Equivalence of shape fibrations
and approximate fibrations

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Abstract

In this paper we give a description of approximate fibrations of arbitrary topological spaces using relations. Our method is to use relations with smaller and smaller images of points. This does not require any extensive knowledge of relations. The main result is that this improved version of the original concept of an approximate fibration of Coram and Duvall is equivalent to the notion of a shape fibration of Mardešić. Hence, our approximate fibrations could be regarded as an intrinsic definition of shape fibrations.

Keywords: Fibration; Weak fibration; Shape fibration; Weak shape fibration; Close; Approximate fibration; Relation; Numerable covering; Homotopy; Approximate polyhedron; Bundle

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1. Introduction

In order to give a proper motivation for this paper we must first recall definitions of fibrations and weak fibrations. For this we need the following notion of the HLP(\(\mathcal{X}\)), the homotopy lifting property with respect to a class \(\mathcal{X}\) of spaces.

A map \(p : E \rightarrow B\) has the HLP(\(\mathcal{X}\)) provided for every \(X \in \mathcal{X}\), every map \(g : X \rightarrow E\), and every homotopy \(h : X \times I \rightarrow B\) with (L1) \(h_0 = p \circ g\), there is a homotopy \(k : X \times I \rightarrow E\) with (L2) \(k_0 = g\) and (L3) \(h = p \circ k\).

Let us use \(\mathcal{T}\) for the class of all topological spaces and \(\mathcal{D}\) for the class of all disks \(D^n\) where \(n \geq 0\). With this notation, a map \(p : E \rightarrow B\) is a fibration if it has the HLP(\(\mathcal{T}\)) and a weak fibration if it has the HLP(\(\mathcal{D}\)).
Fibrations were defined by Hurewicz [7] and weak fibrations by Serre [14]. Both concepts are crucial in homotopy theory and they usually appear in the applications of homotopy theory to geometric problems. However, their usefulness is greatly reduced when their domains and/or codomains are spaces with poor local properties.

In an attempt to correct this problem, Coram and Duvall [3] have introduced approximate fibrations by replacing the HLP(\(X\)) with the AHLP(\(X\)), the approximate homotopy lifting property with respect to the class \(X\). The AHLP(\(X\)) differs from the HLP(\(X\)) only in the fact that the conditions (L1)–(L3) are substituted by their approximate versions. These can be easily described if we agree to write \(f \overset{\varepsilon}{=} g\) for maps \(f, g : U \to V\) and an open covering \(\varepsilon\) of \(V\) provided for every \(u \in U\) some member of \(\varepsilon\) contains \(f(u)\) and \(g(u)\).

A map \(p : E \to B\) has the AHLP(\(X\)) provided for every open covering \(\alpha\) of \(B\) and every open covering \(\delta\) of \(E\) there is an open covering \(\beta\) of \(B\) such that for every \(X \in \mathcal{X}\), every map \(g : X \to E\), and every homotopy \(h : X \times I \to B\) with (\(\beta L1\) \(h_0 \overset{\beta}{=} p \circ g\)), there is a homotopy \(k : X \times I \to E\) with (\(\delta L2\) \(k_0 \overset{\delta}{=} g\)) and (\(\alpha L3\) \(h_0 \overset{\alpha}{=} p \circ k\)).

The AHLP(\(X\)) was utilized by Mardešić and Rushing [9, 11] (for compact metric spaces) and by Mardešić [8] (for arbitrary spaces) to introduce another improved form of fibrations and weak fibrations known as shape fibrations and weak shape fibrations. Their definitions involve inverse systems and resolutions into polyhedra.

The main purpose of this paper is to redefine approximate fibrations so that shape fibrations and weak shape fibrations will become special cases of available choices. This will provide intrinsic definitions of shape fibrations and weak shape fibrations. Of course, approximate fibrations in the sense of Coram and Duvall are also included among these choices.

Our idea is to replace the assumption that \(g, h,\) and \(k\) in the definition of the AHLP(\(X\)) are maps (i.e., single-valued continuous functions) with the assumption that they belong to classes \(\mathcal{G}, \mathcal{H},\) and \(\mathcal{K}\) of relations (i.e., multi-valued functions with nonempty images of points) and that the sizes of their images of points are controlled by open coverings.

At this point we must repeat our statement from the abstract that absolutely no expertise on relations is necessary to follow this paper. Anybody unfamiliar or uncomfortable with relations should replace them with functions and exercise slightly more care at places where inverses appear to get special functional versions of our results. For this the reference [2] might be useful. However, since the inverse of a function is more often a relation rather than a function, insisting on functions is not natural for our approach because it makes statements and proofs more complicated and less general.

In order to unravel our method further, let us observe that the notation \(f \overset{\varepsilon}{=} g\) applies also to relations. Moreover, for an open covering \(\delta\) of a space \(E\), by a \(\delta \mathcal{G}\)-relation we mean a relation \(g : X \to E\) in the class \(\mathcal{G}\) with the property that there is an open covering \(\sigma\) of \(X\) such that for every member \(S\) of \(\sigma\) the image \(g(S)\) lies in some member of \(\delta\).

Let \(\tau = (\mathcal{G}, \mathcal{H}, \mathcal{K})\) be a triple of classes of relations and let \(\mathcal{X}\) be a class of topological spaces.
A map $p : E \to B$ has the AHLP($\mathcal{X}, \tau$) provided for every open covering $\alpha$ of $B$ and every open covering $\delta$ of $E$ there is an open covering $\beta$ of $B$ and an open covering $\varepsilon$ of $E$ such that for every $X \in \mathcal{X}$, every $\varepsilon \mathcal{G}$-relation $g : X \to E$, and every $\beta \mathcal{H}$-relation $h : X \times I \to B$ satisfying $(\beta \mathcal{L} 1)$, there is a $\delta \mathcal{K}$-relation $k : X \times I \to E$ for which $(\delta \mathcal{L} 2)$ and $(\alpha \mathcal{L} 3)$ hold.

We can now describe this paper as a primer on approximate $(\mathcal{X}, \tau)$-fibrations i.e., on maps which have the AHLP($\mathcal{X}, \tau$). The main result is that a map $p : E \to B$ is a shape fibration if and only if it is an approximate $(T, \vartheta)$-fibration, where $\vartheta$ is either $(\mathcal{R}, \mathcal{R}, \mathcal{R})$ or $(\mathcal{S}, \mathcal{S}, \mathcal{S})$ and $\mathcal{R}$ denotes the class of all relations while $\mathcal{S}$ denotes the class of all functions (i.e., single-valued relations). Similarly, a map $p : E \to B$ is a weak shape fibration if and only if it is an approximate $(D, \vartheta)$-fibration. Moreover, a map $p : E \to B$ has the AHLP($\mathcal{X}$) if and only if it has the AHLP($\mathcal{X}, \mu$), where $\mu = (\mathcal{M}, \mathcal{M}, \mathcal{M})$ and $\mathcal{M}$ denotes the class of all maps (i.e., continuous functions).

From this we infer that the condition AHLP($\mathcal{X}, \tau$) is the shape theory version of the condition HLP($\mathcal{X}$) designed for use with arbitrary spaces. In other words, the AHLP($\mathcal{X}, \tau$) provides the unification of theories of shape fibrations [8,9], weak shape fibrations [11], $n$-shape fibrations [10], and approximate fibrations [3].

In conclusion, let us remark that open coverings in this introduction are in fact numerable coverings [5] and that the use of relations with smaller and smaller images of points which is the underlying method in our approach has its origin in Sanjurjo’s article [13]. An important side result which we also prove here using this technique is the characterization of approximate polyhedra [8] as approximately movable spaces.

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2. Small relations

In this preparatory section we shall introduce notions that are required for our theory. We begin by describing our notation related to numerable coverings. These coverings will be used to estimate sizes and distances of relations.

