Minimal partial realization by descriptor systems

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Abstract

In this paper a generalization of Kalman’s partial realization theory is developed using partial realizations defined by descriptor systems. The use of singular system realizations in contrast to regular linear systems enables us to circumvent certain technical difficulties inherent in the standard approach to partial realizations. An existence and uniqueness result for minimal partial descriptor realizations is proven and a simple rank formula for the McMillan degree is derived. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Realization theory is a cornerstone in the structure theory of linear dynamical systems. It plays a fundamental role in engineering areas such as system identification and modelling and provides a powerful link of input–output methods with state space techniques. From a historical perspective, the mathematical foundations of state space realization theory were laid by Kronecker, Sylvester and Frobenius. The present form of the theory has been mainly shaped through Kalman’s pioneering contributions and today the realization theory is seen as being closely connected to...
diverse areas such as for example, the interpolation theory, Padé approximation, continued fractions, factorizations and inversion of Hankel and Toeplitz operators, coding theory, Waring’s problem for binary forms and even spline approximations and neural networks \[1,2,11,14,18\]. Thus realization theory is by no means an isolated topic of linear system theory.

The \textit{minimal realization problem} for time-invariant, discrete-time linear systems is to find for any given strictly proper rational matrix \( G(z) \in \mathbb{K}^{p \times m}(z) \) over a field \( \mathbb{K} \), a linear system
\[
\begin{align*}
  x_{t+1} &= Ax_t + Bu_t, & u_t \in \mathbb{K}^m, & x_t \in \mathbb{K}^n, & y_t \in \mathbb{K}^p, & t = 0, 1, 2, \ldots, (1.1)
\end{align*}
\]
such that the associated transfer matrix \( C(zI_n - A)^{-1}B \) is equal to \( G(z) \) and the \textit{dimension} \( n \) is as small as possible. Alternatively, since any strictly proper rational matrix \( G(z) \in \mathbb{K}^{p \times m}(z) \) can be expanded in a (formal) Laurent series at \( \infty \)
\[
G(z) = \sum_{i=1}^{\infty} H_iz^{-i}, \quad H_i \in \mathbb{K}^{p \times m}, \quad i \in \mathbb{N} := \{1, 2, 3, \ldots\},
\]
the problem of minimal realization may be stated as follows. Given an \textit{infinite} sequence \((H_1, H_2, \ldots)\) of \( p \times m \) matrices over \( \mathbb{K} \), find a linear system \((A, B, C)\) of minimal dimension \( n \) such that \((H_1, H_2, \ldots)\) is the sequence of \textit{Markov parameters} of the system; that is,
\[
H_i = CAi^{-1}B \quad \text{for all } i \in \mathbb{N}.
\]
Kalman \[19\] showed, using a result of Kronecker, that such a realization exists if and only if the rank of the associated infinite block Hankel matrix \( HG = [H_{i+j-1}]_{i,j \in \mathbb{N}} \) is finite. Moreover, the minimal dimension of a realization (1.1) coincides with the \textit{McMillan degree} of \( G(z) \) and is equal to the rank of \( HG \). Any two minimal realizations are similar, and furthermore a realization is minimal if and only if it is controllable and observable.

In contrast, the starting point for the partial realization theory is a \textit{finite} Laurent series
\[
G_\tau(z) = \sum_{i=1}^{\tau} H_iz^{-i}
\]
for some integer \( \tau \in \mathbb{N} \), and the goal is to find state space systems (1.1) of minimal dimension \( n \) such that
\[
C(zI_n - A)^{-1}B = \sum_{i=1}^{\tau} H_iz^{-i} + O(z^{-\tau-1})
\]
holds. Equivalently, given a \textit{finite} sequence \( H^\tau := (H_1, \ldots, H_\tau) \) of \( p \times m \) matrices, systems (1.1) of minimal dimension are sought such that
\[
H_i = CAi^{-1}B \quad \text{for } i = 1, \ldots, \tau. \quad (1.2)
\]
This is the minimal partial realization problem which was introduced and solved by Kalman in the special case of scalar Markov parameters, see [20]. For extensions to matrix Markov parameters see for example, [2,3,10,12,29,31].

The minimal partial realization problem can be viewed as the problem to extend the finite matrix sequence $H^\tau$ to an infinite sequence $(H_i)_{i \in \mathbb{N}}$ such that the associated infinite Hankel matrix is of minimal rank. This minimal rank is called the McMillan degree of $H^\tau$ and is denoted by $\mu(H^\tau)$. It is equal to the smallest possible dimension of a state space system (1.1) realizing $H^\tau$ (that is, satisfying (1.2)).

Unfortunately the McMillan degree $\mu(H^\tau)$ cannot be expressed directly as the rank of one finite Hankel matrix which is only composed of the data $H_1, \ldots, H_\tau$. Instead one has to consider the $\tau$ finite block Hankel matrices associated with the sequence $H^\tau$

$$
H(i, \tau + 1 - i) := \begin{bmatrix}
H_1 & \ldots & H_{\tau+1-i} \\
\vdots & \ddots & \vdots \\
H_i & \ldots & H_\tau
\end{bmatrix}, \quad i = 1, \ldots, \tau.
$$

The following explicit formula for the McMillan degree was derived by Gohberg et al. [12]:

$$
\mu(H^\tau) = \sum_{i+j=\tau+1} \operatorname{rank} H(i, j) - \sum_{i+j=\tau} \operatorname{rank} H(i, j).
$$

It is easily checked that the state space dimension $n$ of a minimal partial realization (1.1) satisfies

$$
n \geq \max \{ \operatorname{rank} H(i, \tau + 1 - i); \ i = 1, \ldots, \tau \}. \tag{1.4}
$$

Since not every finite Hankel matrix has a rank preserving extension to an infinite Hankel matrix, equality in (1.4) can, in general, not be achieved. For example, consider the case of the finite scalar sequence $(0, \ldots, 0, 1) \in \mathbb{K}^\tau$, where $\tau \geq 2$. The associated Hankel matrices $H(i, \tau + 1 - i), i = 1, \ldots, \tau$ have rank 1 whereas the minimal realizations of the form (1.1) are $\tau$-dimensional.

The reason for this discrepancy lies in focusing on realizations of the form (1.1). In order to obtain equality it is a natural idea to consider the partial realization problem for a larger class of systems than the class of state space systems of the form (1.1).

In this paper we extend the concept of partial realization by considering linear descriptor systems of the form

$$
Ex_{t+1} = Ax_t + Bu_t, \quad y_t =Cx_t, \quad t = 0, 1, 2, \ldots,
$$

where $E, A \in \mathbb{K}^{n \times n}, B \in \mathbb{K}^{n \times m}, C \in \mathbb{K}^{p \times n}$ and the following two conditions are satisfied:

$$
\det(zE + wA) \neq 0, \tag{1.5}
$$

$$
EA = AE. \tag{1.6}
$$
Let $S_{n,m,p}$ be the set of all quadruples $(E, A, B, C) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m} \times \mathbb{K}^{p \times n}$ which satisfy (1.5) and (1.6). We say that a descriptor system $(E, A, B, C) \in S_{n,m,p}$ is an $n$-dimensional partial descriptor realization of a given finite sequence $H^\tau = (H_1, \ldots, H_{\tau})$ of $p \times m$-matrices if

$$H_i = CE^\tau_i A^{-1} B, \quad i = 1, \ldots, \tau.$$ 

For a motivation of this formula, see Section 3 (Eqs. (3.18) and (3.19)). A similar realization concept has been considered in the theory of boundary-value descriptor systems [25,26]. The smallest $n \in \mathbb{N}$ such that there exists an $n$-dimensional partial descriptor realization of $H^\tau$ is called the generalized McMillan degree of $H^\tau$ and is denoted by $\delta(H^\tau)$. A partial descriptor realization of dimension $\delta(H^\tau)$ is called minimal.

The main objective of this paper is to characterize the minimal partial descriptor realizations of a given finite sequence $H^\tau = (H_1, \ldots, H_{\tau})$. With such a sequence we associate the central Hankel matrix $\mathcal{H}$, which is defined by

$$\mathcal{H} := \begin{cases} \begin{bmatrix} H_1 & \ldots & H_k \\ \vdots & \ddots & \vdots \\ H_k & \ldots & H_{2k-1} \end{bmatrix} & \text{if } \tau = 2k - 1 \text{ odd} \\ \begin{bmatrix} H_1 & \ldots & H_k & H_{k+1} \\ \vdots & \ddots & \vdots & \ddots \\ H_k & \ldots & H_{2k-1} & H_{2k} \end{bmatrix} & \text{if } \tau = 2k \text{ even.} \end{cases} \quad (1.7)$$

We choose this terminology, since $\mathcal{H}$ lies at the centre of the sequence of Hankel matrices

$$H(1, \tau), \quad H(2, \tau - 1), \ldots, \quad H(\tau, 1).$$

Throughout the paper we assume that the central Hankel matrix (1.7) satisfies the rank condition

$$\text{rank } \mathcal{H} < k \quad (1.8)$$

(for $\tau = 2k - 1$ or $\tau = 2k$).

This condition means that the rank of the associated Hankel matrices is small compared to the length of the sequence $H^\tau$. For scalar sequences, condition (1.8) is not necessary and we obtain a full generalization of Kalman’s results. In the general multivariable case the situation is more complicated and the development of a partial descriptor realization theory without the above condition remains an open problem.

Our approach to solving the partial descriptor realization problem can be described as follows. In analogy to the well-known decomposition of a rational function as sum of a strictly proper rational function and a polynomial there is a corresponding decomposition of finite Hankel matrices. In fact, subject to the rank condition $\text{rank } H < \min\{ M, N \}$, every $M \times N$ Hankel $H$ has a unique decomposition

$$H = H^{(1)} + H^{(2)}$$
into its regular and singular parts $H^{(1)}$, $H^{(2)}$ (see Lemma 2.10). By taking canonical regular and singular realizations of $H^{(1)}$, $H^{(2)}$ one arrives at a minimal descriptor realization for $H$ in Weierstrass form. An important point to show is then that every minimal descriptor realization is equivalent to such a form. This is done in part (iii) of Theorem 4.16.

The paper is organized as follows. In Section 2 some elements of an algebraic structure theory of finite block Hankel matrices are introduced. This material is basic for our approach to the partial descriptor realization problem and is also of independent interest. Sections 3 and 4 contain our main results. Using the well-known Weierstrass form of descriptor systems, some basic properties of minimal partial descriptor realizations are derived in Section 3. An existence result is proved (Proposition 3.11) and the relationship between the structural indices of $H$ and the controllability/observability indices of associated minimal partial realizations is discussed. A counterpart of Kalman’s realization theorem is presented in Section 4. It states that there exists a minimal partial descriptor realization of $H^T$ of dimension $\delta(H^T) = \text{rank } H$ and these realizations are uniquely determined modulo $\tau$-equivalence. Moreover, a minimal partial descriptor realization is controllable and observable.

We conclude the paper with an algorithm for constructing minimal partial descriptor realizations of sequences which satisfy the rank condition (1.8) (Section 5).

2. Regular and singular parts

In this section, we introduce a number of concepts for the structural analysis of finite matrix sequences and block Hankel matrices. Some of the definitions and results can already be found in [24]. We recall them here in order to keep the paper self-contained.

Throughout the paper let $\mathbb{K}$ denote a field of characteristic 0, and let $\mathbb{N} = \{1, 2, \ldots\}$. For integers $m, p, M, N \in \mathbb{N}$ let $\text{Hank}_{p,m}(M \times N)$ denote the vector space of all $M \times N$ block Hankel matrices of the form

$$H = \begin{bmatrix} H_1 & \cdots & H_N \\ \vdots & & \vdots \\ H_M & \cdots & H_{M+N-1} \end{bmatrix}, \quad H_j \in \mathbb{K}^{p \times m}, \quad j = 1, \ldots, M+N-1.$$

With any $H \in \text{Hank}_{p,m}(M \times N)$ we associate the matrices

$$H(i, j) = \begin{bmatrix} H_1 & \cdots & H_j \\ \vdots & & \vdots \\ H_i & \cdots & H_{i+j-1} \end{bmatrix}, \quad i, j \in \mathbb{N}, \quad i + j \leq M + N.$$

Note that these matrices are not necessarily submatrices of $H$ since either $i$ may be larger than $M$ or $j$ larger than $N$. 
In [24] we extended Iohvidov’s [17] notion of the characteristic of a scalar Hankel matrix to the block Hankel case. In order to describe this concept we need the following well-known lemma, see for example, [28, Lemma 6.6.9].

**Lemma 2.1.** Suppose that $H \in \text{Hank}_{p,m}(M \times N)$ and $r \in \mathbb{N}$ are such that $2r \leq M + N - 1$. Then there exists a (unique) infinite block Hankel matrix $G = \left[ G_{i+j-1} \right]_{i,j \in \mathbb{N}}$, $G_{i+j-1} \in \mathbb{K}^{p \times m}$ of rank $r$ such that

$$G(r, r + 1) = H(r, r + 1)$$

if and only if

$$\text{rank } H(r, r) = \text{rank } H(r + 1, r) = \text{rank } H(r, r + 1) = r. \quad (2.1)$$

**Definition 2.2.** The unique Hankel matrix $G$ appearing in Lemma 2.1 is said to be the infinite regular extension of $H(r, r + 1)$.

**Lemma 2.3.** Suppose $H \in \text{Hank}_{p,m}(M \times N)$ and there does not exist $r \in \mathbb{N}$, $2r \leq M + N - 1$, such that (2.1) is satisfied. Then

$$H_1 = H_2 = \cdots = H_{\min\{M,N\}} = 0 \quad \text{or} \quad \text{rank } H \geq \min\{M, N\}.$$

**Rank Assumption 2.4.** In the following we will restrict our analysis to finite block Hankels $H \in \text{Hank}_{p,m}(M \times N)$ whose rank satisfies

$$n := \text{rank } H < \min\{M, N\}.$$

**Definition 2.5.** For every $H \in \text{Hank}_{p,m}(M \times N)$ with rank $H < \min\{M, N\}$ the regularity index $r(H)$ is, by definition, the largest $r \in \mathbb{N}$, $r < \min\{M, N\}$ for which condition (2.1) is satisfied. If such an integer does not exist, we set $r(H) := 0$. If $r = r(H) > 0$, the infinite regular extension $G$ of $H(r, r + 1)$ is called the regular model of $H$. If $r(H) = 0$, the regular model of $H$ is, by definition, the zero matrix $G = \left[ G_{i+j-1} \right]_{i,j \in \mathbb{N}}$, $G_i = 0$, $i \in \mathbb{N}$.

