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The representation dimension of quantum complete intersections

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Abstract

We give a lower and an upper bound for the representation dimension of a quantum complete intersection.

Keywords: Representation dimension; Quantum complete intersections

1. Introduction

In [Au1], Auslander introduced the representation dimension of an Artin algebra in order to study algebras of infinite representation type. He showed that a non-semisimple algebra is of finite type if and only if its representation dimension is exactly two, whereas it is of infinite type if and only if the representation dimension is at least three.

For more than three decades no general method for computing the representation dimension was known, and it was even unclear if this dimension could exceed three. A negative answer to the latter would imply that the finitistic dimension conjecture holds (cf. [IgT]). However, in 2006 Rouquier showed in [Ro2] that the representation dimension of the exterior algebra on a d-dimensional vector space is $d + 1$, using the notion of the dimension of a triangulated category (cf. [Ro1]). In particular, there exist finite dimensional algebras of arbitrarily large representation dimension. Other examples illustrating this were subsequently given in [KrK] and [Op1].

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In this paper, we study the representation dimension of quantum complete intersections, a class of algebras originating from work by Manin (cf. [Man]) and Avramov, Gasharov and Peeva (cf. [AGP]). In particular, under some assumptions we show that the representation dimension of such an algebra is strictly greater than its codimension.

2. Representation dimension

Let Λ be an Artin algebra, and denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules. The *representation dimension* of Λ, denoted $\operatorname{repdim}\Lambda$, is defined as

$$\operatorname{repdim}\Lambda \stackrel{\text{def}}{=} \inf\{\operatorname{gldim}\operatorname{End}_\Lambda(M) \mid M \text{ generates and cogenerates } \operatorname{mod}\Lambda\},$$

where gldim denotes the global dimension of an algebra. A priori we see that the representation dimension might be infinite, but Iyama showed in [Iya] that it is finite for every Artin algebra. Auslander himself showed that the representation dimension of a selfinjective algebra is at most its Loewy length.

When computing the representation dimension of the exterior algebras, Rouquier used the notion of dimensions of triangulated categories, a concept he introduced in [Ro1]. As we will also use this notion when computing a lower bound for the representation dimension of a quantum complete intersection, we recall the definitions. Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{C}$ and $\mathcal{D}$ be subcategories of $\mathcal{T}$. We denote by $\langle\mathcal{C}\rangle$ the full subcategory of $\mathcal{T}$ consisting of all the direct summands of finite direct sums of shifts of objects in $\mathcal{C}$. Furthermore, we denote by $\mathcal{C} * \mathcal{D}$ the full subcategory of $\mathcal{T}$ consisting of objects M such that there exists a distinguished triangle

$$C \to M \to D \to C[1]$$

in $\mathcal{T}$, with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Finally, we denote the subcategory $\langle\mathcal{C} * \mathcal{D}\rangle$ by $\mathcal{C} \diamond \mathcal{D}$. Now define $\langle\mathcal{C}\rangle_0$ and $\langle\mathcal{C}\rangle_1$ to be 0 and $\langle\mathcal{C}\rangle$, respectively, and for each $n \geqslant 2$ define inductively $\langle\mathcal{C}\rangle_n$ to be $\langle\mathcal{C}\rangle_{n-1} \diamond \langle\mathcal{C}\rangle$. The *dimension* of $\mathcal{T}$, denoted $\dim\mathcal{T}$, is defined as

$$\dim\mathcal{T} \stackrel{\text{def}}{=} \inf\{d \in \mathbb{Z} \mid \text{there exists an object } M \in \mathcal{T} \text{ such that } \mathcal{T} = \langle M\rangle_{d+1}\}.$$

From the definition, we see that the dimension of a triangulated category is not necessarily finite. However, the following elementary result provides a method for computing an upper bound in terms of dense subcategories.

Proposition 2.1. *(See [Ro1, Lemma 3.4].) If $F : \mathcal{S} \to \mathcal{T}$ is a functor of triangulated categories whose image is dense in $\mathcal{T}$, then* $\dim\mathcal{T} \leqslant \dim\mathcal{S}$.

We will use this result to show that for any quantum complete intersection of codimension n, there exists a chain of n subalgebras and a corresponding chain of $n-1$ inequalities bounding the representation dimension from below. When proving both this and the main results, the triangulated category we use is the *stable module category* of the algebra. Recall therefore that when Λ is a selfinjective algebra, its stable module category, denoted $\underline{\operatorname{mod}}\,\Lambda$, is defined as follows: the objects of $\underline{\operatorname{mod}}\,\Lambda$ are the same as in $\operatorname{mod}\Lambda$, but two morphisms in $\operatorname{mod}\Lambda$ are equal in $\underline{\operatorname{mod}}\,\Lambda$ whenever their difference factors through a projective Λ-module. The cosyzygy functor

$\Omega_\Lambda^{-1} : \underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda$ is an equivalence of categories, and a triangulation of $\underline{\mathrm{mod}}\,\Lambda$ is given by using this functor as a shift and by letting short exact sequences in $\mathrm{mod}\,\Lambda$ correspond to triangles. Thus $\underline{\mathrm{mod}}\,\Lambda$ is a triangulated category, and its dimension is related to the representation dimension of Λ by the following result.

Proposition 2.2. *(See [Ro2, Proposition 3.7].) If Λ is a non-semisimple selfinjective algebra, then* $\operatorname{repdim} \Lambda \geqslant \dim(\underline{\mathrm{mod}}\,\Lambda) + 2$.

We end this section with the following lemma. It gives a lower bound for the dimension of the stable module category of a selfinjective algebra, in terms of certain subalgebras.

Lemma 2.3. *Let A and B be selfinjective Artin algebras. If there exist algebra homomorphisms*

$$A \xrightarrow{i} B \xrightarrow{\pi} A$$

such that πi is the identity and such that B is a projective A-module, then $\dim(\underline{\mathrm{mod}}\,A) \leqslant \dim(\underline{\mathrm{mod}}\,B)$.

