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The upper and lower solution method for a class of singular nonlinear second order three-point boundary value problems ☆

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Abstract

In this paper we develop the upper and lower solution method and the monotone iterative technique for a class of singular nonlinear second order three-point boundary value problems. A maximum principle is established and some new existence results are obtained. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is well known that a powerful tool for proving existence results for nonlinear problems is the method of upper and lower solutions. And in many cases it is possible to find a minimal and a maximal solution between the lower and the upper solution by the monotone iterative technique (see [8,10,12]). However, in the literature of ordinary differential equations is seldom seen the paper in which the upper and lower solution method has been used to deal with second order three-point boundary value problems, not to mention the monotone iterative technique. For this reason, we are going to develop the upper and lower solution method and the monotone iterative technique for a second order three-point boundary value problem of the form

$$\begin{cases} -u''(t) = f(t, u(t)) \quad \text{a.e. on } (0,1), \\ u(0) = \xi, \quad u(1) - \lambda u(\delta) = \eta. \end{cases} \tag{1.1}$$

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Throughout this paper, we make the following assumptions:

(A1) $\delta \in (0,1)$, $\lambda > 0$, and $\xi, \eta \in \mathbb{R}$ are given.

(A2) $f(t,u)$ is real-valued function that is defined on $(0,1) \times \mathbb{R}$ and satisfies

(i) $f(t,u)$ is measurable on $(0,1)$ for each fixed $u \in \mathbb{R}$,

(ii) $f(t,u)$ is continuous on $\mathbb{R}$ for almost all $t \in (0,1)$, and

(iii) for every given $N > 0$ there exists a function $k_N(t) \in E$ such that

$$|f(t,u)| \leqslant k_N(t) \text{ for almost all } t \in (0,1) \text{ and all } u \in [-N, N],$$

where $E := \{h(t) \in L^1_{\text{loc}}(0,1); \|h\|_E < +\infty\}$ is the Banach space equipped with the norm

$$\|h\|_E := \int_0^\delta s|h(s)|\,\mathrm{d}s + \int_\delta^1 (1-s)|h(s)|\,\mathrm{d}s.$$

(A3) There exist two functions $x(t)$ and $y(t)$ that are lower and upper solutions to (1.1), respectively. Moreover, $x(t) \leqslant y(t)$ on $[0,1]$.

Here a function $x(t)$ is said to be a lower solution to the three-point boundary value problem (1.1), if it belongs to $D[0,1]$ and satisfies

$$\begin{cases} -x''(t) \leqslant f(t,x(t)) & \text{a.e. on } (0,1), \\ x(0) \leqslant \xi, \quad x(1) - \lambda x(\delta) \leqslant \eta, \end{cases} \tag{1.2}$$

where $D[\alpha,1] := \{u(t) \in AC[\alpha,1];\ u'(t) \in L^1[\alpha,1] \cap AC_{\text{loc}}(\alpha,1),\ u''(t) \in E\}$ for any $\alpha \in [0,\delta)$.

Similarly, we say that a function $y(t)$ is an upper solution to (1.1) if it belongs to $D[0,1]$ and satisfies the reversed inequalities in (1.2). And a function $u(t)$ is called a solution to (1.1), provided that it is a lower solution and an upper solution as well.

Remark. Assumption (A2) allows $f(t,u)$ to be singular at $t=0$ and 1. For example, the function

$$f(t,u) = t^{-a}(1-t)^{-b}\mathrm{e}^u(u-8)^3, \quad a,b \in (1,2)$$

satisfies (A2).

As far as second order m-point ($m \geqslant 3$) boundary value problems are concerned, a great deal of existence and uniqueness results have been established up to now. For details, see, for example, [1–7,9,11] and the references therein. However, among the existing results no one can be applied to our problem. This is another reason why we study problem (1.1).

The main results of the present paper are as follows.

Theorem 1. *Let* (A1)–(A3) *be fulfilled. Then the three-point boundary value problem* (1.1) *has a solution $u(t)$ with $x(t) \leqslant u(t) \leqslant y(t)$ on* $[0,1]$.

Theorem 2. *Let* (A1)–(A3) *hold. Assume that there exists a function $k(t) \in E$ such that $F(t,u) := f(t,u) + k(t)u$ is* (*strictly*) *increasing on* $[\min\{x(t);\ 0 \leqslant t \leqslant 1\}, \max\{y(t);\ 0 \leqslant t \leqslant 1\}]$ *for almost all $t \in (0,1)$. Then there exist two sequences $\{x_m(t)\}_{m=0}^{\infty} \nearrow u_*(t)$ and $\{y_m(t)\}_{m=0}^{\infty} \searrow u^*(t)$ uniformly on* $[0,1]$ *with $x_0(t) = x(t)$ and $y_0(t) = y(t)$. Here $u_*(t)$ and $u^*(t)$ are the minimal and the maximal solutions to* (1.1), *respectively, that is, if $u(t)$ is a solution* (1.1) *with $x(t) \leqslant u(t) \leqslant y(t)$ on* $[0,1]$, *then $u_*(t) \leqslant u(t) \leqslant u^*(t)$ on* $[0,1]$.

