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A strictly stationary, N-tuplewise independent counterexample to the Central Limit Theorem

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Abstract

For an arbitrary integer $N \geq 2$, this paper gives the construction of a strictly stationary (and ergodic), N-tuplewise independent sequence of (nondegenerate) bounded random variables such that the Central Limit Theorem fails to hold. The sequence is in part an adaptation of a nonstationary example with similar properties constructed by one of the authors (ARP) in a paper published in 1998.

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1. Introduction and main result

Suppose $X := (X_k, k \in \mathbf{Z})$ is a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$. This sequence X is said to be "strictly stationary" if for all choices of integers j and ℓ and nonnegative integer m, the random vectors $(X_j, X_{j+1}, \ldots, X_{j+m})$ and $(X_\ell, X_{\ell+1}, \ldots, X_{\ell+m})$ have the same distribution. For a given integer $N \geq 2$, the sequence X (stationary or not) is said to satisfy "N-tuplewise independence" if for every choice of N distinct integers $k(1), k(2), \ldots, k(N)$, the random variables $X_{k(1)}, X_{k(2)}, \ldots, X_{k(N)}$ are independent. For $N = 2$ (resp. $N = 3$), the word "N-tuplewise" is also expressed as "pairwise" (resp. "triplewise").

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Etemadi [5] proved a strong law of large numbers for sequences of pairwise independent, identically distributed random variables with finite absolute first moment. Janson [7] showed with several classes of counterexamples that for strictly stationary sequences of pairwise independent, nondegenerate, square-integrable random variables, the Central Limit Theorem (henceforth abbreviated CLT) need not hold. Subsequently, Bradley [2, Theorem 1] constructed another such counterexample, a three-state one that has the additional property of satisfying the absolute regularity (weak Bernoulli) condition. For an arbitrary fixed integer $N \geq 3$, Pruss [9] constructed a (not strictly stationary) sequence of bounded, nondegenerate, N-tuplewise independent, identically distributed random variables for which the CLT fails to hold. In that paper, Pruss left open the question whether, for any integer $N \geq 3$, a strictly stationary counterexample exists. For $N = 3$, Bradley [3, Theorem 1] answered that question by showing that the counterexample in [2, Theorem 1] alluded to above is in fact triplewise independent.

In a similar spirit, for an arbitrary integer $N \geq 2$, Flaminio [6] constructed a strictly stationary, finite-state, N-tuplewise independent random sequence $X := (X_k, k \in \mathbf{Z})$ which also has zero entropy and is mixing (in the ergodic-theoretic sense). That paper explicitly left open the question of whether those examples satisfy the CLT.

In this paper, we shall answer affirmatively the question in [9], by constructing for an arbitrary fixed integer $N \geq 2$ a strictly stationary (and ergodic), N-tuplewise independent sequence of bounded, nondegenerate random variables such that the CLT fails to hold. The construction will be in part an adaptation of the (not strictly stationary) counterexample in [9].

Before the main result is stated, a few notations are needed:

The Borel σ-field on the real number line $\mathbf{R}$ will be denoted by $\mathcal{R}$.

Convergence in distribution will be denoted by $\Rightarrow$.

The set of positive integers will be denoted by $\mathbf{N}$. For a given sequence $X := (X_k, k \in \mathbf{Z})$ of random variables, the partial sums will be denoted for $n \in \mathbf{N}$ by

$$S_n := S(X, n) := X_1 + X_2 + \cdots + X_n. \tag{1.1}$$

Here is our main result.

Theorem 1.1. *Suppose N is an integer such that $N \geq 2$. Then there exists a strictly stationary, ergodic sequence $X := (X_k, k \in \mathbf{Z})$ of random variables such that the following four statements hold:*

(A) *The random variable X_0 is uniformly distributed on the interval* $[-3^{1/2}, 3^{1/2}]$ *(and hence $EX_0 = 0$ and $EX_0^2 = 1$).*

(B) *For every choice of N distinct integers $k(1), k(2), \ldots, k(N)$, the random variables $X_{k(1)}, X_{k(2)}, \ldots, X_{k(N)}$ are independent.*

(C) *The random variables $|X_k|, k \in \mathbf{Z}$ are independent (and identically distributed).*

(D) *For every infinite set $Q \subset \mathbf{N}$, there exist an infinite set $T \subset Q$ and a nondegenerate, non-normal probability measure μ on* $(\mathbf{R}, \mathcal{R})$ *such that* $S_n/n^{1/2} \Rightarrow \mu$ as $n \to \infty, n \in T$.

Here are some comments on Theorem 1.1:

By property (A), the "natural normalization" to consider for the central limit question for the partial sums of this sequence is $S_n/n^{1/2}$.

Property (B) is of course N-tuplewise independence.

By property (D) and the Theorem of Types (see e.g. [1, p. 193, Theorem 14.2]), there do not exist constants $a_n > 0$ and $b_n \in \mathbf{R}$ for $n \in \mathbf{N}$ such that $a_n S_n + b_n \Rightarrow N(0, 1)$, even along a subsequence of the positive integers.

Also by property (D) and an elementary argument, there do not exist constants $b_n \in \mathbf{R}$ for $n \in \mathbf{N}$ such that $n^{-1/2}S_n + b_n \to 0$ in probability, even along a subsequence of the positive integers.

In property (D), the probability measure μ may depend on the set Q.

Remark 1.2. With essentially the same construction, one obtains an analog of Theorem 1.1 with property (A) replaced by the following one: (A′) The random variable X_0 has the $N(0, 1)$ distribution.

Beyond ergodicity itself (as stated in the theorem), we did not investigate further the ergodic-theoretic properties of the sequence X in Theorem 1.1; nor did we try to ascertain the particular class of probability measures μ that can arise in statement (D) there. Bradley [4] gives the construction of a (nondegenerate, two-state) strictly stationary, 5-tuplewise independent random sequence which fails to satisfy the CLT (instead, it satisfies property (D) in Theorem 1.1), and also has the extra properties of being "causal" (for an appropriate use of that term) and therefore "Bernoulli" (i.e. isomorphic to a Bernoulli shift) and also having a trivial double tail σ-field. There does not seem to be an obvious way to build those extra properties into the construction here for Theorem 1.1.

The proof of Theorem 1.1 will be given in Sections 2–5. Refer to properties (A) and (C) in Theorem 1.1. In spirit, the construction will involve taking a sequence of independent, identically distributed random variables uniformly distributed on the interval $[-3^{1/2}, 3^{1/2}]$, and changing the signs of the variables in such a way as to introduce a dependence that preserves N-tuplewise independence but is incompatible with the CLT.

In essence it involves the conversion of the (not strictly stationary) counterexample in [9] to one that is strictly stationary. The main ideas for that conversion were outlined in an e-mail message by one of the authors (Pruss [10]) to the other author (RCB), and are developed in Section 3 here. Section 2 gives some vital "preliminary" information; much of it is taken or adapted from [9], but will be given again here in detail because of extra complications in our context. In Section 4, the use of higher order moments to establish property (D) in Theorem 1.1, is adapted from the analogous use of 6th moments for the same purpose in (an earlier, 2006 version of) the preprint [4]. Underlying all this is the repeated creation of "big" collections of N-tuplewise independent random variables from "smaller" ones; such procedures are well known in the theory of error-correcting codes (see e.g. [8]).

The comments and suggestions by two referees played a major role in this paper. In the original version of this paper, the final sequence X was obtained as an "almost sure limit" of a family of random sequences; and that involved some rather awkward notations and cumbersome arguments. One referee made detailed suggestions on a method to obtain the final sequence X much more quickly and easily as a "limit in distribution" of a family of random sequences. Those suggestions resulted in a substantially shorter and simpler argument here. The other referee made some comments to point out the importance that the property of ergodicity, not dealt with in our original version, might have. In response to those comments, we have worked out and included (in Section 5) a proof of ergodicity, and included that property in the statement of Theorem 1.1.

2. Part 1 of proof of Theorem 1.1: Preliminaries and key random vectors

Sections 2–5 together will give the proof of Theorem 1.1. Sections 2–4 will be divided into several "steps", including some "definitions", some "lemmas", etc. Throughout this proof, the setting is a probability space $(\Omega, \mathcal{F}, P)$, "enlarged" as necessary to accommodate all random variables defined in this proof.

Section 2 here will be devoted to preliminaries and to the construction (in Definition 2.7) and study of a family of random vectors that will play a key role later on.

Step 2.1. Let L be an arbitrary fixed integer such that

$$L \text{ is } \mathbf{even} \text{ and } L \geq 6. \tag{2.1}$$

To prove Theorem 1.1, it suffices to construct a strictly stationary, ergodic, $(L-1)$-tuplewise independent random sequence $X := (X_k, k \in \mathbf{Z})$ that also satisfies properties (A), (C), and (D) in Theorem 1.1. That will be the goal of Sections 2–5.

Step 2.2. The following notations and conventions will be used:

(a) Refer to (2.1). For $n \in \{0, 1, 2, \ldots\}$, when the term L^n appears in a subscript or exponent, it will be written as $L \uparrow n$ for typographical convenience.