Proof. Every B-module is an A-module via the map i. Moreover, if a map in $\mathrm{mod}\,B$ factors through a projective B-module, then it factors through a projective A-module since B is A-projective. Therefore the map i induces a functor $F : \underline{\mathrm{mod}}\,B \to \underline{\mathrm{mod}}\,A$, and this is clearly a functor of triangulated categories. Now let M be an object in $\underline{\mathrm{mod}}\,A$. Then M is a B-module via the map π, hence every object in $\underline{\mathrm{mod}}\,A$ belongs to the image of F. The result now follows from Proposition 2.1. □

3. Quantum complete intersections

Throughout the rest of this paper, let k be a field and n a positive integer. Let $a_1, \dots, a_n$ be a sequence of integers with $a_u \geqslant 2$ for all u, and for each pair (i, j) of integers with $1 \leqslant i < j \leqslant n$, let q_{ij} be a nonzero element of k. Denote by Λ the algebra

$$\Lambda = k\langle X_1, \dots, X_n\rangle / \big(\{X_u^{a_u}\}_{u=1}^n, \{X_i X_j - q_{ij} X_j X_i\}_{1 \leqslant i < j \leqslant n}\big),$$

and by x_i the image of X_i in Λ. This algebra is finite dimensional of dimension $\prod_{u=1}^n a_u$ and Loewy length $1 + \prod_{u=1}^n (a_u - 1)$, and a quantum complete intersection of codimension n. Namely, it is the quotient of the quantum symmetric algebra

$$k\langle X_1, \dots, X_n\rangle / (X_i X_j - q_{ij} X_j X_i \text{ for } 1 \leqslant i < j \leqslant n)$$

by the quantum regular sequence $x_1^{a_1}, \dots, x_n^{a_n}$. Furthermore, these algebras are Frobenius; the codimension two argument in the beginning of [BeE, Section 3] transfers to the general situation. In particular, these algebras are selfinjective.

The class of quantum complete intersections contains some well known algebras. For example, when $q_{ij} = 1$ for $1 \leqslant i < j \leqslant n$ we obtain the truncated polynomial ring

$$k[X_1, \dots, X_n] / \big(X_1^{a_1}, \dots, X_n^{a_n}\big),$$

which is a finite dimensional commutative complete intersection. Moreover, when $a_u = 2$ for all u and $q_{ij} = -1$ for $1 \leqslant i < j \leqslant n$, then the resulting quantum complete intersection is just the exterior algebra on k^n.

As mentioned in the introduction, the notion of quantum complete intersections originates from work by Manin (cf. [Man]), who introduced the concept of *quantum symmetric algebras*. These algebras were used by Avramov, Gasharov and Peeva in [AGP] to study modules behaving homologically as modules over commutative complete intersections. In particular, they introduced *quantum regular sequences* of endomorphisms of modules, thus generalizing the classical notion of regular sequences. In [BEH] a rank variety theory was given for quantum complete intersections satisfying certain conditions, and in [BeE] the Hochschild cohomology and homology of these algebras were studied.

We now prove the first of the main results. Given any subset $\{i_1, \dots, i_t\}$ of $\{1, \dots, n\}$, let $\Lambda_{i_1,\dots,i_t}$ denote the subalgebra of Λ generated by $x_{i_1}, \dots, x_{i_t}$. Thus $\Lambda_{i_1,\dots,i_t}$ is the codimension t quantum complete intersection we obtain when "forgetting" the variables X_i not in the sequence $X_{i_1}, \dots, X_{i_t}$.

Theorem 3.1. *Let $\{i_1, \dots, i_{n-1}\}$ be any subset of $\{1, \dots, n\}$ of size $n-1$. Then*

$$\dim(\underline{\mathrm{mod}}\,\Lambda_{i_1}) \leqslant \dim(\underline{\mathrm{mod}}\,\Lambda_{i_1,i_2}) \leqslant \cdots \leqslant \dim(\underline{\mathrm{mod}}\,\Lambda_{i_1,\dots,i_{n-1}}) \leqslant \dim(\underline{\mathrm{mod}}\,\Lambda).$$

Proof. For any $t \leqslant n$ there are algebra homomorphisms

$$\Lambda_{i_1,\dots,i_{t-1}} \to \Lambda_{i_1,\dots,i_t} \to \Lambda_{i_1,\dots,i_{t-1}}$$

whose composition is the identity. Moreover, the middle term is free as a module over $\Lambda_{i_1,\dots,i_{t-1}}$. Namely, it is isomorphic to $\bigoplus_{j=0}^{a_{i_t}-1} (\Lambda_{i_1,\dots,i_{t-1}}) x_{i_t}^j$. The inequalities now follow from Lemma 2.3. $\square$

We end this section with a result giving an upper bound for the representation dimension of a quantum complete intersection.

Theorem 3.2. *The representation dimension of Λ is at most $2n$.*

Proof. The algebra Λ is $\mathbb{Z}^n$-graded, where the degree of x_i is the ith unit vector $(0, \dots, 1, \dots, 0)$ in $\mathbb{Z}^n$. We prove by induction on n that if Λ is a quantum complete intersection of codimension n, then there exists a finitely generated graded Λ-module M such that the following hold:

(i) Λ is a direct summand of M,
(ii) the global dimension of $\mathrm{End}_\Lambda(M)$ is at most $2n$,
(iii) all the simple $\mathrm{End}_\Lambda(M)$-modules are one-dimensional.

The theorem will then follow from (i) and (ii).

If $n = 1$, then Λ is of the form $k[X]/(X^a)$. In this case, define M by $M = \bigoplus_{i=1}^{a} k[X]/(X^i)$. Then it is well known that (i), (ii) and (iii) hold for M. Next let $n \geqslant 1$, and suppose that the above claim holds for every quantum complete intersection of codimension at most n. Let Λ be a codimension $n+1$ quantum complete intersection

$$\Lambda = k\langle X_1, \dots, X_{n+1}\rangle / \big(\{X_u^{a_u}\}_{u=1}^{n+1}, \{X_iX_j - q_{ij}X_jX_i\}_{1\leqslant i<j\leqslant n+1}\big),$$

and let $\Lambda_{1,\dots,n}$ and Λ_{n+1} be the subalgebras generated by $x_1, \dots, x_n$ and x_{n+1}, respectively. The algebra $\Lambda_{1,\dots,n}$ is $\mathbb{Z}^n$-graded, whereas Λ_{n+1} is $\mathbb{Z}$-graded.

Define a map $g : \mathbb{Z}^n \times \mathbb{Z} \to k \setminus \{0\}$ by

$$\big((z_1, \dots, z_n), z\big) \mapsto \prod_{i=1}^{n} q_{i,n+1}^{-zz_i}.$$

It is straightforward to check that this is a homomorphism of groups. We now use this homomorphism to define a "twisted" tensor product algebra $\Lambda_{1,\dots,n} \otimes_k^g \Lambda_{n+1}$ as follows. As a k-vector space $\Lambda_{1,\dots,n} \otimes_k^g \Lambda_{n+1}$ is just the usual tensor product $\Lambda_{1,\dots,n} \otimes_k \Lambda_{n+1}$, but the multiplication is given by

$$(\lambda_1 \otimes \gamma_1)(\lambda_2 \otimes \gamma_2) \stackrel{\text{def}}{=} g\big(|\lambda_2|, |\gamma_1|\big)\lambda_1\lambda_2 \otimes \gamma_1\gamma_2,$$

where λ_i and γ_i are homogeneous elements of $\Lambda_{1,\dots,n}$ and Λ_{n+1}, respectively. This algebra is $\mathbb{Z}^{n+1}$-graded; the homogeneous elements of degree $(z_1, \dots, z_{n+1})$ are those of the form $\lambda \otimes \gamma$ for homogeneous elements $\lambda \in \Lambda_{1,\dots,n}$ and $\gamma \in \Lambda_{n+1}$ of degrees $(z_1, \dots, z_n)$ and z_{n+1}, respectively. With this ring structure we see that the graded algebras $\Lambda_{1,\dots,n} \otimes_k^g \Lambda_{n+1}$ and Λ are isomorphic.

