Entropy of the brane-world black hole

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Abstract

The entropy in the background of the brane-world black hole with tidal charge has been studied by using the generalized uncertainty principles. Comparing with the four-dimensional black holes, this TeV-size black hole, which contains information of the extra dimension, may increase the Bekenstein–Hawking entropy. We also show that the entropy has a negative logarithmic correction once the generalized uncertainty principles are taken into account.

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Black hole, as one of the most intriguing objects in our universe, has been recently revived since the advent of brane-world gravity theories [1,2]. Indeed, it appears natural to consider the case when the black hole is embedded in a space–time of higher dimensionality since models with extra spatial dimensions deliver a possible solution to the hierarchy problem (between the electroweak scale and the fundamental scale of quantum gravity) without appealing to supersymmetry [3]. The main opinion of the brane-world models is that our physical universe is a D3-brane with the standard model fields localized on it, while gravity, being mediated by closed strings, can propagate both on the brane and in the bulk [3–5]. The size of the extra dimensions may be much larger [3] than the conventional Planck length scale \( L \gg l_p \approx 10^{-33} \text{ cm} \). Another scenario (like the Randall and Sundrum (RS) scenario) even claims that the extra spatial dimensions can be infinitely extended [4]. Due to the large size of extra dimensions the scale of quantum gravity becomes the same order as the electroweak interaction scale (of the order of a few TeVs). This, in turn, leads to a striking consequences that the lowering of the fundamental gravity scale opens up the possibility of TeV-size black hole production in the universe. Such TeV-size black holes are centered on the brane and may have been created in the early universe due to density perturbations and phase transitions. It was also suggested that such TeV-size black holes may be produced at high energy collision processes in cosmic ray and at future colliders such as LHC [6–8].

On the other hand, the microscopic origin of black hole entropy has long been a puzzling problem to most theoretical physicists since Bekenstein and Hawking proposed a beautiful analogy between the laws of black hole dynamics and the laws of thermodynamics [9–12]. Several schools of thoughts have emerged for solving this problem. One of them is the brick wall model proposed by ’t Hooft [13]. In this model, the density of quantum states becomes explosive when approaching the event horizon. In order to get a finite entropy, a cut-off is introduced near the horizon, which is however rather unnatural. Although some authors noted that the explosive entropy in the...
brick wall model is related to the divergence of the one-loop effective action of quantum field theory in curved space–time [14] and managed to regularize the divergence entropy, they failed to solve the black hole entropy puzzle because a conceptual difficulty of the so-called bare entropy introduced. Recently, with the advent of generalized uncertainty principles (GUPs), originating from several studies in string theory approach to quantum gravity [15–18], loop quantum gravity [19], noncommutative space–time algebra [20–22] and black holes gedanken experiments [23,24], the contribution to the entropy of quantum states with momentum above a given scale has been suppressed and the UV divergence completely removed [25,26].

The entropy of a brane-world black hole with GUP was first studied in [27], and then in [28]. It is generally believed that the entropy of a brane-world black hole may carry information of the extra dimensions, this in turn may help us with observing the properties of the extra dimensions, and more important, providing a deep insight of the quantum theory of gravity. The motivation of the present Letter is to study the entropy of the TeV-size black holes in the frame of the ADD brane-world scenario [3]. We start with a black hole localized on 3-brane in 4-dimensional background, which is of the Reissner–Nordström type given as [29]

\[ ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \]  \hspace{1cm} (1)

where \( f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \), \( M_d \) and \( M_P \) are the Planck length for 4-dimensional and \( d \)-dimensional space–time, respectively, and \( Q \) is the ‘tidal charge’ arising from the projection onto the brane of the gravitational field in the bulk. Thus \( q \) contains the information of the extra dimension. For \( q > 0 \), this metric is a direct analogy to the Reissner–Nordström solution with two horizons. For \( q < 0 \), the metric has only one horizon given by

\[ r_+ = \frac{M}{M_d} \left( 1 + \sqrt{1 + \frac{q^2 M_d^2}{M_P^2}} \right), \]

which is larger than the Schwarzschild horizon. So the negative tidal charge increases the entropy (this can be also confirmed latter in this Letter). We will concentrate our attention on \( q < 0 \) (in the following we express the negative tidal charge to be \( Q \)), since it was argued that this negative tidal charge is the physically more natural case [29].