Recall [1] that an open covering $\sigma$ of a space $Y$ is numerable provided it has a partition of unity, or equivalently, provided there is a metric space $M$, a continuous function $f : Y \to M$, and an open covering $\alpha$ of $M$ such that the collection $f^{-1}(\alpha)$ of complete preimages under $f$ of members of $\alpha$ refines $\sigma$. Let $\text{Cov}(Y)$ denote the collection of all numerable coverings of a topological space $Y$. With respect to the refinement relation $\geq$ the set $\text{Cov}(Y)$ is a directed set.

Let $\sigma \in \text{Cov}(Y)$. Let $\sigma^+$ denote the set of all numerable coverings of $Y$ refining $\sigma$ while $\sigma^*$ denotes the set of all numerable coverings $\tau$ of $Y$ such that the star $\text{st}(\tau)$ of $\tau$ refines $\sigma$. Similarly, for a natural number $n$, $\sigma^{\ast n}$ denotes the set of all numerable coverings $\tau$ of $Y$ such that the $n$th star $\text{st}^n(\tau)$ of $\tau$ refines $\sigma$.

We shall now introduce relations and describe our notation related to them.

Let $X$ and $Y$ be topological spaces. By a relation $F : X \to Y$ we mean a rule which associates a nonempty subset $F(x)$ of $Y$ to every point $x$ of $X$. 
Let $G$ be a class of relations. In order to state that a relation $F: X \to Y$ is from the class $G$ we shall say that $F$ is a $G$-relation. Let $G(X, Y)$ denote all $G$-relations from $X$ into $Y$.

Unless stated otherwise, we reserve $R, S, M, U,$ and $L$ for classes of all relations, single-valued relations, single-valued continuous relations, upper semi-continuous relations, and lower semi-continuous relations, respectively. We shall use relation, function, and map instead of $R$-relation, $S$-relation, and $M$-relation, respectively.

For a relation $F: X \to Y$ and a subset $A$ of $X$, let $F(A) = \bigcup\{F(x)|x \in A\}$. We shall denote by $F^{-1}$ the relation from $F(X)$ into $X$ defined by

$$F^{-1}(y) = \{x \in X \mid y \in F(x)\},$$

for every $y \in F(X)$.

The size of images of points of a relation can be controlled in the following two fashions. The first requires that members of a numerable cover of the domain have images in members of a numerable cover of the codomain, while the second requires this only for images of points. This justifies the use of adjective weak for the second.

Let $F: X \to Y$ be a relation and let $\alpha \in \text{Cov}(X)$ and $\beta \in \text{Cov}(Y)$. We shall say that $F$ is an $(\alpha, \beta)$-relation provided for every $A \in \alpha$ there is a $B_A \in \beta$ with $F(A) \subseteq B_A$.

Now, we define that $F$ is a $\beta$-relation provided there is an $\alpha \in \text{Cov}(X)$ such that $F$ is an $(\alpha, \beta)$-relation. On the other hand, $F$ is called a weak $\beta$-relation provided for every $x \in X$ there is a $B_x \in \beta$ with $F(x) \subseteq B_x$.

We shall frequently use the obvious property of maps $f: X \to Y$ that they are $\sigma$-relations for every $\sigma \in \text{Cov}(Y)$. Moreover, if $\sigma \in \text{Cov}(Y)$ and $\alpha \in \text{Cov}(X)$ refines $f^{-1}(\sigma)$, then $f$ is an $(\alpha, \sigma)$-relation.

The fact that a relation $F$ is at the same time from the class $G$ of relations and that it is a (weak) $\beta$-relation will be expressed by saying that it is a (weak) $\beta G$-relation. The term $(\alpha, \beta)G$-relation has an analogous meaning.

The following obvious lemma will be used to estimate the size of the composition of relations. Recall that the composition $G \circ F$ of the relations $F: X \to Y$ and $G: Y \to Z$ is the relation from $X$ into $Z$ defined for every $x \in X$ by

$$G \circ F(x) = \{z \in Z \mid (\exists y \in Y) \ y \in F(x) \& z \in G(y)\}.$$

**Lemma 2.1.** Let $\alpha$, $\beta$, and $\gamma$ be numerable coverings of spaces $X$, $Y$, and $Z$. Let $G: Y \to Z$ be a $(\beta, \gamma)$-relation, let $F: X \to Y$ be a relation, and let $H: X \to Z$ denote the composition of $F$ and $G$.

1. If $F$ is an $(\alpha, \beta)$-relation, then $H$ is an $(\alpha, \gamma)$-relation.
2. If $F$ is a (weak) $\beta$-relation, then $H$ is a (weak) $\gamma$-relation.

3. Proximities for relations

Also important will be the following concepts of closeness and nearness for relations that correspond to two different notions of size for relations.
Let \( F, G : X \to Y \) be relations and let \( \sigma \in \text{Cov}(Y) \). We shall say that \( F \) and \( G \) are \( \sigma \)-close and we write \( F \equiv G \) provided for every \( x \) in \( X \) there is an \( S_x \in \sigma \) with \( F(x) \cup G(x) \subseteq S_x \).

On the other hand, let \( F, G : X \to Y \) be relations, let \( \alpha \in \text{Cov}(X) \), and let \( \sigma \in \text{Cov}(Y) \). We shall say that \( F \) and \( G \) are \((\alpha, \sigma)\)-near and we write \( F \cong^\alpha \sigma G \) provided for every member \( A \) of the covering \( \alpha \) there is a member \( S_A \) of \( \sigma \) with \( F(A) \cup G(A) \subseteq S_A \). Moreover, \( F \) and \( G \) are \( \sigma \)-near and we write \( F \equiv^\sigma G \) provided there is a numerable covering \( \alpha \) of \( X \) such that \( F \) and \( G \) are \((\alpha, \sigma)\)-near.

Observe that \( \sigma \)-near relations are also \( \sigma \)-close. In the next lemma we shall show that the converse is almost true for \( \sigma \)-relations.

**Lemma 3.1.** Let \( \sigma \) be a numerable covering of a space \( Y \). If two \( \sigma \)-relations \( F \) and \( G \) from a space \( X \) into \( Y \) are \( \sigma \)-close, then they are also \( \text{st}(\sigma) \)-near.

**Proof.** Since \( F \) and \( G \) are \( \sigma \)-relations, there are numerable coverings \( \alpha \) and \( \beta \) of \( X \) such that \( F \) is an \((\alpha, \sigma)\)-relation and \( G \) is a \((\beta, \sigma)\)-relation. Let \( \delta \in \text{Cov}(X) \) be a common refinement of \( \alpha \) and \( \beta \). Let \( D \in \delta \). Then there are \( A \in \alpha \) and \( B \in \beta \) with \( D \subseteq A \cap B \). Pick members \( S_A \) and \( S_B \) of \( \sigma \) such that \( F(A) \subseteq S_A \) and \( G(B) \subseteq S_B \). For any \( x \in D \), there is an \( S_x \in \sigma \) with \( F(x) \cup G(x) \subseteq S_x \). It follows that \( F(D) \cup G(D) \subseteq \text{st}(S_x, \sigma) \).

Hence, \( F \cong^\delta \sigma G \).

**Corollary 3.2.** Let \( \sigma \) be a numerable covering of a space \( Y \). If \( f \) and \( g \) are \( \sigma \)-close maps into \( Y \), then they are also \( \text{st}(\sigma) \)-near.

**Lemma 3.3.** Let \( \sigma \) be a numerable covering of a space \( Y \). If \( F \) and \( G \) are \( \sigma \)-close \( \mathcal{U} \)-relations from a paracompact space \( X \) into \( Y \), then they are also \( \sigma \)-near.

**Proof.** For every \( x \in X \) there is an \( S_x \in \sigma \) with \( F(x) \cup G(x) \subseteq S_x \). Since \( F \) and \( G \) are upper semi-continuous, there is an open neighbourhood \( U_x \) of \( x \) in \( X \) with \( F(U_x) \subseteq S_x \) and \( G(U_x) \subseteq S_x \). Hence, \( F(U_x) \cup G(U_x) \subseteq S_x \). Since the space \( X \) is paracompact, the family \( \tau = \{ U_x \mid x \in X \} \) is a numerable covering of \( X \). We conclude that \( F \) and \( G \) are \( \sigma \)-near.