By induction, there exist graded modules $M_1 \in \operatorname{mod} \Lambda_{1,\dots,n}$ and $M_2 \in \operatorname{mod} \Lambda_{n+1}$ satisfying (i), (ii) and (iii) over $\Lambda_{1,\dots,n}$ and Λ_{n+1}, respectively. Consider the graded module $M_1 \otimes_k^g M_2$ over $\Lambda_{1,\dots,n} \otimes_k^g \Lambda_{n+1}$, where the scalar action is defined in the natural way. It is not hard to see that $\Lambda_{1,\dots,n} \otimes_k^g \Lambda_{n+1}$ is a direct summand of $M_1 \otimes_k^g M_2$, and so this module satisfies (i). Now let Γ be its endomorphism ring over $\Lambda_{1,\dots,n} \otimes_k^g \Lambda_{n+1}$. Then $\Gamma \simeq \Gamma_1 \otimes_k^g \Gamma_2$, where $\Gamma_1 = \operatorname{End}_{\Lambda_{1,\dots,n}}(M_1)$ and $\Gamma_2 = \operatorname{End}_{\Lambda_{n+1}}(M_2)$ (note that Γ_1 and Γ_2 are graded, being endomorphism rings of finitely generated graded modules). If S is a simple Γ-module, then it is a quotient of a module of the form $S_1 \otimes_k^g S_2$, where S_i is a simple Γ_i-module for $i = 1, 2$. Since S_1 and S_2 are both one-dimensional, the module S must be isomorphic to $S_1 \otimes_k^g S_2$. Now given graded modules $X \in \operatorname{mod} \Gamma_1$ and $Y \in \operatorname{mod} \Gamma_2$, the minimal projective $\Gamma_1 \otimes_k^g \Gamma_2$-resolution of $X \otimes_k^g Y$ is just the tensor product of the minimal projective resolutions of X and Y over Γ_1 and Γ_2, respectively. This can be seen by the same argument as in the classical case, i.e. when there is no twist. Therefore

$$\operatorname{pd}_{\Lambda_{1,\dots,n} \otimes_k^g \Lambda_{n+1}} S = \operatorname{pd}_{\Lambda_{1,\dots,n}} S_1 + \operatorname{pd}_{\Lambda_{n+1}} S_2 \leqslant 2n + 2.$$

In particular, we see that (ii) and (iii) hold for the module $M_1 \otimes_k^g M_2$. This completes the proof. □

4. Homogeneous quantum complete intersections

We now turn our attention to "homogeneous" quantum complete intersections of codimension n. Throughout the rest of this paper, fix an integer $a \geqslant 2$ and a primitive ath root of unity $q \in k$. We denote by Λ_n^a the quantum complete intersection

$$\Lambda_n^a = k\langle X_1, \dots, X_n\rangle / \big(\{X_u^a\}_{u=1}^{n}, \{X_iX_j - qX_jX_i\}_{1\leqslant i<j\leqslant n}\big),$$

that is, all the defining exponents and commutators are equal to a and q, respectively.

Our aim is to show that the representation dimension of Λ_n^a is at least $n+1$. To do this, we show that given any object $M \in \underline{\mathrm{mod}}\,\Lambda_n^a$, there are objects $\{N_i \in \underline{\mathrm{mod}}\,\Lambda_n^a\}_{i=1}^n$ and morphisms $\{N_i \xrightarrow{f_i} N_{i+1}\}_{i=1}^{n-1}$ satisfying the following: for each i the map

$$\underline{\mathrm{Hom}}_{\Lambda_n^a}(-, N_i) \xrightarrow{(f_i)_*} \underline{\mathrm{Hom}}_{\Lambda_n^a}(-, N_{i+1})$$

vanishes on $\langle M\rangle$, whereas the composition $(f_{n-1})_* \circ \cdots \circ (f_1)_*$ does not vanish on $\underline{\mathrm{mod}}\,\Lambda_n^a$. We may then conclude from [Ro1, Lemma 4.11] that $\dim(\underline{\mathrm{mod}}\,\Lambda_n^a) \geqslant n-1$, and then Proposition 2.2 gives $\operatorname{repdim} \Lambda_n^a \geqslant n+1$.

The first step is the following lemma on the behavior of a linear combination of the generators. Whenever we have a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$, we denote the corresponding element $\alpha_1 x_1 + \cdots + \alpha_n x_n \in \Lambda_n^a$ by σ_α.

Lemma 4.1. *For any n-tuple α in k^n, the following hold.*

(i) $\sigma_\alpha^a = 0$.
(ii) $\sigma_\alpha^{a-1} x_i + \sigma_\alpha^{a-2} x_i \sigma_\alpha + \cdots + \sigma_\alpha x_i \sigma_\alpha^{a-2} + x_i \sigma_\alpha^{a-1} = 0$ *for all i.*

Proof. The first part of the lemma is just [BEH, Lemma 2.3]. To prove the second part, assume that k is infinite, and let ϵ be any nonzero element of k. Then from (i) the element $(\sigma_\alpha + \epsilon x_i)^a$ is zero, and expanding we get

$$0 = \sigma_\alpha^a + \epsilon \left(\sum_{j=0}^{a-1} \sigma_\alpha^j x_i \sigma_\alpha^{a-1-j} \right) + \epsilon^2 z,$$

where z is some element in Λ_n^a depending on ϵ. Therefore the set of all elements ϵ in k for which the equality

$$0 = \epsilon z + \sum_{j=0}^{a-1} \sigma_\alpha^j x_i \sigma_\alpha^{a-1-j}$$

holds contains all the nonzero elements. However, this set is closed in the Zariski topology, and so since k is infinite the zero element must also belong to it. This proves the second part of the lemma in the case when k is infinite. If k is finite, let K/k be an infinite field extension. Then $K \otimes_k \Lambda_n^a$ is the quantum complete intersection we obtain when allowing the scalars to be elements of K, with the same defining relations as in Λ_n^a. Since (ii) holds in $K \otimes_k \Lambda_n^a$ by the above, it also holds in Λ_n^a. $\square$

In order to prove the main result, we also need the following two lemmas. These are versions of [Op1, Lemma 9] and [Op1, Proposition 11], respectively.