The metric on the 4D brane can be rewritten as

\[ f = 1 - \frac{2M}{r} - \frac{Q}{r^2}, \]  \hspace{1cm} (2)

with two roots \( r_{\pm} = M \left( 1 \pm \sqrt{1 + \frac{Q^2}{M_d^2}} \right) \) of \( f = 0 \), \( r_+ \) is the black hole horizon, while \( r_- \) is negative and without physical meaning. Let us begin the formal analysis by presenting the GUP as it typically appears in the literature [19]

\[ \delta x \geq \frac{\hbar}{\delta p} + \alpha L_p^2 \frac{\delta p}{\hbar}, \]  \hspace{1cm} (3)

where \( L_p = (\frac{\hbar G_d}{c})^{\frac{1}{2}} \) is Planck length in extra-dimensional scenario, \( d \) is the total number of space–time dimensions, \( G_d = G_4 L^n \) where \( n = d - 4 \) and \( L \) is extension of \( n \) extra dimensions in ADD model [3], and \( \alpha \), whose value depends on particular model of quantum gravity, is a dimensionless constant of the order unity. By way of some simple manipulations, we can re-express the GUP in the following manner:

\[ \delta x \geq \frac{\delta x \hbar}{2\alpha L_p^2} \left[ 1 - \sqrt{1 - \frac{4\alpha^2 L_p^2}{(\delta x)^2}} \right]. \]  \hspace{1cm} (4)

According to general relativistic result, for a black hole absorbing a classical particle of energy \( E \) and size \( R \), the minimal increase in the horizon area can be expressed as [30]

\[ (\Delta A_d)_{\min} \geq \frac{8\pi L_p^d - 2 E R}{(d - 3)\hbar c}. \]  \hspace{1cm} (5)

Hence we have

\[ (\Delta A_d)_{\min} = \frac{8\pi L_p^2 \delta p \delta x}{\hbar} \quad \text{or} \quad (\Delta A_d)_{\min} = \frac{\epsilon L_p^2 \delta p \delta x}{\hbar} \]  \hspace{1cm} (6)

where we have taken note of the quantum-theory result that \( E \) and \( R \) can never be taken as smaller than \( c \delta p \) and \( \delta x \), respectively. \( \epsilon \) is a parameter to be determined.

Using the value of \( \delta x \) in Eq. (4), one obtains

\[ (\Delta A_d)_{\min} \approx \frac{\epsilon (\delta x)^2}{2\alpha^2} \left[ 1 - \sqrt{1 - \frac{4\alpha^2 L_p^2}{(\delta x)^2}} \right]. \]  \hspace{1cm} (7)

As shown in [31], the particles that we are interested in have a Compton length on the order of the inverse surface gravity [12]. Hence we can take for \( \delta x \) as

\[ \delta x \sim \kappa^{-1} = \frac{2 r_+}{r_+ - r_-}. \]

When \( Q \) is small, it becomes

\[ \delta x \sim 2(r_+ + r_-) = \frac{A}{\sqrt{\pi}} \left( 1 - \frac{4\pi Q}{A} \right), \]  \hspace{1cm} (8)

where \( A \) is the outer horizon area of the black hole.

The information theory [32] showed that the minimal increase of entropy should be, irrespective of the value of the area, simply one “bit” of information. We let \( b \) represent this quantum, then we find

\[ \frac{dS}{dA} \sim \frac{(\Delta S_d)_{\min}}{(\Delta A_d)_{\min}} \approx \frac{2b\alpha^2}{\epsilon (\delta x)^2 [1 - \sqrt{1 - \frac{4\alpha^2 L_p^2}{(\delta x)^2}}]} \]

\[ = \frac{b}{2\alpha L_p^2} \left[ 1 + \sqrt{1 - \frac{4\alpha^2 L_p^2}{(\delta x)^2}} \right]. \]  \hspace{1cm} (9)