4. Homotopies for relations

Next we introduce for relations the notions which correspond to the equivalence relation of homotopy for maps.

Let \( F \) and \( G \) be relations from a space \( X \) into a space \( Y \). It is customary to call a relation \( H \) from the product \( X \times I \) of \( X \) and the unit segment \( I = [0, 1] \) into \( Y \) such that \( F(x) = H(x, 0) \) and \( G(x) = H(x, 1) \) for every \( x \in X \) a homotopy that joins \( F \) and \( G \). We say that \( F \) and \( G \) are homotopic and we write \( F \simeq G \).
Let $D$ be a numerable covering of $Y$. If a homotopy $H : X \times I \to Y$ is a (weak) $\beta$-relation, then we shall say that $H$ is a (weak) $\beta$-homotopy, that $F$ and $G$ are (weakly) $\beta$-homotopic, and we shall write $F \beta \simeq G$ for weak homotopy and $F \beta \simeq G$ for homotopy.

The fact that a homotopy $H$ is at the same time from the class $G$ of relations and that it is a (weak) $\beta$-homotopy will be expressed by saying that it is a (weak) $\beta G$-homotopy.

5. Approximate movability and approximate polyhedra

In this section we shall define proximate movability and approximate movability and relate it to the notion of an approximate polyhedron.

Let $\mathcal{F}$ and $\mathcal{G}$ be classes of relations and let $\mathcal{X}$ be a class of topological spaces. Let $\omega = (\mathcal{F}, \mathcal{G})$. A space $B$ is called proximately $(\mathcal{X}, \omega)$-movable provided for every $\sigma \in \text{Cov}(B)$ there is a $\tau \in \text{Cov}(B)$ such that for every $X \in \mathcal{X}$ and every $\tau \mathcal{F}$-relation $F : X \to B$ there is a $\sigma \mathcal{G}$-relation $G : X \to B$ with $F \simeq G$. On the other hand, a space $B$ is approximately $(\mathcal{X}, \omega)$-movable provided for every $\sigma \in \text{Cov}(B)$ there is a $\tau \in \text{Cov}(B)$ so that for every $X$ in $\mathcal{X}$, every $\tau \mathcal{F}$-relation $F : X \to B$, and every $\varrho \in \text{Cov}(B)$ there is a $\varrho \mathcal{G}$-relation $G : X \to B$ with $F \simeq G$.

One can show that when closeness is replaced by nearness in the above definitions we shall get notions that agree with the original properties.

Recall [8] that an approximate polyhedron is a topological space $Y$ with the property that for every $\varrho \in \text{Cov}(Y)$ there are a polyhedron $P$ and maps $u : Y \to P$ and $d : P \to Y$ with $\text{id}_Y \simeq d \circ u$.

**Theorem 5.1.** Let $\omega = (\mathcal{R}, \mathcal{M})$. Let $\mathcal{T}$ be the class of all topological spaces. A space $Y$ is an approximate polyhedron iff it is approximately $(\mathcal{T}, \omega)$-movable.

**Proof.** ($\Rightarrow$) Let a numerable covering $\sigma$ of $Y$ be given. Let $\lambda \in \sigma^*$ and $\nu \in \lambda^*$. Choose a simplicial polytope $P$ with the CW-topology and maps $u : Y \to P$ and $d : P \to Y$ with $\text{id}_Y \simeq d \circ u$. Let $\varepsilon = d^{-1}(\nu) \in \text{Cov}(P)$. Let $\eta \in \varepsilon^*$. Since $P$ is an ANR [6, p. 106], there is a refinement $\pi$ of $\eta$ with the property that every partial realization in $P$ of a simplicial polytope $K$ with the Whitehead topology relative to $\pi$ defined on a subpolytope $L$ of $K$ which contains all vertices of $K$ extends to a full realization of $K$ in $P$ relative to $\nu$ [6, p. 122]. Let $\xi \in \pi^*$ and let $\tau \in \text{Cov}(Y)$ be a common refinement of $\nu$ and $u^{-1}(\xi)$.

Consider a $\tau$-relation $F : X \to Y$. Choose a covering $\beta \in \text{Cov}(X)$ such that $F$ is a $(\beta, \tau)$-relation. Let $\{\lambda_B|B \in \beta\}$ be a partition of unity subordinated to $\beta$, and let $\{\mu_B|B \in \beta\}$ be a locally finite improvement [5, p. 354]. Let $\varrho \in \{\mu_B^{-1}(\{0, 1\})|B \in \beta\}$. Hence, for every $R \in \varrho$ there is a $T_R \in \tau$, an $N_R \in \nu$, a $K_R \in \xi$, $y_R \in Y$, and a $z_R \in P$ with $F(R) \subset T_R$, $T_R \subset N_R$, $u(T_R) \subset K_R$, $y_R \in T_R$, $z_R \in K_R$, and $z_R = u(y_R)$. Let $p : X \to N(\varrho)$ be a canonical map of $X$ into the nerve $N(\varrho)$ of $\varrho$ (see [5]).

Define a function $\varphi : N(\varrho)^0 \to P$ by the rule $\varphi(R) = z_R$ for every $R \in \varrho$. This function is continuous and it provides a partial realization of $N(\varrho)$ in $P$ relative to the covering $\pi$. 
Indeed, let $\delta = (A, \ldots, Z)$ be a simplex of $N(\varphi)$. We shall find a member of $\pi$ which contains the set $\varphi(N(\varphi)^0 \cap \delta)$, i.e., the set $\{z_A, \ldots, z_Z\}$. Suppose $x \in A \cap \cdots \cap Z$. Since $F(x)$ is nonempty, the sets $T_A, \ldots, T_Z$ and therefore also the sets $K_A, \ldots, K_Z$ have nonempty intersection. Since $\xi$ is a star-refinement of $\pi$, it is clear that some member $P_\xi$ of $\pi$ contains their union.

Let $\psi: N(\varphi) \to P$ be a full realization in $P$ of $N(\varphi)$ relative to $\eta$. Let $f$ denote the composition $d \circ \psi$. Then $f$ is the required continuous single-valued function.

Let $x \in X$ and suppose that $A, \ldots, Z$ are all the members of $\varphi$ which contain the point $x$. Then $p(x)$ lies in the simplex $\delta$ of $N(\varphi)$ determined by these sets. It follows that a member $E_x$ of $\eta$ contains both $\psi \circ p(x)$ and the points $z_A, \ldots, z_Z$. Since $\xi$ refines $\eta$ and $\eta$ is a star-refinement of $\varepsilon$, there is a member $N_x$ of $\nu$ with $d(E_x \cup K_A \cup \cdots \cup K_Z) \subset N_x$.

On the other hand, we have the existence of members $N_A, \ldots, N_Z$ of $\nu$ such that $N_C$ contains both $y_C$ and $d(z_C)$ for every $C = A, \ldots, Z$. It follows that $f \circ p(x) \in N_x$, $d(z_A) \in N_x \cap N_A$, $y_A \in N_A \cap T_A$, and $F(x) \subset T_A$. Hence, some member of $\sigma$ contains both $f \circ p(x)$ and $F(x)$.

Let a numerable covering $\sigma$ of $Y$ be given. Let $\xi \in \sigma^*$. Since the space $Y$ is approximately $(\mathcal{T}, \omega)$-movable, there is a $\tau \in \xi^+$ with the property that every $\tau$-relation into $Y$ is $\xi$-close to a continuous single-valued function. Let $P$ denote the geometric realization $|N(\tau)|$ of the nerve $N(\tau)$ of the numerable covering $\tau$. Let $u: Y \to P$ be a canonical projection map. Let $B_\tau: P \to Y$ be a natural $\tau$-relation defined as follows: For a point $x \in P$, $B_\tau(x)$ is the intersection of all vertices of $N(\tau)$ with respect to which the coordinates of $x$ are positive. Observe that the composition $B_\tau \circ u$ is $\tau$-close to the identity map $id_Y$ on $Y$. By assumption, the relation $B_\tau$ is $\xi$-close to a continuous single-valued function $d: P \to Y$. Clearly, the composition $d \circ u$ is $\sigma$-close to $id_Y$ so that $Y$ is an approximate polyhedron.