Lemma 4.2. *Suppose k is infinite, and let $M \in \mathrm{mod}\,\Lambda_n^a$ be a module. Then there exists a non-empty open subset $U_M \subseteq k^n$ such that for any $\alpha \in U_M$ and any $m \in M$, the following implications hold*:

$$\sigma_\alpha m = 0 \quad \Rightarrow \quad \sigma_\beta m \in \sigma_\alpha M \quad \textit{for all } \beta \in k^n,$$
$$\sigma_\alpha^{a-1} m = 0 \quad \Rightarrow \quad \left(\sum_{i=0}^{a-2} \sigma_\alpha^i \sigma_\beta \sigma_\alpha^{a-2-i} \right) m \in \sigma_\alpha^{a-1} M \quad \textit{for all } \beta \in k^n.$$

Proof. Let U_1 be the set of all elements $\alpha \in k^n$ such that the matrix representing the linear map $M \xrightarrow{\cdot \sigma_\alpha} M$ (multiplication with σ_α from the right) has maximal rank. Then U_1 is non-empty and open, by an argument similar to that of the proof of [Op1, Lemma 9]. For an element $\alpha \in U_1$, choose a basis

$$\{m_1, \dots, m_s, w_1, \dots, w_t\}$$

of M such that $\sigma_\alpha m_i = 0$ and the set $\{\sigma_\alpha w_1, \dots, \sigma_\alpha w_t\}$ is linearly independent. Then for any nonzero $\epsilon \in k$ and any $\beta \in k^n$, the set

$$\left\{ \sigma_{(\alpha+\epsilon\beta)} m_i, \sigma_{(\alpha+\epsilon\beta)} w_1, \dots, \sigma_{(\alpha+\epsilon\beta)} w_t \right\}$$

is linearly dependent by the choice of α. Since $\sigma_\alpha m_i = 0$, the set

$$\left\{ \sigma_\beta m_i, \sigma_{(\alpha+\epsilon\beta)} w_1, \dots, \sigma_{(\alpha+\epsilon\beta)} w_t \right\}$$

is also linearly dependent for any nonzero $\epsilon \in k$ and any $\beta \in k^n$. However, the set of all $\epsilon \in k$ such that this set is linearly dependent is closed, and since it contains all the nonzero elements it must be k itself. Consequently, for any element $\beta \in k^n$ the set

$$\{\sigma_\beta m_i, \sigma_\alpha w_1, \dots, \sigma_\alpha w_t\}$$

is linearly dependent, i.e. $\sigma_\beta m_i \in \sigma_\alpha M$. Therefore, given any elements $\alpha \in U_1$ and $m \in M$, the first implication in the lemma holds.

Next define U_2 to be the set of all elements $\alpha \in k^n$ such that the matrix representing the linear map $M \xrightarrow{\cdot \sigma_\alpha^{a-1}} M$ has maximal rank. Then U_2 is non-empty and open. An argument similar to that for U_1 shows that given any elements $\alpha \in U_2$ and $m \in M$, the second implication in the lemma holds, since

$$\sigma_\alpha^{a-1}\sigma_\beta + \sigma_\alpha^{a-2}\sigma_\beta\sigma_\alpha + \cdots + \sigma_\alpha\sigma_\beta\sigma_\alpha^{a-2} + \sigma_\beta\sigma_\alpha^{a-1} = 0$$

for any $\beta \in k^n$ by Lemma 4.1(ii). Now since U_1 and U_2 are non-empty open sets in the Zariski topology, their intersection $U_M = U_1 \cap U_2$ is also non-empty and open, and this is a set having the properties we are seeking. $\square$

Lemma 4.3. *Suppose k is infinite, and let M be a Λ_n^a-module. Then there exists a non-empty open subset $U \subseteq k^n$ such that for any $\alpha \in U$ and any $1 \leqslant p \leqslant n$, the following hold.*

(i) *For every $j \in \mathbb{Z}$, any composition*

$$\Lambda_n^a/(\sigma_\alpha) \xrightarrow{\cdot \sum_{i=0}^{a-2} \sigma_\alpha^i x_p \sigma_\alpha^{a-2-i}} \Lambda_n^a/\left(\sigma_\alpha^{a-1}\right) \to \Omega_{\Lambda_n^a}^j(M)$$

of left Λ_n^a-homomorphisms is zero in $\underline{\mathrm{mod}}\,\Lambda_n^a$.

(ii) *For every $j \in \mathbb{Z}$, any composition*

$$\Lambda_n^a/\big(\sigma_\alpha^{a-1}\big) \xrightarrow{\cdot x_p} \Lambda_n^a/(\sigma_\alpha) \to \Omega^j_{\Lambda_n^a}(M)$$

of left Λ-homomorphisms is zero in $\underline{\mathrm{mod}}\,\Lambda_n^a$.

Proof. For simplicity, we denote our algebra Λ_n^a by just Λ. First note that the maps $\Lambda/(\sigma_\alpha) \xrightarrow{\cdot \sum_{i=0}^{a-2}\sigma_\alpha^i x_p \sigma_\alpha^{a-2-i}} \Lambda/(\sigma_\alpha^{a-1})$ and $\Lambda/(\sigma_\alpha^{a-1}) \xrightarrow{\cdot x_p} \Lambda/(\sigma_\alpha)$ of left modules are well defined by Lemma 4.1(ii). Choose two non-empty open subsets U_M and $U_{\Omega_\Lambda(M)}$ of k^n satisfying Lemma 4.2, and consider their intersection $U = U_M \cap U_{\Omega_\Lambda(M)}$. Then U is also non-empty and open. Take any $\alpha \in U$ and let $\Lambda/(\sigma_\alpha^{a-1}) \xrightarrow{g} N$ be a homomorphism of left Λ-modules, where N is either M or $\Omega_\Lambda(M)$. Furthermore, fix any $1 \leqslant p \leqslant n$, and let $m = g(1 + (\sigma_\alpha^{a-1}))$. Then $\sigma_\alpha^{a-1}m = 0$, and so by Lemma 4.2 there exists an element $m' \in N$ with the property that $(\sum_{i=0}^{a-2}\sigma_\alpha^i x_p \sigma_\alpha^{a-2-i})m = \sigma_\alpha^{a-1}m'$. This gives the factorization

$$\begin{tikzcd}
\Lambda/(\sigma_\alpha) \arrow[r, "\cdot \sum_{i=0}^{a-2}\sigma_\alpha^i x_p \sigma_\alpha^{a-2-i}"] \arrow[dr, "1\mapsto \sigma_\alpha^{a-1}"'] & \Lambda/(\sigma_\alpha^{a-1}) \arrow[r, "g"] & N \\
 & \Lambda \arrow[ur, "1\mapsto m'"'] &
\end{tikzcd}$$

proving (i) in the case when $j = 0, 1$. A similar argument shows that (ii) holds for $j = 0, 1$.

