_Ek\rangle$. Since E is finite dimensional, any finitely generated E-module is finite dimensional. Clearly, all 1-dimensional modules are in $\langle {}_Ek\rangle$. Let N be a finite dimensional module. Since $\mathrm{soc}(N)\neq 0$, we have an exact sequence

$$0\longrightarrow {}_Ek\longrightarrow N\longrightarrow N/{}_Ek\longrightarrow 0.$$

Since $\dim N/{}_Ek<\dim N$, an induction on the dimension of N implies that N lies in $\langle {}_Ek\rangle$. Hence all finitely generated E-modules are in $\langle {}_Ek\rangle$. □

Corollary 4.6. *Let A be a Koszul DG algebra and E be its Ext-algebra. If ${}_Ak\in\mathcal{D}^c(A)$, then we have an equivalence of triangulated categories*

$$\mathcal{D}^b(\text{mod-}E)\underset{\Psi_{\mathrm{res}}}{\overset{\Phi_{\mathrm{res}}}{\rightleftarrows}}\mathcal{D}^c\big(A^{op}\big).$$

Since E is finite dimensional, the vector space dual $(\ )^*$ induces a duality of triangulated categories

$$\mathcal{D}(\text{mod-}E) \underset{(\ )^*}{\overset{(\ )^*}{\rightleftarrows}} \mathcal{D}\big(\text{mod-}E^{op}\big).$$

Now, we are able to give a version of the Koszul duality for Koszul DG algebras.

Theorem 4.7 *(Koszul duality). Let A be a Koszul DG algebra and E be its Ext-algebra. Suppose ${}_Ak \in \mathcal{D}^c(A)$. Then there is a duality of triangulated categories*

$$\mathcal{D}^b\big(\text{mod-}E^{op}\big) \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{D}^c\big(A^{op}\big).$$

It is easy to see that

$$\mathcal{F}(k_E) = \Phi({}_Ek) = A_A \tag{3}$$

and

$$\begin{aligned}\mathcal{F}(E_E) &= \Phi\big({}_EE^*\big)\\ &= L\big(\big(E^*\big)^{E^*}\big) \otimes^L_{\Omega(E^*)} A\\ &\overset{\text{(a)}}{\cong} k_{\Omega(E^*)} \otimes^L_{\Omega(E^*)} A\\ &\cong k_A,\end{aligned} \tag{4}$$

where the isomorphism (a) holds, because $L((E^*)^{E^*}) = \Omega(E^*; E^*)$ which is quasi-isomorphic to $k_{\Omega(E^*)}$ as a DG $\Omega(E^*)$-module by the narrative below Lemma 1.3.

From the proof of above results, we have proved in fact the following result.

Corollary 4.8. *Let R be a finite dimensional local algebra with the residue field k. Then there is a duality of triangulated categories*

$$\mathcal{D}^b\big(R^{op}\big) \rightleftarrows \mathcal{D}^c\big(\Omega\big(R^*\big)^{op}\big);$$

and under this duality, the trivial module k_R corresponds to $\Omega(R^*)$ and R_R to $k_{\Omega(R^*)}$.

The following corollary was indicated in [Ke2] and [Le]. As an application of Corollary 4.8, we give a proof here.

Corollary 4.9. *Let R be a finite dimensional local algebra with the residue field k. Then the connected DG algebra $\Omega(R^*)$ is a Koszul DG algebra. Moreover, the Ext-algebra $\operatorname{Ext}^*_{\Omega(R^*)}(k,k)$ is isomorphic to R.*

Proof. By Corollary 4.8, $k_{\Omega(R^*)}$ is compact, and

$$\mathrm{Ext}^n_{\Omega(R^*)^{op}}(k,k)=\mathrm{Hom}_{\mathcal{D}^c(\Omega(R^*)^{op})}\big(k,k[-n]\big)\cong\mathrm{Hom}_{\mathcal{D}^b(R^{op})}\big(R[n],R\big)=0$$

if $n\neq 0$. Therefore $\Omega(R^*)$ is a Koszul DG algebra. Moreover, the following are algebra isomorphisms

$$\mathrm{Ext}^*_{\Omega(R^*)}(k,k)\cong\mathrm{Ext}^*_{\Omega(R^*)^{op}}(k,k)^{op}\cong\mathrm{Ext}^*_{R^{op}}(R,R)\cong R. \qquad \square$$

In particular, by Corollary 4.9, if k is algebraically closed, then any finite dimensional local algebra can be viewed as the Ext-algebra of some Koszul DG algebra.

