Exponential Energy Decay Estimate for the Solutions of Internally Damped Wave Equation in a Bounded Domain

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Here we are concerned about the stability of the solution of internally damped wave equation \( y'' = \Delta y + \mu \Delta y' \) with small damping constant \( \mu > 0 \), in a bounded domain \( \Omega \) in \( \mathbb{R}^n \) under mixed undamped boundary conditions. A uniform exponential energy decay rate \( E(t) \leq M e^{-\beta t}E(0) \), where \( M \geq 1 \) and \( \beta = \mu/\mu_0 \), \( \mu_0 \) being a quadratic function of \( \mu \), is proved for the solution of this type of boundary value problem. Earlier authors have considered the undamped wave equation with certain forms of damped boundary conditions proving similar and faster energy decay rates. © 1997 Academic Press

1. INTRODUCTION

Let \( \Omega \) be a bounded connected set in \( \mathbb{R}^n \) and let \( \Gamma \) be its boundary which is piecewise smooth consisting of two parts \( \Gamma^0 \) and \( \Gamma^1 \) such that \( \Gamma = \Gamma^0 \cup \Gamma^1 \) and \( \Gamma^0 \cap \Gamma^1 = \emptyset \). We denote by \( \nu \), the unit normal of \( \Gamma \) pointing towards the exterior of \( \Omega \). Let \( x^0 \) be an arbitrary but fixed point in \( \mathbb{R}^n \) and set

\[
m(x) = x - x^0, \quad x \in \mathbb{R}^n.
\]

Let the two disjoint open subsets \( \Gamma^1 \) and \( \Gamma^0 \) of \( \Gamma \) be defined by

\[
m(x) \cdot \nu(x) > 0 \quad \text{on} \ \Gamma^1 \quad \text{(1.2)}
\]

\[
m(x) \cdot \nu(x) \leq 0 \quad \text{on} \ \Gamma^0. \quad \text{(1.3)}
\]

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The question of uniform exponential decay of energy defined by

\[ E(t) = \frac{1}{2} \int_\Omega (y^2 + |\nabla y|^2) \, dx \]  

(1.4)

of the solution of the undamped wave equation in \( \Omega \) has been studied by a number of authors—Chen [1], Lagnese [7], Lasiecka and Triggiani [9], Triggiani [12], and Lions [11]. They considered the prototype system

\[
\begin{align*}
    y'' &= \Delta y \quad \text{in } \Omega \times (0, \infty) \\
    y &= 0 \quad \text{on } \Gamma^0 \times (0, \infty) \\
    \frac{\partial y}{\partial n} &= -b(x)y' \quad \text{on } \Gamma^1 \times (0, \infty) \\
    y(0) &= y_0 \quad \text{and} \quad y'(0) = y_1 \quad \text{in } \Omega,
\end{align*}
\]

(1.5)-(1.8)

where \( ' \) denotes the time derivative, \( \Delta \) the Laplacian in \( R^n \) taken in the space variables, and \( b(x) \in L^\infty(\Gamma^1), \ b(x) \geq b_0 > 0; \) that is, boundary damping is essential on some portion \( \Gamma^1(\Gamma^1 \neq \emptyset) \) of the boundary \( \Gamma. \) They proved a result of the type

\[ E(t) \leq Me^{-\beta t}E(0), \quad t \geq 0, \]  

(1.9)

\( M \geq 1 \) and \( \beta > 0 \) being some constants. Later, Lagnese [8] and Komornik [6] obtained somewhat faster energy decay rates for certain forms of \( b(x). \) Also Chen [2] demonstrated, a faster energy decay rate than (1.9) when external damping \( 2\gamma y' \) is present in the left hand side of (1.5). In the method of treatment [2, 6, 8] adopt a direct method by constructing suitable functionals related to \( E(t), \) where as [1, 7, 12] employ semigroup theory, in as much as the underlying operator of the system generates a strongly continuous contraction semigroup.

Herein we shall be concerned with the internally damped wave equation

\[ y'' = \Delta y + \mu \Delta y' \quad \text{in } \Omega \times (0, \infty) \]  

(1.10)

with the undamped mixed boundary conditions

\[
\begin{align*}
    y &= 0 \quad \text{on } \Gamma^0 \times (0, \infty) \\
    \frac{\partial y}{\partial n} &= 0 \quad \text{on } \Gamma^1 \times (0, \infty)
\end{align*}
\]

(1.11)-(1.12)

and initial conditions (1.8). \( \mu > 0 \) is the small internal damping constant.