**Definition 2.6.** Suppose $H \in \text{Hank}_{p,m}(M \times N)$ is a block Hankel matrix with rank $H < \min\{M, N\}$. Let $G = \left[ G_{i+j-1} \right]_{i,j \in \mathbb{N}}$ be the regular model of $H$. Then

$$s(H) = \max\{i \in \mathbb{N}; \ H_{M+N-i} \neq G_{M+N-i} \}, \quad \max \emptyset := 0$$

is called the singularity index of $H$ and the pair char $H := (r(H), s(H))$ is said to be the characteristic of $H$.

The singularity index measures the deviation of the finite Hankel matrix $H$ from its regular model.

In the scalar case the sum of the regularity and the singularity index is equal to the rank $n$ of the Hankel matrix; that is, $r + s = n$, if the Rank Assumption 2.4 is satisfied (see [22]). Such an equality does not hold, in general, for block Hankel matrices. (For example, consider a Hankel matrix $H \in \text{Hank}_{p,m}(M \times N)$ whose
elements form a matrix sequence \((0, \ldots, 0, H_{M+N-1})\) with \(2 \leq \text{rank } H_{M+N-1} < \min\{M, N\}\). However, the sum of the regularity and singularity indices is bounded above by the rank of the matrix.

**Lemma 2.7.** Suppose that \(H \in \text{Hank}_{p,m}(M \times N)\) has rank \(n < \min\{M, N\}\) and characteristic \((r, s)\). Then either \((r = n\) and \(s = 0)\) or \((0 \leq r < n\) and \(1 \leq s \leq n - r)\).

A proof of this result can be found in [24].

The next result shows that the regular model of a given finite block Hankel matrix \(H\) can be viewed as the “best strictly proper rational model” of order \(\leq \text{rank } H\).

**Proposition 2.8.** Let \(H = [H_{i+j-1}] \in \text{Hank}_{p,m}(M \times N)\) be a finite block Hankel matrix of rank \(n < \min\{M, N\}\) and characteristic \(\text{char } H = (r, s)\). Then the regular model \(G = [G_{i+j-1}]_{i,j \in \mathbb{N}}\) of \(H\) is Padé-optimal in the sense that

(i) \(G\) satisfies

\[
G_i = H_i \quad \text{for } i = 1, \ldots, M + N - 1 - s. \tag{2.2}
\]

(ii) For any other infinite Hankel \(\tilde{G} = [\tilde{G}_{i+j-1}]_{i,j \in \mathbb{N}}\) of rank less than or equal to \(n\) there exists an integer \(j \in \mathbb{N}\) with \(j < M + N - 1 - s\) and

\[
\tilde{G}_j \neq H_j.
\]

Our proof of Proposition 2.8 is based on the following observation which can be found in [28, Corollary 6.6.3].

**Lemma 2.9.** Let \(G = [G_{i+j-1}]_{i,j \in \mathbb{N}}, \ G_{i+j-1} \in \mathbb{K}^{p \times m}\) and \(\tilde{G} = [\tilde{G}_{i+j-1}]_{i,j \in \mathbb{N}}, \ \tilde{G}_{i+j-1} \in \mathbb{K}^{p \times m}\) be two infinite block Hankel matrices of finite ranks \(n_1\) and \(n_2\), respectively. The following two conditions are equivalent:

(i) \(G \neq \tilde{G}\);

(ii) there is an integer \(i \leq n_1 + n_2\) such that \(G_i \neq \tilde{G}_i\).

We prove Proposition 2.8.

**Proof.** Condition (i) is a consequence of Definition 2.6. In order to prove (ii) let \(\tilde{G} = [\tilde{G}_{i+j-1}]_{i,j \in \mathbb{N}}, \ \tilde{G} \neq G\) be an infinite Hankel matrix of rank \(\tilde{n} \leq n\) and let

\[
j := \min\{i \in \mathbb{N} : \ \tilde{G}_i \neq G_i\}. \tag{2.3}
\]

The integer \(j\) is well defined, since \(\tilde{G} \neq G\). From (2.3), Lemma 2.9 and \(\text{rank } G = r\) we obtain \(j \leq r + \tilde{n}\). Using Lemma 2.7, we deduce

\[
j \leq r + \tilde{n} \leq r + n
\leq 2n - s < 2 \min\{M, N\} - 1 - s
\leq M + N - 1 - s. \tag{2.4}
\]
From (2.4) and (2.2) it follows that $G_j = H_j$, and therefore, according to (2.3), $\tilde{G}_j \neq H_j$. □

In the next lemma a characterization of the pair char $H$ is given. For $i, j \in \mathbb{N}$, we denote by $0_{i \times j}$ the $ip \times jm$ zero matrix, where $0_{0 \times j}$ and $0_{i \times 0}$ are, by definition, empty matrices. Similarly $H(0, 0)$ is the empty matrix. Furthermore let

$$\text{Hank}_{p,m}(n, M \times N) = \{ H \in \text{Hank}_{p,m}(M \times N); \text{rank } H = n \}$$

and

$$\text{Hank}^*_p(n, M \times N) = \{ H \in \text{Hank}_p(n, M \times N); \text{rank } H(n, n) = n \}.$$ (2.5)

**Lemma 2.10.** Suppose $H \in \text{Hank}_{p,m}(n, M \times N)$ and $n < \min\{M, N\}$. The following two conditions are equivalent:

(i) $H$ has characteristic $\text{char}(H) = (r, s)$.

(ii) $H$ has a decomposition of the form

$$H = H^{(1)} + H^{(2)},$$ (2.6)

where

$$H^{(1)} \in \text{Hank}^*_{p,m}(r, M \times N),$$ (2.7)

$$H^{(2)} = \begin{bmatrix} 0_{(M-s) \times (N-s)} & 0_{(M-s) \times s} \\ 0_s \times (N-s) & \tilde{H}(s, s) \end{bmatrix} \in \text{Hank}_{p,m}(M \times N),$$ (2.8)

$$\tilde{H}(s, s) = \begin{bmatrix} 0 & \ldots & 0 & \tilde{H}_{M+N-s} \\ \vdots & \ddots & \vdots \\ \tilde{H}_{M+N-s} & \ldots & H_{M+N-2} & \tilde{H}_{M+N-1} \end{bmatrix}, \quad \tilde{H}_{M+N-s} \neq 0,$$ (2.9)

and

$$\text{rank } H^{(2)} = \text{rank } \tilde{H}(s, s) = n - r \geq s.$$ (2.10)

(If $s = 0$, the matrix $\tilde{H}(s, s)$ is absent and $H^{(2)} = 0$.)

Moreover, the decomposition of the form (2.6)–(2.10) is uniquely determined and satisfies

$$\text{rank } H = \text{rank } H^{(1)} + \text{rank } H^{(2)}.$$ (2.11)

**Proof.** (i) $\implies$ (ii): Suppose (i) and let $G$ be the regular model of $H$. We set $H^{(1)} := G(M,N)$ and $H^{(2)} := H - G(M,N)$. Then (2.7)–(2.9) is a consequence of Definitions 2.5 and 2.6. It is not difficult to verify (see [24, proof of Theorem 2.9]) that there are lower and upper triangular matrices $C$ and $D$ with unit diagonal of sizes $pM \times pM$ and $mN \times mN$, respectively, such that

$$CH^{(1)}D = \begin{bmatrix} H^{(1)}(r, r) & 0_{r \times (N-r)} \\ 0_{(M-r) \times r} & 0_{(M-r) \times (N-r)} \end{bmatrix},$$ (2.12)
\[ C H^{(2)} D = \begin{bmatrix} 0_{(M-s) \times (N-s)} & 0_{(M-s) \times s} \\ 0_{s \times (N-s)} & U(s, s) \end{bmatrix}, \]

where \( U(s, s) \in \text{Hank}_{p, m}(s \times s) \). Since \( s \leq n - r \) (Lemma 2.7), we conclude from (2.12) that (2.11) holds. Whence (2.10) follows.

(ii) \( \implies \) (i): Assuming (ii) we note that (2.10) implies \( 2r + 1 \leq 2n - 2s + 1 < M + N - 2s \leq M + N - s \); hence

\[ H(r + 1, r + 1) = H^{(1)}(r + 1, r + 1). \]  

Using (2.7) and (2.13) it follows that

\[ \text{rank } H(r, r) = \text{rank } H(r + 1, r) = \text{rank } H(r + 1, r + 1) = r. \]

In order to establish that the integer \( r \) is the regularity index of \( H \), it now suffices to check that

\[ \text{rank } H(j, j + 1) > \text{rank } H(j, j) \quad \text{for } M + N - s \leq 2j - 1 \leq 2n - 1. \]  

As before, there exist lower and upper triangular matrices \( C \) and \( D \) with unit diagonal of sizes \( pM \times pM \) and \( mN \times mN \), respectively, such that

\[ CH^{(1)} D = \begin{bmatrix} H^{(1)}(r, r) & 0_{r \times (N-r)} \\ 0_{(M-r) \times r} & 0_{(M-r) \times (N-r)} \end{bmatrix}, \]

\[ CH^{(2)} D = \begin{bmatrix} 0_{(M-s) \times (N-s)} & 0_{(M-s) \times s} \\ 0_{s \times (N-s)} & \tilde{H}(s, s) \end{bmatrix}, \]

where \( \tilde{H}(s, s) \) is the matrix (2.9). Thanks to (2.10) and \( j \leq n \), it follows that

\[ r + j \leq 2n - s < M + N - s. \]

For \( M + N - s \leq 2j - 1 \leq 2n - 1 \), we obtain from (2.6), (2.15) and (2.16) that

\[ (CHD)(j, j + 1) = \begin{bmatrix} H^{(1)}(r, r) & 0_{r \times (j-r+1)} \\ 0_{(j-r) \times r} & V \end{bmatrix}, \]

where

\[ V = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \tilde{H}_{M+N-s} & \tilde{H}_{M+N-s} \neq 0 \end{bmatrix}, \]

This implies (2.14), since \( C \) and \( D \) are lower and upper triangular matrices with unit diagonal. Let \( G \) be the regular model of \( H \) (see Definition 2.5). Since \( G(M, N) = H^{(1)} \) it follows from conditions (2.6)–(2.10) that \( \max \{ i \in \mathbb{N} ; H_{M+N-i} \neq G_{M+N-i} \} = s \); that is, \( H \) has singularity index \( s \).
The uniqueness of the decomposition (2.6)–(2.10) follows from (2.13) and the uniqueness of the infinite regular extension of \( H(r, r + 1) \). Finally, (2.11) is a direct consequence of (2.6)–(2.10).

**Definition 2.11.** Suppose that \( H \in \text{Hank}_{p,m}(n, M \times N) \), \( n < \min\{M, N\} \) and \( \text{char} \, H = (r, s) \). The unique Hankel matrices \( H^{(1)} \in \text{Hank}^*_p(r, M \times N) \) and \( H^{(2)} \in \text{Hank}_{p,m}(n-r, M \times N) \) appearing in the decomposition (2.6) are said to be the **regular** and the **singular parts** of \( H \), respectively.

**Remark 2.12.** Note that the decomposition of \( H \) into a regular and a singular part is based on the Rank Assumption 2.4. Without this assumption Lemma 2.10 does not hold. In fact, if \( p, m \geq 2 \), the regular part, \( H^{(1)} \) will be trivial (that is, \( H^{(1)} = 0 \)) for generic Hankels \( H \in \text{Hank}_{p,m}(M \times N) \), and \( H^{(2)} = H \) will not have the structure (2.8), (2.9).

In order to express the rank of a block Hankel matrix via the characteristic as in the scalar case it is necessary to introduce a multivariable version of the concept of characteristic. Let us begin by introducing the following terminology.

**Definition 2.13.** Given a block Hankel matrix \( H \in \text{Hank}_{p,m}(M \times N) \), a selection of row indices of \( H \)

\[ \alpha = \{\alpha_1, \ldots, \alpha_n\}, \quad 1 \leq \alpha_1 < \ldots < \alpha_n \leq pM, \]

is said to be **saturated from below** if and only if

\[ i \in \alpha \quad \text{and} \quad i > p \implies i - p \in \alpha, \]

and **saturated from above** if and only if

\[ i \in \alpha \quad \text{and} \quad i \leq (M-1)p \implies i + p \in \alpha, \]

respectively. An analogous terminology is used for selections of column indices of \( H \).

The following result can be found in [24].

**Lemma 2.14.** Suppose \( H \in \text{Hank}_{p,m}(n, M \times N) \), \( n < \min\{M, N\} \) and \( \text{char} \, H = (r, s) \). Let \( \alpha = \{\alpha_1, \ldots, \alpha_n\} \) be the selection consisting of the indices of the first \( n \) linearly independent rows of \( H \) written in increasing order, and let

\[ \alpha^{\text{reg}} := \{\alpha_1, \ldots, \alpha_r\}, \quad \alpha^{\text{sing}} := \{\alpha_{r+1}, \ldots, \alpha_n\}. \]

Then the index lists \( \alpha^{\text{reg}} \) and \( \alpha^{\text{sing}} \) are saturated from below and from above, respectively. In particular, there exist uniquely determined nonnegative integers \( a_1, \ldots, a_p \) and \( y_1, \ldots, y_p \) with sum \( a_1 + \cdots + a_p = r \) and \( y_1 + \cdots + y_p = n - r \), respectively, such that
\[ \alpha_{\text{reg}} = \bigcup_{i=1}^{p} \{i, i + p, \ldots, i + (a_i - 1)p\}, \]  
(2.17)

\[ \alpha_{\text{sing}} = \bigcup_{i=1}^{p} \{i + (M - y_i)p, \ldots, i + (M - 2)p, i + (M - 1)p\}. \]  
(2.18)

Similarly, there exists index lists \( \beta_{\text{reg}} \) and \( \beta_{\text{sing}} \) saturated from below and from above, respectively, and uniquely determined nonnegative integers \( b_1, \ldots, b_m \) and \( z_1, \ldots, z_m \) with sum \( b_1 + \cdots + b_m = r \) and \( z_1 + \cdots + z_m = n - r \), respectively, such that

\[ \beta_{\text{reg}} = \bigcup_{i=1}^{m} \{i, i + m, \ldots, i + (b_i - 1)m\}, \]  
(2.19)

\[ \beta_{\text{sing}} = \bigcup_{i=1}^{m} \{i + (N - z_i)m, \ldots, i + (N - 2)m, i + (N - 1)m\}. \]  
(2.20)

Note that if \( r = 0 \) (respectively, \( s = 0 \)) in the previous lemma then \( \alpha_{\text{reg}} = \emptyset \) (respectively, \( \alpha_{\text{sing}} = \emptyset \)) and \( a \) (respectively, \( y \)) is the zero vector.