Example 4.10. Now let $V=kx\oplus ky\oplus kz$ and $R=T(V)/T^{\geqslant 4}(V)$. Clearly, R is a finite dimensional local algebra. Then $B=\Omega(R^*)$ is a Koszul DG algebra with $\mathrm{Ext}^*_B({}_Bk,{}_Bk)=R$. Since $gr(R)\cong R$ is not a Koszul algebra, so R is not a strongly quasi-Koszul algebra. By Corollary 3.10, the cohomology $H(B)$ cannot be a Koszul algebra. Hence the converse of Proposition 2.3 is not true.

5. BGG correspondence

In [BGG], Bernstein–Gelfand–Gelfand established an equivalence of categories

$$\overline{\text{grmod-}}\Lambda(V)\cong\mathcal{D}^b\big(\mathrm{Coh}\,\mathbb{P}^n\big)$$

where $\overline{\text{grmod-}}\Lambda(V)$ is the stable category of finitely generated graded modules over the exterior algebra $\Lambda(V)$ of an $(n+1)$-dimensional space $V=kx_0\oplus kx_1\oplus\cdots\oplus kx_n$, and $\mathcal{D}^b(\mathrm{Coh}\,\mathbb{P}^n)$ is the bounded derived category of coherent sheaves over the n-dimensional projective space $\mathbb{P}^n$. This equivalence is now called the *BGG correspondence* in literature. A sketch of the proof of the BGG correspondence can be found also in [GMa, P. 273, Ex. 1]. The BGG correspondence has been generalized to noncommutative projective geometry by several authors [Jo,MS,Mo]. Let R be a Koszul noetherian AS-Gorenstein algebra with finite global dimension. Then its Ext-algebra $E(R)$ is a Frobenius algebra [Sm]. A version of the noncommutative BGG correspondence was proved in [MS], which was stated as

$$\overline{\text{grmod-}}E(R)\cong\mathcal{D}^b\big(\mathrm{qgr}\,R^{op}\big),$$

where $\overline{\text{grmod-}}E(R)$ is the stable category of finitely generated graded modules over $E(R)$ and $\mathrm{qgr}\,R^{op}$ is the quotient category grmod-$R^{op}/\operatorname{tors}R^{op}$ [AZ]. Let $\mathcal{D}^b_{fd}(\text{grmod-}R^{op})$ be the full subcategory of $\mathcal{D}^b(\text{grmod-}R^{op})$ consisting of objects X with finite dimensional cohomology groups. It is well known that [Mi]

$$\mathcal{D}^b\big(\mathrm{qgr}\,R^{op}\big)=\mathcal{D}^b\big(\text{grmod-}R^{op}\big)/\mathcal{D}^b_{fd}\big(\text{grmod-}R^{op}\big).$$

Hence the above BGG correspondence can be stated as

$$\overline{\text{grmod-}}E(R)\cong\mathcal{D}^b\big(\text{grmod-}R^{op}\big)/\mathcal{D}^b_{fd}\big(\text{grmod-}R^{op}\big). \qquad (5)$$

In this section, we deduce a correspondence similar to (5) for AS-Gorenstein Koszul DG algebras.

First of all we recall the definition of AS-Gorenstein DG algebra. Let A be a connected DG algebra. We say that A is *right AS-Gorenstein* (AS stands for Artin–Schelter) if $\mathrm{RHom}_{A^{op}}(k, A) \cong s^n k$ for some integer n [FHT1,LPWZ,LPWZ2]; A is *right AS-regular* if A is right AS-Gorenstein and $k_A \in \mathcal{D}^c(A^{op})$. Similarly, we define left AS-Gorenstein DG algebra and left AS-regular algebra. We say that A is *AS-Gorenstein* (*resp., regular*) if A is both left and right *AS-Gorenstein* (*resp., regular*).