(b) Suppose $n \in \mathbf{N}$. A vector $x \in \mathbf{R}^n$ will often be represented as $x := (x_0, x_1, \ldots, x_{n-1})$ (instead of $(x_1, x_2, \ldots, x_n)$), for "bookkeeping" convenience. For a given $x := (x_0, x_1, \ldots, x_{n-1}) \in \mathbf{R}^n$, define the two real numbers

$$\operatorname{sum} x := \sum_{i=0}^{n-1} x_i \quad \text{and} \quad \operatorname{prod} x := \prod_{i=0}^{n-1} x_i. \tag{2.2}$$

(c) Sometimes the coordinates of a vector will be permuted. If $n \in \mathbf{N}$, $x := (x_0, x_1, \ldots, x_{n-1}) \in \mathbf{R}^n$, and σ is a permutation of the set $\{0, 1, \ldots, n-1\}$, then define the vector $x_\sigma \in \mathbf{R}^n$ by $x_\sigma := ((x_\sigma)_0, (x_\sigma)_1, \ldots, (x_\sigma)_{n-1}) := (x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(n-1)})$.

(d) If $a \leq b$ are integers and $Y_a, Y_{a+1}, Y_{a+2}, \ldots, Y_b$ are random variables, then the random vector $(Y_a, Y_{a+1}, Y_{a+2}, \ldots, Y_b)$ will also be denoted as $Y[a, b]$. Similarly, if a is an integer and $Y_a, Y_{a+1}, Y_{a+2}, \ldots$ are random variables, then that random sequence $(Y_a, Y_{a+1}, Y_{a+2}, \ldots)$ will also be denoted as $Y[a, \infty)$.

(e) A "measure" on the space $\mathbf{R}$, $\mathbf{R}^n (n \in \mathbf{N})$, $\mathbf{R}^{\mathbf{N}}$, or $\mathbf{R}^{\mathbf{Z}}$ will always mean a measure on the Borel σ-field (denoted as $\mathcal{R}$, $\mathcal{R}^n$, $\mathcal{R}^{\mathbf{N}}$, or $\mathcal{R}^{\mathbf{Z}}$ respectively) on that space.

(f) If η is a random variable/vector/sequence, then the distribution of η on the appropriate space ($\mathbf{R}$, $\mathbf{R}^n$, $\mathbf{R}^{\mathbf{N}}$, or $\mathbf{R}^{\mathbf{Z}}$) will be denoted as $\mathcal{L}(\eta)$. If also F is an event such that $P(F) > 0$, then $\mathcal{L}(\eta \mid F)$ will denote the conditional distribution of η, given F.

(g) For a given $n \in \mathbf{N}$, an "$\mathbf{R}^n$-valued random vector" is simply a random vector with n coordinates.

(h) A couple of notations will be defined here. Suppose $Y := (Y_i, i \in I)$ is a family of random variables, where I is a nonempty (possibly infinite) index set. To avoid any confusion, this family Y is said to satisfy "$(L-1)$-tuplewise independence" (see (2.1)) if either (i) $\operatorname{card} I = 1$, or (ii) $\operatorname{card} I \geq 2$ and for every set $S \subset I$ such that $2 \leq \operatorname{card} S \leq L-1$, the random variables Y_i, $i \in S$ are independent. (The point there is to formally include the case $\operatorname{card} I = 1$ in that terminology.) Here and below, "card" means cardinality. In the case of a random vector $Y := (Y_0, Y_1, \ldots, Y_{n-1})$, where $n \in \mathbf{N}$, the phrase "Y satisfies $(L-1)$-tuplewise independence", simply means that the family of its coordinates $(Y_i, i \in \{0, 1, \ldots, n-1\})$ satisfies $(L-1)$-tuplewise independence.

Remark. The following elementary fact will be used throughout this proof: Suppose that for each $j = 1, 2, 3, \ldots$, the family $\eta^{(j)} := (\eta_1^{(j)}, \eta_2^{(j)}, \eta_3^{(j)}, \ldots)$ of random variables satisfies

$(L-1)$-tuplewise independence. Suppose that every $L-1$ of these families $\eta^{(j)}$ are mutually independent. Then the "combined" family $(\eta_i^{(j)}, j \in \mathbf{N}, i \in \mathbf{N})$ satisfies $(L-1)$-tuplewise independence.

Step 2.3. For each $n \in \mathbf{N}$, define the function $\varphi_n : \mathbf{R}^n \to \mathbf{R}^n$ as follows: For $x \in \mathbf{R}^n$ (see (2.2)),

$$\varphi_n(x) := \begin{cases} x & \text{if sum } x > 0 \\ 0_n & \text{if sum } x = 0 \\ -x & \text{if sum } x < 0 \end{cases} \tag{2.3}$$

where 0_n denotes the origin in $\mathbf{R}^n$.

Thus for example $\varphi_3(5, -7, 4) = \varphi_3(-5, 7, -4) = (5, -7, 4)$.

Remark 1. For each $n \in \mathbf{N}$ and each $x \in \mathbf{R}^n$, $\text{sum}\, \varphi_n(x) = |\text{sum}\, x|$.

Remark 2. Suppose $n \in \mathbf{N}$, $x := (x_0, x_1, \dots, x_{n-1}) \in \mathbf{R}^n$, $y := \varphi_n(x) := (y_0, y_1, \dots, y_{n-1})$, and σ is a permutation of the set $\{0, 1, \dots, n-1\}$. Recall from Step 2.2(c) the notations $x_\sigma := (x_{\sigma(0)}, x_{\sigma(1)}, \dots, x_{\sigma(n-1)})$ and $y_\sigma := (y_{\sigma(0)}, y_{\sigma(1)}, \dots, y_{\sigma(n-1)})$. Then $y_\sigma = \varphi_n(x_\sigma)$. This holds by a careful trivial argument, using the fact that $\text{sum}\, x_\sigma = \text{sum}\, x$. Thus for $0 \le i \le n-1$ (see Step 2.2(c) again), $(\varphi_n(x_\sigma))_i = (y_\sigma)_i = y_{\sigma(i)} = (\varphi_n(x))_{\sigma(i)}$.

Remark 3. If n, x, and y are as in Remark 2 above (with $y = \varphi_n(x)$), and also $\text{sum}\, x \ne 0$, then $|x_i| = |y_i|$ for each $i \in \{0, 1, \dots, n-1\}$.

Remark 4. Suppose $n \in \mathbf{N}$, Y is an $\mathbf{R}^n$-valued random vector (see Step 2.2(f)(g)) such that $\mathcal{L}(-Y) = \mathcal{L}(Y)$ and $\mathcal{L}(Y)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^n$, and V is a random variable independent of Y such that $P(V = -1) = P(V = 1) = 1/2$. Then $P(\text{sum}\, Y = 0) = 0$; and for any Borel set $B \subset \{x \in \mathbf{R}^n : \text{sum}\, x > 0\}$,

$$P(Y \in B) = P(-Y \in B) = (1/2) \cdot P(\varphi_n(Y) \in B).$$

By trivial arguments, one thereby obtains that $\mathcal{L}(\varphi_n(Y))$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^n$, and one also obtains that $\mathcal{L}(V\varphi_n(Y)) = \mathcal{L}(Y)$.

Step 2.4. The notation λ_{unps3} will refer to the uniform distribution on the interval $[-3^{1/2}, 3^{1/2}]$, regarded as a probability measure on $\mathbf{R}$. (The subscript "unps3" stands for **u**niform on the interval from **n**egative to **p**ositive **s**quare root of **3**.) For any positive integer m, the m-fold product measure $\lambda_{\text{unps3}} \times \lambda_{\text{unps3}} \times \dots \times \lambda_{\text{unps3}}$ on $\mathbf{R}^m$ will be denoted as $\lambda_{\text{unps3}}^{[m]}$.

Remark. If U is a random variable such that $\mathcal{L}(U) = \lambda_{\text{unps3}}$ (see Step 2.2(f)), then of course $EU^n = 0$ for odd $n \in \mathbf{N}$ and $EU^n = (n+1)^{-1} \cdot 3^{n/2}$ for even $n \in \mathbf{N}$. Hence, if also Z is a $N(0,1)$ random variable, then (see e.g. [1, p. 275, Eq. (21.7)]) for every $n \in \mathbf{N}$, $EZ^n \ge EU^n \ge 0$ (with equality for odd n and for $n = 2$ with $EZ^2 = EU^2 = 1$, and strict inequality in the other cases).

Step 2.5. Define (see (2.1) and (2.2)) the set $\Upsilon := \{x \in \{-1, 1\}^L : \text{prod}\ x = -1\}$. Then $\text{card}\ \Upsilon = 2^{L-1}$. Let ν denote the uniform distribution on Υ — that is, the probability measure ν on $\mathbf{R}^L$ such that $\nu(\{x\}) = 1/2^{L-1}$ for each $x \in \Upsilon$.

Remark 2.6. Suppose $V := (V_0, V_1, \ldots, V_{L-1})$ is an Υ-valued random vector such that $\mathcal{L}(V) = \nu$ (see (2.1) and Step 2.5). Then by elementary arguments, the following statements hold:

(i) For each $k \in \{0, 1, \ldots, L-1\}$, $P(V_k = -1) = P(V_k = 1) = 1/2$.

(ii) V satisfies $(L-1)$-tuplewise independence.

(iii) $\mathcal{L}(-V) = \nu$.

(iv) For every permutation σ on the set $\{0, 1, \ldots, L-1\}$, the random vector $V_\sigma := (V_{\sigma(0)}, V_{\sigma(1)}, \ldots, V_{\sigma(L-1)})$ (see Step 2.2(c)) satisfies $\mathcal{L}(V_\sigma) = \nu$.

(v) prod $V = -1$ (see (2.2)).