Physically, Eq. (1.10) occurs in the study of vibrations of flexible structures in a bounded domain governed by the Voigt model of viscoelasticity. The motivation for incorporating internal material damping in the wave equation as in (1.10) arises from the fact that, inherent small material
damping, usually uniform of constant measure (µ in the Voigt model), is always present in real materials (cf. Christensen [4]). Hence from the physical point of view we say that internal structural damping force will appear so long as the system vibrates. Establishment of exponential energy decay of the form (1.9) is thus sought under natural boundary conditions (1.11) and (1.12), without having to introduce boundary damping. Here, the exponential decay rate will depend on µ and we adopt a direct method such as in Komornik [6], Lagnese [8] for extracting the functional form of this dependence. In contrast, Chen and Russell [3] considered the generalized operator version of (1.10) of the form $y'' + B y' + A y = 0$ to study the analyticity of the semigroup of contraction over a suitable Hilbert space of the underlying operator. Also, several examples of partial differential equations with boundary or point control have been illustrated in Lasiecka and Triggiani [10] which can be reduced to the abstract form.

We proceed by differentiating Eq. (1.4) with respect to $t$ and replacing $y''$ by $\Delta y + D y'$ to obtain

$$E'(t) = \int_\Omega \left[ y' (\Delta y + \mu \Delta y') + (\nabla y \cdot \nabla y') \right] dx.$$  

Applying Green’s formula we have

$$E'(t) = -\mu \int_\Omega |\nabla y'|^2 dx, \quad (1.13)$$

where the boundary conditions (1.11) and (1.12) have been used. We thus have

$$E'(t) \leq 0 \quad \text{for } t \geq 0. \quad (1.14)$$

Hence energy is nonincreasing with time, i.e.,

$$E(t) \leq E(0) \quad \text{for } t > 0. \quad (1.15)$$

We establish from the negativity of the right hand side of (1.13) that the energy of the system is dissipating due to the presence of internal material damping of the system.

2. MAIN RESULT

The validity of the uniform exponential decay of $E(t)$ for the problem (1.10)–(1.12) follows from the following theorem:

**Theorem.** Let $y$ be a regular solution of (1.10)–(1.12) satisfying the initial conditions (1.8). Then the energy

$$E(t) \leq Me^{-\beta t} E(0), \quad t \geq 0$$
for some reals \( M \geq 1, \beta > 0 \) of the form \( \beta = \mu/(a\mu^2 + b\mu + c) \), \( a, b, c > 0 \) and for all initial states \( y_0 \in H^1(\Omega), y_1 \in H^3(\Omega) \) where \( H^k(\Omega) = \{ y | y \in H^k(\Omega), y = 0 \text{ on } \Gamma^0 \text{ and } H^4(\Omega), k \text{ being positive integer, is the classical Sobolev space of real valued functions } y, \text{ whose partial derivatives defined in the distributional sense of order } \leq k, \text{ lie in } L^2(\Omega) \).

Before proving the above main theorem, we first establish the following lemmas.

**Lemma 2.1.** If \( y \) is a regular solution of Eqs. (1.10)–(1.12) with (1.8), then the function \( F(t) \) defined by

\[
F(t) = \frac{1}{2} \int_{\Omega} (|\nabla y'|^2 + |\Delta y|^2) \, dx
\]

is nonincreasing for \( t \geq 0 \).

**Proof.** Differentiating (2.1) with respect to \( t \) we obtain

\[
F'(t) = \int_{\Omega} [ \langle \nabla y', \nabla y'' \rangle + \Delta y \Delta y' ] \, dx.
\]

Use of (1.10) yields

\[
F'(t) = \int_{\Gamma} y'' \frac{\partial y'}{\partial n} \, d\Gamma - \mu \int_{\Omega} |\Delta y'|^2 \, dx,
\]

where we have used the Green’s formula. Further, using the boundary conditions (1.11) and (1.12) we have

\[
F'(t) = -\mu \int_{\Omega} |\Delta y'|^2 \, dx \leq 0.
\]

Hence \( F(t) \) is nonincreasing for \( t \geq 0 \). We conclude that

\[
F(t) \leq F(0) \quad \text{for } t \geq 0.
\]

**Lemma 2.2.** Let \( y \) be a regular solution of (1.10)–(1.12) with (1.8). If we define a function \( G(t) \) by

\[
G(t) = \int_{\Omega} [ \nabla y' \cdot (m \cdot \nabla) \nabla y ] \, dx
\]

then \( |G(t)| \leq KF(t) \) for \( t \geq 0 \), where \( K \geq 1 \) is a constant, independent of \( t \).
Proof. From Eq. (2.5) we can write
\[ |G(t)| \leq \int_{\Omega} |\nabla y'| |(m.\nabla) \nabla y| \, dx \leq \frac{1}{2} \int_{\Omega} \left( |\nabla y'|^2 + |(m.\nabla) \nabla y|^2 \right) \, dx. \tag{2.6} \]