The proofs of our main results will be given in Section 3. And Section 2 will be devoted to some preliminary propositions, including a maximum principle, which are indispensable to establishing our main results.

2. Preliminaries

In this section, we present some propositions that will be frequently used later on.

Lemma 3. *Assume $h(t) \in E$. Then we have*

$$\lim_{t\downarrow 0} t\int_t^\delta h(s)\,\mathrm{d}s = 0 = \lim_{t\uparrow 1}(1-t)\int_\delta^t h(s)\,\mathrm{d}s.$$

Proof. Let

$$g(t) := t\int_t^\delta h(s)\,\mathrm{d}s, \quad 0 < t \leqslant \delta.$$

Then we have

$$|g(t)| \leqslant \int_0^\delta s|h(s)|\,\mathrm{d}s < +\infty, \quad 0 < t \leqslant \delta,$$

$$g'(t) = \int_t^\delta h(s)\,\mathrm{d}s - th(t) \quad \text{a.e. on } (0,\delta).$$

Integrating the above on $[0,\delta]$ and then interchanging the order of integration, we get

$$\int_0^\delta |g'(t)|\,\mathrm{d}t \leqslant \int_0^\delta \mathrm{d}t \int_t^\delta |h(s)|\,\mathrm{d}s + \int_0^\delta s|h(s)|\,\mathrm{d}s$$
$$= 2\int_0^\delta s|h(s)|\,\mathrm{d}s < +\infty$$

which shows that $g'(t) \in L^1[0,\delta]$ and hence $g(t) \in AC[0,\delta]$.

As a result, we integrate $g'(t)$ from 0 to $r \in (0,\delta]$ to obtain

$$\int_0^r g'(t)\,\mathrm{d}t = \int_0^r \mathrm{d}t \int_t^\delta h(s)\,\mathrm{d}s - \int_0^r th(t)\,\mathrm{d}t \equiv g(r), \quad 0 < r \leqslant \delta,$$

which implies that $g(0) = 0$, i.e., the first equation is valid.

In the same as above, we can deduce the second equation.

Lemma 4. *Let $k(t) \in E$ with $k(t) > 0$ a.e on $(0,1)$. Then initial value problems*

$$\begin{cases} u''(t) = k(t)u(t), & 0 \leqslant \alpha < t < 1, \\ u(\alpha) = 0, \quad u'(\alpha) = 1 \end{cases} \tag{2.1}$$

and

$$\begin{cases} u''(t) = k(t)u(t), & 0 < t < \beta \leqslant 1, \\ u(\beta) = 0, \quad u'(\beta) = -1 \end{cases} \tag{2.2}$$

have unique positive solutions $p_\alpha(t) \in AC[\alpha, 1) \cap C^1[\alpha, 1)$ *and* $q_\beta(t) \in AC[0, \beta] \cap C^1(0, \beta]$, *respectively. Moreover, they are strictly convex on their intervals of definition. As a result, we have*

$$\begin{cases} t - \alpha \leqslant p_\alpha(t) \leqslant p_\alpha(a)(t-\alpha)/(a-\alpha), & \alpha \leqslant t \leqslant a \leqslant 1, \\ \beta - t \leqslant q_\beta(t) \leqslant q_\beta(b)(\beta - t)/(\beta - b), & 0 \leqslant b \leqslant t \leqslant \beta. \end{cases} \tag{2.3}$$

for any $a \in [\alpha, 1]$ *and* $b \in [0, \beta)$.

When $0 \leqslant \alpha < \beta \leqslant 1$, *we have*

$$W_{[\alpha,\beta]}(t) := \begin{vmatrix} q_\beta(t) & p_\alpha(t) \\ q'_\beta(t) & p'_\alpha(t) \end{vmatrix} \equiv q_\beta(\alpha) = p_\alpha(\beta) \quad \text{on } [\alpha, \beta]. \tag{2.4}$$