Proposition 5.1. *Let A be a connected DG algebra. If the cohomology algebra $H(A)$ is a left AS-Gorenstein algebra, then A is a left AS-Gorenstein DG algebra.*

Proof. Consider the Eilenberg–Moore spectral sequence [KM]

$$E_2^{p,q} = \mathrm{Ext}^p_{H(A)}\big(k, H(A)\big)^q \implies \mathrm{Ext}_A^{p+q}(k, A)$$

where the index p in $\mathrm{Ext}^p_{H(A)}(k, H(A))^q$ is the usual homological degree and q is the grading induced from the gradings of ${}_Ak$ and $H(A)$. If the cohomology spectral sequence is regular, then it is complete convergent [We]. If $H(A)$ is AS-Gorenstein, then by definition there exist some integers d and l such that

$$\mathrm{Ext}^n_{H(A)}\big(k, H(A)\big) = \begin{cases} 0 & n \neq d, \\ k[l] & n = d. \end{cases}$$

Then it is routine to see that

$$\mathrm{Ext}^n_A(k, A) = \begin{cases} 0 & n \neq d + l, \\ k & n = d + l. \end{cases}$$

Hence $\mathrm{RHom}_A(k, A) \cong s^n k$ for $n = d + l$. □

We don't know whether the converse of Proposition 5.1 is true or not. If A is a connected graded algebra, viewed as a DG algebra with trivial differential, then A is an AS-Gorenstein DG algebra if and only if A satisfies AS-Gorenstein condition in the usual sense. The AS-Gorenstein property is invariant under quasi-isomorphism.

Proposition 5.2. *Let $f : A \to A'$ be a quasi-isomorphism of connected DG algebras. Then A is left AS-Gorenstein* (*AS-regular*) *if and only if A' is.*

Proof. The proof is similar to that of Proposition 2.8. □

Lemma 5.3. *Let A be an AS-regular DG algebra. Suppose* $\mathrm{RHom}_{A^{op}}(k, A) \cong s^l k$ *for some integer l. Let $P \to k_A$ be a minimal semifree resolution of k_A with a semifree filtration*

$$0 \subseteq P(0) \subseteq P(1) \subseteq \cdots \subseteq P(n)$$

such that $P(n) = P$ and $P(n)/P(n-1) \neq 0$. Then $P(n)/P(n-1) = A[-l]$.

Proof. There are finite dimensional graded vector spaces $V(0), V(1), \dots, V(n)$ such that $P(i)/P(i-1) = V(i) \otimes A$ for all $0 \leqslant i \leqslant n$ ($P(-1) = 0$). As graded A-modules $P = \bigoplus_{i=0}^n V(i) \otimes A$. Hence $\mathrm{Hom}_A(P, A) = \bigoplus_{i=0}^n A \otimes V(i)^\#$ as graded left A-modules. Let $\{x_1, \dots, x_t\}$ be a homogeneous basis of $V(n)$. Let d be the differential of $\mathrm{Hom}_A(P, A)$ induced by the differentials of P and A. For any $1 \leqslant s \leqslant t$, define a graded right A-module morphism

$$f_s : P = \bigoplus_{i=0}^n V(i) \otimes A \longrightarrow A$$

by sending x_s to the identity of A, x_j to zero for $j \neq s$, and sending $V(r)$ to zero for all $r < n$. One can see that $f_1, \dots, f_t$ so defined are cocycles of the cochain complex $\mathrm{Hom}_A(P, A)$. Since P is minimal, $d(g)(x_j) = (d_A g - (-1)^{|g|} g d_P)(x_j) \in A^{\geqslant 1}$ for any homogeneous element $g \in \mathrm{Hom}_A(P, A)$ and x_j. Hence any f_j ($1 \leqslant j \leqslant t$) cannot be a coboundary. By hypothesis $\mathrm{RHom}_{A^{op}}(k, A) \cong s^l k$, which forces $\dim V(n) = 1$ and the degree of non-zero elements in $V(n)$ is $-l$, that is, $P(n)/P(n-1) \cong s^{-l} A$. $\square$

The following proposition is a special case of [LPWZ2, Theorem 9.8].