Consider the two diagrams

$$\begin{tikzcd}
0 \arrow[r] & \Lambda/(\sigma_\alpha) \arrow[r, "\cdot\sigma_\alpha^{a-1}"] \arrow[d, "\cdot \sum_{i=0}^{a-2}\sigma_\alpha^i x_p \sigma_\alpha^{a-2-i}"] & \Lambda \arrow[r] \arrow[d, "\cdot x_p"] & \Lambda/(\sigma_\alpha^{a-1}) \arrow[r] \arrow[d, "\cdot x_p"] & 0 \\
0 \arrow[r] & \Lambda/(\sigma_\alpha^{a-1}) \arrow[r, "\cdot(-\sigma_\alpha)"] & \Lambda \arrow[r] & \Lambda/(\sigma_\alpha) \arrow[r] & 0,
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \Lambda/(\sigma_\alpha^{a-1}) \arrow[r, "\cdot(-\sigma_\alpha)"] \arrow[d, "\cdot x_p"] & \Lambda \arrow[r] \arrow[d, "\cdot \sum_{i=0}^{a-2}\sigma_\alpha^i x_p \sigma_\alpha^{a-2-i}"] & \Lambda/(\sigma_\alpha) \arrow[r] \arrow[d, "\cdot \sum_{i=0}^{a-2}\sigma_\alpha^i x_p \sigma_\alpha^{a-2-i}"] & 0 \\
0 \arrow[r] & \Lambda/(\sigma_\alpha) \arrow[r, "\cdot\sigma_\alpha^{a-1}"] & \Lambda \arrow[r] & \Lambda/(\sigma_\alpha^{a-1}) \arrow[r] & 0
\end{tikzcd}$$

which are commutative by Lemma 4.1(ii). Since the rows are exact, the modules $\Lambda/(\sigma_\alpha)$ and $\Lambda/(\sigma_\alpha^{a-1})$ are both 2-periodic with respect to the syzygy operator, i.e. $\Omega_\Lambda^{2u}(\Lambda/(\sigma_\alpha)) \simeq \Lambda/(\sigma_\alpha)$ and $\Omega_\Lambda^{2u}(\Lambda/(\sigma_\alpha^{a-1})) \simeq \Lambda/(\sigma_\alpha^{a-1})$ for any $u \in \mathbb{Z}$. Moreover, the commutativity of the squares shows that the two maps $\Lambda/(\sigma_\alpha) \xrightarrow{\cdot \sum_{i=0}^{a-2}\sigma_\alpha^i x_p \sigma_\alpha^{a-2-i}} \Lambda/(\sigma_\alpha^{a-1})$ and $\Lambda/(\sigma_\alpha^{a-1}) \xrightarrow{\cdot x_p} \Lambda/(\sigma_\alpha)$ are also 2-periodic with respect to the syzygy operator. Now let $\Lambda/(\sigma_\alpha^{a-1}) \xrightarrow{g} \Omega_\Lambda^j(M)$ and

$\Lambda/(\sigma_\alpha) \xrightarrow{f} \Omega^j_\Lambda(M)$ be any maps, and choose an integer $u \in \mathbb{Z}$ such that $\Omega^{j+2u}_\Lambda(M)$ is isomorphic to either M or $\Omega_\Lambda(M)$ in $\underline{\mathrm{mod}}\,\Lambda$. Then the diagrams

$$\begin{tikzcd}
\Lambda/(\sigma_\alpha) \arrow[r, "\cdot\sum_{i=0}^{a-2}\sigma_\alpha^i x_p \sigma_\alpha^{a-2-i}"] \arrow[d, "\wr"] & \Lambda/(\sigma_\alpha^{a-1}) \arrow[r, "g"] \arrow[d, "\wr"] & \Omega^j_\Lambda(M) \\
\Lambda/(\sigma_\alpha) \arrow[r, "\cdot\sum_{i=0}^{a-2}\sigma_\alpha^i x_p \sigma_\alpha^{a-2-i}"] & \Lambda/(\sigma_\alpha^{a-1}) \arrow[r, "\Omega^{2u}_\Lambda(g)"] & \Omega^{j+2u}_\Lambda(M),
\end{tikzcd}$$

$$\begin{tikzcd}
\Lambda/(\sigma_\alpha^{a-1}) \arrow[r, "\cdot x_p"] \arrow[d, "\wr"] & \Lambda/(\sigma_\alpha) \arrow[r, "f"] \arrow[d, "\wr"] & \Omega^j_\Lambda(M) \\
\Lambda/(\sigma_\alpha^{a-1}) \arrow[r, "\cdot x_p"] & \Lambda/(\sigma_\alpha) \arrow[r, "\Omega^{2u}_\Lambda(f)"] & \Omega^{j+2u}_\Lambda(M)
\end{tikzcd}$$

are commutative in $\underline{\mathrm{mod}}\,\Lambda$, where the vertical maps are isomorphisms. We have already shown that the bottom compositions are zero in $\underline{\mathrm{mod}}\,\Lambda$, but then so are the top compositions, since these are shifts of the bottom compositions. $\square$

We are now ready to prove that the representation dimension of Λ^a_n is at least $n+1$ when n is even and the field k is infinite.

Proposition 4.4. *If k is infinite and n is even, then* $\operatorname{repdim} \Lambda^a_n \geqslant n+1$.

Proof. As in the previous proof, we denote our algebra Λ^a_n by just Λ. Let $\alpha \in k^n$ be an n-tuple, and denote the element

$$x_{n-1}\left(\sum_{i=0}^{a-2}\sigma_\alpha^i x_n \sigma_\alpha^{a-2-i}\right)x_{n-3}\left(\sum_{i=0}^{a-2}\sigma_\alpha^i x_{n-2} \sigma_\alpha^{a-2-i}\right)\cdots x_3\left(\sum_{i=0}^{a-2}\sigma_\alpha^i x_4 \sigma_\alpha^{a-2-i}\right)x_2$$

by w_α. In the first (and longest) part of this proof, we show that the set of all $\alpha \in k^n$ such that w_α does not belong to $\sigma_\alpha\Lambda + \Lambda\sigma_\alpha$ contains a non-empty open set.

Fix any n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n$ with $\alpha_1 \neq 0$, and, for simplicity, denote the corresponding σ_α and w_α by σ and w, respectively. Every element $\lambda \in \Lambda$ admits a unique decomposition $\lambda = N_{x_1}(\lambda) + \sigma R_{x_1}(\lambda)$, in which x_1 does not occur in any of the monomials in $N_{x_1}(\lambda)$. With this notation, we see that

$$\begin{aligned}
w \in \sigma\Lambda + \Lambda\sigma \quad &\Leftrightarrow \quad N_{x_1}(w) \in \sigma\Lambda + \Lambda\sigma \\
&\Leftrightarrow \quad N_{x_1}(w) + h\sigma \in \sigma\Lambda \quad \text{for some } h \in \Lambda \\
&\Leftrightarrow \quad N_{x_1}(w) + h\sigma \in \sigma\Lambda \quad \text{for some } h \in \Lambda \text{ with } N_{x_1}(h) = h \\
&\Leftrightarrow \quad N_{x_1}\big(N_{x_1}(w) + h\sigma\big) = 0 \quad \text{for some } h \in \Lambda \text{ with } N_{x_1}(h) = h,
\end{aligned}$$

where we may assume that the h occurring is homogeneous. Now note that the degree of w is $(\frac{n}{2}-1)a+1$. This implies that any homogeneous h satisfying the above implications is of degree $(\frac{n}{2}-1)a$. If in addition $N_{x_1}(h)=h$, i.e. if x_1 does not occur in any of the monomials in h, then $hx_1 = q^{-(\frac{n}{2}-1)a}x_1h = x_1h$. Writing $\sigma' = \sigma - \alpha_1 x_1$ we then get

$$h\sigma = \alpha_1 h x_1 + h\sigma' = \alpha_1 x_1 h + h\sigma' = \sigma h - \sigma' h + h\sigma'.$$