Definition 2.7. For each $n \in \{0, 1, 2, \ldots\}$, an $\mathbf{R}^{L\uparrow n}$-valued random vector $W^{(n)} := W^{(n)}[0, L^n - 1] := (W_0^{(n)}, W_1^{(n)}, \ldots, W_{(L\uparrow n)-1}^{(n)})$ (see Step 2.2(a)(d)) will be defined (together with some other useful related random vectors). The definition is recursive and is as follows:

To start with $n = 0$, let $W^{(0)} := (W_0^{(0)})$ be a random variable uniformly distributed on the interval $[-3^{1/2}, 3^{1/2}]$.

Now suppose $n \geq 0$, and the $\mathbf{R}^{L\uparrow n}$-valued random vector $W^{(n)} := W^{(n)}[0, L^n - 1]$ is already defined. Define the $\mathbf{R}^{L\uparrow(n+1)}$-valued random vector $W^{(n+1)} := W^{(n+1)}[0, L^{n+1} - 1]$ as follows:

First let $\zeta^{(n+1,j)} := \zeta^{(n+1,j)}[0, L^n - 1]$, for $j \in \{0, 1, \ldots, L-1\}$, be L independent $\mathbf{R}^{L\uparrow n}$-valued random vectors, with each having the same distribution (on $\mathbf{R}^{L\uparrow n}$) as the random vector $W^{(n)}$. Let $V^{(n+1)} := V^{(n+1)}[0, L-1]$ be an Υ-valued random vector with distribution ν (see Step 2.5), such that $V^{(n+1)}$ is independent of $(\zeta^{(n+1,j)}, j \in \{0, 1, \ldots, L-1\})$. Then define the random vector $W^{(n+1)} := W^{(n+1)}[0, L^{n+1} - 1]$ as follows (see (2.3)):

$$\forall j \in \{0, 1, \ldots, L-1\}, \quad W^{(n+1)}[jL^n, (j+1)L^n - 1] := V_j^{(n+1)} \varphi_{L\uparrow n}(\zeta^{(n+1,j)}). \quad (2.4)$$

That is, $W_{j(L\uparrow n)+i}^{(n+1)} := V_j^{(n+1)} (\varphi_{L\uparrow n}(\zeta^{(n+1,j)}))_i$ for $0 \leq j \leq L-1$ and $0 \leq i \leq L^n - 1$, with the representation (used here and below) $\varphi_{L\uparrow n}(\zeta^{(n+1,j)}) := (\varphi_{L\uparrow n}(\zeta^{(n+1,j)}))[0, L^n - 1]$.

Lemma 2.8. *For each $n \geq 0$, $\mathcal{L}(-W^{(n)}) = \mathcal{L}(W^{(n)})$, and the distribution of $\mathcal{L}(W^{(n)})$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^{L\uparrow n}$.*

Proof. For $n = 0$, this is trivial since $\mathcal{L}(W^{(0)}) = \lambda_{\text{unps3}}$ (see Step 2.4). Now for induction, suppose Lemma 2.8 holds for a given $n \geq 0$.

From (2.4), one directly obtains an obvious representation

$$W^{(n+1)} := g(V^{(n+1)}; \zeta^{(n+1,0)}, \zeta^{(n+1,1)}, \ldots, \zeta^{(n+1,L-1)})$$

in terms of a particular function $g : \{-1, 1\}^L \times (\mathbf{R}^{L\uparrow n})^L \to \mathbf{R}^{L\uparrow(n+1)}$. Multiplying both sides of (2.4) by -1, one trivially obtains the representation $-W^{(n+1)} := g(-V^{(n+1)}; \zeta^{(n+1,0)}, \ldots, \zeta^{(n+1,L-1)})$. By Remark 2.6(iii) and the conditions in Definition 2.7,

$$\mathcal{L}(-V^{(n+1)}; \zeta^{(n+1,0)}, \ldots, \zeta^{(n+1,L-1)}) = \mathcal{L}(V^{(n+1)}; \zeta^{(n+1,0)}, \ldots, \zeta^{(n+1,L-1)}).$$

It follows that $\mathcal{L}(-W^{(n+1)}) = \mathcal{L}(W^{(n+1)})$.

Next, by the induction assumption, the conditions in Definition 2.7, and Remark 4 in Step 2.3, for each $j \in \{0, 1, \ldots, L-1\}$, $\mathcal{L}(\varphi_{L\uparrow n}(\zeta^{(n+1,j)})) = \mathcal{L}(\varphi_{L\uparrow n}(W^{(n)}))$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^{L\uparrow n}$. Hence by (2.4) and the conditions in Definition 2.7, for each $v \in \Upsilon$, one has that $\mathcal{L}(W^{(n+1)} \mid V^{(n+1)} = v)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^{L\uparrow(n+1)}$; and hence that is true for $\mathcal{L}(W^{(n+1)})$ as well. That completes the induction step and the proof. $\square$

Lemma 2.9. *Suppose $n \geq 0$. Then the $\mathbf{R}^{L\uparrow n}$-valued random vectors $W^{(n+1)}[jL^n, (j+1)L^n - 1]$, $j \in \{0, 1, \ldots, L-1\}$ have the following two properties:* (i) *Every $L-1$ of them are mutually independent.* (ii) *Each one has the same distribution (on $\mathbf{R}^{L\uparrow n}$) as the random vector $W^{(n)}$.*

Proof. In (2.4), consider first the value $j = 0$. By Lemma 2.8 and the conditions in Definition 2.7, $\mathcal{L}(-\zeta^{(n+1,0)}) = \mathcal{L}(-W^{(n)}) = \mathcal{L}(W^{(n)}) = \mathcal{L}(\zeta^{(n+1,0)})$, and $\mathcal{L}(\zeta^{(n+1,0)})$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^{L\uparrow n}$. By Remark 2.6(i), the conditions in Definition 2.7, and Remark 4 in Step 2.3,

$$\mathcal{L}(W^{(n+1)}[0, L^n - 1]) = \mathcal{L}(V_0^{(n+1)} \varphi_{L\uparrow n}(\zeta^{(n+1,0)})) = \mathcal{L}(\zeta^{(n+1,0)}) = \mathcal{L}(W^{(n)}). \tag{2.5}$$

Now by Remark 2.6(i)(ii) and the conditions in Definition 2.7, the $((\{-1, 1\} \times \mathbf{R}^{L\uparrow n})$-valued) random vectors $(V_j^{(n+1)}, \zeta^{(n+1,j)})$, $j \in \{0, 1, \ldots, L-1\}$ are identically distributed, and every $L-1$ of them are mutually independent. The lemma now follows from (2.4) and (2.5). □

Lemma 2.10. *For each $n \geq 0$, the following three statements hold:*

(a) $\mathcal{L}(W_k^{(n)}) = \lambda_{\text{unps3}}$ *for all $k \in \{0, 1, \ldots, L^n - 1\}$ (see Step 2.4).*

(b) *The (family of coordinates of the) random vector $W^{(n)}$ satisfies $(L-1)$-tuplewise independence.*

(c) *The random variables $|W_k^{(n)}|$, $k \in \{0, 1, \ldots, L^n - 1\}$ are independent.*

Proof. Since $\mathcal{L}(W_0^{(0)}) = \lambda_{\text{unps3}}$ (see Definition 2.7), property (a) holds by Lemma 2.9(ii) and induction on n. Also property (b) holds by Lemma 2.9(i), the Remark after Step 2.2(h), and induction on n. Property (c) will take slightly more work to verify.

Suppose $n \geq 0$. In (2.4), let us first consider the value $j = 0$, and use the representations $\zeta^{(n+1,0)} := (\zeta_0^{(n+1,0)}, \zeta_1^{(n+1,0)}, \ldots, \zeta_{(L\uparrow n)-1}^{(n+1,0)})$ and $\varphi_{L\uparrow n}(\zeta^{(n+1,0)}) := (\eta_0, \eta_1, \ldots, \eta_{(L\uparrow n)-1})$. Then by (2.4), Remark 2.6(i), and then Remark 3 in Step 2.3 (together with Definition 2.7 and Lemma 2.8), for each $k \in \{0, 1, \ldots, L^n - 1\}$, $|W_k^{(n+1)}| = |V_0^{(n+1)} \eta_k| = |\eta_k| = |\zeta_k^{(n+1,0)}|$ a.s. That is, for $j = 0$, the absolute values of the coordinates of $W^{(n+1)}[jL^n, (j+1)L^n - 1]$ are (a.s.) respectively the absolute values of the coordinates of $\zeta^{(n+1,j)}$. The same holds for all $j \in \{0, 1, \ldots, L-1\}$ by essentially the same argument. Hence (see Definition 2.7 again) if the random variables $|W_k^{(n)}|$, $0 \leq k \leq L^n - 1$ are independent (or if $n = 0$), then the random variables $|W_k^{(n+1)}|$, $0 \leq k \leq L^{n+1} - 1$ are independent. Property (c) in Lemma 2.10 now follows by induction on n. □

Lemma 2.11. *Suppose m and n are integers such that $m > n \geq 0$. Then the $\mathbf{R}^{L\uparrow n}$-valued random vectors $W^{(m)}[jL^n, (j+1)L^n - 1]$, $j \in \{0, 1, \ldots, L^{m-n} - 1\}$ have the following two properties:* (i) *Every $L-1$ of them are mutually independent.* (ii) *Each one has the same distribution (on $\mathbf{R}^{L\uparrow n}$) as the random vector $W^{(n)}$.*

Proof. Suppose $n \geq 0$. Then the conclusion (i.e. the last three sentences) of Lemma 2.11 holds for $m = n + 1$ by Lemma 2.9.