Proposition 5.4. *Let A be a Koszul DG algebra with Ext-algebra $E = \mathrm{Ext}_A^*(k, k)$. Then A is right AS-regular if and only if E is Frobenius.*

Proof. Suppose that A is right AS-regular. Then $k_A \in \mathcal{D}^c(A^{op})$, which is equivalent to ${}_A k \in \mathcal{D}^c(A)$. Hence E is finite dimensional. Since A is Koszul, $\mathrm{RHom}_{A^{op}}(k, A) \cong k$ by Lemma 5.3. By Theorem 4.7,

$$\begin{aligned}
\mathrm{Ext}^n_{E^{op}}(k, E) &= \mathrm{Hom}_{\mathcal{D}^b(\text{mod-}E^{op})}\big(k, E[-n]\big) \\
&\cong \mathrm{Hom}_{\mathcal{D}^c(A^{op})}\big(\mathcal{F}(E), \mathcal{F}(k)[-n]\big) \\
&\cong \mathrm{Hom}_{\mathcal{D}^c(A^{op})}\big(k, A[-n]\big) \\
&= \mathrm{Ext}^n_{A^{op}}(k, A).
\end{aligned}$$

Hence $\mathrm{Ext}^n_{E^{op}}(k, E) = 0$ for $n \neq 0$. Let

$$0 \longrightarrow E_E \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

be a minimal injective resolution of E_E. Since E is finite dimensional and local, all the injective modules I^n's are finite dimensional. Hence $0 = \mathrm{Ext}^n_{E^{op}}(k, A) \cong \operatorname{soc} I^n$ for all $n \geqslant 1$, and $\mathrm{Hom}_{E^{op}}(k, E) = k$. We get $I^n = 0$ for all $n \geqslant 1$ and $I^0 = E^*$. Therefore we have a right E-module isomorphism $E \cong E^*$, that is, E is a Frobenius algebra.

Conversely, if E is Frobenius, then it is finite dimensional, and hence $k_A \in \mathcal{D}^c(A^{op})$. Since E itself is injective and local, it follows $\mathrm{Ext}^n_{E^{op}}(k, E) = 0$ for $n \geqslant 1$ and $\mathrm{Ext}^0_{E^{op}}(k, E) = k$. Hence $\mathrm{Ext}^n_{A^{op}}(k, A) \cong \mathrm{Ext}^n_{E^{op}}(k, E) = 0$ for $n \neq 0$ and $\mathrm{Ext}^0_{A^{op}}(k, A) \cong k$. Then $\mathrm{RHom}_{A^{op}}(k, A) \cong k$, and hence A is AS-regular. $\square$

In [LPWZ2, Theorem 9.8], a more general case of the above proposition is proved with some locally finite conditions.

Corollary 5.5. *Let A be a Koszul DG algebra. Then A is right AS-regular if and only if A is left AS-regular.*

Proof. Note that $E^{op} \cong \operatorname{Ext}^*_{A^{op}}(k,k)$. $\square$

Next we are going to deduce a result similar to the classical BGG correspondence.

Lemma 5.6. *Let A be a connected DG algebra such that $k_A \in \mathcal{D}^c(A^{op})$. Then the full triangulated subcategory $\langle k_A \rangle$ of $\mathcal{D}^c(A^{op})$ generated by k_A, is equal to $\mathcal{D}_{fd}(A^{op})$, the full subcategory of $\mathcal{D}^c(A^{op})$ consisting of DG modules M such that* $\dim H(M) < \infty$.