What we have so far gives

$$\begin{aligned}
w \in \sigma\Lambda + \Lambda\sigma \quad &\Leftrightarrow \quad N_{x_1}\big(N_{x_1}(w) + h\sigma\big) = 0 \quad \text{with } N_{x_1}(h) = h \\
&\Leftrightarrow \quad N_{x_1}\big(N_{x_1}(w) + \sigma h - \sigma' h + h\sigma'\big) = 0 \quad \text{with } N_{x_1}(h) = h \\
&\Leftrightarrow \quad N_{x_1}\big(N_{x_1}(w) - \sigma' h + h\sigma'\big) = 0 \quad \text{with } N_{x_1}(h) = h \\
&\Leftrightarrow \quad N_{x_1}(w) - \sigma' h + h\sigma' = 0 \quad \text{with } N_{x_1}(h) = h,
\end{aligned}$$

where the last implication follows from the fact that x_1 does not occur in any of the monomials in $N_{x_1}(w) - \sigma'h + h\sigma'$. Thus w belongs to $\sigma\Lambda + \Lambda\sigma$ if and only if $N_{x_1}(w) = \sigma'h - h\sigma'$ for some homogeneous element h satisfying $N_{x_1}(h) = h$.

Let λ be the element $x_2x_3^{a-1}x_4\cdots x_{n-1}^{a-1}x_n$, which has the same degree as w. For any homogeneous element h of degree one less than that of w, the coefficient of λ in $\sigma'h - h\sigma'$ is easily seen to be zero. Therefore $w \notin \sigma\Lambda + \Lambda\sigma$ if the coefficient of λ in $N_{x_1}(w)$ is nonzero. Note that the set of tuples $(\alpha_2, \dots, \alpha_n) \in k^{n-1}$ for which this holds is open.

Consider the tuple $(\alpha_1, 0, 1, \dots, 1, 0) \in k^n$, that is, we take as σ the element

$$\sigma = \alpha_1 x_1 + x_3 + \cdots + x_{n-1}.$$

Define the element $w' \in \Lambda$ by

$$w' = x_{n-1}\sum_{i=0}^{a-2}\sigma^i x_n \sigma^{a-2-i},$$

i.e. w' is the "first part" of w. Since $x_n\sigma = q^{-1}\sigma x_n$, we see that

$$\begin{aligned}
w' &= x_{n-1}\beta\sigma^{a-2}x_n \\
&= \beta\big(q^{-1}\alpha_1x_1 + \cdots + q^{-1}x_{n-3} + x_{n-1}\big)^{a-2}x_{n-1}x_n
\end{aligned}$$

for some nonzero element $\beta \in k$. Using the equality $N_{x_1}(zy) = N_{x_1}(N_{x_1}(z)y)$, which holds for any elements $z, y \in \Lambda$, an induction argument gives

$$N_{x_1}\big(\big(q^{-1}\alpha_1x_1 + \cdots + q^{-1}x_{n-3} + x_{n-1}\big)^i\big) = \left(\prod_{j=1}^{i}\big(1 - q^{-j}\big)\right)x_{n-1}^i.$$

Applying N_{x_1} to the above expression for w' then gives

$$\begin{aligned} N_{x_1}(w') &= \beta N_{x_1}\big(N_{x_1}\big[\big(q^{-1}\alpha_1 x_1 + \cdots + q^{-1}x_{n-3} + x_{n-1}\big)^{a-2}\big]x_{n-1}x_n\big) \\ &= \beta N_{x_1}\left(\left[\prod_{j=1}^{a-2}\big(1-q^{-j}\big)\right]x_{n-1}^{a-1}x_n\right) \\ &= \beta' x_{n-1}^{a-1}x_n, \end{aligned}$$

where β' is some nonzero element in k. Now define w'' by

$$w'' = x_{n-3}\left(\sum_{i=0}^{a-2}\sigma^i x_{n-2}\sigma^{a-2-i}\right)\cdots x_3\left(\sum_{i=0}^{a-2}\sigma^i x_4\sigma^{a-2-i}\right)x_2,$$

that is, the element w is given by $w = w'w''$. By what we have just shown, the equality

$$N_{x_1}(w) = N_{x_1}\big(N_{x_1}(w')w''\big) = \beta' N_{x_1}\big(x_{n-1}^{a-1}x_n w''\big)$$

holds. Now every monomial in w'' containing x_{n-1} does not "contribute" to $x_{n-1}^{a-1}x_n w''$, therefore we may replace every σ in w'' by $\sigma - x_{n-1}$. Then if we repeat the above process, we see that

$$N_{x_1}(w) = \delta x_2 x_3^{a-1}x_4\cdots x_{n-1}^{a-1}x_n = \delta\lambda,$$

where δ is some nonzero element in k. This shows that the coefficient of the monomial λ in $N_{x_1}(w)$ is nonzero, i.e. $w \notin \sigma\Lambda + \Lambda\sigma$.

Define the set $V \subseteq k^n$ by

$$V = \big\{\alpha \in k^n \bigm| w_\alpha \notin \sigma_\alpha\Lambda + \Lambda\sigma_\alpha\big\},$$

and consider the set

$$W = \big\{(\alpha_2,\dots,\alpha_n) \in k^{n-1} \bigm| (\alpha_1,\alpha_2,\dots,\alpha_n) \in V \text{ for all } 0 \neq \alpha_1 \in k\big\}.$$

We have just shown that W is open and non-empty, hence the subset

$$\big\{(\alpha_1,\alpha_2,\dots,\alpha_n) \in k^n \bigm| \alpha_1 \neq 0 \text{ and } (\alpha_2,\dots,\alpha_n) \in W\big\}$$

of V is non-empty and open in k^n.