Proof. We first prove that (2.1) has a unique positive solution. Define a mapping $\boldsymbol{B}: Z \to Z$ by

$$(\boldsymbol{B}w)(\alpha) = 1, \quad (\boldsymbol{B}w)(t) = 1 + \frac{1}{t-\alpha}\int_\alpha^t (t-s)(s-\alpha)k(s)w(s)\,\mathrm{d}s, \quad \alpha < t \leqslant 1,$$

where $Z := \{w(t) \in C[\alpha, 1]; \|w\|_z < +\infty\}$ is the Banach space endowed with the norm

$$\|w\|_z := \begin{cases} \max\left\{|w(t)| \exp\left(-2\int_0^t r(1-r)k(r)\,\mathrm{d}r\right);\ 0 \leqslant t \leqslant 1\right\} & \text{if } \alpha = 0, \\ \max\left\{|w(t)| \exp\left(-2\int_\alpha^t (1-r)k(r)\,\mathrm{d}r\right);\ \alpha \leqslant t \leqslant 1\right\} & \text{if } \alpha > 0. \end{cases}$$

It is easy to see that the $\boldsymbol{B}$ is well defined. Furthermore, we can prove that $\boldsymbol{B}$ is a contraction mapping. In fact, for any $w_1(t),\ w_2(t) \in \mathbb{Z}$, we have, if $\alpha = 0$

$$\begin{aligned} |(\boldsymbol{B}w_1)(t) - (\boldsymbol{B}w_2)(t)| &\leqslant \int_0^t \left(1 - \frac{s}{t}\right) sk(s)|w_1(s) - w_2(s)|\,\mathrm{d}s \\ &\leqslant \|w_1 - w_2\|_z \int_0^t (1-s)sk(s)\exp\left(2\int_0^s r(1-r)k(r)\,\mathrm{d}r\right)\mathrm{d}s \\ &\leqslant \frac{1}{2}\|w_1 - w_2\|_z \left\{\exp\left(2\int_0^t r(1-r)k(r)\,\mathrm{d}r\right) - 1\right\}, \quad 0 \leqslant t \leqslant 1, \end{aligned}$$

if $\alpha > 0$

$$\begin{aligned} |(\boldsymbol{B}w_1)(t) - (\boldsymbol{B}w_2)(t)| &\leqslant \int_\alpha^t (1-s)k(s)|w_1(s) - w_2(s)|\,\mathrm{d}s \\ &\leqslant \|w_1 - w_2\|_z \int_\alpha^t (1-s)k(s)\exp\left(2\int_\alpha^s (1-r)k(r)\,\mathrm{d}r\right)\mathrm{d}s \\ &\leqslant \frac{1}{2}\|w_1 - w_2\|_z \left\{\exp\left(2\int_\alpha^t (1-r)k(r)\,\mathrm{d}r\right) - 1\right\}, \quad \alpha \leqslant t \leqslant 1 \end{aligned}$$

and hence

$$\|\boldsymbol{B}w_1 - \boldsymbol{B}w_2\|_z \leqslant \tfrac{1}{2}\|w_1 - w_2\|_z \quad \text{for any } w_1, w_2 \in Z.$$

The Banach contraction principle tells us that $\boldsymbol{B}$ has a unique fixed point in Z. Let $w^*(t)$ be the fixed point. Then

$$w^*(\alpha) = 1, \quad w^*(t) = 1 + \int_\alpha^t \frac{(t-s)(s-\alpha)k(s)w^*(s)\,\mathrm{d}s}{t-\alpha}, \quad \alpha < t \leqslant 1.$$

Put

$$\begin{aligned} p_\alpha(t) &:= (t-\alpha)w^*(t) \\ &= (t-\alpha) + \int_\alpha^t (t-s)k(s)p_\alpha(s)\,\mathrm{d}s, \quad \alpha \leqslant t \leqslant 1. \end{aligned} \tag{2.5}$$

Then

$$\begin{cases} p'_\alpha(t) = 1 + \displaystyle\int_\alpha^t k(s)p_\alpha(s)\,\mathrm{d}s, \quad \alpha \leqslant t \leqslant 1, \\ p''_\alpha(t) = k(t)p_\alpha(t) \quad \text{a.e. on } (\alpha, 1), \\ p_\alpha(\alpha) = 0, \quad p'_\alpha(\alpha) = 1. \end{cases} \tag{2.6}$$

This shows that $p_\alpha(t)$ is a unique solution to (2.1), $p_\alpha(t) \in AC[\alpha, 1]$ and $p'_\alpha(t) \in AC_{\mathrm{loc}}[\alpha, 1)$.

We now claim that $p_\alpha(t) > 0$ for all $t \in (\alpha, 1]$. If the claim were false, then there would be a $t_0 \in (\alpha, 1]$ such that

$$p_\alpha(t) > 0 \text{ in } (\alpha, t_0) \quad \text{and} \quad p_\alpha(\alpha) = p_\alpha(t_0) = 0.$$

Since $p'_\alpha(\alpha) = 1$ along with $p_\alpha(\alpha) = 0$ implies that $p_\alpha(t)$ is positive in a right neighborhood of $t = \alpha$. By Roll's theorem, we know that there exists a $\sigma \in (\alpha, t_0)$ such that $p'_\alpha(\sigma) = 0$. On the other hand, from (2.6) we lead to

$$0 = p'_\alpha(\sigma) = 1 + \int_\alpha^\sigma k(s)p_\alpha(s)\,\mathrm{d}s > 1,$$

a contradiction. Therefore, the claim is true.