Proof. For any DG module M, temporarily we write

$$\ell(M) = \sup\{i \mid H^i(M) \neq 0\} - \inf\{i \mid H^i(M) \neq 0\}$$

and

$$\lambda(M) = \sup\{i \mid M^i \neq 0\} - \inf\{i \mid M^i \neq 0\}.$$

We prove the lemma by an induction on $\ell(M)$. Let M be a DG A-module with $\dim H(M) < \infty$. Without loss of generality, we may assume that $H^i(M) = 0$ for $i < 0$ or $i > n$ for $n = \ell(M)$. Since A is connected, by suitable truncations, we may assume that M is concentrated in degrees $0 \leqslant i \leqslant n$. If $\ell(M) = 0$, then M is isomorphic in $\mathcal{D}^c(A^{op})$ to a DG module N with $\lambda(N) = 0$, which is a direct sum of finite copies of k_A, and hence is in $\langle k_A \rangle$. Now suppose that each DG module M with $\dim H(M) < \infty$ and $\ell(M) < n$ is in $\langle k_A \rangle$. If M is a DG module with $\dim H(M) < \infty$ and $\ell(M) = n$, without loss of generality, we may assume $M^0 \neq 0$ and $M^n \neq 0$, and $M^i = 0$ for $i < 0$ and $i > n$. Then the vector space M^n has a decomposition $M^n = d(M^{n-1}) \oplus K$ for some subspace K. Since $\dim H(M) < \infty$, then $\dim K < \infty$. Taking K as a DG A-module concentrated on degree zero, then we have an exact sequence of DG modules

$$0 \longrightarrow s^n K \longrightarrow M \longrightarrow M/s^n K \longrightarrow 0.$$

Now $\ell(s^n K) = 0$ and $\ell(M/s^n K) \leqslant n - 1$. By the induction hypothesis, both $s^n K$ and $M/s^n K$ are objects in $\langle k_A \rangle$, and hence M is in $\langle k_A \rangle$. Therefore $\langle k_A \rangle = \mathcal{D}_{fd}(A^{op})$. $\square$

Theorem 5.7 *(BGG correspondence). Let A be a Koszul DG AS-regular algebra with Ext-algebra $E = \operatorname{Ext}^*_A(k,k)$. Then there is a duality of triangulated categories*

$$\overline{\text{mod-}E^{op}} \rightleftarrows \mathcal{D}^c\big(A^{op}\big)/\mathcal{D}_{fd}\big(A^{op}\big).$$

Proof. By the Koszul duality (Theorem 4.7), There is a duality of triangulated categories

$$\mathcal{D}^b\big(\text{mod-}E^{op}\big) \rightleftarrows \mathcal{D}^c\big(A^{op}\big);$$

and under this duality the object $E_E \in \mathcal{D}^b(\text{mod-}E^{op})$ is corresponding to the object $k_A \in \mathcal{D}^c(A^{op})$ by (4). Hence there is a duality

$$\mathcal{D}^b\big(\text{mod-}E^{op}\big)/\langle E_E \rangle \rightleftarrows \mathcal{D}^c\big(A^{op}\big)/\langle k_A \rangle,$$

where $\langle E_E\rangle$ is the full triangulated subcategory of $\mathcal{D}^b(\text{mod-}E^{op})$ generated by E_E. Since E is a finite dimensional local algebra with $E/J(E)\cong k$ (Theorem 3.5), all finitely generated projective E-modules are free. Therefore $\langle E_E\rangle = \mathcal{D}^b(\text{proj}\, E^{op})$, where $\text{proj}\, E^{op}$ is the category of all finitely generated right projective E-modules. Hence

$$\mathcal{D}^b(\text{mod-}E^{op})/\langle E_E\rangle = \mathcal{D}^b(\text{mod-}E^{op})/\mathcal{D}^b(\text{proj}\, E^{op}).$$

By Proposition 5.4, E is Frobenius, and hence [Be1]

$$\mathcal{D}^b(\text{mod-}E^{op})/\mathcal{D}^b(\text{proj}\, E^{op}) \cong \overline{\text{mod-}}E^{op}.$$

On the other hand, by Lemma 5.6

$$\mathcal{D}^c(A^{op})/\langle k_A\rangle = \mathcal{D}^c(A^{op})/\mathcal{D}_{fd}(A).$$

In summary, there is an a duality of triangulated categories

$$\overline{\text{mod-}}E^{op} \rightleftarrows \mathcal{D}^c(A^{op})/\mathcal{D}_{fd}(A^{op}). \qquad \square$$

Since E is finite dimensional, there is an equivalence form of the BGG correspondence.