We now define a constant \( K \geq 1 \) so that we can write
\[ \int_{\Omega} |(m.\nabla) \nabla y|^2 \, dx \leq K \int_{\Omega} |\Delta y|^2 \, dx. \tag{2.7} \]

It follows from (2.6) then
\[ |G(t)| \leq KF(t) \quad \text{for } t > 0. \]

Hence the lemma follows.

Remark. The inequality of the form (2.7) can be written due to the fact that the expressions \( m_{ij}(\partial^2 y/\partial x_j) \) can be reduced to the form \( M_{ij}(\partial^2 y/\partial x_j) \) by suitable orthogonal transformation of the axes \( x_j, (j, k = 1, 2, \ldots, n) \) and \( \Delta y \) is invariant with respect to orthogonal transformation (cf. [5]), the usual summation convention of repeated indices being used.

**Lemma 2.3.** For every \( u \in H^1(\Omega), \)
\[ \int_{\Omega} \left[ 2u.(m.\nabla)u + n|u|^2 \right] \, dx = \int_{\Gamma} m.\nu |u|^2 \, d\Gamma. \tag{2.8} \]

Proof. We have
\[ \int_{\Omega} \left[ 2u.(m.\nabla)u + n|u|^2 \right] \, dx = \int_{\Omega} \left[ (m.\nabla)|u|^2 \right] + n|u|^2 \right] \, dx \]
\[ = \int_{\Omega} \text{div}(m|u|^2) \, dx \]
\[ = \int_{\Gamma} m.\nu |u|^2 \, d\Gamma. \]

Hence the lemma.

**Lemma 2.4.** If \( y \) is a regular solution of (1.10)–(1.12) with initial conditions (1.8), then
\[ \rho'(t) + \mu \rho_0'(t) + 2\mu G(t) + 2E(t) \]
\[ \leq \mu^2 \int_{\Gamma} m.\nu |\nabla y'|^2 \, d\Gamma + \int_{\Gamma} m.\nu y'^2 \, d\Gamma, \tag{2.9} \]
where
\[
\rho(t) = \int_{\Omega} \left[ 2y'(m,\nabla y) + (n - 1)y' y \right] \, dx \tag{2.10}
\]
and
\[
\rho_0(t) = \frac{n + 1}{2} \int_{\Omega} |\nabla y|^2 \, dx. \tag{2.11}
\]

**Proof.** Differentiating (2.10) with respect to \( t \) and replacing \( y'' \) by \( \Delta y + \mu \Delta y' \) we have
\[
\rho'(t) = \int_{\Omega} \left[ 2(\Delta y + \mu \Delta y')(m,\nabla y) + (n - 1)(\Delta y + \mu \Delta y')y' 
+ 2y'(m,\nabla y') + (n - 1)y'^2 \right] \, dx.
\]

Applying Green's formula we obtain
\[
\rho'(t) = \int_{\Gamma} \left[ [(2(m,\nabla y) + (n - 1)y] \frac{\partial y'}{\partial \nu} + \mu \frac{\partial y'}{\partial \nu} \right] \, d\Gamma 
- \int_{\Omega} \left[ 2\nabla (m,\nabla y) + (n - 1) \nabla y \right] \nabla (y + \mu y') \, dx 
+ \int_{\Omega} \left[ 2y'(m,\nabla y') + (n - 1)y'^2 \right] \, dx.
\]

Using the boundary conditions (1.11) and (1.12) we have
\[
\rho'(t) = \int_{\Gamma} 2 \left[ \frac{\partial y}{\partial \nu} + \mu \frac{\partial y'}{\partial \nu} \right] (m,\nabla y) \, d\Gamma 
- \int_{\Omega} \left[ 2(\nabla y + (m,\nabla) \nabla y) + (n - 1) \nabla y \right] \nabla (y + \mu y') \, dx 
+ \int_{\Omega} \left[ 2y'(m,\nabla y') + (n - 1)y'^2 \right] \, dx 
= \int_{\Gamma} 2 \left[ \frac{\partial y}{\partial \nu} + \mu \frac{\partial y'}{\partial \nu} \right] (m,\nabla y) \, d\Gamma 
- \int_{\Omega} \left[ 2(\nabla y.(m,\nabla) \nabla y) + n|\nabla y|^2 \right] \, dx 
+ \int_{\Omega} \left[ 2y'(m,\nabla) y' + ny'^2 \right] \, dx - 2\mu \int_{\Omega} (\nabla y'.(m,\nabla) \nabla y) \, dx
- \mu(n + 1) \int_{\Omega} (\nabla y.\nabla y') \, dx - \int_{\Omega} (|\nabla y|^2 + y'^2) \, dx.
Applying Lemma 2.3, we obtain