In the same way as above, we can prove that (2.2) has a unique positive solution $q_\beta(t) \in AC[0, \beta] \cap C^1(0, \beta]$, which can be represented as

$$q_\beta(t) = \beta - t + \int_t^\beta (s-t)k(s)q_\beta(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant \beta. \tag{2.7}$$

Moreover, we have

$$\begin{cases} q'_\beta(t) = -1 - \displaystyle\int_t^\beta k(s)q_\beta(s)\,\mathrm{d}s \leqslant 1, \quad 0 < t \leqslant \beta, \\ q''_\beta(t) = k(t)q_\beta(t) \quad \text{a.e. on } (0, \beta), \\ q_\beta(\beta) = 0, \quad q'_\beta(\beta) = -1. \end{cases} \tag{2.8}$$

Since $p''_\alpha(t) > 0$ a.e. on $(\alpha, 1)$ and $q''_\beta(t) > 0$ a.e. on $(0, \beta)$, $p_\alpha(t)$ and $q_\beta(t)$ are strictly convex on their intervals of definition, which implies (2.3).

Finally, we show (2.4).

Differentiating $W_{[\alpha,\beta]}(t)$ and then using (2.6) and (2.8), we obtain

$$W'_{[\alpha,\beta]}(t) = \begin{vmatrix} q'_\beta(t) & p'_\alpha(t) \\ q'_\beta(t) & p'_\alpha(t) \end{vmatrix} + \begin{vmatrix} q_\beta(t) & p_\alpha(t) \\ k(t)q_\beta(t) & k(t)p_\alpha(t) \end{vmatrix} \equiv 0 \quad \text{on } (\alpha, \beta).$$

Eq. (2.4) follows from (2.3), (2.5)–(2.8) and Lemma 3.

The proof is thus complete. □

Theorem 5. *Let $\delta \in (0,1)$, $\lambda > 0$, ξ^*, $\eta^* \in \mathbb{R}$, and $k(t)$, $h(t) \in E$ with $k(t) > 3\lambda/(1-\delta)^2$ a.e. on $(0,1)$. Then the linear three-point boundary value problem*

$$\begin{cases} -w''(t) = k(t)w(t) = h(t), & 0 \leqslant \alpha < t < 1, \\ w(\alpha) = \xi^*, \quad w(1) - \lambda w(\delta) = \eta^* \end{cases} \tag{2.9}$$

has a unique solution

$$w(t) = \begin{cases} \dfrac{p_\alpha(1)}{p_\alpha(1) - \lambda p_\alpha(\delta)} \left(\displaystyle\int_\alpha^1 G_{[\alpha,1]}(t,s)h(s)\,ds + \xi^* \frac{q_1(\delta)}{p_\alpha(1)} + \eta^* \frac{p_\alpha(\delta)}{p_\alpha(1)} \right), & t = \delta, \\ \displaystyle\int_\alpha^\delta G_{[\alpha,\delta]}(t,s)h(s)\,ds + w(\delta)\frac{p_\alpha(t)}{p_\alpha(\delta)} + \xi^* \frac{q_\delta(t)}{q_\delta(\alpha)}, & \alpha \leqslant t \leqslant \delta, \\ \displaystyle\int_\delta^1 G_{[\delta,1]}(t,s)h(s)\,ds + w(\delta)\frac{q_1(t) + \lambda p_\delta(t)}{q_1(\delta)} + \eta^* \frac{p_\delta(t)}{p_\delta(1)}, & \delta \leqslant t \leqslant 1, \end{cases} \tag{2.10}$$

where

$$G_{[\alpha,\beta]}(t,s) := \begin{cases} q_\beta(t)\dfrac{p_\alpha(s)}{p_\alpha(\beta)}, & \alpha \leqslant s \leqslant t \leqslant \beta \\ p_\alpha(t)\dfrac{q_\beta(s)}{q_\beta(\alpha)}, & \alpha \leqslant t \leqslant s \leqslant \beta \end{cases} \qquad (p_\alpha(\beta) = q_\beta(\alpha)).$$

Proof. Since $k(t) > 3\lambda/(1-\delta)^2$ a.e. on $(0,1)$, from Lemma 4 we know that

$$\begin{aligned} p_\alpha(1) - \lambda_\alpha(\delta) &= q_1(\delta)p'_\alpha(\delta) - q'_1(\delta)p_\alpha(\delta) - \lambda p_\alpha(\delta) \\ &> p_\alpha(\delta)\left(1 + \int_\delta^1 k(s)q_1(s)\,ds - \lambda\right) \\ &> p_\alpha(\delta)\left(\int_\delta^1 \frac{3\lambda}{(1-\delta)^2}(1-s)\,ds - \lambda\right) \\ &> 0. \end{aligned}$$