6. Fibrations and approximate fibrations

In this section we shall define various kinds of approximate fibrations and study how they relate to each other.

Unless stated otherwise, we shall use $\mu$, $\varphi$, and $\sigma$ to denote the triples $(\mathcal{M}, \mathcal{M}, \mathcal{M})$, $(\mathcal{R}, \mathcal{R}, \mathcal{R})$, and $(\mathcal{S}, \mathcal{S}, \mathcal{S})$, respectively. The class of all topological spaces is denoted by $\mathcal{T}$ while $\mathcal{P}$ stands for the class of all polyhedra.

Let $\tau = (\mathcal{G}, \mathcal{H}, \mathcal{K})$ be a triple of classes of relations and let $\mathcal{X}$ be a class of topological spaces. A map $p: E \to B$ is said to be an $(\mathcal{X}, \tau)$-fibration provided for every $\alpha \in \text{Cov}(B)$ and every $\delta \in \text{Cov}(E)$ there are $\beta \in \alpha^+$ and $\varepsilon \in \delta^+$ such that for every $X \in \mathcal{X}$, every $\varepsilon \mathcal{G}$-relation $G: X \to E$, and every $\beta \mathcal{H}$-relation $H: X \times I \to B$ with $H_0 = p \circ G$ there is a $\delta \mathcal{K}$-relation $K: X \times I \to E$ with $K_0 = G$ and $p \circ K = H$.

Observe that a map $p: E \to B$ is a Hurewicz fibration if and only if it is a $(\mathcal{T}, \mu)$-fibration while it is a Serre fibration if and only if it is a $(\mathcal{P}, \mu)$-fibration.

On the other hand, a map $p: E \to B$ is called an approximate $(\mathcal{X}, \tau)$-fibration provided for every $\alpha \in \text{Cov}(B)$ and every $\delta \in \text{Cov}(E)$ there is a $\beta \in \text{Cov}(B)$ and an $\varepsilon \in \text{Cov}(E)$...
such that for every member \(X\) of \(\mathcal{X}\), every \(\varepsilon G\)-relation \(G: X \to E\), and every \(\beta H\)-relation \(H: X \times I \to B\) with \(H_0 \beta = p \circ G\) there is a \(\delta K\)-relation \(K: X \times I \to E\) with \(K_0 \delta = G\) and \(p \circ K \alpha H\).

An approximate \((\mathcal{T}, \rho)\)-fibration will be called simply an approximate fibration. One can prove that approximate fibrations agree with approximate \((\mathcal{T}, \sigma)\)-fibrations.

There are three other forms called approximate \((\mathcal{X}, \tau)C\)-fibrations, approximate \((\mathcal{X}, \tau)D\)-fibrations, and approximate \((\mathcal{X}, \tau)CD\)-fibrations. We get them from the above definition by replacing either only the first condition, only the second condition, or both the first and the second conditions on closeness of relations with the equality of relations, respectively.

Observe that a map \(p: E \to P\) is an approximate fibration in the sense of Coram and Duvall [3] if and only if it is an approximate \((\mathcal{T}, \mu)CD\)-fibration.

Finally, by replacing in the above definitions the relation of closeness with the relation of nearness we get four additional versions which we denote in the same way using the word proximate instead of the word approximate.

Our first task will be to show that these two groups of approximate fibrations agree. In other words, closeness and nearness for relations give the same concepts of approximate fibrations.

**Theorem 6.1.** Let \(\tau = (\mathcal{G}, \mathcal{H}, \mathcal{K})\) be a triple of classes of relations and let \(\mathcal{X}\) be a class of topological spaces. A map \(p: E \to B\) is an approximate \((\mathcal{X}, \tau)\)-fibration if and only if it is a proximate \((\mathcal{X}, \tau)\)-fibration.

**Proof.** (\(\Leftarrow\)) Let an \(\alpha \in \text{Cov}(B)\) and a \(\delta \in \text{Cov}(E)\) be given. Let \(\xi \in \alpha^*\) and \(\pi \in \delta^* \cap p^{-1}(\xi)^+\). Since \(p\) is an approximate \((\mathcal{X}, \tau)\)-fibration, there is a \(\beta \in \xi^+\) and an \(\varepsilon \in \pi^+\) with the property that for every space \(X \in \mathcal{X}\), every \(\varepsilon G\)-relation \(G: X \to E\), and every \(\beta H\)-relation \(H: X \times I \to B\) with \(H_0 \beta = p \circ G\) there is a \(\pi K\)-relation \(K: X \times I \to E\) such that \(G \delta K_0\) and \(H \xi p \circ K\).

Let \(X, G,\) and \(H\) as above with \(H_0 \equiv p \circ G\) be given. Since \(H_0 \equiv p \circ G\) implies \(H_0 \beta = p \circ G\), by assumption, there is a \(K\) as above. However, it follows from Lemma 3.1 that \(G \delta K_0\) and \(H \xi p \circ K\). Hence, \(p\) is a proximate \((\mathcal{X}, \tau)\)-fibration.

(\(\Rightarrow\)) Let an \(\alpha \in \text{Cov}(B)\) and a \(\delta \in \text{Cov}(E)\) be given. Since the map \(p\) is a proximate \((\mathcal{X}, \tau)\)-fibration, there is a \(\xi \in \text{Cov}(B)\) and a \(\pi \in \text{Cov}(E)\) with the property that for every space \(X \in \mathcal{X}\), every \(\pi G\)-relation \(G: X \to E\), and every \(\xi H\)-relation \(H: X \times I \to B\) with \(H_0 \equiv p \circ G\) there is a \(\delta K\)-relation \(K: X \times I \to E\) such that \(G \delta K_0\) and \(H \alpha p \circ K\). Let \(\beta \in \xi^*\) and \(\varepsilon \in \pi^* \cap p^{-1}(\beta)^+\).

Let \(X, G,\) and \(H\) as above with \(H_0 \equiv p \circ G\) be given. Since both \(H_0\) and \(p \circ G\) are \(\beta\)-relations, it follows from Lemma 3.1 that \(H_0 \equiv p \circ G\). By assumption, there is a \(K\) as above. But, since the nearness relation is stronger than the closeness relation, we get \(K_0 \delta G\) and \(H \alpha p \circ K\). Hence, the map \(p\) is an approximate \((\mathcal{X}, \tau)\)-fibration. \(\square\)
In our terminology, the content of Lemmas 1.2 in [3] and 2.5 in [4] can be stated as follows: for a map $p : E \to B$ between absolute neighbourhood retracts the following are equivalent:

1. $p$ is an approximate $(\mathcal{T}, \mu)C$-fibration,
2. $p$ is an approximate $(\mathcal{T}, \mu)D$-fibration, and
3. $p$ is an approximate $(\mathcal{T}, \mu)CD$-fibration.

In the next theorem which resembles the above statement we shall see that all four versions of approximate fibrations coincide if we use relations instead of maps.

**Theorem 6.2.** Let $\vartheta$ be either $(\mathcal{R}, \mathcal{R}, \mathcal{R})$ or $(\mathcal{S}, \mathcal{S}, \mathcal{S})$. For a map $p : E \to B$ the following are equivalent:

1. $p$ is an approximate $(\mathcal{T}, \vartheta)$-fibration,
2. $p$ is an approximate $(\mathcal{T}, \vartheta)C$-fibration,
3. $p$ is an approximate $(\mathcal{T}, \vartheta)D$-fibration, and
4. $p$ is an approximate $(\mathcal{T}, \vartheta)CD$-fibration.