$$\rho'(t) = \int_{\Gamma^0} 2\left( \frac{\partial y}{\partial v} + \mu \frac{\partial y'}{\partial v} \right)(m.\nabla y) \, d\Gamma + \int_{\Gamma^1} m.\nu(y^2 - |\nabla y|^2) \, d\Gamma$$

$$- 2\mu G(t) - \mu \rho_0(t) - 2E(t). \tag{2.12}$$

Since $y = 0$ on $\Gamma^0$, $\nabla y = \nu(\partial y/\partial v)$ and $|\nabla y|^2 = |\partial y/\partial v|^2$ on $\Gamma^0$. Also $m.\nu > 0$ on $\Gamma^1$. Hence we have from (2.12)

$$\rho'(t) + \mu \rho_0(t) + 2\mu G(t) + 2E(t)$$

$$\leq \int_{\Gamma^0} m.\nu|\nabla y|^2 \, d\Gamma + 2\mu \int_{\Gamma^0} m.\nu(\nabla y.\nabla y') \, d\Gamma + \int_{\Gamma^1} m.\nu y^2 \, d\Gamma$$

$$\leq \int_{\Gamma^0} m.\nu|\nabla y|^2 \, d\Gamma + \int_{\Gamma^0} |m.\nu||\nabla y|^2 + \mu^2|\nabla y'|^2 \, d\Gamma + \int_{\Gamma^1} m.\nu y^2 \, d\Gamma$$

$$\leq \mu^2 \int_{\Gamma^0} |m.\nu||\nabla y'|^2 \, d\Gamma + \int_{\Gamma^1} m.\nu y^2 \, d\Gamma$$

since $m.\nu \leq 0$ on $\Gamma^0$. Hence the lemma.

We are now ready to prove the main result.

**Proof of the Theorem.** We define a function $H(t)$ by

$$H(t) = \lambda E(t) + \mu\left[ \rho(t) + \mu \rho_0(t) + F(t) \right], \tag{2.13}$$

where $\lambda$ is a positive constant defined by

$$\int_{\Gamma^1} m.\nu y^2 \, d\Gamma \leq \lambda \int_{\Omega} |\nabla y'|^2 \, dx \tag{2.14}$$

for all $y \in H^2_0(\Omega)$. We also define the positive constants $\lambda_0$, $\lambda_1$, and $\lambda_2$ by

$$E(t) \leq \lambda_0 F(t) \tag{2.15}$$

$$\int_{\Omega} y^2 \, dx \leq \lambda_1 \int_{\Omega} |\nabla y|^2 \, dx \quad (\lambda_2 > 1) \tag{2.16}$$

and

$$\int_{\Gamma^0} |m.\nu||\nabla y'|^2 \, d\Gamma \leq \lambda_2 \int_{\Omega} |\Delta y|^2 \, dx \tag{2.17}$$

for all $y \in H^2_0(\Omega)$. Inequalities (2.15) and (2.16) arise due to Poincare. Inequalities (2.14) and (2.17) follow from the combination of the Poincare inequality with the Trace inequality in $H^2(\Omega)$. Here all $\lambda$, $\lambda_0$, $\lambda_1$, and $\lambda_2$
are independent of $t$, depending only on the set $\Omega$ in $R^n$ and eventually on $x^0$. They are also independent of the initial value of $\{y_0, y_1\}$. Their explicit determination is in general very difficult.

Now we have from (2.10)

$$|\rho(t)| \leq R_0 \int_\Omega (y^2 + |\nabla y|^2) \, dx + \frac{(n - 1)}{2} \int_\Omega (y^2 + y'^2) \, dx$$

$$\leq [2R_0 + (n - 1)\lambda_1]E(t) = C_0 E(t), \quad (2.18)$$

where $R_0 = \sup |m(x)| : x \in \Omega$ and $C_0 = [2R_0 + (n - 1)\lambda_1]$. From (2.11) we also have

$$0 \leq \rho_0(t) \leq (n + 1)E(t). \quad (2.19)$$

With the help of (2.15), (2.18), and (2.19), it follows from (2.13) that

$$(\lambda + \mu/\lambda_0 - \mu C_0)E(t) \leq H(t)$$

$$\leq [\lambda + \mu(C_0 + \mu(n + 1))]E(t) + \mu F(t).$$

(2.20)