From Lemma 3, we know that

$$w(\alpha) = \xi^* \quad \text{and} \quad w(1) - \lambda w(\delta) = \eta^*.$$

Differentiating (2.10), we obtain

$$w'(t) = \begin{cases} q'_\delta(t)\displaystyle\int_\alpha^t \frac{p_\alpha(s)}{p_\alpha(\delta)}\, h(s)\,\mathrm{d}s + p'_\alpha(t)\int_t^\delta \frac{q_\delta(t)}{q_\delta(\alpha)}\, h(s)\,\mathrm{d}s & \\ \quad + w(\delta)\,\dfrac{p'_\alpha(t)}{p_\alpha(\delta)} + \xi^*\,\dfrac{q'_\delta(t)}{q_\delta(\alpha)}, & \alpha < t < \delta, \\ q'_1(t)\displaystyle\int_\delta^t \frac{p_\delta(s)}{p_\delta(1)}\, h(s)\,\mathrm{d}s + p'_\delta(t)\int_t^1 \frac{q_1(s)}{q_1(\delta)}\, h(s)\,\mathrm{d}s & \\ \quad + w(\delta)\,\dfrac{q'_1(t) + \lambda p'_\delta(t)}{q_1(\delta)} + \eta^*\,\dfrac{p'_\delta(t)}{p_\delta(1)}, & \delta < t < 1 \end{cases} \tag{2.11}$$

which together with (2.10) implies that $w'(\delta-) = w^*(\delta+)$, that is to say, $w'(t) \in AC_{\mathrm{loc}}(\alpha, 1)$.

From (2.10) and (2.11), we know that

$$\begin{aligned} |w'(t)| &\leqslant -q'_\delta(t)\int_\alpha^\delta \frac{p_\alpha(s)}{p_\alpha(\delta)}\,|h(s)|\,\mathrm{d}s + p'_\alpha(t)\int_t^\delta \frac{q_\delta(s)}{q_\delta(\alpha)}\,|h(s)|\,\mathrm{d}s \\ &\quad + |w(\delta)|\,\frac{p'_\alpha(t)}{p_\alpha(\delta)} + |\xi^*|\,\frac{-q'_\delta(t)}{q_\delta(\alpha)} \\ &\leqslant -q'_\delta(t)\int_\alpha^\delta (s-\alpha)|h(s)|\,\mathrm{d}s + p'_\alpha(t)\int_t^\delta |h(s)|\,\mathrm{d}s \\ &\quad + |w(\delta)|\,\frac{p'_\alpha(t)}{p_\alpha(\delta)} + |\xi_*|\,\frac{-q'_\delta(t)}{q_\delta(\alpha)}, \quad \alpha < t \leqslant \delta, \end{aligned}$$

$$\begin{aligned} |w'(t)| &\leqslant -q'_1(t)\int_\delta^t |h(s)|\,\mathrm{d}s + p'_\delta(t)\int_\delta^1 (1-s)|h(s)|\,\mathrm{d}s \\ &\quad + |w(\delta)|\,\frac{-q'_1(t) + \lambda p'_\delta(t)}{q_1(\delta)} + |\eta^*|\,\frac{p'_\delta(t)}{p_\delta(1)}, \quad \delta \leqslant t < 1 \end{aligned}$$

and hence

$$\begin{aligned} \int_\alpha^1 |w'(t)|\,\mathrm{d}t &\leqslant q_\delta(\alpha)\int_\alpha^\delta (s-\alpha)|h(s)|\,\mathrm{d}s + \int_\alpha^\delta p'_\alpha(t)\,\mathrm{d}t \int_t^\delta |h(s)|\,\mathrm{d}s \\ &\quad + |w(\delta)| + |\xi^*| + |w(\delta)|(\lambda+1) + |\eta^*| \\ &\quad + \int_\delta^1 -q'_1(t)\,\mathrm{d}t\int_\delta^t |h(s)|\,\mathrm{d}s + p_\delta(1)\int_\delta^1 (1-s)|h(s)|\,\mathrm{d}s \\ &= 2p_\alpha(\delta)\int_\alpha^\delta (s-\alpha)|h(s)|\,\mathrm{d}s + 2q_1(\delta)\int_\delta^1 (1-s)|h(s)|\,\mathrm{d}s \\ &\quad + |w(\delta)|(2+\lambda) + |\xi^*| + |\eta^*|, \end{aligned}$$

$$|w(\delta)| \leqslant \frac{p_\alpha(1)}{p_\alpha(1) - \lambda p_\alpha(\delta)} \left(q_1(\delta) \int_\alpha^\delta (s-\alpha)|h(s)| \, \mathrm{d}s \right.$$
$$\left. + \, p_\alpha(\delta) \int_\delta^1 (1-s)|h(s)| \, \mathrm{d}s + |\xi^*| \frac{q_1(\delta)}{p_\alpha(1)} + |\eta^*| \frac{p_\alpha(\delta)}{p_\alpha(1)} \right),$$

which means that $w'(t) \in L^1[\alpha, 1]$ and hence $w(t) \in AC[\alpha, 1]$. Here we have used Lemma 4.