Now for induction on m, keeping $n \geq 0$ fixed, suppose the conclusion of Lemma 2.11 holds for a given $m \geq n + 1$. By Lemma 2.9, the ($\mathbf{R}^{L\uparrow m}$-valued) random vectors $W^{(m+1)}[jL^m, (j+1)L^m - 1]$, $j \in \{0, 1, \ldots, L-1\}$ are identically distributed (with the distribution of $W^{(m)}$), and every $L-1$ of them are mutually independent. Hence by the induction assumption and the Remark after Step 2.2(h) (trivially adapted with the $\eta_i^{(j)}$'s there being $\mathbf{R}^{L\uparrow n}$-valued random vectors), the conclusion of Lemma 2.11 holds with m replaced by $m + 1$. The lemma now holds by induction on m. □

3. Part 2 of proof of Theorem 1.1: The random sequence X

This section will build directly on the work already done in Section 2. In this section, after some further preliminary work, the random sequence $X := (X_k, k \in \mathbf{Z})$ for Theorem 1.1 will be constructed (in Step 3.11), and it will be proved that this sequence X is strictly stationary and satisfies properties (A), (B), and (C) in Theorem 1.1. The proof of property (D) will be given in Section 4, and the proof that X is ergodic will be given in Section 5.

Step 3.1. Refer again to the integer L in (2.1), and refer to Step 2.2(d). For each $n \geq 0$, let $Y^{(n)} := (Y_k^{(n)}, k \in \mathbf{Z})$ be a random sequence with the following properties: The $\mathbf{R}^{L\uparrow n}$-valued random vectors $Y^{(n)}[jL^n, (j+1)L^n - 1]$, $j \in \mathbf{Z}$ are independent, and each of them has the same distribution (on $\mathbf{R}^{L\uparrow n}$) as the random vector $W^{(n)}$ in Definition 2.7.

Remark 3.2. Suppose $n \geq 0$. By Step 3.1, Lemma 2.10, and trivial arguments, the following three statements hold:

(i) For each $k \in \mathbf{Z}$, $\mathcal{L}(Y_k^{(n)}) = \lambda_{\text{unps3}}$.

(ii) The sequence $Y^{(n)}$ satisfies $(L-1)$-tuplewise independence.

(iii) The random variables $|Y_k^{(n)}|$, $k \in \mathbf{Z}$ are independent.

Remark 3.3. Suppose $n \geq 0$.

(i) Refer to Step 2.2(d). As a trivial consequence of the conditions in Step 3.1, the random sequences $Y^{(n)}[jL^n, \infty)$, $j \in \mathbf{Z}$ all have the same distribution (on $\mathbf{R}^{\mathbf{N}}$).

(ii) By (i), if h and j are integers such that $h \equiv j \mod L^n$, then (see Step 2.2(d)(f) again) $\mathcal{L}(Y^{(n)}[h, \infty)) = \mathcal{L}(Y^{(n)}[j, \infty))$.

(iii) As a further simple consequence, for any set $A \in \mathcal{R}^{\mathbf{N}}$ and any integer j,

$$\sum_{u=j}^{j+(L\uparrow n)-1} P(Y^{(n)}[u, \infty) \in A) = \sum_{u=0}^{(L\uparrow n)-1} P(Y^{(n)}[u, \infty) \in A).$$

Remark 3.4. Suppose m and n are integers such that $m > n \geq 0$.

(i) By Step 3.1 (with n there replaced by m here) and Lemma 2.11, the $\mathbf{R}^{L\uparrow n}$-valued random vectors $Y^{(m)}[jL^n, (j+1)L^n - 1]$, $j \in \mathbf{Z}$ have the following two properties: (a) They all have the same distribution as the random vector $W^{(n)}$ in Definition 2.7. (b) Every $L-1$ of them (and hence in particular by (2.1), *every five of them*) are mutually independent.

(ii) By (i)(a)(b) and a trivial argument, if h and j are integers such that $h \equiv j \mod L^n$, then $\mathcal{L}(Y^{(m)}[h, h+L^n-1]) = \mathcal{L}(Y^{(m)}[j, j+L^n-1])$.

(iii) By (i)(b) and a trivial argument, for any two integers j and ℓ such that $|j-\ell| \geq 2L^n$, the random vectors $Y^{(m)}[j, j+L^n-1]$ and $Y^{(m)}[\ell, \ell+L^n-1]$ are independent.

Step 3.5. For each $n \geq 0$, let $\tau(n)$ be a random variable which takes its values in the set $\{0, 1, \ldots, L^n - 1\}$ and is uniformly distributed on that set (that is, $P(\tau(n) = j) = L^{-n}$ for j in that set), such that $\tau(n)$ is independent of the random sequence $Y^{(n)}$ in Step 3.1.

Step 3.6. For each $n \geq 0$, define the random sequence $X^{(n)} := (X_k^{(n)}, k \in \mathbf{Z})$ as follows: For each $k \in \mathbf{Z}$,

$$X_k^{(n)} := Y_{k+\tau(n)}^{(n)}.$$

Remark 3.7. Suppose $n \geq 0$. For any $j \in \mathbf{Z}$ and any set $C \in \mathcal{R}^{\mathbf{N}}$, by Steps 3.5 and 3.6 and Remark 3.3(iii),

$$\begin{aligned} P(X^{(n)}[j,\infty) \in C) &= \sum_{i=0}^{(L\uparrow n)-1} P(X^{(n)}[j,\infty) \in C \mid \tau(n) = i) \cdot P(\tau(n) = i) \\ &= \sum_{i=0}^{(L\uparrow n)-1} P(Y^{(n)}[j+i,\infty) \in C \mid \tau(n) = i) \cdot 1/L^n \\ &= (1/L^n) \sum_{i=0}^{(L\uparrow n)-1} P(Y^{(n)}[j+i,\infty) \in C) \\ &= (1/L^n) \sum_{i=0}^{(L\uparrow n)-1} P(Y^{(n)}[i,\infty) \in C). \end{aligned} \tag{3.1}$$

Of course since the last term does not depend on j, the sequence $X^{(n)}$ is strictly stationary.

Remark 3.8. If $m > n \geq 0$, then for any $B \in \mathcal{R}^n$, by (3.1) (with n replaced by m) and Remark 3.4(ii),

$$\begin{aligned} L^m \cdot P(X^{(m)}[0, L^n - 1] \in B) &= \sum_{i=0}^{(L\uparrow m)-1} P(Y^{(m)}[i, i + L^n - 1] \in B) \\ &= L^{m-n} \sum_{i=0}^{(L\uparrow n)-1} P(Y^{(m)}[i, i + L^n - 1] \in B). \end{aligned}$$

Hence if $m > n \geq 0$, $B \in \mathcal{R}^n$, and $h \in \mathbf{Z}$, then by Remark 3.4(ii) again,

$$P(X^{(m)}[0, L^n - 1] \in B) = L^{-n} \sum_{i=0}^{(L\uparrow n)-1} P(Y^{(m)}[i + h, i + h + L^n - 1] \in B). \tag{3.2}$$

Lemma 3.9. *Suppose $n \geq 0$. Then the following statements hold:*

(i) *The random sequence $X^{(n)}$ is strictly stationary.*

(ii) $\mathcal{L}(X_0^{(n)}) = \lambda_{\text{unps3}}$.

(iii) *The sequence $X^{(n)}$ satisfies $(L-1)$-tuplewise independence.*

(iv) *The random variables $|X_k^{(n)}|$, $k \in \mathbf{Z}$ are independent.*

Proof. Property (i) was already noted in the last sentence of Remark 3.7.

Property (ii) holds by Remark 3.2(i) and Eq. (3.1), applied with $j = 0$ to sets C of the form $C = A \times \mathbf{R} \times \mathbf{R} \times \cdots$ where $A \in \mathcal{R}$.

To verify property (iii), note that by Remark 3.2(i)(ii), for (say) any $L - 1$ distinct positive integers $k(1) < k(2) < \cdots < k(L-1)$ and any integer i, one has (in the terminology of Step 2.4) that $\mathcal{L}(Y_{i+k(1)}^{(n)}, Y_{i+k(2)}^{(n)}, \ldots, Y_{i+k(L-1)}^{(n)}) = \lambda_{\text{unps3}}^{[L-1]}$; and by applying (3.1) to $j = 0$ and to sets $C \in \mathcal{R}^{\mathbf{N}}$ that depend only on the coordinates $k(1), k(2), \ldots, k(L-1)$, one obtains that $\mathcal{L}(X_{k(1)}^{(n)}, X_{k(2)}^{(n)}, \ldots, X_{k(L-1)}^{(n)}) = \lambda_{\text{unps3}}^{[L-1]}$. Property (iii) now follows easily from properties (i) and (ii).

The proof of property (iv) is quite similar to that of property (iii), but involves an arbitrarily high number of (say) arbitrary distinct positive integers $k(1) < k(2) < \cdots < k(\ell)$ and the use of Remark 3.2(i)(iii). The details are left to the reader. $\square$

Definition 3.10. The following notation will be handy: For any random sequence $\eta := (\eta_k, k \in \mathbf{Z})$ and any nonempty finite set $G \subset \mathbf{Z}$, the notation η_G will refer to the random vector $(\eta_{g(1)}, \eta_{g(2)}, \ldots, \eta_{g(\ell)})$ where $\ell := \operatorname{card} G$ and $g(1), g(2), \ldots, g(\ell)$ are in strictly increasing order the elements of G.