Under the hypotheses of Lemma 2.14 the index list \( \alpha_{\text{reg}} \) (respectively, \( \alpha_{\text{sing}} \)) is precisely the index list of the first \( r \) (respectively, \( n - r \)) linearly independent rows of the regular (respectively, singular) part of \( H \). This motivates the following terminology.

**Definition 2.15.** The vectors \( a = (a_1, \ldots, a_p) \) and \( y = (y_1, \ldots, y_p) \) defined by (2.17) and (2.18) are said to be the regular and the singular row index lists of \( H \), respectively. The regular and the singular column index lists \( b = (b_1, \ldots, b_m) \) and \( z = (z_1, \ldots, z_m) \) of \( H \) are defined analogously.

The pairs \((a_1, y_1), \ldots, (a_p, y_p)\) may be viewed as row characteristics and the pairs \((b_1, z_1), \ldots, (b_m, z_m)\) as column characteristics of the rectangular \( p \times m \) block Hankel matrix \( H \). Together they form a vector valued characteristic of \( H \)

\[ \text{CHAR}(H) = ((a_1, y_1), \ldots, (a_p, y_p); (b_1, z_1), \ldots, (b_m, z_m)) \]  
(2.21)

which gives a more detailed information about the structure of \( H \) than the overall characteristic \( \text{char} H = (r(H), s(H)) \). There are reasons to consider the vector characteristic as the proper generalization of Iohvidov’s concept to block Hankel matrices. In fact, we obtain -- as a direct consequence of Lemma 2.14 -- the following result which shows that the rank of \( H \) can be calculated from the row (respectively,
column) characteristics whereas this is not possible from the overall characteristic of $H$.

**Corollary 2.16** [24]. Suppose $H \in \text{Hank}_{p,m}(n, M \times N)$, $n < \min\{M, N\}$ and $\text{CHAR}(H)$ is given by (2.21). Then

$$\text{rank } H = \sum_{i=1}^{p} (a_i + y_i) = \sum_{j=1}^{m} (b_j + z_j).$$

This corollary may be viewed as the proper extension of Iohvidov’s Fundamental Rank Theorem ([17, Theorem 11.1]) to block Hankel matrices.

We end this section by transferring some of the constructions introduced above for Hankel matrices to the underlying sequence of their block entries. For this it is necessary to show that the corresponding constructions yield the same result when applied to any of the Hankel matrices $H(i, \tau + 1 - i)$, $i = 1, \ldots, \tau$, satisfying the Rank Assumption 2.4.

**Lemma 2.17.** Given any sequence $H^\tau = (H_1, \ldots, H_\tau) \in (\mathbb{K}^{p \times m})^\tau$, if one of the associated Hankel matrices $H(i, \tau + 1 - i)$, $i = 1, \ldots, \tau$, say $H := H(M, \tau + 1 - M)$, satisfies the rank condition

$$n := \text{rank } H < \min\{M, \tau + 1 - M\},$$

then the associated central Hankel matrix (1.7) satisfies

$$\text{rank } \mathcal{H} = n < k \quad \text{and} \quad \text{char } \mathcal{H} = \text{char } H,$$

where $k = [\frac{\tau + 1}{2}]$ is the largest integer $\leq \frac{\tau + 1}{2}$.

**Proof.** Let us establish the lemma for the case when $M \leq \tau + 1 - M$. In the case that $N \leq \tau + 1 - N$ the proof follows the same lines. We may assume that

$$M < k,$$

(2.23)

since for $M = k$ there is nothing to prove. Let $(r, s) = \text{char } H$ and observe that Lemma 2.7 implies

$$r + s \leq n.$$

(2.24)

According to Lemma 2.10 we have a decomposition of $H$ into its regular and singular parts

$$H = H^{(1)} + H^{(2)}$$

$$= \begin{bmatrix} H_1^{(1)} & \cdots & H_{\tau+1-M}^{(1)} \\ \vdots & \ddots & \vdots \\ H_M^{(1)} & \cdots & H_{\tau}^{(1)} \end{bmatrix}$$
\[
\begin{bmatrix}
0 & \ldots & 0 & \ldots & 0 \\
\vdots & & \vdots \\
0 & \ldots & 0 & \ldots & H_{\tau+1-s}^{(2)} \\
\vdots & & \vdots \\
0 & \ldots & H_{\tau+1-s}^{(2)} & \ldots & H_{\tau}^{(2)}
\end{bmatrix} +
\begin{bmatrix}
0 & \ldots & 0 & \ldots & 0 \\
\vdots & & \vdots \\
0 & \ldots & 0 & \ldots & H_{\tau+1-s}^{(2)} \\
\vdots & & \vdots \\
0 & \ldots & H_{\tau+1-s}^{(2)} & \ldots & H_{\tau}^{(2)}
\end{bmatrix},
\tag{2.25}
\]

\[
\begin{aligned}
\text{rank } H^{(1)} &= \text{rank } H^{(1)}(r, r) = r, \\
\text{rank } H^{(2)} &= n - r, \quad H_{\tau+1-s}^{(2)} \neq 0.
\end{aligned}
\tag{2.26}
\]

Since \( k > M > r \) and \( k < \tau + 1 - s \), (2.25) induces the following decomposition of the central Hankel:

\[
H = H^{(1)} + H^{(2)} =
\begin{bmatrix}
H_{1}^{(1)} & \ldots & H_{l}^{(1)} \\
\vdots & & \vdots \\
H_{k}^{(1)} & \ldots & H_{\tau}^{(1)}
\end{bmatrix} +
\begin{bmatrix}
0 & \ldots & 0 & \ldots & 0 \\
\vdots & & \vdots \\
0 & \ldots & 0 & \ldots & H_{\tau+1-s}^{(2)} \\
\vdots & & \vdots \\
0 & \ldots & H_{\tau+1-s}^{(2)} & \ldots & H_{\tau}^{(2)}
\end{bmatrix},
\tag{2.27}
\]

where \( l = k \) if \( \tau = 2k - 1 \), and \( l = k + 1 \) if \( \tau = 2k \) (see (1.7)). Now by (2.25)–(2.27)

\[
\begin{aligned}
\text{rank } H^{(1)} &= \text{rank } H^{(1)}(r, r) = r, \\
\text{rank } H^{(2)} &= n - r, \quad H_{\tau+1-s}^{(2)} \neq 0.
\end{aligned}
\tag{2.28}
\]

Taking into account that \( M \leq \tau + 1 - M \), it follows that

\[
r^{(2,24)} \leq (2.22) < (2.23) \leq \frac{n - s}{\tau - s} < k - s,
\tag{2.29}
\]

and hence

\[
\text{rank } H^{(2,27)-(2,29)} = \text{rank } H^{(1)} + \text{rank } H^{(2)} = n \leq (2.22) < (2.23) \leq k.
\tag{2.30}
\]

Finally, using Lemma 2.10 and (2.27)–(2.30), we obtain char \( H = (r, s) = \text{char } H \).

**Remark 2.18.** Suppose \( H := H(i, \tau + 1 - i) \) and \( \tilde{H} := H(j, \tau + 1 - j) \) are two Hankel matrices associated with a given sequence \( H^{\tau} = (H_1, \ldots, H_{\tau}) \). If \( H \) and \( \tilde{H} \) satisfy the Rank Assumption 2.4, then by Lemma 2.17 they are of the same rank and have the same characteristic. Moreover their regular models coincide (see Definition 2.5) and induce by (2.6)–(2.11) the same decomposition of the sequence \( H^{\tau} \) as the central Hankel \( H \)

\[
(H_1, \ldots, H_{\tau}) = (H_1^{(1)}, \ldots, H_{\tau}^{(1)}) + (H_1^{(2)}, \ldots, H_{\tau}^{(2)}).
\tag{2.31}
\]
Definition 2.19.
(i) A sequence $H^\tau = (H_1, \ldots, H_\tau) \in (\mathbb{K}^{p \times m})^\tau$ satisfies the rank assumption if one of the associated Hankel matrices or, equivalently, the central Hankel $\mathcal{H}$ satisfies the Rank Assumption 2.4.

(ii) Let $H^\tau$ be a sequence which satisfies the rank assumption. $H^\tau$ has regularity index $r$, singularity index $s$ and regular model $G = [G_{i+j-1}]_{i,j \in \mathbb{N}}$ if $r$, $s$ and $G$ are the regularity index, the singularity index and the regular model of $\mathcal{H}$, respectively. The sequences $(H_1^{(1)}, \ldots, H_\tau^{(1)})$, $(H_1^{(2)}, \ldots, H_\tau^{(2)})$ appearing in (2.31) are said to be the regular and the singular part of $H^\tau$, respectively, and $\text{rank } H^{(1)}, \text{rank } H^{(2)}$ are referred to as the regular and the singular degree of $H^\tau$, respectively.

Note that the regular part $(H_1^{(1)}, \ldots, H_\tau^{(1)})$ of $H^\tau$ coincides with the subsequence $(G_1, \ldots, G_\tau)$ of the regular model $[G_{i+j-1}]_{i,j \in \mathbb{N}}$. Moreover, the regular degree of $H^\tau$ coincides with the regularity index, while the singular degree is in general different from the singularity index.

Lemma 2.20. Suppose $H^\tau = (H_1, \ldots, H_\tau) \in (\mathbb{K}^{p \times m})^\tau$ satisfies the rank assumption and has regular and singular degree $r$ and $t$, respectively, then

$$\text{rank } H(M, \tau + 1 - M) \leq r + t$$

for all $M = 1, \ldots, \tau$. In particular, if $\mathcal{H}$ is the associated central Hankel matrix, then

$$\text{rank } \mathcal{H} = \max \{ \text{rank } H(i, \tau + 1 - i); i = 1, \ldots, \tau \} = r + t.$$  \hfill (2.32)

Proof. Let $M \in \{1, \ldots, \tau\}$ and $H := H(M, \tau + 1 - M)$. The decomposition (2.31) of $H^\tau$ into its regular part $(H_1^{(1)}, \ldots, H_\tau^{(1)})$ and singular part $(H_1^{(2)}, \ldots, H_\tau^{(2)})$ induces the following decomposition of $H$:

$$H = H^{(1)} + H^{(2)} = \left[ H_1^{(1)} \right]_{i=1, j=1}^{M, \tau+1-M} + \left[ H_1^{(2)} \right]_{i=1, j=1}^{M, \tau+1-M}.$$  

Since $H^{(1)}$ is a submatrix of the regular model $G := [G_{i+j-1}]_{i,j \in \mathbb{N}}$ of $H^\tau$ and rank $G = r$, it follows that:

$$\text{rank } H^{(1)} \leq r.$$  

Similarly, let $\hat{H}^{(2)}$ be the $M \times (\tau + 1 - M)$ Hankel matrix associated with the reflected sequence $(H_\tau^{(2)}, H_{\tau-1}^{(2)}, \ldots, H_1^{(2)})$ and $\hat{G}$ the infinite Hankel matrix associated with the infinite sequence $(H_\tau^{(2)}, H_{\tau-1}^{(2)}, \ldots, H_1^{(2)}, 0, 0, \ldots)$. Then $\hat{H}^{(2)}$ is a submatrix of $\hat{G}$ and

$$\text{rank } \hat{G} = \text{rank } H^{(2)} = \text{rank } \mathcal{H}^{(2)} = t.$$
Hence rank $H^{(2)} = \text{rank} \hat{H}^{(2)} \leq \text{rank} \hat{G} = t$ and
\[
\text{rank} \; H \leq \text{rank} \; H^{(1)} + \text{rank} \; H^{(2)} \leq r + t.
\]
By (2.11) and definition of $r, t$ we have $\mathcal{H} = \text{rank} \; \mathcal{H}^{(1)} + \text{rank} \; \mathcal{H}^{(2)} = r + t$, and so (2.32) follows. □

If the rank assumption is not satisfied, then (2.32) does, in general, not hold.

**Example 2.21.** Let $\tau = 3$ and
\[
H^\tau = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
then rank $H(1, 3) = \text{rank} \; H^\tau = 3$, although rank $\mathcal{H} = \text{rank} \; H(2, 2) = 2$.

### 3. Partial descriptor realizations

First we briefly recall some basic facts about the realization theory of linear descriptor systems of the form
\[
\begin{align*}
\dot{x}_t &= Ax_t + Bu_t, \\
y_t &= Cx_t, & t &= 0, 1, 2, \ldots,
\end{align*}
\]
where $E, A \in \mathbb{K}^{n \times n}, B \in \mathbb{K}^{n \times m}, C \in \mathbb{K}^{p \times n}$ and the pencil $(E, A)$ is regular; that is, \(\det(zE + wA) \neq 0\).

Two descriptor systems $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$ and $(E, A, B, C)$ of the form (3.1), (3.2) are said to be *restricted system equivalent* if $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) = (SET^{-1}, SAT^{-1}, SB, CT^{-1})$ for some nonsingular matrices $S, T$. By the Hautus criterion for descriptor systems [7] a system (3.1), (3.2) is *controllable* if and only if
\[
\text{rank} [\alpha E - \beta A, B] = n \quad \text{for all } \alpha, \beta \in \overline{\mathbb{K}}, \; (\alpha, \beta) \neq (0, 0)
\]
(3.3)
where $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$) and *observable* if and only if
\[
\text{rank} \begin{pmatrix}
\alpha E - \beta A \\
C
\end{pmatrix} = n \quad \text{for all } \alpha, \beta \in \overline{\mathbb{K}}, \; (\alpha, \beta) \neq (0, 0).
\]
(3.4)
A descriptor system (3.1), (3.2) is said to be an $n$-dimensional *descriptor realization* of a given rational matrix $G(z) \in \mathbb{K}^{p \times m}(z)$ if
\[
G(z) = C(zE - A)^{-1}B.
\]
(3.5)
A descriptor realization of $G(z)$ is said to be *minimal* if it is of minimal dimension amongst all descriptor realizations of $G(z)$. Recall that a rational matrix $G(z) \in \mathbb{K}^{p \times m}(z)$
\( K^{p \times m}(z) \) is said to be proper if \( G(\infty) = \lim_{z \to \infty} G(z) \) exists in \( K^{p \times m} \). \( G(z) \) is said to be strictly proper if \( G(\infty) = 0 \). The McMillan degree \( \mu(G(z)) \) of a strictly proper rational matrix \( G(z) \in K^{p \times m}(z) \) is by definition the degree of the least common denominator of all minors of \( G(z) \). In classical realization theory it is shown that the dimension of a minimal state space realization of \( G(z) \) is equal to \( \mu(G(z)) \). For general rational matrices (not necessarily strictly proper) there are different concepts of McMillan degree in the literature, see for example, Verghese et al. [30] and Cobb [8]. In this paper we follow [8] and define the McMillan degree of a rational matrix \( G(z) \in K^{p \times m}(z) \) by

\[
\delta(G(z)) = \mu(G^{(1)}(z)) + \mu(z^{-1}G^{(2)}(z^{-1})), \tag{3.6}
\]

where \( G^{(1)}(z) \) and \( G^{(2)}(z) \) are the strictly proper and polynomial parts of \( G(z) \), respectively. With this definition one obtains the following counterpart of Kalman’s Realization Theorem for regular systems (see Section 1).