Theorem 5.8. *Let A be a Koszul DG AS-regular algebra with Ext-algebra $E = \text{Ext}_A^*(k,k)$. Then there is an equivalence of triangulated categories*

$$\overline{\text{mod-}}E \cong \mathcal{D}^c(A^{op})/\mathcal{D}_{fd}(A^{op}).$$

6. BGG correspondence on Adams connected DG algebras

Many examples of the DG algebra from algebraic geometry and algebraic topology admit an extra grading. Let $A = \bigoplus_{i,j\in\mathbb{Z}} A^i_j$ be a bigraded space. An element $a \in A^i_j$ is of degree (i,j). The second grading is usually called *Adams grading* [KM,LPWZ]. A DG algebra (A,d) is called a DG algebra with Adams grading if A is bigraded and the differential d is of degree $(1,0)$ (i.e., d preserves Adams grading). A DG module over a DG algebra with Adams grading is bigraded and the differential preserves the second grading. A DG algebra A with Adams grading is *augmented* if there is an augmentation map $\varepsilon : A \to k$ of degree $(0,0)$. A DG algebra with Adams grading A is said to be *Adams connected* if (1) $A^i_j = 0$ for $i < 0$ or $j < 0$, and (2) $A^0_0 = k$, $A^0_j = 0$ and $A^i_0 = 0$ for $i, j \neq 0$. All Adams connected DG algebras are augmented.

Similarly, we define coaugmented DG coalgebras with Adams grading.

In this section, all the DG algebras and DG coalgebras involved are with Adams grading. For simplicity, we call a DG algebra (coalgebra) with Adams grading an *Adams DG algebra* (*coalgebra*).

It is not hard to see that the bar (cobar) construction (see Section 1) of an (a) (co)augmented Adams DG algebra (coalgebra) is an Adams DG coalgebra (algebra). The canonical twisting cochain (see Section 4) τ_0 from a cocomplete Adams DG coalgebra C to $\Omega(C)$ is of degree $(1,0)$.

Let A be an Adams DG algebra, and let $\mathcal{AC}_{dg}(A)$ ($\mathcal{AC}_{dg}(A^{op})$) be the category of left (right) DG A-modules with morphisms of degree $(0,0)$. We use $s^i = [i]$ to denote the ith shift functor on the first grading and use $s^{-(j)} = (j)$ to denote the jth shift functor on the Adams grading. Let $\mathcal{AD}_{dg}(A)$ be the derived category of $\mathcal{AC}_{dg}(A)$. Denote $\mathcal{AD}^c(A)$ ($\mathcal{AD}^c(A^{op})$) as the full triangulated subcategory of $\mathcal{AD}_{dg}(A)$ ($\mathcal{AD}_{dg}(A^{op})$) generated by ${}_AA$ (A_A). Let M and N be objects in $\mathcal{AC}_{dg}(A)$, we use

$$\mathcal{E}xt_A^{i,j}(M,N) = \mathrm{Hom}_{\mathcal{AD}_{dg}(A)}\big(M, N[-i](j)\big)$$

to denote the derived functor. Then

$$\mathcal{E}xt_A^{*,*}(M,N) = \bigoplus_{i,j\in\mathbb{Z}} \mathcal{E}xt_A^{i,j}(M,N)$$

is a bigraded space. In particular, if A is an augmented Adams DG algebra, then $\mathcal{E} = \mathcal{E}xt_A^{*,*}(k,k)$ is a bigraded algebra. For convenience, we usually write $\mathcal{E}_j^i = \mathcal{E}xt_A^{i,j}(k,k)$.

The results obtained in previous sections can be easily generalized to Adams DG algebras. Hence in this section, we only state the results without giving proofs. More general results can be found in [LPWZ2, Section 10], with some locally finite conditions.

Let A be an Adams connected DG algebra and M be a bounded below DG module over A. Then there is a minimal semifree resolution (the construction is similar to [KM, Theorem IV.3.7]) $P \to M$ in $\mathcal{AC}_{dg}(A)$ (see also [MW]).

Definition 6.1. Let A be an Adams connected DG algebra. It is called a Koszul Adams DG algebra if $\mathcal{E}xt_A^{i,*}(k,k) = \bigoplus_{j\in\mathbb{Z}} \mathcal{E}xt_A^{i,j}(k,k) = 0$ for all $i \neq 0$.