**Proof.** Let $\vartheta = (\mathcal{S}, \mathcal{S}, \mathcal{S})$. We shall prove that (1) and (4) are equivalent. The equivalence of (1) with (2) and (3) can be established in a similar fashion. The case when $\vartheta = (\mathcal{R}, \mathcal{R}, \mathcal{R})$ is analogous.

$(\Rightarrow)$ Let $\alpha \in \text{Cov}(B)$ and $\delta \in \text{Cov}(E)$ be given. Let $\eta \in \delta^{*2}$. Since $p$ is an approximate $(\mathcal{T}, \vartheta)$-fibration, there are $\beta \in \alpha^{+}$ and $\varepsilon \in \eta^{+}$ such that for every space $X$, every $\varepsilon$-function $g : X \to E$, and every $\beta$-function $h : X \times I \to B$ with $p \circ g = h_{0}$, there is an $\eta$-function $k : X \times I \to E$ with $k_{0} = g$ and $p \circ k \equiv h$.

Consider a space $X$, an $\varepsilon$-function, $g : X \to E$, and a $\beta$-function $h : X \times I \to B$ with $h_{0} = p \circ g$. Since the equality of two functions into a space $Z$ implies that they are $\sigma$-close for every enumerable covering $\sigma$ of $Z$, there is a function $\delta$ as above. Define a function $m : X \times I \to E$ by the rule

$$m(x, t) = \begin{cases} g(x), & t = 0, \\ k(x, t), & t \neq 0. \end{cases}$$

Clearly, $m$ is a $\delta$-function, $m_{0} = g$, and $p \circ m = h$.

$(\Leftarrow)$ Let $\alpha \in \text{Cov}(B)$ and $\delta \in \text{Cov}(E)$ be given. Let $\pi \in \alpha^{*}$. Since $p$ is an approximate $(\mathcal{T}, \vartheta)CD$-fibration, there are $\eta \in \pi^{+}$ and $\varrho \in \text{Cov}(E)$ such that for every space $X$, every $\varrho$-function $g : X \to E$, and every $\eta$-function $m : X \times I \to B$ with $p \circ g = m_{0}$, there is a $\delta$-function $k : X \times I \to E$ with $k_{0} = g$ and $p \circ k \equiv m$. Let $\beta \in \eta^{*2}$ and $\varepsilon \in \varrho^{+} \cap p^{-1}(\beta)^{+}$.

Consider a space $X$, an $\varepsilon$-function $g : X \to E$, and a $\beta$-function $h : X \times I \to B$ with $h_{0} = p \circ g$. Now we can define a function $m : X \times I \to m$ by the rule

$$m(x, t) = \begin{cases} p \circ g(x), & t = 0, \\ h(x, t), & t \neq 0. \end{cases}$$
Observe that $m$ is an $\eta$-function and $m_0 = p \circ g$. Select a function $k$ as above. Clearly, $k_0 = g$ and $p \circ k = h$. □

7. Shape fibrations

In this section we shall recall the definition of shape fibrations via AP-resolutions of maps from [8]. Those who are familiar with shape fibrations and resolutions can skip this section and proceed directly to the next section. Here we shall collect only those notions and results from this area necessary for the proof of our main result which establishes the equivalence of approximate fibrations and shape fibrations.

An AP-system $\mathcal{E} = (E_a, q_a^b, A)$ consists of a directed set $(A, \leq)$, a collection $\{E_a\}_{a \in A}$ of approximate polyhedra, and a collection $\{q_a^b: (a, b) \in \leq\}$ of maps $q_a^b: E_b \to E_a$ such that $q_a^b \circ q_c^b = q_a^c$ and $q_a^a = \text{id}$ whenever $a \leq b \leq c$.

A map $q: E \to \mathcal{E}$ of a space $E$ into an AP-system $\mathcal{E} = (E_a, q_a^b, A)$ is a collection $\{q^a: a \in A\}$ of maps $q^a: E \to E_a$ such that $q_a^b \circ q^a = q^b$ whenever $b \geq a$.

A map $q = \{q^a\}: E \to \mathcal{E}$ of a space $E$ into an AP-system $\mathcal{E} = (E_a, q_a^b, A)$ is called an AP-resolution of $E$ provided the following two conditions hold.

(R1) For every approximate polyhedron $P$, every $\sigma \in \text{Cov}(P)$, and every map $f: E \to P$, there are an $a \in A$ and a map $f_a: E_a \to P$ such that $f_a \circ q^a = f$.

(R2) For every approximate polyhedron $P$ and every $\sigma \in \text{Cov}(P)$, there is a $\tau \in \text{Cov}(P)$ such that for every $a \in A$ and all maps $f, g: E_a \to P$ with $f \circ q^a \succeq g \circ q^a$ there is a $b \geq a$ with $f \circ q_b^a \succeq g \circ q_b^a$.

It was observed in [8, Theorem 6] that an AP-resolution $q$ of a space $E$ also has the following property.

(B1) For every $\sigma \in \text{Cov}(E)$, there are an $a \in A$ and a $\tau \in \text{Cov}(E_a)$ such that $(q^a)^{-1}(\sigma)$ refines $\sigma$.

A map $p: \mathcal{E} \to \mathcal{B}$ between AP-systems $\mathcal{E} = (E_a, q_a^b, A)$ and $\mathcal{B} = (B_m, q_m^n, M)$ is a pair $p = (\pi, \{p_m\}_{m \in M})$ consisting of an increasing function $\pi: M \to A$ and a collection of maps $p_m: E_{\pi(m)} \to B_m$ such that $r_m \circ p_m = p_{\pi(m)} \circ q_{\pi(m)}$ whenever $n \geq m$.

A pair of indices $(a, m) \in A \times M$ is called admissible provided $a \geq \pi(m)$. For every admissible pair we define a map $p^a_m: E_a \to B_m$ by $p^a_m = p_m \circ q_{\pi(m)}^a$.

A map of AP-systems $p = (\pi, \{p_m\}): \mathcal{E} \to \mathcal{B}$ has the AHLP(\mathcal{X}) (the approximate homotopy lifting property with respect to a class of spaces $\mathcal{X}$) provided for every admissible pair $(b, m) \in A \times M$, every $\tau \in \text{Cov}(E_b)$, and every $\eta \in \text{Cov}(B_m)$, there exist an admissible pair of indexes $(j, i)$ (called lifting indexes) and a $\lambda \in \text{Cov}(B_i)$ (called lifting mesh) such that $(j, i) \geq (b, m)$ (i.e., $j \geq b$ and $i \geq m$) and for an arbitrary space $X \in \mathcal{X}$ and for arbitrary maps $f: X \to E_j$, $h: X \times I \to B_i$ with $h_0 = p_j^i \circ f$, there exists a homotopy $k: X \times I \to E_b$ such that $k_0 = q_j^b \circ f$ and $p_b^m \circ k \succeq r_i^m \circ h$.

An AP-resolution of a map $p: E \to B$ consists of an AP-resolution $q: E \to \mathcal{E}$ of $E$, of an AP-resolution $r: \mathcal{E} \to \mathcal{B}$ of $\mathcal{B}$, and of a map of AP-systems $p: \mathcal{E} \to \mathcal{B}$ such that $r_m \circ p = p_m \circ q_{\pi(m)}$ for every index $m \in M$.

Now we can define shape $X$-fibrations as was done in [8].
Let $\mathcal{X}$ be a class of topological spaces. A map $p : E \to B$ between topological spaces is a shape $\mathcal{X}$-fibration provided there exists an AP-resolution $(q, r, p)$ of $p$ such that $p : E \to B$ has the AHLP($\mathcal{X}$). When $\mathcal{X}$ is the class $\mathcal{T}$ of all topological spaces, then we use the term shape fibration for shape $\mathcal{T}$-fibration.

To assess fully this definition one must bear in mind these two facts:

(i) Every map admits an AP-resolution (see [8, Theorems 11 and 13]).

(ii) In order to decide whether a map $p$ is a shape $\mathcal{X}$-fibration, one can use any AP-resolution of $p$ (see [8, Theorem 4]).