**Theorem 3.1** [9].

(i) Any rational matrix \( G(z) \in K^{p \times m}(z) \) has a descriptor realization of dimension \( \delta(G(z)) \).

(ii) A descriptor realization is minimal if and only if it is controllable and observable.

(iii) Minimal descriptor realizations \((E, A, B, C)\) and \((\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})\) of \( G(z) \) are restricted system equivalent.

(iv) The McMillan degree (3.6) is precisely the minimal dimension of all descriptor realizations of \( G(z) \).

In classical realization theory Laurent expansions and Hankel matrices play a vital role. A state space system realizes a strictly proper rational matrix \( G(z) \) if and only if its Markov parameters coincide with the coefficients of the Laurent expansion of \( G(z) \) at \( \infty \), which form the entries of the associated Hankel matrix \( H_G \). This reformulation of the realization concept in terms of Laurent expansions and Hankel matrices is fundamental for Kalman’s partial realization problem.

We will now develop a similar framework for descriptor realizations and in this way prepare the ground for studying partial realizations by descriptor systems. Let \( G(z) \) be an arbitrary \( p \times m \) rational matrix and consider its Laurent expansion

\[
G(z) = G^{(1)}(z) + G^{(2)}(z) = \sum_{i=1}^{\infty} H_{i}^{(1)}z^{-i} + \sum_{i=1}^{\nu} H_{i}^{(2)}z^{-i}, \tag{3.7}
\]

where \( G^{(1)}(z) \) and \( G^{(2)}(z) \) are the strictly proper and polynomial parts of \( G(z) \), respectively, and \( \nu = \nu(G(z)) \) is the index of the last nonzero coefficient in the expansion of \( G^{(2)}(z) \), if \( G^{(2)}(z) \neq 0 \). We define \( \nu = 0 \), if \( G^{(2)}(z) = 0 \). The Laurent expansion (3.7) gives rise to two infinite block Hankel matrices of finite ranks,
and
\[ H_G^{(1)} = \begin{bmatrix}
H_1^{(1)} & H_2^{(1)} & H_3^{(1)} & \cdots \\
H_2^{(1)} & H_3^{(1)} & & \\
H_3^{(1)} & & & \\
& & & \\
\vdots & & & \\
\end{bmatrix} \quad (3.8) \]

and
\[ H_G^{(2)} = \begin{bmatrix}
H_1^{(2)} & H_2^{(2)} & H_3^{(2)} & \cdots \\
H_2^{(2)} & H_3^{(2)} & & \\
H_3^{(2)} & & & \\
& & & \\
\vdots & & & \\
\end{bmatrix}, \quad \text{where } H_i^{(2)} := 0 \text{ for } i > \nu. \quad (3.9) \]

Conversely, any pair of finite rank Hankel matrices of the form (3.8), (3.9) defines a rational matrix (3.7). It turns out that there is a bijective correspondence between the set of rational matrices (3.7) and the set of pairs of finite rank infinite block Hankel matrices of the form (3.8), (3.9). Moreover, since \( z^{-1}G^{(2)}(z^{-1}) \) is strictly proper and the McMillan degree of a strictly proper rational matrix is equal to the rank of the associated infinite Hankel matrix, the McMillan degree (3.6) is given by
\[ \delta(G(z)) = \text{rank } H_G^{(1)} + \text{rank } H_G^{(2)}. \quad (3.10) \]

**Definition 3.2.** A descriptor system (3.1) is said to be in Weierstrass form if
\[
E = \begin{bmatrix}
I_r & 0 \\
0 & E_2
\end{bmatrix}, \quad A = \begin{bmatrix}
A_1 & 0 \\
0 & I_{n-r}
\end{bmatrix}, \\
B = \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}, \quad C = [C_1 \ C_2], \quad (3.11)
\]

where \( 0 \leq r \leq n \) and \( E_2 \in \mathcal{K}^{(n-r)\times(n-r)} \) is nilpotent. The least integer \( \nu \) satisfying \( E_2^\nu = 0 \) is referred to as the index of nilpotency of the matrix \( E_2 \). The regular system \((I_r, A_1, B_1, C_1)\) is called the slow subsystem and the descriptor system \((E_2, I_{n-r}, B_2, C_2)\) the fast subsystem of \((E, A, B, C)\).

Note that every descriptor system in Weierstrass form (3.11) is an element of the system space
\[
S_{n,m,p} = \{ (E, A, B, C) \in \mathcal{K}^{n\times n} \times \mathcal{K}^{n\times n} \times \mathcal{K}^{n\times m} \times \mathcal{K}^{p\times n}; \det(zE + wA) \neq 0, \ EA = AE \}
\]

(see Section 1). Furthermore, the following properties are well-known (see for example, [9]).
Lemma 3.3.  
(i) Each descriptor system \((\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \in \mathcal{S}_{n,m,p}\) is restricted system equivalent to a descriptor system in Weierstrass form (3.11) with \(r = \text{deg} \det(\tilde{E} - \tilde{A})\).
(ii) If \((\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}), (E, A, B, C) \in \mathcal{S}_{n,m,p}\) are restricted system equivalent and both are in Weierstrass form, then the slow (fast) subsystem of \((\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})\) is similar to the slow (fast) subsystem of \((E, A, B, C)\).
(iii) A descriptor system \((E, A, B, C) \in \mathcal{S}_{n,m,p}\) in Weierstrass form is controllable and observable if and only if both its slow and fast subsystems are controllable and observable.
(iv) The transfer function of a descriptor system \((E, A, B, C) \in \mathcal{S}_{n,m,p}\) in Weierstrass form has Laurent expansion
\[
C(zE - A)^{-1}B = \sum_{i=1}^{\infty} C_1 A_1^{i-1} B_1 z^{-i} - \sum_{i=1}^{v} C_2 E_2^{i-1} B_2 z^{i-1},
\]
where \(v\) is the index of nilpotency of \(E_2\).

Lemma 3.4. Let \(G(z) = \sum_{i=1}^{v} H_i z^{i-1}\) with \(H_v \neq 0\) be a \(p \times m\) polynomial matrix. If \((E, I, B, C)\) is a minimal descriptor realization of \(G(z)\), or equivalently, \((E, B, C)\) is a minimal state space realization of \(-z^{-1}G(z^{-1})\), then \(E\) is nilpotent and \(v\) is the index of nilpotency of \(E\).

Proof. Since \(-z^{-1}G(z^{-1})\) has only poles at zero, \(E\) has only zero eigenvalues. Thus, \(E\) is nilpotent. From
\[
\sum_{i=1}^{\infty} C E^{i-1} B z^{-i} = C(zI - E)^{-1}B = -z^{-1}G(z^{-1})
\]
\[
= -\sum_{i=1}^{v} H_i z^{-i}, \quad H_v \neq 0
\]
(3.12)
it follows that
\[
CE^{v-1}B \neq 0, \quad CE^i B = 0, \quad i \geq v. \quad (3.13)
\]
Using (3.13), we observe that the infinite block Hankel matrix
\[
\begin{bmatrix}
C \\
CE \\
CE^2 \\
\vdots
\end{bmatrix}
E^v \begin{bmatrix}
B, EB, EB^2, \ldots
\end{bmatrix}
\]
(3.14)
is the zero matrix. Since \((E, B, C)\) is minimal, according to the classical realization theory, the observability and the controllability matrices in factorization (3.14) both have full rank; hence \(E^v = 0\). Since \(E^{v-1} \neq 0\), the index of nilpotency of \(E\) coincides with \(v\). □
Lemma 3.5. Let $G(z)$ be a $p \times m$ rational matrix as in (3.7) with associated Hankel matrices (3.8), (3.9), and let $(E, A, B, C) \in S_{n,m,p}$ be a descriptor system in Weierstrass form. The following statements are equivalent:
(i) $(E, A, B, C)$ is a (minimal) descriptor realization of $G(z)$.
(ii) The slow subsystem $(I_r, A_1, B_1, C_1)$ is a (minimal) state space realization of $G^{(1)}(z)$ and the fast subsystem $(E_2, I_{n-r}, B_2, C_2)$ is a (minimal) descriptor realization of $G^{(2)}(z)$.

\[ H_i^{(1)} = C_1 A_1^{i-1} B_1, \quad i \in \mathbb{N}, \quad (\text{rank } H_i^{(1)} = r), \]  
\[ H_i^{(2)} = -C_2 E_2^{i-1} B_2, \quad 1 \leq i \leq v, \quad (\text{rank } H_i^{(2)} = n - r). \]  

Moreover, if $(E, A, B, C)$ is a minimal descriptor realization of $G(z)$ the index of nilpotency of $E_2$ is equal to $v$.

Proof. (i) $\iff$ (ii): By (3.11), $G(z) = C(zE - A)^{-1} B$ is equivalent to $G^{(1)}(z) = C_1(zI_r - A_1)^{-1} B_1$ and $G^{(2)}(z) = C_2(zE_2 - I_{n-r})^{-1} B_2$. From part (ii) of Theorem 3.1 and part (iii) of Lemma 3.3 it follows that $(E, A, B, C)$ is minimal if and only if its slow and fast subsystems are minimal.

(ii) $\iff$ (i): According to the classical realization theory, (3.15) holds if and only if $G^{(1)}(z) = C_1(zI_r - A_1)^{-1} B_1$ (and $(I_r, A_1, B_1, C_1)$ is minimal). Similarly, (3.16) holds if and only if $-z^{-1} G^{(2)}(z^{-1}) = C_2(zI_{n-r} - E_2)^{-1} B_2$ (see (3.12), (3.13)) and $(I_{n-r}, E_2, B_2, C_2)$ is minimal). Since $-z^{-1} G^{(2)}(z^{-1}) = C_2(zI_{n-r} - E_2)^{-1} B_2$ is equivalent to $G^{(2)}(z) = C_2(zE_2 - I_{n-r})^{-1} B_2$, the result follows.

The last assertion is a consequence of Lemma 3.4. \hfill \Box

Part (iii) of Lemma 3.5 indicates how to define the concept of “partial descriptor realization in Weierstrass form”. Suppose $H^r = (H_1, \ldots, H_\tau) \in (\mathbb{K}^{p \times m})^r$ is a finite sequence which satisfies the rank assumption (see Definition 2.19), and let

\[(H_1, \ldots, H_\tau) = (H_1^{(1)}, \ldots, H_\tau^{(1)}) + (0, \ldots, 0, H_1^{(2)}, \ldots, H_\tau^{(2)}),\]

\[H_{\tau+1-s}^{(2)} \neq 0,\]

be the decomposition of $H^r$ into its regular and singular parts, where $s$ denotes the singularity index of $H^r$. Let $H^{(1)} = [H_{i+j-1}^{(1)}]_{i,j \in \mathbb{N}}$ be the regular model of $H^r$ (see Definitions 2.5 and 2.19). With $H^r$ we associate the rational matrix

\[G_{H^r}(z) := G^{(1)}_{H^r}(z) + G^{(2)}_{H^r}(z) = \sum_{i=1}^{\infty} H_i^{(1)} z^{-i} - \sum_{i=1}^{s} H_{\tau+1-i}^{(2)} z^{i-1}.\]  

We say that a descriptor system $(E, A, B, C)$ in Weierstrass form is a partial descriptor realization of $H^r$ if

\[H_i^{(1)} = C_1 A_1^{i-1} B_1, \quad H_{\tau+1-i}^{(2)} = C_2 E_2^{i-1} B_2, \quad 1 \leq i \leq \tau\]  

\[\text{(3.18)}\]
is satisfied. Note that any state space system \((I_n, A, B, C)\) is a descriptor system in Weierstrass form. Therefore the new concept extends Kalman’s original definition of partial realizations which was developed for regular state space systems. Moreover, observe that condition (3.18) is equivalent to

\[
H_i = CE^{\tau-i} A^{i-1} B, \quad 1 \leq i \leq \tau.
\]  

(3.19)

In order to get rid of the restriction to descriptor systems in Weierstrass form we introduce the following definition.

**Definition 3.6.** A linear system \((E, A, B, C) \in S_{n,m,p}\) is called an \(n\)-dimensional partial descriptor realization of \(H^\tau = (H_1, \ldots, H_\tau) \in (\mathbb{K}^{p \times m})^\tau\) if (3.19) is satisfied. If \((E, A, B, C) \in S_{n,m,p}\) is of minimal dimension with this property, \((E, A, B, C)\) is said to be a minimal partial descriptor realization of \(H^\tau\). In this case its dimension is called the generalized McMillan degree of \(H^\tau\), which we denote by \(\delta(H^\tau)\).

The next proposition justifies Definition 3.6.

**Proposition 3.7.** Suppose \(H^\tau = (H_1, \ldots, H_\tau) \in (\mathbb{K}^{p \times m})^\tau\) satisfies the rank assumption and has regular and singular degree \(r\) and \(t\), respectively. For any \((E, A, B, C) \in S_{n,m,p}\) in Weierstrass form the following statements are equivalent:

(i) \((E, A, B, C)\) is a (minimal) partial descriptor realization of \(H^\tau\).

(ii) \((E, A, B, C)\) is a (minimal) descriptor realization of \(GH^\tau(z)\) defined by (3.17).

(iii) The subsystems \((A_1, B_1, C_1)\) and \((E_2, B_2, C_2)\) are (minimal) state space realizations of \(G^{(1)}_{H^\tau}(z)\) and \(-z^{-1}G^{(2)}_{H^\tau}(z^{-1})\), respectively.