Now differentiating (2.13) with respect to $t$ and applying (1.13), (2.3), and Lemma 2.4, we have

$$H'(t) = \lambda E'(t) + \mu(\rho'(t) + \mu \rho_0'(t) + F'(t))$$

$$\leq -\lambda \mu \int_\Omega |\nabla y|^2 \, dx + \mu \left[ \mu^2 \int_{\Gamma^0} |\mathbf{m} \cdot \nu||\nabla y'|^2 \, d\Gamma + \int_{\Gamma^1} \mathbf{m} \cdot \nu y'^2 \, d\Gamma \right.$$  

$$-2\mu G(t) - 2E(t) - \mu \int_\Omega |\Delta y'|^2 \, dx \right].$$

Applying the inequalities (2.14), (2.17), and Lemma 2.2, we obtain

$$H'(t) \leq \mu^2(\mu \lambda_2 - 1)\int_\Omega |\Delta y'|^2 \, dx + 2\mu[\mu KF(t) - E(t)]. \quad (2.21)$$

Now since $\{y_0, y_1\} \in H^2_0(\Omega) \times H^4(\Omega)$, therefore, from the inequalities (2.4) and (1.15), we have $F(t) \leq F(0) < \infty$ and $E(t) \leq E(0) < \infty$. Hence there exists a positive constant $K_0$ such that

$$F(t) \leq K_0 E(t). \quad (2.22)$$
Inequality (2.21) can then be written as

$$H'(t) \leq \mu^2 (\mu \lambda_2 - 1) \int_{\Omega} |\Delta y|^2 \, dx + \mu (2 \mu KK_0 - 1) E(t) - \mu E(t).$$

(2.23)

Let

$$\mu \leq \min\{1/\lambda_2, 1/2KK_0, \lambda/C_0\},$$

(2.24)

which determines here an upper bound of the value of $\mu$ consistent with stability. We then have from (2.23)

$$H'(t) \leq -\mu E(t)$$

(2.25)

and at the same time we have from (2.20)

$$\frac{\mu}{\lambda_0} E(t) \leq H(t) \leq [\lambda + \mu (C_0 + \mu(n + 1) + K_0)] E(t)$$

$$= \mu_0 E(t),$$

(2.26)

where the positive constant

$$\mu_0 = a \mu^2 + b \mu + c$$

(2.27)

is a quadratic function of $\mu$, and $a = n + 1, \; b = C_0 + K_0, \; c = \lambda$ are independent of it. Use of (2.26) in (2.25) yields

$$H'(t) + \beta H(t) < 0,$$

(2.28)

where $\beta = \mu/\mu_0$. Multiplying (2.28) by $e^{\beta t}$ and integrating from zero to $t$, we get

$$H(t) \leq e^{-\beta t} H(0).$$

Thus it follows from (2.26) that

$$E(t) \leq Me^{-\beta t} E(0) \quad \text{for} \; t \geq 0,$$

(2.29)

where

$$M = \frac{\mu_0 \lambda_0}{\mu} \geq 1$$

(2.30)

by virtue of Eq. (2.26).
The expression for $\beta$ as a function of the visco-elastic damping parameter $\mu$ shows that the decay rate is maximum for $\mu = \sqrt{\lambda/(n+1)}$. Hence from (2.24), the maximum decay rate is attained for

$$\mu = \min\{1/\lambda_2, 1/2KK_0, \lambda/C_0, \sqrt{\lambda/(n+1)}\}.$$  \hfill (2.31)

Properties of the decay rate are restricted by the lack of explicit knowledge in general, of the constants $\lambda$, $\lambda_2$, $K$, $K_0$, $C_0$ appearing in the expression.

**Remark.** This study deals with the exponential decay of the solution of the internally damped wave equation (1.10) together with boundary conditions (1.11) and (1.12) and initial conditions (1.8) in the sense of decay of the total energy according to the stated theorem. The problem considered here is a generalization of the abstract dynamical system with the internal damping term. This is the main interest of this analysis, since internal structural damping is always present in actual systems (cf. Christensen [4]). The boundary conditions are standard without boundary damping. For establishing the stability theory, recourse is taken to the available methods of functional analysis (as adopted in the references), basing the main result on the necessary Sobolev spaces for the initial values of the system. Finally we conclude that systems of such type ultimately go to rest due to their own material damping property.

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**REFERENCES**