Step 3.11. The random sequence X for Theorem 1.1 will now be constructed, with a standard argument.

For any nonempty finite set $G \subset \mathbf{Z}$, the family of distributions of the random vectors $X_G^{(n)}$, $n \ge 0$ is (trivially) tight by Lemma 3.9(i)(ii), and hence every subsequence of those distributions has a further subsequence that converges weakly to some probability measure (on $\mathbf{R}^{(\operatorname{card} G)}$). Of course there exist only countably many finite subsets of $\mathbf{Z}$. Employing a standard "Cantor diagonal" procedure, one obtains an (henceforth fixed) infinite set $\mathbf{Q} \subset \mathbf{N}$ and a family of probability measures μ_G, for nonempty finite sets $G \subset \mathbf{Z}$ (with μ_G being a probability measure on $\mathbf{R}^{(\operatorname{card} G)}$ for each such G), such that for *every* nonempty finite set $G \subset \mathbf{Z}$, $X_G^{(n)} \Rightarrow \mu_G$ as $n \to \infty, n \in \mathbf{Q}$. By an elementary argument, that family of probability measures μ_G satisfies the Kolmogorov consistency condition. Applying the Kolmogorov Existence (Consistency) Theorem, let $X := (X_k, k \in \mathbf{Z})$ be a sequence of random variables such that for every nonempty finite set $G \subset \mathbf{Z}$,

$$X_G^{(n)} \Rightarrow X_G \quad \text{as } n \to \infty, n \in \mathbf{Q}. \tag{3.3}$$

Step 3.12. Our task now is to verify for the sequence X in (3.3) the properties stated in Theorem 1.1.

If h and j are integers and m is a nonnegative integer, then for the sets $G(1) := \{h, h+1, \ldots, h+m\}$ and $G(2) := \{j, j+1, \ldots, j+m\}$, one has that $\mathcal{L}(X_{G(1)}^{(n)}) = \mathcal{L}(X_{G(2)}^{(n)})$ for each $n \ge 0$ by Lemma 3.9(i), and hence $\mathcal{L}(X_{G(1)}) = \mathcal{L}(X_{G(2)})$ by (3.3). Consequently, the sequence X is strictly stationary.

Next, by Lemma 3.9(ii) and Eq. (3.3) with $G := \{0\}$, one obtains that $\mathcal{L}(X_0) = \lambda_{\text{unps3}}$. Thus property (A) in Theorem 1.1 holds.

Next, for any set $G \subset \mathbf{Z}$ with $\operatorname{card} G = L - 1$, one obtains from Lemma 3.9(i)(ii)(iii) and Eq. (3.3) that $\mathcal{L}(X_G) = \lambda_{\text{unps3}}^{[L-1]}$ (in the terminology of Step 2.4). Property (B) in Theorem 1.1 (with $N = L - 1$ — see the sentence after (2.1)) now follows from property (A) (and strict stationarity).

The proof of property (C) in Theorem 1.1 is similar to that of property (B). It involves Lemma 3.9(i)(ii)(iv) and Eq. (3.3) with finite sets $G \subset \mathbf{Z}$ of arbitrarily high cardinality.

The only remaining items to prove now are property (D) (in Theorem 1.1) and ergodicity. Their proofs will be given in Sections 4 and 5 respectively.

4. Part 3 of proof of Theorem 1.1: Property (D)

The proof of property (D) will proceed through a series of lemmas.

Lemma 4.1. *Suppose $n \geq 0$. Then for every $h \in \{0, 1, \ldots, L^n - 1\}$, there exists a permutation σ on $\{0, 1, \ldots, L^n - 1\}$ such that* (i) $\sigma(0) = h$ *and* (ii) *in the notations of* Step 2.2(c)(f), $\mathcal{L}(W_\sigma^{(n)}) = \mathcal{L}(W^{(n)})$.

Proof. For $n = 0$ (which forces $h = 0$), this is trivial. Now for induction, suppose $n \geq 0$, and the conclusion (i.e. the second sentence) of Lemma 4.1 holds for this n. Our task is to show that it holds for $n + 1$.

Suppose H is an element of $\{0, 1, \ldots, L^{n+1} - 1\}$. Our task is to prove the conclusion of Lemma 4.1 for $n + 1$ and H.

Let $J \in \{0, 1, \ldots, L - 1\}$ and $I \in \{0, 1, \ldots, L^n - 1\}$ be such that $H = JL^n + I$. Applying the induction assumption, let σ be a permutation on $\{0, 1, \ldots, L^n - 1\}$ such that $\sigma(0) = I$ and $\mathcal{L}(W_\sigma^{(n)}) = \mathcal{L}(W^{(n)})$.

Refer to (2.4) and the other details in Definition 2.7. Define the random vector $\eta := \eta[0, L^{n+1} - 1]$ as follows: First, define

$$\begin{aligned}\eta[0, L^n - 1] &:= V_J^{(n+1)} \varphi_{L\uparrow n}(\zeta_\sigma^{(n+1,J)}) \\ &= (W^{(n+1)}_{J(L\uparrow n)+\sigma(0)}, W^{(n+1)}_{J(L\uparrow n)+\sigma(1)}, \ldots, W^{(n+1)}_{J(L\uparrow n)+\sigma((L\uparrow n)-1)});\end{aligned}$$

the second equality holds by (2.4) (see the sentence after it), Remark 2 in Step 2.3, and a careful trivial argument. Next, if $J \neq 0$, define (see (2.4) again)

$$\eta[JL^n, (J+1)L^n - 1] := V_0^{(n+1)} \varphi_{L\uparrow n}(\zeta^{(n+1,0)}) = W^{(n+1)}[0, L^n - 1].$$

(That line is omitted if $J = 0$.) Finally, for every $j \in \{0, 1, \ldots, L - 1\} - \{0, J\}$, define

$$\eta[jL^n, (j+1)L^n - 1] := V_j^{(n+1)} \varphi_{L\uparrow n}(\zeta^{(n+1,j)}) = W^{(n+1)}[jL^n, (j+1)L^n - 1].$$

Now by Remark 2.6(iv) and the conditions stipulated in Definition 2.7, the joint distribution of $V_0^{(n+1)}, V_1^{(n+1)}, \ldots, V_{L-1}^{(n+1)}; \zeta^{(n+1,0)}, \zeta^{(n+1,1)}, \ldots, \zeta^{(n+1,L-1)}$ is invariant under the interchange of any two of the $V_j^{(n+1)}$'s, under the interchange of any two of the $\zeta^{(n+1,j)}$'s, and under the replacing of $\zeta^{(n+1,j)}$ by $\zeta_\sigma^{(n+1,j)}$ (in the notations in Step 2.2(c)) for a given j (where σ is as specified above). Hence (regardless of whether $J = 0$ or $J \neq 0$) by (2.4) the random vector η has the same distribution (on $\mathbf{R}^{L\uparrow(n+1)}$) as the random vector $W^{(n+1)}$. Also (regardless of whether $J = 0$ or $J \neq 0$) the L^{n+1} coordinates of η include, exactly once, every coordinate of $W^{(n+1)}$, with $\eta_0 = W^{(n+1)}_{J(L\uparrow n)+\sigma(0)} = W^{(n+1)}_{J(L\uparrow n)+I} = W^{(n+1)}_H$. Thus this scheme implicitly but directly gives a permutation ρ on $\{0, 1, \ldots, L^{n+1} - 1\}$, with $\eta = W_\rho^{(n+1)}$ (again in the notations of Step 2.2(c)) and with $\rho(0) = JL^n + \sigma(0) = JL^n + I = H$, such that $\mathcal{L}(W_\rho^{(n+1)}) = \mathcal{L}(\eta) = \mathcal{L}(W^{(n+1)})$. That completes the induction step and the proof of Lemma 4.1. □

Lemma 4.2. *Suppose $n \geq 0$. Then the random vector $\xi := \xi[0, L^n - 1] := \varphi_{L\uparrow n}(W^{(n)})$ has the following three properties:*

(i) $E(\text{sum}\,\xi) = E|\text{sum}\,W^{(n)}| \geq (1/2)L^{n/2}$.

(ii) $\mathcal{L}(\xi_0) = \mathcal{L}(\xi_1) = \cdots = \mathcal{L}(\xi_{(L\uparrow n)-1})$.

(iii) *For any $k \in \{0, 1, \ldots, L^n - 1\}$,* $E\xi_k = L^{-n} E|\text{sum}\,W^{(n)}|$.

Proof. For convenience, throughout the proof of this lemma, let $m := L^n$.

Let us first prove (i). Trivially $\text{sum}\,\xi = |\text{sum}\,W^{(n)}|$ by Remark 1 in Step 2.3. We just need to prove the ("latter") inequality in (i).

Refer to (2.1) and Lemma 2.10. By simple calculations, including the standard argument in [1, p. 85, proof of Theorem 6.1] (which uses only 4-tuplewise independence), one has that for each $k \in \{0, 1, \ldots, m-1\}$, $\mathcal{L}(W_k^{(n)}) = \lambda_{\text{unps3}}$, $EW_k^{(n)} = 0$, $E(W_k^{(n)})^2 = 1$, and $E(W_k^{(n)})^4 = 9/5$, and hence $E(\text{sum } W^{(n)})^2 = m$ and

$$E(\text{sum } W^{(n)})^4 = m \cdot E(W_0^{(n)})^4 + 3m(m-1) \cdot (E(W_0^{(n)})^2)^2 < 3m^2.$$

Hence by Hölder's inequality,

$$\begin{aligned} m = E(\text{sum } W^{(n)})^2 &\le \left\| |\text{sum } W^{(n)}|^{2/3} \right\|_{3/2} \cdot \left\| |\text{sum } W^{(n)}|^{4/3} \right\|_3 \\ &= \left[E|\text{sum } W^{(n)}| \right]^{2/3} \cdot \left[E(\text{sum } W^{(n)})^4 \right]^{1/3} \\ &\le \left[E|\text{sum } W^{(n)}| \right]^{2/3} \cdot \left(3m^2 \right)^{1/3}. \end{aligned}$$

Hence $m^{3/2} \le E|\text{sum } W^{(n)}| \cdot (3m^2)^{1/2}$. Hence (since $m = L^n$) the ("latter") inequality in (i) holds.