Moreover, the generalized McMillan degree of \(H^\tau\) coincides with the McMillan degree of \(GH^\tau\) and

\[
\delta(H^\tau) = \delta(G_{H^\tau}(z)) = r + t, \quad r = \mu(G^{(1)}_{H^\tau}(z)), \quad t = \mu(-z^{-1}G^{(2)}_{H^\tau}(z^{-1})).
\]

(3.20)

**Proof.** (i) \(\iff\) (ii): We set \(\tilde{G}(z) := G_{H^\tau}(z)\) and observe that the associated infinite Hankel matrices \(H^{(1)}_{\tilde{G}}\) and \(H^{(2)}_{\tilde{G}}\) defined by (3.8) and (3.9), respectively, have the following shape:

\[
H^{(1)}_{\tilde{G}} := \begin{bmatrix} H^{(1)}_{i+j-1} \end{bmatrix}_{i,j \in \mathbb{N}},
\]

\[
H^{(2)}_{\tilde{G}} := \begin{bmatrix} -H^{(2)}_{\tau} & \ldots & -H^{(2)}_{\tau+1-s} & 0 & \ldots \\ \vdots \end{bmatrix},
\]

\[
\begin{bmatrix} -H^{(2)}_{\tau+1-s} \\ \vdots \end{bmatrix}.
\]
Thus, using Lemma 3.5, it follows that \( \tilde{G}(z) = C(zE - A)^{-1}B \) is equivalent to (3.18). By part (iv) of Theorem 3.1 we conclude that \( \delta(H^\tau) = \delta(\tilde{G}(z)) \).

(ii) \( \iff \) (iii) follows from Lemma 3.5. To prove (3.20), we get from (3.10) that \((E, A, B, C) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m} \times \mathbb{K}^{p \times n} \) is a minimal descriptor realization of \( \tilde{G}(z) \) if and only if \( n = \text{rank } H^{(1)}_G + \text{rank } H^{(2)}_G \). By definition of \( r \) and \( t \) (see (2.27) and Definition 2.19), we have

\[
    r = \text{rank } H^{(1)}_G = \mu(\tilde{G}^{(1)}(z)), \quad t = \text{rank } H^{(2)}_G = \mu(-z^{-1}\tilde{G}^{(2)}(z^{-1}))
\]

and, therefore, (3.20) follows. \( \square \)

**Remark 3.8.**

(i) If \( H^\tau(X, Y) \) is the binary form associated with \( H^\tau = (H_1, \ldots, H_\tau) \):

\[
    H^\tau(X, Y) = \sum_{i=1}^\tau \binom{\tau - 1}{i - 1} H_i X^{\tau - i} Y^{i - 1},
\]

then the partial realization condition (3.19) can be expressed succinctly as follows:

\[
    H^\tau(X, Y) = C(XE + YA)^{\tau - 1}B.
\]

(ii) If a state space system \((I_n, A, B, C)\) is a minimal partial descriptor realization of \( H^\tau \), then it is also a minimal partial realization in the usual sense. In this case the generalized McMillan degree of \( H^\tau \) coincides with the McMillan degree of \( H^\tau \). Note that the converse is not true. A minimal partial realization in the usual sense is not necessarily a minimal partial descriptor realization.

(iii) It is easily seen that every sequence \( H^\tau \in (\mathbb{K}^{p \times m})^\tau \) has a minimal partial descriptor realization with \( \delta(H^\tau) \leq \min\{m, p\} \cdot \tau \).

The classical approach to the solution of the minimal partial realization problem for regular state space systems is based on an analysis of the associated finite Hankel matrices (1.3), see [20]. In order to solve the partial realization problem for descriptor systems \((E, A, B, C) \in S_{n,m,p}\) we also choose a Hankel approach. It is convenient to associate with any pair of integers \( M, N \in \mathbb{N} \) and any system \((E, A, B, C) \in S_{n,m,p}\) the \( M \times N \) block Hankel matrix

\[
    H_{MN}(E, A, B, C) := O_M(E, A, C)R_N(E, A, B),
\]

where

\[
    R_N(E, A, B)
    := \begin{bmatrix} E^{N-1}B, E^{N-2}AB, \ldots, EA^{N-2}B, A^{N-1}B \end{bmatrix} \in \mathbb{K}^{n \times Nm},
\]
By definition,

\[ H_{MN}(E, A, B, C) = \begin{bmatrix}
    C \ E^{M+N-2} B & C \ E^{M+N-3} A B & \cdots & C \ E^{M-1} A^{N-1} B \\
    C \ E^{M+N-3} A B & C \ E^{M+N-4} A^2 B & \cdots & C \ E^{M-2} A^N B \\
    \vdots & \vdots & \ddots & \vdots \\
    C \ E^{N-1} A^{M-1} B & C \ E^{N-2} A^M B & \cdots & C A^{M+N-2} B
\end{bmatrix} \]  

(3.24)

**Remark 3.9.** In terms of the finite Hankel matrices (1.3) and (3.24), one may restate Definition 3.6 as follows. A linear system \((E, A, B, C) \in S_{n,m,p}\) is an \(n\)-dimensional partial descriptor realization of \(H^\tau = (H_1, \ldots, H_\tau)\) if for some \(M \in \{1, \ldots, \tau\}\) and \(N = \tau + 1 - M\)

\[ H_{MN}(E, A, B, C) = H(M, N). \]

We need the following rank tests for controllability and observability.

**Proposition 3.10.** A descriptor system \((E, A, B, C) \in S_{n,m,p}\) is controllable (respectively, observable) if and only if one of the following two equivalent conditions is satisfied:

(i) \(\text{rank } R_N(E, A, B) = n\) (respectively, \(\text{rank } O_M(E, A, C) = n\)) for all \(N \geq n\) (respectively, \(M \geq n\)).

(ii) \(\text{rank } R_i(E, A, B) = n\) (respectively, \(\text{rank } O_j(E, A, C) = n\)) for some \(i \in \mathbb{N}\) (respectively, \(j \in \mathbb{N}\)).

**Proof.** In [23] it was shown that the assertion of the proposition holds for descriptor systems \((E, A, B, C) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m} \times \mathbb{K}^{m \times n} \times \mathbb{K}^{p \times n}\) which satisfy the condition

\[ I_n \in \mathbb{K} \cdot E + \mathbb{K} \cdot A. \] (3.25)

Now any system \((E, A, B, C) \in S_{n,m,p}\) can be transformed to a system satisfying (3.25) by a transformation \((E, A, B, C) \mapsto (VE, VA, VB, C)\), where \(V := (\tilde{\alpha} E + \tilde{\beta} A)^{-1}\) and \(\tilde{\alpha}, \tilde{\beta} \in \mathbb{K}\) are such that \(\det(\tilde{\alpha} E + \tilde{\beta} A) \neq 0\). Hence the required result follows from (3.3), (3.4) and the observation that, for any \(\alpha, \beta \in \mathbb{K}\), \((\alpha, \beta) \neq (0, 0)\),

\[ \text{rank } [\alpha VE - \beta VA, VB] = \text{rank } [\alpha E - \beta A, B], \]

\[ \text{rank } R_N(VE, VA, VB) = \text{rank } R_N(E, A, B), \]
\[
\begin{align*}
\text{rank} \begin{bmatrix} \alpha V E - \beta V A \\ C \end{bmatrix} &= \text{rank} \begin{bmatrix} \alpha E - \beta A \\ C \end{bmatrix}, \\
\text{rank} O_M(V E, V A, C) &= \text{rank} O_M(E, A, C).
\end{align*}
\]

We now turn to properties of partial descriptor realizations.

\textbf{Proposition 3.11.} Given any sequence \( H^\tau = (H_1, \ldots, H_\tau) \in \mathbb{R}^{p \times m}_\tau \), let \( H(i, \tau + 1 - i), i = 1, \ldots, \tau \) be the associated Hankel matrices, defined as in (1.3), and let

\[
d(H^\tau) := \max\{\text{rank} H(i, \tau + 1 - i); \ i = 1, \ldots, \tau\}. \tag{3.26}
\]

Then the following statements hold:

(i) There is no partial descriptor realization of \( H^\tau \) of dimension smaller than \( d(H^\tau) \).

(ii) If \( (E, A, B, C) \) is a \( d(H^\tau) \)-dimensional partial descriptor realization of \( H^\tau \), then \( (E, A, B, C) \) is minimal, controllable and observable.

Assuming additionally that \( H^\tau \) satisfies the rank assumption, we obtain:

(iii) There exists a \( d(H^\tau) \)-dimensional partial descriptor realization in Weierstrass form of \( H^\tau \).

(iv) A partial descriptor realization of \( H^\tau \) is minimal if and only if it is controllable, observable and of dimension \(<[\frac{\tau + 1}{2}] \), where \([\frac{\tau + 1}{2}] \) is the largest integer \( \leq \frac{\tau + 1}{2} \).

\textbf{Proof.} \textit{We first choose an integer} \( M \in \{1, \ldots, \tau\} \) \textit{such that} \( \text{rank} H(M, N) = d(H^\tau) \) \textit{with} \( N := \tau + 1 - M \).

(i) \textit{Let} \( (E, A, B, C) \) \textit{be an} \( l \)-dimensional partial descriptor realization of \( H^\tau \). By Remark 3.9, condition (3.19) can be written in the form \( H_{MN}(E, A, B, C) = H(M, N) \), and hence, by (3.21)–(3.23), \( d(H^\tau) = \text{rank} H_{MN}(E, A, B, C) \leq \text{rank} O_M(E, A, C) \leq l \), which implies \( d(H^\tau) \geq d(H^\tau) \).

(ii) \textit{If} \( (E, A, B, C) \) \textit{is a} \( d(H^\tau) \)-dimensional partial descriptor realization of \( H^\tau \), then it is minimal by (i). Eqs. (3.21)–(3.23) imply \( d(H^\tau) = \text{rank} H_{MN}(E, A, B, C) \leq \text{rank} O_M(E, A, C) \leq d(H^\tau) \). Analogously we obtain \( d(H^\tau) = \text{rank} R_N(E, A, B) \), and hence (ii) follows from part (ii) of Proposition 3.10.

(iii) Suppose \( H^\tau \) satisfies the rank assumption and has regular and singular degree \( r \) and \( t \), respectively (see Definition 2.19). By Kalman’s Realization Theorem the associated strictly proper rational matrices \( G(1)(H^\tau(z)) \) and \( -z^{-1}G(2)(H^\tau(z^{-1})) \) (see (3.17)) have minimal state space realizations \( (A_1, B_1, C_1) \) and \( (E_2, B_2, C_2) \) of dimensions \( r \) and \( t \), respectively. According to Proposition 3.7, \( H^\tau \) has a minimal partial descriptor realization in Weierstrass form of dimension \( r + t \). Using Lemma 2.20, we obtain \( r + t = d(H^\tau) \).

(iv) Since \( H^\tau \) satisfies the rank assumption, Lemma 2.20 implies that

\[
d(H^\tau) = \text{rank} \mathcal{H}, \tag{3.27}
\]
where $\mathcal{H}$ is the central Hankel matrix (1.7). By (i) and (iii), every minimal partial descriptor realization of $H^\tau$ is $d(H^\tau)$-dimensional; hence controllable and observable, according to (ii). Using (1.7), (1.8) and (3.27) we see that $d(H^\tau) < \left\lceil \frac{\tau + 1}{2} \right\rceil$. Conversely, if $(E, A, B, C)$ is a controllable and observable partial descriptor realization of dimension $l < \left\lceil \frac{\tau + 1}{2} \right\rceil$ of $H^\tau$, then it follows from (i) that $l \geq d(H^\tau)$. By Remark 3.9, condition (3.19) can be written in the form

$$\mathcal{H} = H_{\frac{\tau + 1}{2}} (E, A, B, C) \quad \text{or} \quad \mathcal{H} = H_{\frac{\tau + 2}{2}} (E, A, B, C),$$

according to $\tau$ is odd or even (see (1.7)). Then (3.21), inequality $l < \left\lceil \frac{\tau + 1}{2} \right\rceil$ and Proposition 3.10 imply that $l = \text{rank } \mathcal{H}$. Thanks to (3.26), we have $l \leq d(H^\tau)$. Hence $l = d(H^\tau)$ follows; that is, $(E, A, B, C)$ is minimal. □

For any $n, m, p \in \mathbb{N}$ let

$$S_{c,o}^{n,m,p} := \{(E, A, B, C) \in S_{n,m,p}; \ (E, A, B, C) \text{ is controllable and observable}\}.$$

We end this section by introducing the concept of regular and singular controllability/observability indices for descriptor systems $(E, A, B, C) \in S_{c,o}^{n,m,p}$ in Weierstrass form and relate them to the regular and singular column/row indices of the associated Hankels.

**Lemma 3.12.** Suppose $(E, A, B, C) \in S_{c,o}^{n,m,p}$ is a descriptor system in Weierstrass form (3.11) with fast subsystem $(E_2, I_{n-r}, B_2, C_2) \in S_{c,o}^{n-r,m,p}$ and let $\nu$ be the degree of nilpotency of $E_2$. From the rank $n-r$ matrix

$$\begin{bmatrix} C_2E_2^{\nu-1} \\ \vdots \\ C_2E_2 \\ C_2 \end{bmatrix} \in \mathbb{K}^{\nu p \times (n-r)}$$

(3.28)

we pick the first (that is, starting from the top) $n-r$ linearly independent rows. Then the resulting selection of row indices $\tilde{\alpha} = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p\}$ is saturated from above (see Definition 2.13). In particular, there exist uniquely determined nonnegative integers $\tilde{y}_1, \ldots, \tilde{y}_p$ with sum $\tilde{y}_1 + \cdots + \tilde{y}_p = n-r$ such that

$$\tilde{\alpha} = \bigcup_{i=1}^{p} \{i + (v - \tilde{y}_i)p, \ldots, i + (v - 2)p, i + (v - 1)p\}. \quad (3.29)$$

**Proof.** Let $c_1, \ldots, c_p$ denote the row vectors of the matrix $C_2$. We prove that if the row vector $c_iE_2^{\nu-1}$ of matrix (3.28) is linearly independent from the preceding rows, then so is the row vector $c_iE_2^{\nu-1}$. Suppose the contrary; then there would be scalars $r_{kl} \in \mathbb{K}$ such that

$$c_iE_2^{\nu-1} = \sum_{l=1}^{i-1} r_{il}c_lE_2^{\nu-1} + \sum_{k=1}^{\nu-j} \sum_{l=1}^{p} r_{kl}c_lE_2^{\nu-1+k}. \quad (3.30)$$
But since $E_2^{\nu} = 0$ it would follow that
\[
c_iE_2^{j} = \sum_{l=1}^{i-1} r_{i} c_i E_2^{j} + \sum_{k=1}^{\nu - j - 1} \sum_{l=1}^{p} r_{kl} c_i E_2^{j+k},
\]
contradicting the independence of $c_iE_2^{j}$. This proves that the selection $\tilde{\alpha}$ is saturated from above. □

**Definition 3.13.** Given a descriptor system $(E, A, B, C) \in S_{n,m,p}^{c,o}$ in Weierstrass form with slow and fast subsystems $(I_r, A_1, B_1, C_1)$ and $(E_2, I_{n-r}, B_2, C_2)$, respectively. The usual observability (respectively, controllability) index lists of $(C_1, A_1)$ (respectively, $(A_1, B_1)$) are called the regular observability (respectively, controllability) index lists of $(E, A, B, C)$. The index list $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_p)$, defined by (3.29), is said to be the singular observability index list of $(E, A, B, C)$. Under the hypothesis of Lemma 3.12 a similar selection procedure can be applied to the columns of the matrix
\[
[E_2^{\nu-1}B_2, \ldots, E_2B_2, B_2] \in \mathbb{K}^{(n-r) \times vm}
\]
(starting on the left) and a singular controllability index list $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_m)$ can be defined.