Now let M be a Λ-module, and let $U \subseteq k^n$ be a non-empty open set having the properties stated in Lemma 4.3. Since V contains a non-empty open subset, the intersection $U \cap V$ is not empty, and therefore contains an n-tuple $\alpha \in k^n$. For each $1 \leqslant i \leqslant n-1$, define maps $f_i \in \underline{\operatorname{mod}}\,\Lambda$ by

$$f_i = \begin{cases} \Lambda/(\sigma_\alpha^{a-1}) \xrightarrow{\cdot x_{n-i}} \Lambda/(\sigma_\alpha) & \text{when } i \text{ is odd and } i \leqslant n-3, \\ \Lambda/(\sigma_\alpha) \xrightarrow{\cdot\sum_{i=0}^{a-2}\sigma_\alpha^i x_{n+2-i}\sigma_\alpha^{a-2-i}} \Lambda/(\sigma_\alpha^{a-1}) & \text{when } i \text{ is even}, \\ \Lambda/(\sigma_\alpha^{a-1}) \xrightarrow{\cdot x_2} \Lambda/(\sigma_\alpha) & \text{when } i = n-1. \end{cases}$$

By Lemma 4.3 each induced map $(f_i)_*$ vanishes on the subcategory $\langle M\rangle$ of $\underline{\mathrm{mod}}\,\Lambda$. However, the composition $(f_{n-1})_* \circ \cdots \circ (f_1)_*$ does not vanish: if it did then the map $\Lambda/(\sigma_\alpha^{a-1}) \xrightarrow{\cdot w_\alpha} \Lambda/(\sigma_\alpha)$ would be zero in $\underline{\mathrm{mod}}\,\Lambda$. However, the injective envelope of $\Lambda/(\sigma_\alpha^{a-1})$ is $\Lambda/(\sigma_\alpha^{a-1}) \xrightarrow{\cdot\sigma_\alpha} \Lambda$, whereas the projective cover of $\Lambda/(\sigma_\alpha)$ is the projection $\Lambda \xrightarrow{\pi} \Lambda/(\sigma_\alpha)$. Therefore, if the map $\Lambda/(\sigma_\alpha^{a-1}) \xrightarrow{\cdot w_\alpha} \Lambda/(\sigma_\alpha)$ was zero in $\underline{\mathrm{mod}}\,\Lambda$, then it would factor in a commutative diagram

$$\begin{tikzcd}
\Lambda/(\sigma_\alpha^{a-1}) \arrow[r, "\cdot w_\alpha"] \arrow[d, "\cdot\sigma_\alpha"] & \Lambda/(\sigma_\alpha) \\
\Lambda \arrow[r] & \Lambda \arrow[u, "\pi"']
\end{tikzcd}$$

in $\mathrm{mod}\,\Lambda$. Since any homomorphism $\Lambda \to \Lambda$ is right multiplication with an element in Λ, this would mean that there exists an element $\lambda \in \Lambda$ such that $w_\alpha = \sigma_\alpha \lambda$ in $\Lambda/(\sigma_\alpha)$. In other words, the element w_α would belong to $\sigma_\alpha \Lambda + \Lambda\sigma_\alpha$, and this is not the case since $\alpha \in V$. From [Ro1, Lemma 4.11] we see that $\dim(\underline{\mathrm{mod}}\,\Lambda) \geqslant n-1$, and so Proposition 2.2 gives $\operatorname{repdim}\Lambda \geqslant n+1$. $\square$

Remark. It is also possible to prove Proposition 4.4 using [Op2, Theorem 1(b)]. However, as in the proof we just gave, the key ingredient in such a proof is deciding whether or not w_α belongs to $\sigma_\alpha\Lambda + \Lambda\sigma_\alpha$.

Having established the case when n is even and the field k is infinite, we prove the main theorem in this paper, namely the general case.

Theorem 4.5. *The representation dimension of Λ_n^a satisfies*

$$n+1 \leqslant \operatorname{repdim}\Lambda_n^a \leqslant 2n.$$

Proof. For the inequality $n+1 \leqslant \operatorname{repdim}\Lambda_n^a$, we first assume k to be infinite. The case when n is even was treated in Proposition 4.4, so it remains to give a proof for odd n. To do so, we keep n even, and prove the inequality $n \leqslant \operatorname{repdim}\Lambda_{n-1}^a$. We denote our algebra Λ_n^a by Λ. Furthermore, denote by $\tilde{\Lambda}$ the algebra $\Lambda_{2,3,\dots,n}$, where the notation is the same as that used in Theorem 3.1. In other words, the algebra $\tilde{\Lambda}$ is the codimension $n-1$ quantum complete intersection subalgebra of Λ generated by $x_2,\dots,x_n$, and therefore isomorphic to Λ_{n-1}^a. In this notation, our aim is to show that $\operatorname{repdim}\tilde{\Lambda} \geqslant n$. We may assume $n \geqslant 4$, since we know that the representation dimension of the truncated polynomial ring $k[X]/(X^a)$, which is of finite representation type, is two.

Given an n-tuple $\alpha \in k^n$, define as before the corresponding element $\sigma_\alpha \in \Lambda$, and denote by w_α the element

$$x_{n-1}\left(\sum_{i=0}^{a-2}\sigma_\alpha^i x_n \sigma_\alpha^{a-2-i}\right)x_{n-3}\left(\sum_{i=0}^{a-2}\sigma_\alpha^i x_{n-2}\sigma_\alpha^{a-2-i}\right)\cdots x_3\left(\sum_{i=0}^{a-2}\sigma_\alpha^i x_4 \sigma_\alpha^{a-2-i}\right)x_2.$$

We showed in the proof of the previous result that the set

$$V = \left\{\alpha \in k^n \;\middle|\; w_\alpha \notin \sigma_\alpha\Lambda + \Lambda\sigma_\alpha\right\}$$

contains a non-empty open subset. Now for any $(n-1)$-tuple $\tilde{\alpha} = (\alpha_2, \dots, \alpha_n) \in k^{n-1}$, define the element $\tilde{\sigma}_{\tilde{\alpha}} \in \tilde{\Lambda}$ by $\tilde{\sigma}_{\tilde{\alpha}} = \alpha_2 x_2 + \cdots + \alpha_n x_n$, and denote by $\tilde{w}_{\tilde{\alpha}}$ the element

$$x_{n-1}\left(\sum_{i=0}^{a-2} \tilde{\sigma}_{\tilde{\alpha}}^{i} x_n \tilde{\sigma}_{\tilde{\alpha}}^{a-2-i}\right) x_{n-3}\left(\sum_{i=0}^{a-2} \tilde{\sigma}_{\tilde{\alpha}}^{i} x_{n-2} \tilde{\sigma}_{\tilde{\alpha}}^{a-2-i}\right) \cdots x_3\left(\sum_{i=0}^{a-2} \tilde{\sigma}_{\tilde{\alpha}}^{i} x_4 \tilde{\sigma}_{\tilde{\alpha}}^{a-2-i}\right)$$

in $\tilde{\Lambda}$. Furthermore, consider the subset

$$\tilde{V} = \left\{\tilde{\alpha} \in k^{n-1} \;\middle|\; \tilde{w}_{\tilde{\alpha}} \notin \tilde{\sigma}_{\tilde{\alpha}} \tilde{\Lambda} + \tilde{\Lambda} \tilde{\sigma}_{\tilde{\alpha}}^{a-1}\right\}$$

of k^{n-1}. We show that this set contains a non-empty subset which is open.