Now let us prove (ii). Suppose $j \in \{0, 1, \ldots, m-1\}$. Referring to Lemma 4.1, let σ be a permutation of the set $\{0, 1, \ldots, m-1\}$ such that $\sigma(0) = j$ and $\mathcal{L}(W_\sigma^{(n)}) = \mathcal{L}(W^{(n)})$. Then $\mathcal{L}(\varphi_m(W_\sigma^{(n)})) = \mathcal{L}(\varphi_m(W^{(n)})) = \mathcal{L}(\xi)$. Also, $\xi_\sigma = \varphi_m(W_\sigma^{(n)})$ by Remark 2 in Step 2.3, and hence $\mathcal{L}(\xi_\sigma) = \mathcal{L}(\xi)$. Hence $\mathcal{L}(\xi_j) = \mathcal{L}(\xi_{\sigma(0)}) = \mathcal{L}(\xi_0)$. Since $j \in \{0, 1, \ldots, m-1\}$ was arbitrary, (ii) follows.

Statement (iii) follows trivially from statements (i) and (ii). □

Lemma 4.3. *For every $n \ge 0$,*

$$E \prod_{j=0}^{L-1} \left(\text{sum } W^{(n+1)}[jL^n, (j+1)L^n - 1] \right) \le -2^{-L} \cdot (L^n)^{L/2}. \tag{4.1}$$

Proof. Suppose $n \ge 0$. By (2.4) and the various conditions in Definition 2.7,

$$[\text{LHS of } (4.1)] = \left[E \prod_{j=0}^{L-1} V_j^{(n+1)} \right] \cdot \left[\prod_{j=0}^{L-1} E \left(\text{sum } \varphi_{L \uparrow n}(\zeta^{(n+1,j)}) \right) \right].$$

By Remark 2.6(v), the first factor in the right hand side equals -1; and by Lemma 4.2(i), the second factor there is positive and bounded below by $2^{-L} \cdot (L^n)^{L/2}$. Lemma 4.3 follows. □

Lemma 4.4. *Suppose $n \ge 0$. Suppose that for each $j \in \{0, 1, \ldots, L-1\}$, $k(j)$ is an element of the set $\{jL^n, jL^n + 1, \ldots, (j+1)L^n - 1\}$. Then $E \prod_{j=0}^{L-1} W_{k(j)}^{(n+1)} < 0$.*

Proof. For each $j \in \{0, 1, \ldots, L-1\}$, define the integer $h(j) \in \{0, 1, \ldots, L^n - 1\}$ by $h(j) = k(j) - jL^n$. By the conditions in Definition 2.7, for each $j \in \{0, 1, \ldots, L-1\}$, one has that $W_{k(j)}^{(n+1)} = V_j^{(n+1)} \cdot (\varphi_{L \uparrow n}(\zeta^{(n+1,j)}))_{h(j)}$, and also $E(\varphi_{L \uparrow n}(\zeta^{(n+1,j)}))_{h(j)} > 0$ by Lemma 4.2(i)(iii). Lemma 4.4 now follows by an argument analogous to that of Lemma 4.3. □

Lemma 4.5. *For every $n \ge 0$ and every integer h,*

$$\begin{aligned} &E\left[\text{sum } Y^{(n)}[hL^{n+1}, (h+1)L^{n+1} - 1] \right]^L - E\left[\text{sum } Y^{(n+1)}[hL^{n+1}, (h+1)L^{n+1} - 1] \right]^L \\ &\quad \ge L! \cdot 2^{-L} \cdot (L^n)^{L/2}. \end{aligned} \tag{4.2}$$

Proof. Applying Remark 3.3(ii) (to both indices n and $n+1$), we assume without loss of generality that $h = 0$. Then

$$[\text{LHS of (4.2)}] = \sum \Bigg[E \prod_{i=0}^{L-1} \Big(\text{sum}\, Y^{(n)}[j_i L^n, (j_i + 1)L^n - 1] \Big) - E \prod_{i=0}^{L-1} \Big(\text{sum}\, Y^{(n+1)}[j_i L^n, (j_i + 1)L^n - 1] \Big) \Bigg], \tag{4.3}$$

where in the right hand side, the sum is taken over all L-tuples $(j_0, j_1, \dots, j_{L-1}) \in \{0, 1, \dots, L-1\}^L$.

Now by Remark 3.4(i) and the conditions in Step 3.1, the joint distribution of any $L-1$ of the random vectors $Y^{(n+1)}[jL^n, (j+1)L^n - 1]$, $j \in \mathbf{Z}$ is identical to the joint distribution of any $L-1$ of the random vectors $Y^{(n)}[jL^n, (j+1)L^n - 1]$. Hence for any L-tuple $(j_0, j_1, \dots, j_{L-1})$ in which two or more of the j_i's are equal, the summand in the right hand side of (4.3) equals 0. Next, for any integer j, one has that $E(\text{sum}\, Y^{(n)}[jL^n, (j+1)L^n - 1]) = 0$ by Remark 3.2(i). For any L-tuple $(j_0, j_1, \dots, j_{L-1}) \in \{0, 1, \dots, L-1\}^L$ such that the indices j_i are distinct (and hence together they give each element of $\{0, 1, \dots, L-1\}$ exactly once), one now has by the conditions in Step 3.1 (for both indices n and $n+1$) that (i) $E \prod_{i=0}^{L-1} (\text{sum}\, Y^{(n)}[j_i L^n, (j_i + 1)L^n - 1]) = 0$, and hence (ii) by Lemma 4.3, the summand in the right hand side of (4.3) is bounded below by $2^{-L} \cdot (L^n)^{L/2}$. Since there are $L!$ such L-tuples (of distinct indices), (4.2) follows. □

Lemma 4.6. *Suppose $k(1), k(2), \dots, k(L)$ are (not necessarily distinct) integers, and $n \geq 0$. Then*

$$E \prod_{i=1}^{L} Y_{k(i)}^{(n+1)} \leq E \prod_{i=1}^{L} Y_{k(i)}^{(n)}. \tag{4.4}$$

Proof. If the integers $k(i)$ are not all distinct, and hence include at most $L-1$ distinct ones, then as a simple consequence of Remark 3.2(i)(ii),

$$\mathcal{L}(Y_{k(1)}^{(n+1)}, Y_{k(2)}^{(n+1)}, \dots, Y_{k(L)}^{(n+1)}) = \mathcal{L}(Y_{k(1)}^{(n)}, Y_{k(2)}^{(n)}, \dots, Y_{k(L)}^{(n)}), \tag{4.5}$$

and (4.4) holds trivially (with equality). Therefore, we henceforth assume that the integers $k(i)$ are distinct.

If there does *not* exist an integer h such that $k(i) \in \{hL^{n+1}, hL^{n+1} + 1, \dots, (h+1)L^{n+1} - 1\}$ for all $i \in \{1, 2, \dots, L\}$, then (4.5) holds (and hence trivially (4.4)), with the common distribution in (4.5) being $\lambda_{\text{unps3}}^{[L]}$ (see Step 2.4), by Remark 3.2(i)(ii) and the conditions in Step 3.1 (applied to both indices n and $n+1$). Therefore, we henceforth assume that such an integer h does exist. By Remark 3.3(i) (or Step 3.1), we can assume without loss of generality that $h = 0$, i.e. each $k(i)$ belongs to $\{0, 1, \dots, L^{n+1} - 1\}$.

Now by Step 3.1 and Lemma 2.9, if for some $j \in \{0, 1, \dots, L-1\}$, the set $\{jL^n, jL^n + 1, \dots, (j+1)L^n - 1\}$ has *none* of the integers $k(i)$, then again (4.5) holds and hence trivially also (4.4). Hence we can now assume that each of the sets $\{jL^n, jL^n + 1, \dots, (j+1)L^n - 1\}$, $j \in \{0, 1, \dots, L-1\}$, has exactly one of the $k(i)$'s.

Under that assumption, the random variables $Y_{k(i)}^{(n)}$, $i \in \{1, 2, \dots, L\}$ are independent by the conditions in Step 3.1, and hence by Remark 3.2(i), the right hand side of (4.4) equals 0. The left hand side of (4.4) is negative by Lemma 4.4 and the conditions in Step 3.1 with index $n+1$. Thus (4.4) holds. This completes the proof of the lemma. □

Lemma 4.7. (i) *Suppose $n \geq 0$, and G is a nonempty finite subset of* $\mathbf{Z}$. *Then (in the notations in Definition 3.10)* $E[\text{sum}\, Y_G^{(n+1)}]^L \leq E[\text{sum}\, Y_G^{(n)}]^L$.