It is not hard to see that, if A is a Koszul Adams DG algebra, then its Ext-algebra $\mathcal{E} = \mathcal{E}xt_A^{*,*}(k,k)$ has the property that $\mathcal{E}_j^i = 0$ for $i \neq 0$ or $j > 0$. Hence $\mathcal{E}$ is a negatively graded algebra. Comparing with Theorem 3.3, we have the following.

Proposition 6.2. *Let A be a Koszul Adams DG algebra, and let $S_j = \mathcal{E}_{-j}^0$. Then $S = \bigoplus_{j\geqslant 0} S_j$ is a connected graded algebra. If in addition ${}_Ak \in \mathcal{AD}_{dg}^c(A)$, then S is a finite dimensional graded algebra.*

We also have the following form of Lefèvre-Hasegawa's theorem.

Theorem 6.3. *Let C be a cocomplete Adams DG coalgebra, B an augmented Adams DG algebra and $\tau : C \to B$ is a twisting cochain of degree $(1,0)$. The following are equivalent*

(i) *The map τ induces a quasi-isomorphism $\Omega(C) \to B$;*
(ii) *The adjunction map*

$$B \otimes_\tau C \otimes_\tau B \to B$$

is a quasi-isomorphism;
(iii) *There is an equivalence of triangulated categories*

$$\mathcal{AD}_{dg}(C) \rightleftarrows \mathcal{AD}_{dg}\big(B^{op}\big)$$

where $\mathcal{AD}_{dg}(C)$ is the coderived category over the cocomplete Adams DG algebra C.

If ${}_Ak \in \mathcal{AD}^c(A)$, then $\mathcal{E}$ is finite dimensional, and hence the graded vector space dual $\mathcal{E}^\#$ is finite dimensional coalgebra. By applying the above theorem and notice that a DG comodule over the Adams DG coalgebra $\mathcal{E}^\#$ is exactly a complex of graded comodules over $\mathcal{E}^\#$, we have the following proposition which is analogous to Theorem 4.7.

Proposition 6.4. *Let A be a Koszul Adams DG algebra. If ${}_Ak \in \mathcal{AD}^c(A)$, then there is a duality of triangulated categories*

$$\mathcal{D}^b\big(\text{grmod-}\mathcal{E}^{op}\big) \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{AD}^c\big(A^{op}\big).$$

It is convenient for us to deal with the positively graded algebra S, rather than the negatively graded algebra $\mathcal{E}$. The abelian category grmod-$\mathcal{E}^{op}$ is equivalent to grmod-S^{op} of finitely generated right S-modules. We have the following Koszul duality theorem of Adams DG algebras.

Theorem 6.5. *Let A be a Koszul Adams DG algebra. Let S be the graded algebra such that $S_j = \mathcal{E}xt_A^{0,-j}(k,k)$. If ${}_Ak \in \mathcal{AD}^c(A)$, then there is a duality of triangulated categories*

$$\mathcal{D}^b\big(\text{grmod-}S^{op}\big) \underset{\phi}{\overset{\psi}{\rightleftarrows}} \mathcal{AD}^c\big(A^{op}\big).$$

To establish a version of the BGG correspondence, we need the concept of AS-Gorenstein Adams DG algebra which is first introduced in [LPWZ].

Definition 6.6. Let A be an Adams connected DG algebra. It is called an *AS-Gorenstein* Adams DG algebra if $\mathrm{RHom}_{A^{op}}(k, A) \cong k[r](s)$. Moreover if $k_A \in \mathcal{AD}^c(A^{op})$, then A is called an *AS-regular* Adams DG algebra.

The following proposition is proved in [LPWZ] by using A_∞-algebra. Also one can give a proof by using Theorem 6.5.

Proposition 6.7. *Let A be a Koszul Adams DG algebra. Then A is AS-regular if and only if its Ext-algebra $\mathcal{E}$ is Frobenius.*

Now we can state the BGG correspondence on Adams DG algebras.