Differentiating (2.11) again, we obtain

$$w''(t) = k(t)w(t) - h(t) \quad \text{a.e. on } (\alpha, 1).$$

Summarizing the above discussion, we know that the function $w(t)$ defined by (2.10) is a solution to (2.9).

Finally, we prove the uniqueness of the solution. Let $w_1(t)$ and $w_2(t)$ be both solutions to (2.8) and let $w(t) := w_1(t) - w_2(t)$. Then $w(t)$ satisfies

$$-w''(t) + k(t)w(t) = 0 \quad \text{a.e. on } (\alpha, 1),$$
$$w(\alpha) = 0, \quad w(1) - \lambda w(\delta) = 0.$$

Note that the homogeneous equation has a general solution

$$w(t) = A p_\alpha(t) + B q_1(t),$$

where A and B are arbitrary constants. It is easy to see that if and only if $A = B = 0$ the function $w(t)$ satisfies the homogeneous boundary value conditions. This means that the solution to (2.9) is unique. □

Theorem 6. *Let $\xi^*, \eta^* \in \mathbb{R}$ and $k(t), h(t) \in E$ with $k(t) > 0$ a.e. on $(0,1)$. Then the linear two-point boundary value problem*

$$-w''(t) + k(t)w(t) = h(t), \quad 0 \leqslant \alpha < t < \beta \leqslant 1,$$
$$w(\alpha) = \xi^*, \quad w(\beta) = \eta^*$$

has a unique solution

$$w(t) = \int_\alpha^\beta G_{[\alpha,\beta]}(t,s)h(s) \, \mathrm{d}s + \xi^* \frac{q_\beta(t)}{q_\beta(\alpha)} + \eta^* \frac{p_\alpha(t)}{p_\alpha(\beta)}, \quad \alpha \leqslant t \leqslant \beta.$$

Proof. The proof is similar to that of Theorem 5 and hence omitted here. □

The following maximum principles are consequences of Theorems 5 and 6.

Theorem 7. *Assume that $k(t) \in E$ with $k(t) > 3\lambda/(1-\delta)^2$ a.e. on $(\alpha, 1)$, $0 \leqslant \alpha < \delta$, and $w(t) \in D[\alpha, 1]$ satisfies*

$$-w''(t) + k(t)w(t) \geqslant 0 \quad \text{a.e. on } (\alpha, 1),$$
$$w(\alpha) \geqslant 0, \quad w(1) - \lambda w(\delta) \geqslant 0.$$

Then $w(t) \geqslant 0$ on $[0, 1]$.

Theorem 8. *Assume that $k(t) \in E$ with $k(t) > 0$ a.e. on $(0,1)$ and $w(t) \in D[\alpha,\beta]$ satisfies*

$$-w''(t) + k(t)w(t) \geqslant 0 \quad \text{a.e. on } (\alpha,\beta),$$

$$w(\alpha) \geqslant 0, \ \ w(\beta) \geqslant 0, \quad 0 \leqslant \alpha < \beta \leqslant 1.$$

Then $w(t) \geqslant 0$ on $[\alpha,\beta]$.

3. Proofs of main results

In the present section, we always assume that

$$k(t) > 3\lambda/(1-\delta)^2 \quad \text{a.e. on } (0,1).$$

Otherwise, we can replace the $k(t)$ by

$$k^*(t) := k(t) + 3\lambda/(1-\delta)^2.$$

To prove Theorem 1, we consider the modified three-point boundary value problem

$$\begin{cases} -u''(t) + k(t)u(t) = F^*(t,u(t)) & \text{a.e. on } (0,1), \\ u(0) = \xi, \quad u(1) - \lambda u(\delta) = \eta, \end{cases} \tag{3.1}$$

where

$$F^*(t,u) := \begin{cases} F(t,x(t)) & \text{if } u < x(t), \\ F(t,u) & \text{if } x(t) \leqslant u \leqslant y(t), \\ F(t,y(t)) & \text{if } u > y(t). \end{cases}$$

$$F(t,u) := f(t,u) + k(t)u.$$

Let $N := \max\{|x(t)| + |y(t)|;\ 0 \leqslant t \leqslant 1\}$. Then we have

$$|F^*(t,u)| \leqslant k_N(t) + k(t)N =: h^*(t) \text{ for almost all } (t,u) \in (0,1) \times \mathbb{R}. \tag{3.2}$$