**Remark 3.14.** In general the singular controllability index list is different from the usual controllability index list of $(E_2, B_2)$. As an example, consider the controllable matrix pair
\[
E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\]
We have
\[
[E_2^3B_2, E_2B_2, B_2] = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\]
Hence, $\tilde{z} = (0, 3, 0)$ is the singular controllability index list; but the usual controllability index list of $(E, B)$ is $(1, 1, 1)$.

Our next proposition connects the regular and singular row (respectively, column) indices of block Hankel matrices (see Definition 2.15) with the regular and singular observability (respectively, controllability) indices of the corresponding partial descriptor realizations in Weierstrass form.

**Proposition 3.15.** Suppose $(E, A, B, C) \in S_{n,m,p}^{c,o}$ is a (minimal) partial descriptor realization in Weierstrass form of a given sequence $H^\tau = (H_1, \ldots, H_\tau) \in (\mathbb{K}^{p \times m})^\tau$, and suppose that an associated Hankel matrix, say $H := H(M, N), M + N = \tau + 1$,
satisfies the Rank Assumption 2.4. Then the regular and singular row index lists of \( H \) coincide with the regular and singular observability index lists of \((E, A, B, C)\), respectively. Similarly the regular and singular column index lists of \( H \) coincide with the regular and singular controllability index lists of \((E, A, B, C)\), respectively.

**Proof.** We decompose \( H \) into its regular and its singular part

\[
H = H_{MN}(I_r, A_1, B_1, C_1) + H_{MN}(E_2, I_{n-r}, B_2, C_2),
\]

where \((I_r, A_1, B_1, C_1)\) and \((E_2, I_{n-r}, B_2, C_2)\) are the slow and the fast subsystem of \((E, A, B, C)\), respectively (see Definition 2.11). We may factor \( H_{MN}(I_r, A_1, B_1, C_1) \) and \( H_{MN}(E_2, I_{n-r}, B_2, C_2) \) as follows:

\[
H_{MN}(I_r, A_1, B_1, C_1) = O_M(I_r, A_1, C_1)R_N(I_r, A_1, B_1) \quad (3.30)
\]

\[
H_{MN}(E_2, I_{n-r}, B_2, C_2) = O_M(E_2, I_{n-r}, C_2)R_N(E_2, I_{n-r}, B_2), \quad (3.31)
\]

where the controllability and observability matrices are defined by (3.22) and (3.23), respectively. The matrix \( E_2 \) is nilpotent with index of nilpotency equal to \( \nu \). Hence \( R_N(E_2, I_{n-r}, B_2) \) and \( O_M(E_2, I_{n-r}, C_2) \) have the following shape:

\[
R_N(E_2, I_{n-r}, B_2) = \begin{bmatrix} 0, \ldots, 0, E_2^{\nu-1}B_2, \ldots, B_2 \end{bmatrix} \in \mathbb{K}^{(n-r)\times Nm},
\]

\[
O_M(E_2, I_{n-r}, C_2) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_2E_2^{\nu-1} \\ \vdots \\ C_2 \end{bmatrix} \in \mathbb{K}^{Mp \times (n-r)}.
\]

Since each of the factors in (3.31) has full rank \( n - r \), the linear dependence relations between the rows (respectively, columns) of \( H_{MN}(E_2, I_{n-r}, B_2, C_2) \) and the rows of \( O_M(E_2, I_{n-r}, C_2) \) (respectively, columns of \( R_N(E_2, I_{n-r}, B_2) \)) are identical. Therefore the singular row (respectively, column) index list of \( H \) coincides with the singular observability (respectively, controllability) index list of \((E, A, B, C)\) (see Lemmas 2.14 and 3.12). Using the factorization (3.30), the proof for the regular index lists is completely analogous. \( \Box \)

### 4. Main results

It is well known that, if two regular minimal state-space systems realize the same infinite sequence, they must be similar. In this section, we will establish a counterpart of this uniqueness result for partial descriptor realizations \((E, A, B, C) \in \mathcal{S}^{c,o}_{n,m,p}\) of finite sequences \( H^\tau \)
It is convenient to introduce Möbius transforms of linear descriptor systems and Fischer–Frobenius transforms of Hankel matrices, since they can be used in order to regularize partial descriptor realizations.

**Definition 4.1.** For every \((E, A, B, C) \in S_{n,m,p}\), \(g = [g_{ij}] \in \text{Gl}(2)\), the descriptor system

\[
g \cdot (E, A, B, C) := (g_{11}E + g_{12}A, g_{21}E + g_{22}A, B, C)
\]  

is said to be a Möbius transform of \((E, A, B, C)\).

It is easily verified that, if \((E, A, B, C) \in S_{n,m,p}\) and \(g \in \text{Gl}(2)\), then \(g \cdot (E, A, B, C) \in S_{n,m,p}\). A straightforward proof then yields the following lemma.

**Lemma 4.2.** The mapping

\[
\text{Gl}(2) \times S_{n,m,p} \rightarrow S_{n,m,p},
\]

\([g, (E, A, B, C)] \mapsto g \cdot (E, A, B, C)\)  

is a left group action of \(\text{Gl}(2)\) on \(S_{n,m,p}\).

We will now describe an interesting relationship between Möbius transforms of descriptor systems and Fischer–Frobenius transforms of finite block Hankel matrices. For this we need the following definition.

**Definition 4.3.** The Hankel mapping is defined by

\[
H_{MN} : S_{n,m,p} \rightarrow \text{Hank}_{p,m}(M \times N),
\]

\((E, A, B, C) \mapsto H_{MN}(E, A, B, C)\).

Möbius transforms of descriptor systems induce via the Hankel mapping transforms of the associated finite Hankel matrices, called Fischer–Frobenius transforms. Fischer–Frobenius transforms have their origin in the theory of scalar Hankel and Toeplitz matrices as well as in the theory of quadratic and Hermitian forms, see [17, Section 19]. Brockett [4] was the first to use Fischer–Frobenius transforms in a system theoretic context. Fischer–Frobenius transforms of block Hankel matrices have been studied in [13, 24].

We will now recall some basic facts of Fischer–Frobenius transforms of finite sequences and block Hankel matrices. They are defined as follows. Let \(B_{p,m}(\tau)\) denote the \(\mathbb{K}\)-vector space of binary forms of degree \(\tau - 1\) with coefficients in \(\mathbb{K}^{p \times m}\):

\[
H^\tau(X, Y) = \sum_{i=1}^{\tau} \binom{\tau - 1}{i - 1} H_i X^{\tau-i} Y^{i-1}, \quad H_i \in \mathbb{K}^{p \times m}.
\]
Associating with every finite sequence
\[ H^\tau = (H_1, \ldots, H_\tau) \in (\mathbb{K}^{p \times m})^\tau \]
the binary form \( H^\tau(X, Y) \) with coefficients \( H_i \) yields a linear isomorphism
\[ \phi_\tau : (\mathbb{K}^{p \times m})^\tau \rightarrow B_{p,m}(\tau), \]
\[ H^\tau \mapsto H^\tau(X, Y). \quad (4.4) \]
For any \( \tau \geq 2 \), the general linear group \( \text{Gl}(2) \) of all invertible \( 2 \times 2 \) matrices \( g = \begin{bmatrix} g_{ij} \end{bmatrix} \) over \( \mathbb{K} \) acts on \( B_{p,m}(\tau) \) via the linear transformation
\[ \rho_{p,m,\tau}(g) : H^\tau(X, Y) \mapsto H^\tau(g_{11}X + g_{21}Y, g_{12}X + g_{22}Y). \quad (4.5) \]
(4.5) defines a left action of \( \text{Gl}(2) \) on the vector space \( B_{p,m}(\tau) \).

**Definition 4.4.** For every finite sequence \( H^\tau \in (\mathbb{K}^{p \times m})^\tau \) and matrix \( g \in \text{Gl}(2) \), the sequence
\[ g \cdot H^\tau := \phi_\tau^{-1} \left( \rho_{p,m,\tau}(g) \phi_\tau(H^\tau) \right) \]
is said to be a Fischer–Frobenius transform of \( H^\tau \).

Note that (4.6) induces a left action of \( \text{Gl}(2) \) on \( (\mathbb{K}^{p \times m})^\tau \):

\[ \text{Gl}(2) \times (\mathbb{K}^{p \times m})^\tau \rightarrow (\mathbb{K}^{p \times m})^\tau, \]
\[ (g, H^\tau) \mapsto g \cdot H^\tau. \quad (4.7) \]
For any pair \( (M, N) \) of integers, satisfying \( M + N = \tau + 1 \), we have a linear isomorphism
\[ (\mathbb{K}^{p \times m})^\tau \cong \text{Hank}_{p,m}(M \times N). \quad (4.8) \]
Using (4.8), the action (4.7) induces the following left \( \text{Gl}(2) \)-action on \( \text{Hank}_{p,m}(M \times N) \):
\[ \text{Gl}(2) \times \text{Hank}_{p,m}(M \times N) \rightarrow \text{Hank}_{p,m}(M \times N), \quad (g, H) \mapsto g \cdot H. \quad (4.9) \]

**Definition 4.5.** \( g \cdot H \) is referred to as a Fischer–Frobenius transform of the Hankel matrix \( H \in \text{Hank}_{p,m}(M \times N) \).

By straightforward (but messy) calculations (see [24]) it can be shown that an equivalent description of (4.9) is
\[ g \cdot H = \left[ U_M(g)^T \otimes I_p \right] H \left[ U_N(g) \otimes I_m \right], \]
\[ H \in \text{Hank}_{p,m}(M \times N), \quad g \in \text{Gl}(2), \]
where \( U_\tau(g) \in \text{Gl}(\tau) \) is the matrix of the linear operator \( \rho_{1,1,\tau}(g^T) \) (see (4.5)) with respect to the canonical basis \( \{ X^{\tau-1}, X^{\tau-2}Y, \ldots, Y^{\tau-1} \} \) of \( B_{1,1}(\tau) \), \( A^T \) is the trans-
pose of a matrix $A$, and $\otimes$ denotes the Kronecker product. It follows that the Fischer–Frobenius transformation (4.9) leaves the rank of Hankel matrices invariant; that is, for arbitrary $n \leq \min\{M_p, N_m\}$,

$$g \cdot \text{Hank}_{p,m}(n, M \times N) = \text{Hank}_{p,m}(n, M \times N), \quad g \in \text{GL}(2). \quad (4.10)$$

**Remark 4.6.** Note that Definition 4.4 of a Fischer–Frobenius transform is slightly different from that we used in [24,23]. In the present paper we introduce Fischer–Frobenius transforms via binary forms without any use of matrix representations of linear maps involved. In our previous work we defined Fischer–Frobenius transforms via matrix representations in such a way that a right $\text{GL}(2)$-action was obtained whereas our present definition (4.6) yields a left $\text{GL}(2)$-action. Consequently the formulae in [23,24] correspond to formulae in the present paper with $g \in \text{GL}(2)$ replaced by its transpose.

The next proposition shows that the diagram

$$
\begin{array}{ccc}
S_{n,m,p} & \xrightarrow{g} & S_{n,m,p} \\
H_{MN} \downarrow & & \downarrow H_{MN} \\
\text{Hank}_{p,m}(M \times N) & \xrightarrow{g} & \text{Hank}_{p,m}(M \times N)
\end{array}
$$

is commutative for all $g \in \text{GL}(2)$ and hence the Fischer–Frobenius transformation (4.9) corresponds to the Möbius transformation (4.2).

**Proposition 4.7.** The Hankel mapping (4.3) is an intertwining mapping for the Fischer–Frobenius transformation (4.9) and the Möbius transformation (4.2); that is, for $g \in \text{GL}(2)$, $(E, A, B, C) \in S_{n,m,p}$

$$H_{MN}(g \cdot (E, A, B, C)) = g \cdot H_{MN}(E, A, B, C).$$

**Proof.** Let $g = \begin{bmatrix} g_{ij} \end{bmatrix} \in \text{GL}(2)$, $(E, A, B, C) \in S_{n,m,p}$ and let $\tau = M + N - 1$. Using (3.24) and isomorphisms (4.8) and (4.4), we obtain

$$H_{MN}(E, A, B, C) = \phi_{\tau}^{-1}\left(\sum_{i=1}^{\tau} \binom{\tau - i}{i - 1} C E^{\tau - i} A^{i - 1} B X^{\tau - i} Y^{i - 1}\right) = \phi_{\tau}^{-1}\left(C(XE + YA)^{\tau - 1} B\right). \quad (4.11)$$

Hence by (4.4)–(4.6), (4.11) and (4.1) it follows that

$$g \cdot H_{MN}(E, A, B, C) = \phi_{\tau}^{-1}\left(C[(g_{11}X + g_{21}Y)E + (g_{12}X + g_{22}Y)A]^{\tau - 1} B\right) = \phi_{\tau}^{-1}\left(C[X(g_{11}E + g_{12}A) + Y(g_{21}E + g_{22}A)]^{\tau - 1} B\right) = H_{MN}(g \cdot (E, A, B, C)). \quad \square$$
Corollary 4.8. Controllability and observability are preserved under the Moebius transformation (4.2).