(ii) *If also there exists an integer j such that $\{j, j+1, \ldots, j+2L^{n+1}-1\} \subset G$, then*

$$E[\text{sum}\, Y_G^{(n+1)}]^L \leq E[\text{sum}\, Y_G^{(n)}]^L - L! \cdot 2^{-L} \cdot (L^n)^{L/2}.$$

Proof. For any nonempty finite set $H \subset \mathbf{Z}$,

$$E[\text{sum}\, Y_H^{(n)}]^L - E[\text{sum}\, Y_H^{(n+1)}]^L = \sum \left[E \prod_{i=1}^{L} Y_{k(i)}^{(n)} - E \prod_{i=1}^{L} Y_{k(i)}^{(n+1)} \right], \tag{4.6}$$

where in the right hand side, the sum is taken over all L-tuples $(k(1), k(2), \ldots, k(L)) \in H^L$. Part (i) of Lemma 4.7 now follows from Lemma 4.6 and Eq. (4.6) with $H = G$.

Now suppose also the hypothesis of part (ii) of Lemma 4.7 holds. Then there exists an integer $h \in \mathbf{Z}$ such that $D := \{hL^{n+1}, hL^{n+1}+1, \ldots, (h+1)L^{n+1}-1\} \subset G$. By (4.6) (applied to both G and D) and Lemma 4.6, followed by Lemma 4.5,

$$[\text{LHS of (4.6) with } H = G] \geq [\text{LHS of (4.6) with } H = D] \geq L! \cdot 2^{-L} \cdot (L^n)^{L/2}.$$

Thus Lemma 4.7(ii) holds. □

Step 4.8. Let Z and Z_k, $k \in \mathbf{Z}$ be independent $N(0,1)$ random variables.

Recall that the random variables $Y_k^{(0)}$, $k \in \mathbf{Z}$ are independent (by Step 3.1). Consequently, by Remark 3.2(i) and the Remark in Step 2.4, for any positive integer m, any m-tuple $(k(1), k(2), \ldots, k(m))$ of distinct integers, and any (not necessarily distinct) positive integers $h(1), h(2), \ldots, h(m)$, one has that $E \prod_{i=1}^{m} (Y_{k(i)}^{(0)})^{h(i)} \leq E \prod_{i=1}^{m} Z_{k(i)}^{h(i)}$. Thus for any L-tuple $(k(1), k(2), \ldots, k(L))$ of (not necessarily distinct) integers, $E \prod_{i=1}^{L} Y_{k(i)}^{(0)} \leq E \prod_{i=1}^{L} Z_{k(i)}$. Adding up both sides over appropriate L-tuples, one obtains that for any nonempty finite set $G \subset \mathbf{Z}$,

$$E[\text{sum}\, Y_G^{(0)}]^L \leq E\left[\sum_{k \in G} Z_k\right]^L = E[(\text{card}\, G)^{1/2} Z]^L.$$

Hence by Lemma 4.7(i) and induction on n, one has that for every $n \geq 0$ and every nonempty finite set $G \subset \mathbf{Z}$,

$$E[\text{sum}\, Y_G^{(n)}]^L \leq E[(\text{card}\, G)^{1/2} Z]^L. \tag{4.7}$$

Now if $n \geq 0$, h is an integer such that $h \geq 2L^{n+1}$, and j is any integer, then

$$\forall m \geq n+1, \quad E(\text{sum}\, Y^{(m)}[j+1, j+h])^L \leq E[h^{1/2} Z]^L - L! \cdot 2^{-L} \cdot (L^n)^{L/2}; \tag{4.8}$$

that holds for $m = n+1$ by (4.7) and Lemma 4.7(ii); and hence it holds for all $m \geq n+1$ by Lemma 4.7(i) and induction.

Now recall Steps 3.5 and 3.6 and the notations in (1.1). If $n \geq 0$ and $h \geq 2L^{n+1}$, then for any $m \geq n+1$ and any $i \in \{0, 1, \ldots, L^m - 1\}$,

$$E\left[\left(S(X^{(m)}, h)\right)^L \middle| \tau(m) = i\right] = E\left[\left(\text{sum}\, Y^{(m)}[1+i, h+i]\right)^L \middle| \tau(m) = i\right]$$
$$\leq E[h^{1/2}Z]^L - L! \cdot 2^{-L} \cdot (L^n)^{L/2};$$

here the equality comes from Step 3.6, and then the inequality comes from (4.8) and Step 3.5. It follows that for any $n \geq 0$, any $h \geq 2L^{n+1}$, and any $m \geq n+1$,

$$E[S(X^{(m)}, h)]^L \leq E[h^{1/2}Z]^L - L! \cdot 2^{-L} \cdot (L^n)^{L/2}.$$

Hence by (3.3) and [1, Theorem 29.2 and Theorem 25.11], for any $n \geq 0$ and any $h \geq 2L^{n+1}$,

$$E[S(X, h)]^L \leq E[h^{1/2}Z]^L - L! \cdot 2^{-L} \cdot (L^n)^{L/2}. \tag{4.9}$$

From that, one has the following lemma.

Lemma 4.9. *For any $h \geq 2L$, one has that $E[S(X, h)/h^{1/2}]^L \leq EZ^L - 2^{-3L/2} \cdot L! \cdot L^{-L}$, where Z is a $N(0, 1)$ random variable.*

Proof. Let n be the nonnegative integer such that $2L^{n+1} \leq h < 2L^{n+2}$. Then the first inequality ($2L^{n+1} \leq h$) yields (4.9), and the second inequality ($h < 2L^{n+2}$) yields $h^{-L/2} > (2L^{n+2})^{-L/2} = (2L^n)^{-L/2} \cdot L^{-L}$ and hence $L! \cdot 2^{-L} \cdot (L^n)^{L/2} \cdot h^{-L/2} \geq 2^{-3L/2} \cdot L! \cdot L^{-L}$. Hence, dividing both sides of (4.9) by $h^{L/2}$, one obtains Lemma 4.9. □

Step 4.10. Now we are ready to prove property (D) in Theorem 1.1. Recall the notations in Eq. (1.1).

By property (A) (proved in Step 3.12) in Theorem 1.1, and a routine calculation, $EX_0 = 0$, $EX_0^2 = 1$, and $EX_0^4 = 9/5$. Recall Eq. (2.1) and the property of $(L-1)$-tuplewise independence (proved in Step 3.12) for the sequence X. One has that

$$\forall n \geq 1, \quad E\left(n^{-1/2}S(X, n)\right) = 0 \quad \text{and} \tag{4.10}$$

$$\forall n \geq 1, \quad E\left(n^{-1/2}S(X, n)\right)^2 = 1. \tag{4.11}$$

Further, by the standard argument in [1, p. 85, proof of Theorem 6.1] (which requires only 4-tuplewise independence), one has that for each $n \in \mathbf{N}$,

$$E(S(X, n))^4 = n \cdot EX_0^4 + 3n(n-1)(EX_0^2)^2 = n \cdot (9/5) + (3n^2 - 3n) \cdot 1 \leq 3n^2.$$

Hence

$$\forall n \geq 1, \quad E\left(n^{-1/2}S(X, n)\right)^4 \leq 3. \tag{4.12}$$

By (say) (4.11) and Chebyshev's inequality, the family of distributions of the normalized partial sums $n^{-1/2}S(X, n)$, $n \in \mathbf{N}$ is tight.

Now for the proof of property (D) in Theorem 1.1, suppose Q is an infinite subset of $\mathbf{N}$. Because of tightness, there exists an infinite set $T \subset Q$ and a probability measure μ on $\mathbf{R}$ (both T and μ henceforth fixed) such that

$$n^{-1/2}S(X, n) \Rightarrow \mu \quad \text{as } n \to \infty, n \in T. \tag{4.13}$$

To complete the proof of property (D) in Theorem 1.1, our task now is to show that μ is neither degenerate nor normal.

Because of (4.12) and (4.13), and the Corollary in [1, p. 338], one has by (4.10) and (4.11) that

$$\int_{x \in \mathbf{R}} x \mu(\mathrm{d}x) = 0 \quad \text{and} \quad \int_{x \in \mathbf{R}} x^2 \mu(\mathrm{d}x) = 1. \tag{4.14}$$

Hence the probability measure μ has positive variance and is therefore nondegenerate.

If μ were normal, then by (4.14) it would have to be the $N(0, 1)$ distribution. One would then have by (2.1), (4.13), and [1, Theorem 25.7 and Theorem 25.11] that $\liminf_{n\to\infty, n\in T} E[n^{-1/2} S(X, n)]^L \geq EZ^L$ where Z is a $N(0, 1)$ random variable. But that contradicts Lemma 4.9; hence μ cannot be normal. That completes the proof of property (D).

5. Part 4 of proof of Theorem 1.1: Ergodicity

In this section, we shall carry out the final task in the proof of Theorem 1.1: to show that the sequence X is ergodic.

In what follows, the notation $G \doteq H$ (for events G and H) means $P(G \triangle H) = 0$, where $\triangle$ denotes the "symmetric difference".

Let $\mathbf{T}$ denote the usual shift operator on $\mathbf{R}^{\mathbf{Z}}$: For $x := (x_k, k \in \mathbf{Z}) \in \mathbf{R}^{\mathbf{Z}}$, the sequence $(\mathbf{T}x) := ((\mathbf{T}x)_k, k \in \mathbf{Z})$ is defined by $(\mathbf{T}x)_k := x_{k+1}$ for $k \in \mathbf{Z}$. The (strictly stationary) sequence $X := (X_k, k \in \mathbf{Z})$ is (by definition) ergodic if the *only* sets $A \in \mathcal{R}^{\mathbf{Z}}$ (see Step 2.2(e)) such that $\{\mathbf{T}X \in A\} = \{X \in \mathbf{T}^{-1}A\} \doteq \{X \in A\}$ holds, are the sets $A \in \mathcal{R}^{\mathbf{Z}}$ such that $P(X \in A) = 0$ or 1.