8. Approximate fibrations and shape fibrations

Recall that $q = (\mathcal{R}, \mathcal{R}, \mathcal{R})$, where $\mathcal{R}$ is the class of all relations. Let $\mathcal{X}$ be a class of spaces. In this section we shall prove that approximate ($\mathcal{X}, q$)-fibrations agree with Mardešić’s shape $\mathcal{X}$-fibrations [8]. In particular, we get that approximate fibrations agree with shape fibrations.

**Lemma 8.1.** Let $q : E \to \mathcal{E}$ be an AP-resolution of a space $E$, where $\mathcal{E} = \{E_a, q_a^*, A\}$ and $q = \{q^a\}$. Then for every $\varepsilon \in \text{Cov}(E)$, every $a \in A$, every $\alpha \in \text{Cov}(E_a)$, and every $b \geq a$ there is a $c \geq b$ and an $\varepsilon$-relation $S_c : E_c \to E$ such that $S_c \circ q^c \equiv \text{id}_E$ and $q^a \circ S_c = q_a^c$.

**Proof.** Let $\delta \in \varepsilon^*$ and $\eta \in \alpha^*$. Since the AP-resolution $q$ of the space $E$ satisfies the condition (R2), there is a $\xi \in \eta^+$ such that for any $w \in A$ and for any two maps $g, h : E_w \to E_a$ for which $g \circ q^\omega = h \circ q^\omega$, there exists a $u \geq w$ such that $g \circ q^u = h \circ q^u$ for any $v \geq u$. On the other hand, observe that

$$q^a \circ q^\eta = q^\eta$$

for all $d \geq c \geq b$ and that

$$q^\varepsilon \equiv q^\varepsilon \circ q^\eta$$

for every $c \geq b$ and every $\varepsilon \in \text{Cov}(E_a)$. Let $\zeta \in \xi^*$. By Theorem 5.1, there is a refinement $\pi$ of $\zeta$ with the property that every $\pi$-relation into $E_a$ is $\zeta$-close to a map. Let $\nu \in \text{Cov}(E)$ be a common refinement of the numerable coverings $\delta$ and $(q^\omega)^{-1}(\pi)$. Let $Z$ denote a geometric realization $|N(\nu)|$ of the nerve of $\nu$. Let $p : E \to Z$ be a canonical projection. Define a relation $B_\nu : Z \to E$ by the rule that the image of a point $z \in Z$ is the intersection of all members of $\nu$ with respect to which the coordinates of $z$ are positive. Let $\sigma$ denote the numerable covering $\star^\nu$ of $Z$ by open stars of vertices of $Z$. Next, we use the condition (R1) for the AP-resolution $q$ with respect to the polyhedron $Z$, the covering $\sigma$, and the mapping $p$ to select an index $d \geq b$ such that there exists a mapping $g_d : E_d \to Z$ satisfying $g_d \circ q^d = p$. Let $\bar{q} = (g_d)^{-1}(\sigma)$. Observe that

$$q^d \equiv q_d^\bar{q} \circ q^\bar{q}$$

(3)
for every \( c \geq d \). Moreover, since \( p \) is a canonical map, \( B_\nu \) is a \((\sigma, \nu)\)-relation; since \( \nu \) refines \((q^a)^{-1}(\pi)\), the map \( q^a \) is a \((\nu, \pi)\)-relation; and since \( g_d \) is a map, it is a \( \sigma \)-relation. Hence, by Lemma 2.1, the composition \( q^a \circ B_\nu \circ g_d \) is a \( \pi \)-relation. By the choice of \( \pi \), there is a mapping \( f^a_d : E_d \to E_a \) with

\[
f^a_d \equiv q^a \circ B_\nu \circ g_d.
\]  

(4)

It follows that

\[
f^a_d \circ q^d \equiv q^a \circ B_\nu \circ g_d \circ q^d.
\]  

(5)

But,

\[
g_d \circ q^d \equiv p
\]  

(6)

and since \( B_\nu \) is a \((\sigma, \nu)\)-relation, we get

\[
q^a \circ B_\nu \circ g_d \circ q^d \equiv q^a \circ B_\nu \circ p.
\]  

(7)

However,

\[
B_\nu \circ p \equiv \text{id}_E,
\]  

(8)

so that

\[
q^a \circ B_\nu \circ p \equiv q^a.
\]  

(9)

From (5), (7), (9), and (2) we get \( f^a_d \circ q^d \equiv q^a \circ q^d \). The way in which the covering \( \xi \) was selected implies the existence of an index \( c \geq d \) such that

\[
f^a_d \circ q^d \equiv q^a \circ q^d.
\]  

(10)

From (4), (10), and (1) we have

\[
q^a \circ B_\nu \circ q^d \equiv q^a.
\]  

Therefore, if we put \( S_c = B_\nu \circ g_d \circ q^d \) and \( \beta = (q^d)^{-1}(\beta) \), then the relation \( S_c \) will be a \((\beta, \epsilon)\)-relation and \( q^a \circ S_c \equiv q^a \).

On the other hand, it follows from (3) that

\[
S_c \circ q^a \equiv B_\nu \circ g_d \circ q^a.
\]  

(11)

Moreover, from the relation (6) and the fact that \( B_\nu \) is a \((\sigma, \delta)\)-relation because \( \nu \) refines \( \delta \), we obtain

\[
B_\nu \circ g_d \circ q^d \equiv B_\nu \circ p.
\]  

(12)

Finally, from the relations (8), (11), and (12) we arrive at the desired estimate \( S_c \circ q^a \equiv \text{id}_E \). □

**Remark 8.2.** The above lemma is also true for the approximate AP-resolutions of Mardešić and Watanabe [12]. The proof for this case is almost identical with the above argument. In order to get the relation (1), we use the condition (A2) from [12] with
respect to $a$ and $\eta$ which will give us an $m \geq b$ so that (1) holds for all $d \geq c \geq m$.
On the other hand, to get the relation (2), we utilize the condition (AS) from [12] with
respect to $a$ and $\zeta$ which will give us an $n \geq m$ so that (2) is true for every $c \geq n$.
Next, we select $d \geq n$ as above. Finally, for the relation (3), we use (AS) with respect
to $d$ and $p$ and get an $e \geq d$ so that (3) holds for every $c \geq e$. The index $c$ must now be
selected so that $c \geq e$.

The following theorem is the main result in this paper. Notice that in this section we
can suppress the class from which the relation emanates because only $\mathcal{R}$-relations will
be used and we call these simply relations.

**Theorem 8.3.** Let $\mathcal{X}$ be a class of topological spaces. A mapping $p : E \to B$ is an
approximate $(\mathcal{X}, \varphi)$-fibration if and only if it is a shape $\mathcal{X}$-fibration.

**Proof.** $(\Rightarrow)$ Let $(q, r, p)$ be an AP-resolution of $p$. Then

$q = \{q^n\} : E \to \mathcal{E} = \{E_m, q^n_\mu : N\}$

is an AP-resolution of $E$, $r = \{r^m\} : B \to \mathcal{B} = \{B_m, r^m_\mu : M\}$ is an AP-resolution of
$B$, and $p = (\pi, p_m) : \mathcal{E} \to \mathcal{B}$ is a map of AP-systems such that $p \circ q = r \circ p$, i.e., such
that $r^m \circ p = p_m \circ q^\mu_\mu$ for every index $m \in M$. In order to prove that $p$ is a shape
$\mathcal{X}$-fibration, we shall show that the map of AP-systems $p$ has the approximate homotopy
lifting property (AHLP) with respect to the class $\mathcal{X}$.