Proof. Suppose \((E, A, B, C) \in S_{n,m,p}\) is controllable, \(g \in \text{Gl}(2)\) and \((\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) := g \cdot (E, A, B, C)\). For arbitrary \((\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}\), let \((\tilde{\alpha}, -\tilde{\beta}) = (\alpha, -\beta) \cdot g\). Then \((\tilde{\alpha}, -\tilde{\beta}) \neq (0, 0)\) and by (3.3)

\[
\text{rank}[\alpha \tilde{E} - \beta \tilde{A}, B] = \text{rank}[\alpha(g_{11}E + g_{12}A) - \beta(g_{21}E + g_{22}A), B] = \text{rank}[\tilde{\alpha}E - \tilde{\beta}A, B] = n.
\]

Hence \((\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})\) is controllable. Preservation of observability is shown similarly. □

Lemma 4.9. Let \((E, A, B, C) \in S^{c,o}_{n,m,p}\) and \(n \leq \min\{M, N\} \cdot 2n < M + N\). Then \(E\) is nonsingular if and only if \(H_{MN}(E, A, B, C) \in \text{Hank}^*_{p,m}(n, M \times N)\) (see (2.5)).

Proof. Suppose \(E\) is nonsingular and \(\chi_{E^{-1}A}(z) = z^n - p_n z^{n-1} - \cdots - p_1\) is the characteristic polynomial of \(E^{-1}A\). Then by the Cayley–Hamilton theorem

\[
E^{-n}A^n = \sum_{i=1}^{n} p_i E^{-(i-1)} A^{i-1}.
\]

Multiplying this equation by \(CEM^{M+N-2-j}\) on the left and by \(A^j B\) on the right for \(j = 0, 1, \ldots, M + N - 2 - n\), we obtain

\[
CEM^{M+N-2-n-j} A^{n+j} B = \sum_{i=1}^{n} p_i CEM^{M+N-2-(i-1)-j} A^{i-1+j} B,
\]

\[
0 \leq j \leq M + N - 2 - n.
\] (4.12)

Thus all columns of the Hankel matrix \(H_{MN}(E, A, B, C) \in \text{Hank}^*_{p,m}(n, M \times N)\) must be linear combinations of the first \(mn\) columns, that is, the columns appearing in the first \(n\) blocks. Moreover, we conclude from (4.12) that all rows of \(H_{MN}(E, A, B, C)\) must be linear combinations of the first \(pn\) rows. Hence \(H_{MN}(E, A, B, C) \in \text{Hank}^*_{p,m}(n, M \times N)\).

Conversely, let \(H := H_{MN}(E, A, B, C) \in \text{Hank}^*_{p,m}(n, M \times N)\) and suppose that \(E\) is singular. Then by (3.21) and since \(EA = AE\)

\[
H(n, n) = \begin{bmatrix}
CEM^{M-1} \\
CEM^{M-2} A \\
\vdots \\
CEM^{M-n} A^{n-1}
\end{bmatrix} \begin{bmatrix}
E^{N-1} B, E^{N-2} AB, \ldots, E^{N-n} A^{n-1} B
\end{bmatrix}
\]

\[
= O_n(E, A, C) E^{M+N-2n} R_n(E, A, B).
\]
Since \( M + N - 2n > 0 \), the last equation implies \( \text{rank} \, H(n, n) \leq \text{rank} \, E < n \), which yields a contradiction to \( H \in \text{Hank}^*_p,m(n, M \times N) \). □

Making use of minimal partial descriptor realizations we are now in a position to obtain a new and very simple proof of the following Regularity Theorem [4,24] which is an important element of the algebraic structure theory of finite block Hankel matrices.

**Regularity Theorem 4.10.** Let \( H \in \text{Hank}_p,m(n, M \times N) \), \( n < \min\{M, N\} \) and

\[
g_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in \text{Gl}(2), \quad t \in \mathbb{K}.
\]  

(4.13)

Then \( g_t \cdot H \in \text{Hank}^*_p,m(n, M \times N) \) for all but a finite number of \( t \in \mathbb{K} \).

**Proof.** Applying Lemma 2.17 and Eqs. (2.32) and (3.26), it follows that \( n = d (H^{M+N-1}) \), where \( H^{M+N-1} \) is the associated sequence \((H_1, \ldots, H_{M+N-1})\). Hence, by part (iii) of Proposition 3.11, there exists an \( n \)-dimensional descriptor system \((E, A, B, C)\) in Weierstrass form such that \( H = H_{MN}(E, A, B, C) \). Proposition 4.7 shows that

\[
g_t \cdot H = g_t \cdot H_{MN}(E, A, B, C) = H_{MN}(E + tA, A, B, C),
\]

and from (3.2) we conclude that the polynomial \( \det(E + tA) \) is nonzero. It follows that the matrices \( E + tA, t \in \mathbb{K} \), are nonsingular for all but a finite number of \( t \). By Corollary 4.8, \((E + tA, A, B, C)\) is controllable and observable. Hence the assertion is a consequence of Lemma 4.9. □

**Remark 4.11.**

(i) In the scalar case, Theorem 4.10 was first established by Brockett [4] and for block Hankels the theorem was first derived in [24], with a much more complicated proof than the above.

(ii) In [4] (scalar case) and [24] (block case) Theorem 4.10 has been instrumental in constructing an analytic manifold structure on the space \( \text{Hank}^*_p,m(n, M \times N) \), \( n < \min\{M, N\} \). Moreover, Theorem 4.10 has been used in order to determine the topological closures of \( \text{Hank}^*_p,m(n, M \times N) \) in \( \text{Hank}^*_{p,m}(M \times N) \), see [15,22] for the scalar case, and [24] for the case of block Hankels.

(iii) There are examples of Hankel matrices \( H \in \text{Hank}^*_p,m(n, M \times N) \) with \( n > \min\{M, N\} \) for which the assertion of Theorem 4.10 does not hold. Hence the rank condition \( n < \min\{M, N\} \) can in general not be dispensed with. However, it can be shown that Theorem 4.10 also holds for \( n = \min\{M, N\} \), see [22] (scalar case) and [21] (block case).

We turn now to the study of the relationship between two minimal partial descriptor realizations of the same finite sequence.
Definition 4.12. We say that the descriptor system \((E, A, B, C) \in S_{n,m,p}\) is \(\tau\)-equivalent to \((\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \in S_{n,m,p}\), denoted
\[
(E, A, B, C) \sim_\tau (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})
\]
if there exist \(K, L, T \in \text{Gl}(n)\) such that \(K \tilde{E} = \tilde{E} K, K \tilde{A} = \tilde{A} K, LE = EL, LA = AL\) and
\[
(K \tilde{E}, K \tilde{A}, K \tilde{B}, \tilde{C} K^{-\tau}) = (T LET^{-1}, TLAT^{-1}, T LB, CL^{-\tau} T^{-1}). \tag{4.14}
\]
Hence, \((E, A, B, C)\) is \(\tau\)-equivalent to \((\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})\) if there exist nonsingular matrices \(K\) and \(L\) commuting with \(\tilde{E}, \tilde{A}\) and \(E, A\), respectively, such that \((K \tilde{E}, K \tilde{A}, K \tilde{B}, \tilde{C} K^{-\tau})\) and \((LE, LA, LB, CL^{-\tau})\) are similar.

A proof of the following technical lemma can be found in [16].

Lemma 4.13. Let \((E, A, B, C), (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \in S_{n,m,p}\) and let \(t \in \mathbb{K}\) be such that \(E_i := E + tA\), \(\tilde{E}_i := \tilde{E} + t\tilde{A}\) are nonsingular. The following three conditions are equivalent:

(i) \((E, A, B, C) \sim_\tau (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})\);

(ii) \((I_n, E_i^{-1} A, E_i^{-1} B, CE_i^{-1}) \sim_\tau (I_n, \tilde{E}_i^{-1} \tilde{A}, \tilde{E}_i^{-1} \tilde{B}, \tilde{C} E_i^{-1})\);

(iii) \((E_i^{-1} A, E_i^{-1} B, CE_i^{-1})\) and \((\tilde{E}_i^{-1} \tilde{A}, \tilde{E}_i^{-1} \tilde{B}, \tilde{C} E_i^{-1})\) are similar.

As a consequence we obtain that \(\sim_\tau\) is indeed an equivalence relation and is compatible with the Moebius action (4.2).


(i) \(\sim_\tau\) is an equivalence relation on \(S_{n,m,p}\).

(ii) Let \(g \in \text{Gl}(2)\). Then \((E, A, B, C), (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \in S_{n,m,p}\) are \(\tau\)-equivalent if and only if \(g \cdot (E, A, B, C), g \cdot (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})\) are \(\tau\)-equivalent.

Proof. (i) Given a finite number of systems \((E_i, A_i, B_i, C_i) \in S_{n,m,p}, i = 1, \ldots, l\), by (1.5), there is a \(t \in \mathbb{K}\) such that \(\det(E_i + tA_i) \neq 0\), \(i = 1, \ldots, l\). Hence the assertion is an immediate consequence of Lemma 4.13.

(ii) Suppose that \((K \tilde{E}, K \tilde{A}, K \tilde{B}, \tilde{C} K^{-\tau}) = (T LET^{-1}, TLAT^{-1}, T LB, CL^{-\tau} T^{-1})\) with nonsingular \(K, L, T\) and \(K \tilde{E} = \tilde{E} K, K \tilde{A} = \tilde{A} K, LE = EL, LA = AL\). Then we obtain
\[
(K(g_{11} \tilde{E} + g_{12} \tilde{A}), K(g_{21} \tilde{E} + g_{22} \tilde{A}))
\]
\[
= (TL(g_{11} E + g_{12} A)T^{-1}, TL(g_{21} E + g_{22} A)T^{-1}).
\]
The matrix \(K\) commutes with \((g_{11} \tilde{E} + g_{12} \tilde{A})\) and \((g_{21} \tilde{E} + g_{22} \tilde{A})\), and the matrix \(L\) commutes with \((g_{11} E + g_{12} A)\) and \((g_{21} E + g_{22} A)\). Hence \(g \cdot (E, A, B, C) \sim_\tau g \cdot (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})\). Since (4.2) is a group action, the result follows. \(\Box\)

Remark 4.15.
(i) Given \((E, A, B, C) \in S_{n,m,p}\), the matrices \(E_t := E + tA\) are nonsingular for all but a finite number of \(t \in \mathbb{K}\).
(ii) Lemma 4.13 shows that in Definition 4.12 \(\tilde{E}^{-1}\) may be replaced by \(\tilde{E}^{-1} - t\tilde{L}\) and \(L\) by \(E^{-1} - tE\) for any \(t \in \mathbb{K}\) such that \(\tilde{E}_t, E_t\) are nonsingular.
(iii) If two descriptor systems \((E, A, B, C), (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \in S_{n,m,p}\) are \(\tau\)-equivalent, then the triplets \((E, A, B)\) and \((\tilde{E}, \tilde{A}, \tilde{B})\) are restricted system equivalent; that is, \((\tilde{E}, \tilde{A}, \tilde{B}) = (SET^{-1}, SAT^{-1}, SB)\) for some nonsingular matrices \(S, T\).

We now prove the main result of the paper.

Theorem 4.16. Given any sequence \(H^\tau = (H_1, \ldots, H_\tau) \in (\mathbb{K}^{p \times m})^\tau\) and \(k \in \mathbb{N}\) such that \(\tau = 2k - 1\) or \(\tau = 2k\). Suppose that the associated central Hankel matrix (1.7) satisfies \(n := \text{rank } \mathcal{H} < k\). Then:
(i) \(H^\tau\) has a minimal partial descriptor realization \((E, A, B, C) \in S_{n,m,p}^{c,o}\). The generalized McMillan degree of \(H^\tau\) is \(\delta(H^\tau) = \text{rank } \mathcal{H} = n\).
(ii) Any minimal partial descriptor realization of \(H^\tau\) is controllable and observable. A controllable and observable partial descriptor realization of \(H^\tau\) is minimal if it is of dimension \(n\).
(iii) Any two minimal partial descriptor realizations of \(H^\tau\) are \(\tau\)-equivalent. Conversely, two \(\tau\)-equivalent descriptor systems realize the same finite sequence.

Proof. Conditions (i) and (ii) follow from Proposition 3.11, since \(\text{rank } \mathcal{H} = d(H^\tau)\) (see (3.26) and Lemma 2.20). To establish (iii) suppose that \((E, A, B, C), (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \in S_{n,m,p}^{c,o}\) are minimal partial descriptor realizations of \(H^\tau\); that is,
\[
H_{kl}(E, A, B, C) = H_{kl}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) = \mathcal{H},
\]
where \(l = k\) if \(\tau = 2k - 1\), and \(l = k + 1\) if \(\tau = 2k\) (see (1.7) and Remark 3.9). By virtue of Proposition 4.7,
\[
\begin{align*}
g_t \cdot H_{kl}(E, A, B, C) &= H_{kl}(E + tA, A, B, C) \\
&= g_t \cdot H_{kl}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \\
&= H_{kl}(\tilde{E} + t\tilde{A}, \tilde{A}, \tilde{B}, \tilde{C})
\end{align*}
\]
for
\[
g_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in \text{Gl}(2), \quad t \in \mathbb{K}.
\]
The polynomials $\det(E + tA), \det(\tilde{E} + t\tilde{A})$ are nonzero, controllability and observability are preserved under the Moebius transformation (4.2) (see Corollary 4.8) and the Fischer–Frobenius transformation (4.9) leaves the rank of Hankel matrices invariant (see (4.10)). Hence we may assume that $E, \tilde{E}$ are nonsingular. But then (4.15) implies

$$H = H_{kl}(I_n, E^{-1}A, E^{-1}B, CE^\tau) = H_{kl}(E, A, B, C) = H_{kl}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) = H_{kl}(I_n, \tilde{E}^{-1}\tilde{A}, \tilde{E}^{-1}\tilde{B}, \tilde{C}\tilde{E}^\tau).$$

(4.16)

Since $n < k \leq l$ we obtain from Lemma 4.9

$$H \in \text{Hank}^e_{p,m}(n, M \times N).$$

(4.17)

Using (4.16) and (4.17), it follows from classical realization theory that there exists a unique $T \in \text{Gl}(n)$ with $(I_n, \tilde{E}^{-1}\tilde{A}, \tilde{E}^{-1}\tilde{B}, \tilde{C}\tilde{E}^\tau) = (I_n, TE^{-1}AT^{-1}, TE^{-1}B, CE^\tau T^{-1})$ and this implies (4.14) with $K = \tilde{E}^{-1}$ and $L = E^{-1}$.

Conversely, suppose that $(E, A, B, C), (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \in S_{n,m,p}$ are $\tau$-equivalent.