We first prove that (3.1) has a solution. To this end, let us define a mapping $\Phi : C[0,1] \to C[0,1]$ by

$$(\Phi u)(t) := \begin{cases} \dfrac{p_0(1)}{p_0(1) - \lambda p_0(\delta)} \left(\displaystyle\int_0^1 G_{[0,1]}(\delta,s) F^*(s,u(s))\,\mathrm{d}s + \xi \frac{q_1(\delta)}{p_0(1)} + \eta \frac{p_0(\delta)}{p_0(1)} \right), \\ \displaystyle\int_0^\delta G_{[0,\delta]}(t,s) F^*(s,u(s))\,\mathrm{d}s + (\Phi u)(\delta) \frac{p_0(t)}{p_0(\delta)} \\ \quad + \xi \dfrac{q_\delta(t)}{q_\delta(t)}, \quad 0 \leqslant t \leqslant \delta, \\ \displaystyle\int_\delta^1 G_{[\delta,1]}(t,s) F^*(s,u(s))\,\mathrm{d}s + (\Phi u)(\delta) \frac{q_1(t) + \lambda p_\delta(t)}{q_1(\delta)} \\ \quad + \eta \dfrac{p_\delta(t)}{p_\delta(t)}, \quad \delta \leqslant t \leqslant 1. \end{cases} \tag{3.3}$$

From the proof of Theorem 5 and (3.2), we know that for each fixed $u(t) \in C[0,1]$

$$(\Phi u)(t) \in D[0,1], \quad (\Phi u)(0) = \xi, \quad (\Phi u)(1) - \lambda(\Phi u)(\delta) = \eta,$$

$$(\Phi u)''(t) = k(t)(\Phi u)(t) - F^*(t,u(t)) \quad \text{a.e. on } (0,1),$$

$$|(\Phi u)|(\delta)| \leqslant \frac{p_0(1)}{p_0(1) - \lambda p_0(\delta)} \left(q_1(\delta) \int_0^\delta s h^*(s)\,\mathrm{d}s + p_0(\delta) \int_\delta^1 (1-s) h^*(s)\,\mathrm{d}s \right.$$
$$\left. + |\xi| \frac{q_1(\delta)}{p_0(1)} + |\eta| \frac{p_0(\delta)}{p_0(1)} \right),$$

$$\int_0^1 |(\Phi u)'(t)|\,\mathrm{d}t \leqslant 2 p_0(\delta) \int_0^\delta s h^*(s)\,\mathrm{d}s + 2 q_1(\delta) \int_\delta^1 (1-s) h^*(s)\,\mathrm{d}s$$
$$+ |(\Phi u)(\delta)|(2+\lambda) + |\xi| + |\eta|,$$

which tells us that $\Phi(C[0,1])$ is a uniformly bounded and equicontinuous subset of $C[0,1]$, i.e., a compact subset of $C[0,1]$. From the Schauder fixed point theorem, we know that Φ has a fixed point in $C[0,1]$. Let $u(t) \in C[0,1]$ be a fixed point of Φ. Then the $u(t)$ is a solution to (3.1).

As far as the solution $u(t)$ is concerned, we now claim that $x(t) \leqslant u(t) \leqslant y(t)$ on $[0,1]$. If the second inequality were false, then

$$\Sigma := |t \in (0,1);\ w(t) := y(t) - u(t) < 0|$$

would be nonempty and open, and hence

$$\begin{cases} -w''(t) + k(t)w(t) =:\ h^*(t) \geqslant F(t,y(t)) - F^*(t,u(t)) = 0 & \text{a.e. on } \Sigma, \\ w(0) =:\ \xi^* \geqslant 0, \quad w(1) - \lambda w(\delta) =:\ \eta^* \geqslant 0. \end{cases} \tag{3.4}$$

Let (α,β) be a connect component of Σ. Then we can consider only four cases.

Case (i): $0 \leqslant \alpha < \beta < 1$. In this case, $w(\alpha) = 0 = w(\beta)$. From Theorem 7, we know that $w(t) \geqslant 0$ on $[\alpha,\beta]$, which contradicts the fact that $w(t) < 0$ in (α,β).

Case (ii): $\delta \leqslant \alpha < \beta = 1$ and $\delta \notin \Sigma$. In this case, $w(\alpha) = 0$ and $w(1) = \eta^* + \lambda w(\delta) \geqslant 0$. As Case (i), we can lead to a contradiction.