Let $(n, m) \in N \times M$ be an admissible pair of indices (i.e., $n \geq \pi(m)$). Let $\mu \in
\mathrm{Cov}(B_m)$ and $\nu \in \mathrm{Cov}(E_n)$. Let $\xi \in \nu^*$. Let $\eta \in \nu^*$ refine the numerable covering
$(p^n_\mu)^{-1}(\xi)$. Since $E_n$ is an approximate polyhedron, according to Theorem 5.1, there is
an $\omega \in \eta^+$ such that $\omega$-relations into $E_n$ are $\eta$-close to maps. Let $\alpha = (r^n_\mu)^{-1}(\xi)$ and
$\delta = (q^n)^{-1}(\omega)$. Now we utilize the assumption that $p$ is an approximate $(\mathcal{X}, \varphi)$-fibration
to select numerable coverings $\beta \in \mathrm{Cov}(B)$ and $\varepsilon \in \mathrm{Cov}(E)$ such that for every space
$X \in \mathcal{X}$, every $\varepsilon$-relation $F : X \to E$, and every $\beta \mathcal{R}$-homotopy $H : X \times I \to B$ with

$$H_0 \overset{\delta}{\approx} p \circ F$$  (13)

there is a $\delta \mathcal{R}$-homotopy $K : X \times I \to E$ with

$$K_0 \overset{\delta}{\approx} F$$  (14)

and

$$H \overset{\varepsilon}{\approx} p \circ K.$$  (15)

Let $\gamma \in \beta^*$.  

Next, we apply the condition (B1) for the AP-resolution $r$ to get an index $i \geq m$ and
a numerable covering $\lambda \in \mathrm{Cov}(B_i)$ such that the numerable covering $\varphi = (r^i)^{-1}(\lambda)$ of
$B$ refines the covering $\gamma$. Let $\psi \in \lambda^*$ be a refinement of the covering $(r^i_\mu)^{-1}(\xi)$. Let
$\kappa \in \mathrm{Cov}(B)$ be a common refinement of $\varphi$ and $(r^i)^{-1}(\psi)$.
Now we use Lemma 8.1 for the AP-resolution $\tau$ with respect to $\kappa$, $i$, $\psi$, and $i$ to choose an index $a \geq i$, a numerable covering $\tau \in \text{Cov}(B_a)$, and a $(\tau, \kappa)$-relation $T_a : B_a \rightarrow B$ such that

$$T_a \circ r^\kappa \cong \text{id}_B.$$ (16)

and

$$r^i \circ T_a \cong r^i_a.$$ (17)

Let $\zeta \in \text{Cov}(B)$ be a common refinement of $\varphi$ and $(r^\kappa)^{-1}(\tau)$. Let $\varrho$ be a common refinement of $\epsilon$ and $p^{-1}(\zeta)$. Let $j \geq n, \pi(i)$. Let $\chi$ be a common refinement of $(q_j^\sigma)^{-1}(\omega)$ and $(p_j^\psi)^{-1}(\psi)$.

We use Lemma 8.1 once again for the AP-resolution $q$ with respect to $\varrho$, $j$, $\chi$, and $\pi(a)$ to get an index $b \geq \pi(a)$, a numerable covering $\sigma \in \text{Cov}(E_b)$, and a $(\sigma, \varrho)$-relation $S_b : E_b \rightarrow E$ such that

$$S_b \circ q^b \cong \text{id}_E.$$ (18)

and

$$q^i \circ S_b \cong q^i_b.$$ (19)

Then $a$ and $b$ are the required lifting indexes and $\tau$ is the required lifting mesh. Indeed, consider a space $X \in \mathcal{X}$ and mappings $f : X \rightarrow E_b$ and $h : X \times I \rightarrow B_a$ with

$$p_b^\varrho \circ f \cong h_b.$$ (20)

Let $F$ and $H$ denote the compositions $S_b \circ f$ and $T_a \circ h$.

Observe that $\varrho$ refines $\epsilon$ and $S_b$ is a $(\sigma, \varrho)$-relation so that $F$ is an $\epsilon$-relation. Similarly, $H$ is a $\beta$-relation. Moreover, we shall now prove that the relation (13) holds.

In order to do this, it suffices to show that

$$r^i \circ H_0 \cong r^i \circ p \circ F.$$ (21)

Indeed, since $(r^i)^{-1} : r^i(B) \rightarrow B$ is a $(\lambda)(r^i(B), \varphi)$-relation and the relations appearing on both sides of (21) have values in $r^i(B)$, from (21) we obtain

$$(r^i)^{-1} \circ r^i \circ H_0 \cong (r^i)^{-1} \circ r^i \circ p \circ F.$$ (22)

Notice that $(r^i)^{-1} \circ r^i$ is $(\varphi, \varphi)$-near to the identity function $\text{id}_B$. Moreover, $H_0$ is a $\varphi$-relation because $H$ is a $\kappa$-relation and $\kappa$ refines $\varphi$. Finally, $p \circ F$ is also a $\varphi$-relation because $F$ is a $\varrho$-relation, $\varrho$ refines $p^{-1}(\zeta)$, and $\zeta$ refines $\varphi$. It follows that

$$H_0 \cong (r^i)^{-1} \circ r^i \circ H_0,$$ (23)

and

$$p \circ F \cong (r^i)^{-1} \circ r^i \circ p \circ F.$$ (24)

Combining relations (22), (23), and (24) we conclude that (13) is true.

On the way of establishing (21), first notice that (20) can be used to show that

$$r^i \circ H_0 = r^i \circ T_a \circ h_0 \cong r^i \circ T_a \circ p_b^\varrho \circ f.$$ (25)
The estimate (17) gives
\[ r_i^i \circ T_a \circ p_{b}^a \circ f = r_{a}^i \circ p_{b}^a \circ f. \] (26)

Observe that \( r_{a}^i \circ p_{b}^a = p_{j}^i \circ q_{b}^j \) because for a map between AP-resolutions all diagrams are strictly commutative. This equality and (19) imply
\[ r_{a}^i \circ p_{b}^a \circ f = p_{j}^i \circ q_{b}^j \circ S_{b} \circ f = r_{j}^i \circ p \circ F. \] (27)

The relations (25)–(27) together give the relation (21).

Once we know that (13) is true, our choices imply that there is a \( \delta R \)-homotopy \( K : X \times I \to E \) so that (14) and (15) hold. The composition \( q_{n}^n \circ K \) is an \( \omega \)-relation \( \eta \)-close to a map \( k : X \times I \to E_{n} \). We claim that \( k \) is the required approximate lifting of the homotopy \( h \).

Indeed, by our selections the relation \( k \overset{\eta}{=} q_{n}^n \circ K \) implies
\[ k_{0} \overset{\eta}{=} q_{n}^n \circ K_{0}, \] (28)

and
\[ p_{n}^m \circ k \overset{\xi}{=} p_{n}^m \circ q_{n}^n \circ K = r_{m}^m \circ p \circ K. \] (29)

From (14) we obtain
\[ q_{n}^n \circ K_{0} \overset{\eta}{=} q_{n}^n \circ F = q_{i}^j \circ q_{b}^j \circ S_{b} \circ f. \] (30)

while (19) implies
\[ q_{i}^j \circ q_{b}^j \circ S_{b} \circ f = q_{b}^j \circ q_{b}^j \circ f. \] (31)

Since \( \omega \) refines \( \eta \) and \( st(\eta) \) refines \( \nu \), from (28), (30), and (31) we now get the estimate \( k_{0} \overset{\nu}{=} q_{n}^n \circ F \).

On the other hand, from (15) it follows that
\[ r_{m}^m \circ p \circ K \overset{\xi}{=} r_{m}^m \circ H = r_{m}^i \circ r_{i}^m \circ T_{a} \circ h; \] (32)

while (17) gives
\[ r_{m}^i \circ r_{i}^m \circ T_{a} \circ h \overset{\xi}{=} r_{m}^i \circ r_{a}^m \circ h = r_{a}^m \circ h. \] (33)

The relations (29), (32) and (33) imply \( p_{n}^m \circ k \overset{\nu}{=} r_{a}^m \circ h \).