Since \( L_p \) is normally viewed as an ultraviolet cutoff on the space–time geometry, it should be safe to regard the dimensionless ratio \( L_p^2/(\delta x)^2 \) as small relative to unity. Hence, it is natural to Taylor expand the square root and obtain

\[ \frac{dS}{dA} \approx \frac{b}{\epsilon L_p^2} \left[ 1 - \left( \frac{\alpha^2 L_p^2}{(\delta x)^2} \right) - \left( \frac{\alpha^2 L_p^2}{(\delta x)^2} \right)^2 + \cdots \right]. \]  \hspace{1cm} (10)

Eq. (8) has shown that \( \delta x \) is closely related to the horizon area of the black hole. When \( Q \ll A \) is taken into account, one can
The coefficient of the logarithmic term is $-\frac{1}{2}$, which indicates $\alpha$ should be $\sqrt{\frac{2}{\pi}}$. In spite of the current lack of knowledge about the exact value of $\alpha$, our calculation show that the quantum-corrected black hole entropy should include such logarithmic correction. It also enlightens us with some new properties of the full, but current unknown, quantum gravity model.

Further study on the behavior of the correction terms can be performed by observing their variation of $\Delta S = S - S_{BH}$ with the area $A$, as shown in Fig. 2. The most outstanding conclusion can be drawn from this figure is that the correction terms increase with $A$, in spite of the decreasing velocity. It behaves as the minus of the logarithmic function. The main reason of this behavior is the logarithmic term in Eq. (12) dominates the correction terms when $Q$ is small. Undoubtedly, it could be totally different when $Q$ is large, which is out of our discussion in this Letter.

Fig. 3 shows the impact of parameter $\alpha$ on the black hole entropy with negative tidal charge. The most outstanding character in this picture is the entropy decreases with $\alpha$, regardless of $Q$. However, the tidal charge also affects the relations between entropy and $\alpha$. Generally, the bigger the charge is, the more slowly the entropy decreases with $\alpha$. Since $Q$ refers to the information of the extra dimensions of the space–time (its value determines the gravitational field strength in the bulk), large $Q$ corresponds to strong gravitational field in the bulk.
This indicates that gravitational field in the bulk will increase the entropy of the black hole on the brane. In other words, the gravitational field in the bulk will increase the freedom degrees of the black hole localized in the brane. This argument can be further confirmed by observing the variation of the black hole entropy with the tidal charge as shown in Fig. 4, where entropy increases as $Q$ increases, regardless of the value of $\alpha$. Therefore, in any quantum gravity models (they are related to the value of $\alpha$), gravitational field in the bulk will increase the black hole entropy on the brane. What is the origin of this bulk effect? A possible reason is that, according to the brane-world scenarios [3,4], gravity can propagate both on the brane and in the bulk (in spite of the gauge fields are confined on the brane), while the black hole entropy is closely related to the gravity through the black hole area. We see from the definition of $r_+$ that negative tidal charge will enlarge the horizon of the black hole localized on the brane. It is thus reasonable for the black hole entropy to observe the effects of the gravitational field in the bulk. What more surprised us is a weird phenomena that the entropy-charge variation curves with different $\alpha$ approximately converge at a point $Q_c$ ($\approx 7.3$ in the case as shown in Fig. 4). For different sets of $A$ and $n$, the same behavior appears except for different converging points as listed in Table 1. A question immediately coming to mind is why these curves may converge at a point, or more specifically, why they converge at these points. To explain this phenomena, we notice that the first two terms of the correction terms in Eq. (12) have the same coefficient, i.e., $\pi^2 Q^2$, indicating the sum of this two terms vanishes as appropriate $Q$ is chosen for certain fixed $A$. Although other terms in Eq. (12) have some impact on the convergent effect, their influence is tiny since $Q/A$ is small quantum in our discussion. This can be also confirmed by Fig. 5, where we plotted $A-Q$ relation of equation

\[
-\ln \frac{A}{4L_p^2} + \left( \frac{8\pi Q}{A} + \frac{24\pi^2 Q^2}{A^2} \right) = 0, \tag{14}
\]

and data listed in Table 1 as well. Data obtained through observing different converging points have a good agreement with values calculated by solving Eq. (14).

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