Suppose the sequence $X := (X_k, k \in \mathbf{Z})$ is *not* ergodic. That is, suppose there exists a set $A \in \mathcal{R}^{\mathbf{Z}}$ (henceforth fixed) such that

$$\{\mathbf{T}X \in A\} = \{X \in \mathbf{T}^{-1}A\} \doteq \{X \in A\} \quad \text{and} \quad 0 < P(X \in A) < 1. \tag{5.1}$$

We shall aim for a contradiction.

By (5.1), $[P(X \in A)]^2 < P(X \in A)$. Let $\varepsilon > 0$ be such that

$$[P(X \in A) + \varepsilon]^2 < P(X \in A) - 2\varepsilon. \tag{5.2}$$

By the standard measure theory, there exist integers $i(1)$ and $i(2)$ with $i(1) \leq i(2)$, and a set $B_0 \in \mathcal{R}^{i(2)-i(1)+1}$, such that (see Step 2.2(d))

$$P(\{X \in A\} \triangle \{X[i(1), i(2)] \in B_0\}) < \varepsilon. \tag{5.3}$$

Of course by (5.1) and strict stationarity, $P(\{\mathbf{T}^{j+1}X \in A\} \triangle \{\mathbf{T}^j X \in A\}) = 0$ for every integer j, and hence by induction, $\{\mathbf{T}^j X \in A\} \doteq \{X \in A\}$ for every integer j. Hence by (5.3) and strict stationarity,

$$\begin{aligned} P(\{X \in A\} \triangle \{X[0, i(2) - i(1)] \in B_0\}) &= P(\{\mathbf{T}^{i(1)}X \in A\} \triangle \{X[i(1), i(2)] \in B_0\}) \\ &= P(\{X \in A\} \triangle \{X[i(1), i(2)] \in B_0\}) < \varepsilon. \end{aligned}$$

Let d be a positive integer such that $i(2) - i(1) + 1 < L^d$, and then define the set $B_1 \subset \mathbf{R}^{L\uparrow d}$ by $B_1 := B_0 \times \mathbf{R}^{(L\uparrow d)-[i(2)-i(1)+1]}$. Then $B_1 \in \mathcal{R}^{L\uparrow d}$ and

$$\begin{aligned} &P(\{X \in A\} \triangle \{X[0, L^d - 1] \in B_1\}) \\ &\quad = P(\{X \in A\} \triangle \{X[0, i(2) - i(1)] \in B_0\}) < \varepsilon. \end{aligned} \tag{5.4}$$

A quick review of some basic measure theory will be useful. If η is a probability measure on $\mathbf{R}^{L\uparrow d}$, then in the "symmetric-difference" pseudo-metric $(C, D) \mapsto \eta(C \triangle D)$ for members of $\mathcal{R}^{L\uparrow d}$, the set B_1 can be approximated arbitrarily closely by a union B_2 of countable many "open rectangles" (that is, "open rectangular L^d-dimensional solids"), hence by the union B_3 of finitely many such "open rectangles", and hence by the union B of finitely many "open rectangles" whose boundaries have η-measure 0. (Recall that any "open rectangle" can be approximated from within by a "continuum" of concentric "open rectangles", at most countably many of which can possess boundaries having positive η-measure.) For such a set B, its boundary ∂B is a subset of the union of the boundaries of those (finitely many) "open rectangles", and hence $\eta(\partial B) = 0$. (If $z \in \partial B$, and $z_1, z_2, z_3, \dots$ is a sequence in B that converges to z, then some subsequence is entirely inside one of those finitely many "open rectangles".) Letting η denote the distribution of $X[0, L^d - 1]$, one now has from (5.4) that there exists a set $B \in \mathcal{R}^{L\uparrow d}$ (henceforth fixed) such that

$$P(X[0, L^d - 1] \in \partial B) = 0 \quad \text{and} \quad P(\{X \in A\} \triangle \{X[0, L^d - 1] \in B\}) < \varepsilon. \tag{5.5}$$

Of course by (5.5), strict stationarity, and the sentence after (5.3), one has that for any integer j,

$$P(\{X \in A\} \triangle \{X[j, j + L^d - 1] \in B\}) < \varepsilon. \tag{5.6}$$

Next, the Cartesian product $B \times B \subset \mathbf{R}^{L\uparrow d} \times \mathbf{R}^{L\uparrow d}$ satisfies $\partial(B \times B) \subset (\partial B \times \mathbf{R}^{L\uparrow d}) \cup (\mathbf{R}^{L\uparrow d} \times \partial B)$ by a trivial argument. Hence for (say) any integer $j \geq 0$, by (5.5) and strict stationarity,

$$P\Big((X[0, L^d - 1], X[j + 2L^d, j + 3L^d - 1]) \in \partial(B \times B)\Big) = 0. \tag{5.7}$$

Now recall the basic inequalities $|P(G) - P(H)| \leq P(G \triangle H)$ and $P((G_1 \cap G_2) \triangle (H_1 \cap H_2)) \leq P(G_1 \triangle H_1) + P(G_2 \triangle H_2)$, involving symmetric differences of events. Applying (5.5) and (5.6), and these inequalities with $G_1 = G_2 = \{X \in A\}$ and $G = G_1 \cap G_2$ and $H = H_1 \cap H_2$ for appropriate H_1 and H_2, one has that for (say) any integer $j \geq 0$,

$$\Big| P(X \in A) - P\Big(\{X[0, L^d - 1] \in B\} \cap \{X[j + 2L^d, j + 3L^d - 1] \in B\}\Big)\Big| < 2\varepsilon. \tag{5.8}$$

For (say) any $j \geq 0$, by (3.3) and (5.7),

$$\begin{aligned} &\lim_{n\to\infty, n\in\mathbf{Q}} P\Big(\{X^{(n)}[0, L^d - 1] \in B\} \cap \{X^{(n)}[j + 2L^d, j + 3L^d - 1] \in B\}\Big) \\ &\quad = P\Big(\{X[0, L^d - 1] \in B\} \cap \{X[j + 2L^d, j + 3L^d - 1] \in B\}\Big). \end{aligned} \tag{5.9}$$

Similarly by (5.5) and (3.3), $\lim_{n\to\infty, n\in\mathbf{Q}} P(X^{(n)}[0, L^d - 1] \in B) = P(X[0, L^d - 1] \in B)$ and (see (5.5) again) $|P(X \in A) - P(X[0, L^d - 1] \in B)| < \varepsilon$. Applying those inequalities, (5.9) and (5.8), let $M \in \mathbf{Q}$ be an integer sufficiently large that

$$M > d \quad \text{and} \quad P(X^{(M)}[0, L^d - 1] \in B) < P(X \in A) + \varepsilon \tag{5.10}$$

and

$$\begin{aligned} &\sum_{j=0}^{(L\uparrow d)-1} P\Big(\{X^{(M)}[0, L^d - 1] \in B\} \cap \{X^{(M)}[j + 2L^d, j + 3L^d - 1] \in B\}\Big) \\ &\quad > L^d \cdot [P(X \in A) - 2\varepsilon]. \end{aligned} \tag{5.11}$$

Now because $M > d$ (see (5.10)), the following equations and inequalities hold respectively by (3.1), Remark 3.4(iii), algebra, (3.2), (3.1) again, (5.10) and (5.2):

$$\sum_{j=0}^{(L\uparrow d)-1} P\Big(\{X^{(M)}[0, L^d - 1] \in B\} \cap \{X^{(M)}[j + 2L^d, j + 3L^d - 1] \in B\}\Big)$$

$$= \sum_{j=0}^{(L\uparrow d)-1} (1/L^M) \sum_{i=0}^{(L\uparrow M)-1} P\Big(\{Y^{(M)}[i, i + L^d - 1] \in B\}$$
$$\cap \{Y^{(M)}[i + j + 2L^d, i + j + 3L^d - 1] \in B\}\Big)$$
$$= \sum_{j=0}^{(L\uparrow d)-1} (1/L^M) \sum_{i=0}^{(L\uparrow M)-1} \Big[P(Y^{(M)}[i, i + L^d - 1] \in B)$$
$$\times P(Y^{(M)}[i + j + 2L^d, i + j + 3L^d - 1] \in B)\Big]$$
$$= (1/L^M) \sum_{i=0}^{(L\uparrow M)-1} \Big[P(Y^{(M)}[i, i + L^d - 1] \in B)$$
$$\times \sum_{j=0}^{(L\uparrow d)-1} P(Y^{(M)}[i + j + 2L^d, i + j + 3L^d - 1] \in B)\Big]$$
$$= (1/L^M) \sum_{i=0}^{(L\uparrow M)-1} \Big[P(Y^{(M)}[i, i + L^d - 1] \in B) \cdot L^d \cdot P(X^{(M)}[0, L^d - 1] \in B)\Big]$$
$$= L^d \cdot P(X^{(M)}[0, L^d - 1] \in B) \cdot P(X^{(M)}[0, L^d - 1] \in B)$$
$$< L^d \cdot [P(X \in A) + \varepsilon]^2$$
$$< L^d \cdot [P(X \in A) - 2\varepsilon].$$

But this contradicts (5.11). Hence the sequence X must be ergodic after all. This completes the proof of ergodicity, and of Theorem 1.1.

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