Recall from the previous proof that the set

$$W = \left\{(\alpha_2, \dots, \alpha_n) \in k^{n-1} \;\middle|\; (\alpha_1, \alpha_2, \dots, \alpha_n) \in V \text{ for all } 0 \neq \alpha_1 \in k\right\}$$

is non-empty and open in k^{n-1}. Furthermore, let W' be the set of all $(n-1)$-tuples $(\alpha_2, \dots, \alpha_n)$ in which α_2 is nonzero. Since W' is non-empty and open, the intersection $\tilde{W} = W \cap W'$ is also non-empty and open. Pick therefore an element $\tilde{\alpha} = (\alpha_2, \dots, \alpha_n) \in \tilde{W}$, let $\alpha \in k^n$ be the n-tuple $\alpha = (\alpha_2, \alpha_2, \dots, \alpha_n)$, and define a map $f : \tilde{\Lambda} \to \Lambda$ by

$$x_i \mapsto \begin{cases} x_1 + x_2 & \text{when } i = 2, \\ x_i & \text{when } i \neq 2. \end{cases}$$

Since $f(x_u^a) = 0 = f(x_i x_j - q x_j x_i)$ for $2 \leqslant u \leqslant n$ and $2 \leqslant i < j \leqslant n$, this is a well defined algebra homomorphism. Note that $f(\tilde{\sigma}_{\tilde{\alpha}}) = \sigma_\alpha$ and that α belongs to V. Moreover, note that $f(\tilde{w}_{\tilde{\alpha}})x_2 = w_\alpha$, and so if $\tilde{w}_{\tilde{\alpha}}$ belongs to $\tilde{\sigma}_{\tilde{\alpha}} \tilde{\Lambda} + \tilde{\Lambda} \tilde{\sigma}_{\tilde{\alpha}}^{a-1}$ then w_α belongs to $\sigma_\alpha \Lambda + \Lambda \sigma_\alpha^{a-1} x_2$. However, from Lemma 4.1(ii) we see that $\sigma_\alpha \Lambda + \Lambda \sigma_\alpha^{a-1} x_2$ is contained in $\sigma_\alpha \Lambda + \Lambda \sigma_\alpha$, and we know that $w_\alpha \notin \sigma_\alpha \Lambda + \Lambda \sigma_\alpha$ since $\alpha \in V$. Therefore $\tilde{w}_{\tilde{\alpha}}$ cannot belong to $\tilde{\sigma}_{\tilde{\alpha}} \tilde{\Lambda} + \tilde{\Lambda} \tilde{\sigma}_{\tilde{\alpha}}^{a-1}$, and this shows that $\tilde{W}$ is a subset of $\tilde{V}$. Consequently, the set $\tilde{V}$ contains a non-empty subset which is open.

The arguments we applied at the end of the previous proof now shows that $\operatorname{repdim} \tilde{\Lambda} \geqslant n$. We have therefore proved that $\operatorname{repdim} \Lambda_n^a \geqslant n+1$ for any n, when the field k is infinite. However, when k is finite, the strategy applied in [Op1, Section 4] carries over to our algebra and shows that the inequality still holds. Therefore $\operatorname{repdim} \Lambda_n^a \geqslant n+1$ regardless of whether the field k is infinite. This proves the first inequality in the theorem, the other is Theorem 3.2. $\square$

Corollary 4.6. *Let Λ be a general quantum complete intersection, i.e.*

$$\Lambda = k\langle X_1, \dots, X_n\rangle / \left(\left\{X_u^{a_u}\right\}_{u=1}^{n}, \{X_i X_j - q_{ij} X_j X_i\}_{1 \leqslant i < j \leqslant n}\right)$$

where $a_u \geqslant 2$ and q_{ij} is nonzero. If there exists a subset $\{i_1, \dots, i_t\}$ of $\{1, \dots, n\}$ such that the subalgebra $\Lambda_{i_1,\dots,i_t}$ is a homogeneous quantum complete intersection of the form Λ_t^a, then $\operatorname{repdim} \Lambda \geqslant t+1$.

Proof. By Theorem 3.1 the inequality $\dim(\underline{\mathrm{mod}}\, \Lambda_t^a) \leqslant \dim(\underline{\mathrm{mod}}\, \Lambda)$ holds, and in the previous proof we showed that $\dim(\underline{\mathrm{mod}}\, \Lambda_t^a)$ is at least $t-1$. Proposition 2.2 now gives $\operatorname{repdim} \Lambda \geqslant t+1$. $\square$

Remarks. (i) In the main results we have only considered homogeneous quantum complete intersections of the form

$$k\langle X_1, \dots, X_n\rangle / \left(\left\{X_u^a\right\}_{u=1}^n, \{X_i X_j - q X_j X_i\}_{1 \leqslant i < j \leqslant n}\right),$$

that is, algebras where the defining exponents are all equal to a and where q is a primitive ath root of unity. However, the proofs are also valid if we relax the requirement that the defining exponents of the "even" indeterminates $X_2, X_4, \dots$ are equal to a, as long as we require that these exponents belong to $\{2, \dots, a\}$. That is, the main results apply to quantum complete intersections of the form

$$k\langle X_1, \dots, X_n\rangle / \left(\left\{X_u^a\right\}_{u \text{ odd}}, \left\{X_v^{a_v}\right\}_{v \text{ even}}, \{X_i X_j - q X_j X_i\}_{1 \leqslant i < j \leqslant n}\right),$$

where q is a primitive ath root of unity and $a_v \in \{2, \dots, a\}$. This is because the key ingredient in the proof when n is even is to show that the coefficient of the element $x_2 x_3^{a-1} x_4 \cdots x_{n-1}^{a-1} x_n$ in

$$x_{n-1}\left(\sum_{i=0}^{a-2} \sigma_\alpha^i x_n \sigma_\alpha^{a-2-i}\right) x_{n-3}\left(\sum_{i=0}^{a-2} \sigma_\alpha^i x_{n-2} \sigma_\alpha^{a-2-i}\right) \cdots x_3\left(\sum_{i=0}^{a-2} \sigma_\alpha^i x_4 \sigma_\alpha^{a-2-i}\right) x_2$$

is nonzero for every α belonging to a certain non-empty open subset of k^n.

(ii) Work in progress by Avramov and Iyengar (cf. [AvI]) shows that the dimension of the stable derived category of a commutative local complete intersection of codimension c is at least $c-1$. Consequently, the representation dimension of an Artin complete intersection is strictly greater than its embedding dimension. In particular, they have shown that the representation dimension of $k[X_1, \dots, X_n]/(X_1^{a_1}, \dots, X_n^{a_n})$ is at least $n+1$ when $a_i \geqslant 2$.

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