From (4.14) and $K \tilde{E} = \tilde{E} K, K \tilde{A} = \tilde{A} K$ it follows that

$$K^{-1}(TLET^{-1}) = (TLET^{-1})K^{-1}, \quad K^{-1}(TLAT^{-1}) = (TLAT^{-1})K^{-1}.$$

Hence

$$\tilde{C} \tilde{E}^\tau i\tilde{A}^{-1}\tilde{B}$$

$$= (CL^{-\tau}T^{-1}K^\tau)(K^{-1}TLET^{-1})^{-i}(K^{-1}TLAT^{-1})^{-i-1}(K^{-1}TLB)$$

$$= (CL^{-\tau}T^{-1}K^\tau)K^{-i+1}(TLET^{-1})^{-i}(TLAT^{-1})^{-i+1}K^{-1}L$$

$$= CL^{-\tau}(LE)^{-i}(LA)^{-i+1}LB$$

$$= CE^\tau i\tilde{A}^{-1}B, \quad i = 1, \ldots, \tau;$$

(4.18)

that is, $(E, A, B, C)$ and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$ realize the same finite sequence (see (3.19)).

□

Remark 4.17.

(i) The calculation (4.18) shows: Given integers $M, N \in \mathbb{N}$ with $M + N - 1 = \tau$, the Hankel mapping (4.3) is an invariant for $\tau$-equivalence; that is,

$$(E, A, B, C) \sim_\tau (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \implies H_{MN}(E, A, B, C) = H_{MN}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$$

for any $(E, A, B, C), (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \in S_{n,m,p}$. If in addition $n < \min\{M, N\}$ holds, it follows from part (iii) of the preceding theorem that the restriction of the Hankel mapping (4.3) to $S_{n,m,p}$ is a complete invariant for $\tau$-equivalence; that is,
\[(E, A, B, C) \sim \tau \ (\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) \iff H_{MN}(E, A, B, C) = H_{MN}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})\]
on Sc,o

(ii) Note that in contrast with the associated Hankel matrices (1.3) the transfer function (3.5) is not invariant with respect to \(\tau\)-equivalence. In particular, if \(G_{H^\tau}(z)\) is the transfer function associated with the sequence \(H^\tau\) according to (3.17), a partial descriptor realization of \(H^\tau\) will in general not be a descriptor realization of \(G_{H^\tau}(z)\). This will however be the case if the partial descriptor realization is in Weierstrass form, see Proposition 3.7.

(iii) Every \(\tau\)-equivalence class contains a system in Weierstrass form and this is uniquely determined modulo similarity.

In the scalar case a better result can be shown. In fact, for \(m = p = 1\) the rank condition \(\text{rank } H < k\) is not needed.

**Corollary 4.18** (Scalar case). Let \(h^\tau = (h_1, \ldots, h_\tau) \in \mathbb{K}^\tau\) be a finite scalar sequence and let \(n\) be the rank of the associated central Hankel \(\mathcal{H}\).

(i) \(h^\tau\) has a minimal partial descriptor realization of dimension \(n\). In particular, \(\delta(h^\tau) = \text{rank } H\).

(ii) Any minimal partial descriptor realization of \(h^\tau\) is controllable and observable. A controllable and observable partial descriptor realization of \(h^\tau\) is minimal if it is of dimension \(n\).

(iii) Assume \(2n < \tau + 1\). Then any two minimal generalized partial realizations of \(h^\tau\) are \(\tau\)-equivalent.

**Proof.** In view of Theorem 4.16 it remains only to consider the case when \(n = k\), where \(k \in \mathbb{N}\) is such that \(\tau = 2k - 1\) or \(\tau = 2k\) (see (1.7)).

(i) If \(\tau = 2k - 1\), then \(\mathcal{H}\) can be (in infinitely many ways) extended to an infinite Hankel matrix of rank \(k\). Hence, applying the standard realization theory, we find \((I_k, A, b, c) \in S_{k,1,1}\) such that \(\mathcal{H} = H_{kk}(I_k, A, b, c)\). If \(\tau = 2k\) there exists by part (iii) of Remark 4.11 a nonzero scalar \(g\) such that \(gt \cdot H_{kk}(I_k, A, b, c) \in \text{Hank}_{1,1}(k, k \times (k + 1))\), where \(g\) is given by (4.13). Since \(gt \cdot \mathcal{H}\) can be (uniquely) extended to an infinite Hankel matrix of rank \(k\), there exists \((I_k, \tilde{A}, \tilde{b}, \tilde{c}) \in S_{k,1,1}\) such that \(gt \cdot \mathcal{H} = H_{kk+1}(I_k, \tilde{A}, \tilde{b}, \tilde{c})\). Then \(\mathcal{H} = g^{-1} \cdot H_{kk+1}(I_k, \tilde{A}, \tilde{b}, \tilde{c})\). Hence \(\mathcal{H} = H_{kk+1}(I_k - t\tilde{A}, \tilde{A}, \tilde{b}, \tilde{c})\) by Proposition 4.7 and so \((E, A, b, c) \equiv (I_k - t\tilde{A}, \tilde{A}, \tilde{b}, \tilde{c})\) is a \(k\)-dimensional partial descriptor realization of \(h^\tau\). Finally part (i) of Proposition 3.11 shows that \((I_k, A, b, c)\) and \((E, A, b, c)\), respectively, are minimal.

(ii) Since we are dealing with the scalar case, it is clear that \(k = d(h^\tau)\) (see (3.26)). Using (i), it is found that \(d(h^\tau) = \delta(h^\tau)\). Hence, by part (ii) of Proposition 3.11, a partial descriptor realization of \(h^\tau\) is minimal if and only if it is controllable, observable and of dimension \(k\).

(iii) Since by assumption \(2n < \tau + 1\), Lemma 4.9 can be applied, and (iii) follows by the same arguments as statement (iii) of Theorem 4.16. □
5. Construction of minimal partial descriptor realizations

In this section we present an algorithm for the construction of minimal partial descriptor realizations in Weierstrass form. The algorithm is based on the use of the following standard procedure for constructing state space realizations from Hankel matrices, see for example, [6,27] or [5].

Let $H = [H_{i+j-1}]_{i,j=1}^{\infty}$ be an infinite block Hankel matrix of finite rank $n$, or equivalently (see Lemma 2.1), a block Hankel matrix $[H_{i+j-1}]_{i,j=1}^{n^2+1}$ of finite block size $(n + 1) \times (n + 1)$ which satisfies the condition

$$\text{rank } H(n, n) = \text{rank } H(n + 1, n + 1) = n.$$  

For each pair of index lists $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_l\}$ of increasing positive integers denote by $H(I,J)$ the $k \times l$ submatrix of $H$ consisting of those rows and columns which are indexed by the elements of $I$ and $J$, respectively.

A regular minimal realization $(A,B,C)$ of $H$ (that is, $H_{i} = CA^{i-1}B$ for all $i$) can be obtained by means of the following steps.

Step 1. Find the indices $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and $\beta = \{\beta_1, \ldots, \beta_n\}$ (written in increasing order) of the first $n$ linearly independent rows and columns of $H$, respectively.

Step 2. Construct the four submatrices $H(\alpha, \beta)$, $H(\alpha + p, \beta)$, $H(\alpha, m)$ and $H(p, \beta)$ of $H$, where $p$, $m$ and $\alpha + p$ are short for $\{1, 2, \ldots, p\}$, $\{1, 2, \ldots, m\}$ and $\{\alpha_1 + p, \ldots, \alpha_n + p\}$, respectively.

Step 3. Compute

$$(A, B, C) = (H(\alpha + p, \beta)H(\alpha, \beta)^{-1}, H(\alpha, m), H(p, \beta)H(\alpha, \beta)^{-1}).$$

$(A, B, C)$ is a minimal realization of $H$.

The following algorithm is a procedure for computing a minimal partial descriptor realization in Weierstrass form for a finite matrix sequence $H^\tau$ of length $\tau$. It is assumed that $H^\tau$, or equivalently, the associated central Hankel matrix (1.7) satisfies the rank assumption (see Definition 2.19).

**Algorithm 5.1 (Minimal partial descriptor realization).** Let $H^\tau = (H_1, \ldots, H_\tau)$ be a sequence of $p \times m$ matrices, and assume that the associated central Hankel matrix $\mathcal{H}$ satisfies

$$n := \text{rank } \mathcal{H} < k,$$

where $k = [(\tau + 1)/2]$ is the largest integer $\leq (\tau + 1)/2$.

Step 1. Find the indices $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and $\beta = \{\beta_1, \ldots, \beta_n\}$ of the first $n$ linearly independent rows and columns of the central Hankel matrix $\mathcal{H}$, respectively.

Step 2. Determine the regularity index $r$ of $\mathcal{H}$ and form the sublists $\alpha^{\text{reg}} = \{\alpha_1, \ldots, \alpha_r\}$, $\beta^{\text{reg}} = \{\beta_1, \ldots, \beta_r\}$ (both saturated from below) and $\alpha^{\text{sing}} = \{\alpha_{r+1}, \ldots, \alpha_n\}$, $\beta^{\text{sing}} = \{\beta_{r+1}, \ldots, \beta_n\}$ (both saturated from above). Determine the singular row and column index lists $(y_1, \ldots, y_p)$ and $(z_1, \ldots, z_m)$ of $\mathcal{H}$ by (2.18) and (2.20), respectively.
Step 3. Construct a minimal realization of the regular part of $\mathcal{H}$ by

$$(A_1, B_1, C_1) := (\mathcal{H}(\alpha_{\text{reg}} + p, \beta_{\text{reg}})\mathcal{H}(\alpha_{\text{reg}}, \beta_{\text{reg}})^{-1},$

$$\mathcal{H}(\alpha_{\text{reg}}, m), \mathcal{H}(p, \beta_{\text{reg}})\mathcal{H}(\alpha_{\text{reg}}, \beta_{\text{reg}})^{-1}).$$

Step 4. Determine the singular part $\mathcal{H}^{(2)} = [H_{i+j-1}^{(2)}]$ of $\mathcal{H} = [H_{i+j-1}]$ by

$$H_j^{(2)} := H_j - C_1 A_1^{-1} B_1, \quad j = 1, \ldots, \tau.$$

Step 5. Define the “reversed” singular part $\hat{\mathcal{H}}^{(2)} = [\hat{H}_{\tau+1-j}]$ of $\mathcal{H}$ by

$$\hat{H}_j^{(2)} := H_{\tau-j+1}^{(2)}, \quad j = 1, \ldots, \tau.$$

Step 6. Form the associated “reversed” index lists $\hat{\alpha}^{\text{sing}}$ and $\hat{\beta}^{\text{sing}}$ as

$$\hat{\alpha}^{\text{sing}} := \bigcup_{j=1}^{p} \{j, j + p, \ldots, j + (y_j - 1)p\},$$

$$\hat{\beta}^{\text{sing}} := \bigcup_{j=1}^{m} \{j, j + m, \ldots, j + (z_j - 1)m\}$$

(written in increasing order).

Step 7. Construct a minimal realization of the reversed singular part $\hat{\mathcal{H}}^{(2)}$ by

$$E_2 := \hat{\mathcal{H}}^{(2)}(\hat{\alpha}^{\text{sing}} + p, \hat{\beta}^{\text{sing}})\hat{\mathcal{H}}^{(2)}(\hat{\alpha}^{\text{sing}}, \hat{\beta}^{\text{sing}})^{-1},$$

$$B_2 := \hat{\mathcal{H}}^{(2)}(\hat{\alpha}^{\text{sing}}, m),$$

$$C_2 := \hat{\mathcal{H}}^{(2)}(p, \hat{\beta}^{\text{sing}})\hat{\mathcal{H}}^{(2)}(\hat{\alpha}^{\text{sing}}, \hat{\beta}^{\text{sing}})^{-1}.$$

Step 8. Put the realizations $(A_1, B_1, C_1)$ and $(E_2, B_2, C_2)$ together to obtain

$$E := \begin{bmatrix} I_r & 0 \\ 0 & E_2 \end{bmatrix}, \quad A := \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C := [C_1 \ C_2].$$

$(E, A, B, C)$ is a minimal partial descriptor realization of $H^\tau$.

A few comments are in order. Let

$$\mathcal{H} = \begin{bmatrix} H_1^{(1)} & \cdots & H_1^{(1)} \\ \vdots & \ddots & \vdots \\ H_k^{(1)} & \cdots & H_k^{(1)} \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & H_{\tau+1-s}^{(2)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & H_{\tau+1-s}^{(2)} & \cdots & H_2^{(2)} \end{bmatrix}$$
(l = k if τ = 2k − 1, and l = k + 1 if τ = 2k) be the decomposition of the central Hankel matrix \( \mathcal{H} \) into its regular and singular parts, where \( s \) denotes the singularity index of \( \mathcal{H} \). By (3.18), a descriptor system

\[
\begin{bmatrix}
I_r & 0 \\
0 & E_2
\end{bmatrix}
, \begin{bmatrix}
A_1 & 0 \\
0 & I_{n-r}
\end{bmatrix}
, \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
, \begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
\]

in Weierstrass form is a minimal partial descriptor realization of \( H^\tau \) if and only if the subsystems \((A_1, B_1, C_1)\) and \((E_2, B_2, C_2)\) are minimal state space realizations of

\[
\mathcal{H}^{(1)} = \begin{bmatrix}
H_1^{(1)} & \cdots & H_l^{(1)} \\
& \vdots & \\
H_k^{(1)} & \cdots & H_\tau^{(1)}
\end{bmatrix}
\]

\[
\hat{\mathcal{H}}^{(2)} = \begin{bmatrix}
H_1^{(2)} & \cdots & H_{\tau+1-s}^{(2)} & 0 & \cdots & 0 \\
& \vdots & \\
& \vdots \\
H_{\tau+1-s}^{(2)} & 0 & \\
& \vdots \\
& \vdots \\
& \vdots \\
0 & \cdots & 0
\end{bmatrix}
\]

respectively; that is, \( H_i^{(1)} = C_1 A_1^{i-1} B_1 \) and \( H_{\tau+1-i}^{(2)} = C_2 E_2^{i-1} B_2 \) for \( i = 1, \ldots, \tau \).

Moreover, the index lists \( \hat{\alpha}^{\text{sing}} \) and \( \hat{\beta}^{\text{sing}} \) are the lists of the first \( n - r \) linearly independent rows and columns of \( \hat{\mathcal{H}}^{(2)} \), respectively. Therefore Step 7 yields a minimal realization of \( \hat{\mathcal{H}}^{(2)} \). This shows that the above algorithm in fact constructs a minimal partial descriptor realization of \( H^\tau \).

References