Case (iii): $\delta < \alpha < \beta = 1$ and $\delta \in \Sigma$. Since $\delta \in \Sigma$ and $\alpha \notin \Sigma$, we can find (α^*,β^*), another connect component of Σ, such that $0 \leqslant \alpha^* < \beta^* \leqslant \alpha < 1$. As Case (i), we lead to a contradiction again.

Case (iv): $0 \leqslant \alpha < \delta < \beta = 1$. In this situation, from Theorem 6 we know that $w(t) \geqslant 0$ on $[\alpha,\beta]$, which contradicts the fact that $w(t) < 0$ in (α,β).

Up to now, we show that $u(t) \leqslant y(t)$ on $[0,1]$.

In the same way as before, we can show that $x(t) \leqslant u(t)$ on $[\alpha,\beta]$.

From the claim, we know that the solution $w(t)$ is also a solution to (1.1). The proof of Theorem 1 is thus complete.

We are now in the position to prove Theorem 2. Let us define

$$[x,y] := \{u(t) \in D[0,1];\ x(t) \leqslant u(t) \leqslant y(t) \text{ on } [0,1]\}$$

and $w = \Phi u$ as the solution of the linear three-point boundary value problem

$$-w''(t) + k(t)w(t) = F(t,u(t)) \quad \text{a.e. on } (0,1),$$

$$w(0) = \xi, \quad w(1) - \lambda w(\delta) = \eta$$

for each fixed $u \in [x, y]$. Then the mapping Φ is continuous and monotone increasing, since $F(t, u)$ is continuous and strictly increasing on $[\min\{x(t);\ 0 \leqslant t \leqslant 1\}, \max\{y(t);\ 0 \leqslant t \leqslant 1\}]$ for almost all $t \in (0, 1)$.

We first prove that Φ is well defined. Let $u \in [x, y]$ and $w = \Phi u$. Set $\rho(t) = w(t) - x(t)$. Then,

$$-\rho''(t) + k(t)\rho(t) \geqslant F(t, w(t)) - F(t, x(t)) \geqslant 0 \quad \text{a.e. on } (0, 1),$$

$$\rho(0) \geqslant 0, \quad \rho(1) - \lambda\rho(\delta) \geqslant 0.$$

By the maximum principle, we deduce $\rho(t) \geqslant 0$ on $[0, 1]$, that is, $x(t) \leqslant w(t)$ on $[0, 1]$. Analogously, we have $w(t) \leqslant y(t)$ on $[0, 1]$.

We now show that Φ is monotone increasing, that is, if $x(t) \leqslant u_1(t) \leqslant u_2(t) \leqslant y(t)$ on $[0, 1]$, then

$$w_1(t) := (\Phi u_1)(t) \leqslant (\Phi u_2)(t) =: w_2(t) \quad \text{on } [0, 1].$$

Indeed, let $\rho(t) := w_2(t) - w_1(t)$. Then

$$-\rho''(t) + k(t)\rho(t) = F(t, u_2(t)) - F(t, u_1(t)) \geqslant 0 \quad \text{a.e. on } (0, 1),$$

$$\rho(0) = 0, \quad \rho(1) - \lambda\rho(\delta) = 0.$$

Hence $\rho(t) \geqslant 0$ on $[0, 1]$, in other words, $w_1(t) \leqslant w_2(t)$ on $[0, 1]$.

Let $x_0(t) = x(t)$ and define $x_{m+1}(t) = (\Phi x_m)(t)$ for $m \geqslant 0$. By the properties of Φ we have that

$$x(t) = x_0(t) \leqslant x_1(t) \leqslant y(t) \quad \text{on } [0, 1].$$

Now, by induction it is easily seen that

$$x(t) \leqslant x_m(t) \leqslant x_{m+1}(t) \leqslant y(t) \quad \text{on } [0, 1]$$

for every $m \geqslant 0$.

Therefore, $\{x_m(t)\}_{m=0}^{\infty}$ is increasing and we have

$$\begin{aligned} u_*(t) &=: \lim_{m \to \infty} x_{m+1}(t) \\ &=: \lim_{m \to \infty} (\Phi x_m)(t) = (\Phi u_*)(t) \quad \text{uniformly on } [0, 1], \end{aligned}$$

by the dominated convergence theorem. This means that $u_*(t)$ is a solution to the problem (1.1).

Analogously, defining $y_0(t) = y(t)$ and $y_{m+1}(t) = (\Phi y_m)(t)$ for $m \geqslant 0$. We have that the sequence $\{y_m(t)\}_{m=0}^{\infty} \searrow u^*(t)$ uniformly on $[0, 1]$. Also, $u^*(t)$ is a solution to problem (1.1).

Finally, it follows from the maximum principle that $u_*(t)$ and $u^*(t)$ are the minimum and maximum solutions to (1.1) in $[x, y]$, respectively. □

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