(\( \Leftarrow \)) Let an \( \alpha \in Cov(B) \) and a \( \delta \in Cov(E) \) be given. Let \( \chi \in \alpha^{*} \). Let \( \xi \in \delta^{*} \) and \( \theta \in \xi^{*} \). It follows from the property (B1) for the AP-resolution \( r \) that we can find an index \( m \in M \) and a numerable covering \( \mu \in Cov(B_{m}) \) such that \( \kappa = (r_{m}^{m})^{-1}(\mu) \) refines \( \chi \). Let \( \eta \in \mu^{*} \). In a similar way we select an index \( n \geq \pi(m) \) and a numerable covering \( \nu \in Cov(E_{n}) \) such that \( \nu \) refines \( (p_{n}^{m})^{-1}(\eta) \) and \( \gamma = (q_{n}^{n})^{-1}(\nu) \) refines both \( \theta \) and \( p^{-1}(\kappa) \).

Let us apply Lemma 8.1 to the AP-resolution \( q \) with respect to \( \gamma, n, \nu, \) and \( n \) to choose an index \( b \geq n, \) a numerable covering \( \sigma \in Cov(E_{b}) \), and a \( (\sigma, \gamma) \)-relation \( S_{b} : E_{b} \to E \) such that
\[ S_{b} \circ q_{b}^{h} \overset{\gamma}{=} id_{E}. \] (34)
and

$$q^n \circ S_b \equiv q^n_b.$$  \hfill (35)

Let $\tau \in \text{Cov}(E_b)$ be a common refinement of $\sigma$ and $(q^n_b)^{-1}(\nu)$.

Since $p$ is a shape $\mathcal{X}$-fibration, the mapping $p$ has the AHLP with respect to the class $\mathcal{X}$, so that for the admissible pair $(b, m)$ and numerable coverings $\tau \in \text{Cov}(E_b)$ and $\eta \in \text{Cov}(B_m)$ there exists an admissible pair $(j, i) \geq (b, m)$ and a numerable covering $\lambda \in \text{Cov}(B_j)$ such that for every space $X \in \mathcal{X}$ and for arbitrary maps $f : X \rightarrow E_j$ and $h : X \times I \rightarrow B_i$ with

$$p_j^i \circ f \equiv h_0$$  \hfill (36)

there exists a homotopy $k : X \times I \rightarrow E_b$ such that

$$q_j^h \circ f \equiv k_0$$  \hfill (37)

and

$$p_b^m \circ k \equiv \tau_j^m \circ h.$$  \hfill (38)

Let $\zeta \subset \lambda^*$ refine $(r_i^m)^{-1}(\eta)$.

Since $B_i$ is an approximate polyhedron, by Theorem 5.1, there is an $\omega \in \zeta^+$ such that $\omega$-relations into $B_i$ are $\zeta$-close to maps. Let $\beta$ be a common refinement of $(r_i^j)^{-1}(\omega)$ and $\kappa$.

Let $\varphi \in \text{Cov}(E_j)$ be a common refinement of $(p_j^i)^{-1}(\omega)$ and $(q_j^b)^{-1}(\tau)$. Since $E_j$ is also an approximate polyhedron, we can find a numerable covering $\psi \in \text{Cov}(E_j)$ such that $\psi$-relations into $E_j$ are $\varphi$-close to maps.

Let us observe that both $S_b \circ q^b$ and $\text{id}_E$ are $\theta$-relations so that the relation (34) with the help of Lemma 3.1 implies that $S_b \circ q^b \equiv \text{id}_E$. Choose a numerable covering $\varphi \in \gamma^+$ such that

$$S_b \circ q^b \varphi \equiv \text{id}_E.$$  \hfill (39)

Let $\varepsilon \in \text{Cov}(E)$ be a common refinement of $\varphi$ and $(q^j)^{-1}(\psi)$. Then $\beta$ and $\varepsilon$ are the required numerable coverings.

Indeed, consider a space $X$ from the class $\mathcal{X}$, an $\varepsilon$-relation $F : X \rightarrow E$, and a $\beta$-relation $H : X \times I \rightarrow B$ with

$$H_0 \beta \equiv p \circ F.$$  \hfill (40)

The composition $q^j \circ F$ is a $\psi$-relation of $X$ into $E_j$. By assumption, there is a map $f : X \rightarrow E_j$ with

$$f \equiv q^j \circ F.$$  \hfill (41)

Also, for a similar reason, there is a map $h : X \times I \rightarrow B_i$ with

$$h \equiv \tau^i \circ H.$$  \hfill (42)

We claim that (36) holds. In order to prove this, observe that (41) implies

$$p_j^i \circ f \equiv p_j^i \circ q^j \circ F.$$
But, $p_j^i q^j = r^i \circ p$ so that
\[ p_j^i f \cong r^i \circ p \circ F. \tag{43} \]

On the other hand, the relation (40) implies
\[ r^i \circ H_0 \cong r^i \circ p \circ F, \tag{44} \]
while (42) gives
\[ h_0 \cong r^i \circ H_0. \tag{45} \]

The relations (43), (44), and (45) together imply (36).

Now we infer that there is a homotopy $k : X \times I \to E_b$ such that (37) and (38) hold. Let $K = S_b \circ k$. Since $S_b$ is a $\gamma$-relation, it follows that $K$ is a $\delta$-relation. We claim that $K$ is the required approximate lifting of $H$, i.e., that
\[ F \cong K_0 \tag{46} \]
and
\[ H \cong p \circ K. \tag{47} \]

On the way of checking (46), let us observe that (37) implies
\[ K_0 = S_b \circ k_0 \cong S_b \circ q_j^b \circ f, \tag{48} \]
while (41) gives
\[ S_b \circ q_j^b \circ f \cong S_b \circ q_j^b \circ q^j \circ F = S_b \circ q^b \circ F. \tag{49} \]

Since $F$ is a $\varphi$-relation, from (39) we obtain
\[ S_b \circ q^b \circ F \cong F. \tag{50} \]

The last three relations together imply the relation (46).

In order to establish the relation (47), it suffices to prove that
\[ r^m \circ H \cong r^m \circ p \circ K. \tag{51} \]

Indeed, since $(r^m)^{-1} : r^m(B) \to B$ is a $(\mu, r^m(B), \chi)$-relation and relations which appear on both sides of (51) have values in $r^m(B)$, from (51) we obtain
\[ (r^m)^{-1} \circ r^m \circ H \cong (r^m)^{-1} \circ r^m \circ p \circ K. \tag{52} \]

But, $(r^m)^{-1} \circ r^m$ is $(\kappa, \kappa)$-near to the identity function $\text{id}_B$ and both $H$ and $p \circ K$ are $\kappa$ relations so that
\[ H \cong (r^m)^{-1} \circ r^m \circ H, \tag{53} \]
and
\[ p \circ K \cong (r^m)^{-1} \circ r^m \circ p \circ K. \tag{54} \]

Combining relations (52), (53), and (54) we conclude that (47) is true.
In order to verify the relation (51), first observe that
\[ r^{nm} \circ p \circ K = r^{nm} \circ p \circ S_b \circ k = p^m_0 \circ q^n_0 \circ S_b \circ k, \tag{55} \]
since \( r^{nm} \circ p = p^m_0 \circ q^n_0 \). Also, from (35), we obtain
\[ p^m_0 \circ q^n_0 \circ S_b \circ k = p^m_0 \circ q^n_0 \circ k = p^m_0 \circ k, \tag{56} \]
while (42) has as a consequence
\[ r^{nm}_i \circ h = r^{nm}_i \circ r^i \circ H = r^m \circ H. \tag{57} \]

The combination of relations (55), (56), (38), and (57) implies the relation (51). \( \square \)

Recall that \( \mathcal{D} \) and \( \mathcal{D}^n \) denote the class of all \( k \)-dimensional disks for all \( k \geq 0 \) and for all \( n \geq k \geq 0 \), respectively. In a similar way one can also prove the following.

**Theorem 8.4.** A map \( p : E \to B \) is an approximate \( (\mathcal{D}, \varrho) \)-fibration if and only if it is a weak shape fibration [11].

**Theorem 8.5.** A map \( p : E \to B \) is an approximate \( (\mathcal{D}^n, \varrho) \)-fibration if and only if it is an \( n \)-shape fibration [10].

**References**