Theorem 6.8. *Let A be a Koszul AS-regular Adams DG algebra and S be the connected graded such that $S_j = \mathcal{E}xt_A^{0,-j}(k,k)$. Then there is a duality of triangulated categories*

$$\overline{\text{grmod-}}S^{op} \cong \mathcal{AD}^c\big(A^{op}\big)/\mathcal{AD}_{fd}\big(A^{op}\big), \tag{6}$$

where $\mathcal{AD}_{fd}(A^{op})$ is the full triangulated subcategory of $\mathcal{AD}^c(A^{op})$ consisting of objects M such that $\dim H(M) < \infty$.

Now let R be a noetherian connected graded algebra. Let A be the Adams connected DG algebra with trivial differential by taking $A^i_i = R_i$ and $A^i_j = 0$ if $i \neq j$. If R is a Koszul algebra, then it is not hard to see that A is a Koszul Adams DG algebra. Moreover, $\mathcal{E}xt_A^{0,-j}(k,k) = R^!_j$ for all $j \geqslant 0$, i.e., $S = R^! = E(R) = \mathrm{Ext}^*_R(k,k)$. Suppose that $\mathrm{gl.dim}\, R < \infty$. Then $S = E(R)$ is finite dimensional. Since $E(R)$ is finite dimensional grmod-$E(R)^{op}$ is dual to grmod-$E(R)$. Hence $\mathcal{D}^b(\text{grmod-}S^{op}) = \mathcal{D}^b(\text{grmod-}E(R)^{op})$ is dual to $\mathcal{D}^b(\text{grmod-}E(R))$. Let us inspect the category $\mathcal{AD}^c(A^{op})$ in Theorem 6.5. Since the differential of A is trivial and A is concentrated in the diagonal of the first quadrant, the triangulated category $\mathcal{AD}_{dg}(A^{op})$ is naturally equivalent to the derived category $\mathcal{D}(\text{Grmod-}R^{op})$ of the category Grmod-R of right graded R-modules. Under this equivalence, A_A is corresponding to R_R in $\mathcal{D}(\text{Grmod-}R^{op})$. Hence $\mathcal{AD}^c(A^{op})$ is equivalent to the full triangulated subcategory of $\mathcal{D}(\text{Grmod-}R^{op})$ generated by R_R (closed under the shifts on the grading of R_R), which is equivalent to $\mathcal{D}^b(\text{proj}\, R^{op})$, the bounded derived category of finitely generated graded projective right R-modules. Since R is noetherian and has finite global dimension, $\mathcal{D}^b(\text{proj}\, R^{op})$ is equivalent to $\mathcal{D}^b(\text{grmod-}R^{op})$, the bounded derived category of finitely generated graded right R-modules. In summary we have the equivalence (which is established in [BGS]) of triangulated categories if R is noetherian and of finite global dimension

$$\mathcal{D}^b\big(\text{grmod-}E(R)\big) \cong \mathcal{D}^b\big(\text{grmod-}R^{op}\big).$$

Moreover, we assume that R is a noetherian Koszul AS-regular algebra. Then the Adams connected DG algebra A is Koszul Adams AS-regular DG algebra. Hence in the left hand of (6), $\overline{\text{grmod-}}S^{op}$ is dual to $\overline{\text{grmod-}}E(R)$. Since $\mathcal{AD}^c(A^{op})$ is equivalent to $\mathcal{D}^b(\text{grmod-}R^{op})$, the full triangulated subcategory $\mathcal{AD}_{fd}(A^{op})$ is equivalent to $\mathcal{D}^b_{fd}(\text{grmod-}R^{op})$, the triangulated subcategory consisting of objects X such that HX is finite dimensional. Hence in the right hand of (6),

$$\mathcal{AD}^c\big(A^{op}\big)/\mathcal{AD}_{fd}\big(A^{op}\big) \cong \mathcal{D}^b\big(\text{grmod-}R^{op}\big)/\mathcal{D}^b_{fd}\big(\text{grmod-}R^{op}\big)$$

which is equivalent to $\mathcal{D}^b(\text{qgr}\, R^{op})$ by [Mi]. In summary we get the BGG correspondence established in [MS]

$$\overline{\text{grmod-}}E(R) \cong \mathcal{D}^b\big(\text{qgr}\, R^{op}\big).$$

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