GLOBAL OPTIMIZATION ALGORITHMS FOR LINEARLY CONSTRAINED INDEFINITE QUADRATIC PROBLEMS

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Abstract—In this paper we consider the problem of finding the constrained global optimum of an indefinite quadratic function. Since such a function may have many local optima, finding the global optimum is a computationally difficult problem. We give an overview of the most important methods used, which include Benders decomposition, concave programming approaches, enumerative techniques, and bilinear programming.

1. INTRODUCTION

Global optimization of constrained quadratic problems has been the subject of active research during the last two decades. Quadratic programming is a very old and important problem of optimization. It has numerous applications in many diverse fields of science and technology and plays a key role in many nonlinear programming methods. Nonconvex quadratic programming refers to problems where the objective function is concave or indefinite. A substantial literature exists for the problem of finding the global minimum of a concave function subject to linear and nonlinear constraints. A survey of deterministic methods for global concave minimization problems can be found in [32,59] and in the recent monograph by Pardalos and Rosen [61].

In this paper we consider the problem of finding a globally optimal solution to nonconvex quadratic problems of the form

$$\min_{x \in \mathbb{R}^n} f(x) = c^T x + \frac{1}{2} x^T Q x,$$

where $Q_{n \times n}$ is an indefinite symmetric matrix, $c, x \in \mathbb{R}^n$, and $P$ is a bounded polyhedron in $\mathbb{R}^n$. The matrix $Q$ is indefinite iff it has at least one negative and one positive eigenvalue.

Apart from its importance as a mathematical programming problem, indefinite quadratic programming arises in several practical applications including production planning, microeconomic theory, and transportation problems. Recently it found applications in VLSI chip design [36,41,85]. Certain aspects of physical chip design can be formulated as an indefinite quadratic problem of special structure.

We are concerned here with several aspects of indefinite quadratic programming. Methods that have been proposed include Benders decomposition, concave programming approaches, bilinear programming, enumeration, and gradient projection methods. Algorithms for structured problems such as linear complementarity, product of linear forms, and problems with network constraints have also been developed.
2. GENERAL PROPERTIES

2.1. Problem complexity

Although many nonlinear programming algorithms are capable of finding a local optimum to problem (1.1), in global optimization there is, in general, no local criterion in deciding whether a given local solution is the global one. For nonconvex functions there may be many local optima, whose function values differ substantially from the global. In fact as it is shown by the next example, an indefinite quadratic program may have $2^m$ local minima, where $m$ is the number of negative eigenvalues of $Q$.

**Example.** Consider the problem

$$
\text{global min} \quad f(x) = -\sum_{i=1}^{m} (c_i x_i + x_i^2) + \sum_{i=m+1}^{n} (x_i + 1)^2,
$$

subject to $-1 \leq x_i \leq 1$, $i = 1, \ldots, n$, \hfill (2.1)

where $c_i$, $i = 1, \ldots, m$ are sufficiently small negative numbers. Let $x = (y, z)$, where $y = (x_1, \ldots, x_m)$ and $z = (x_{m+1}, \ldots, x_n)$. Then $y \in \{-1, 1\}^m$ and $z \in \{-1, 1\}^{n-m}$. Any point of the form $v = (y, -1, \ldots, -1)$ for any $y \in \{-1, 1\}^m$ is a local minimum point, with $z = (-1, \ldots, -1)$ the global minimum. Therefore, this problem has $2^m$ local minima.

From the computational complexity point of view problem (1.1) is an NP-hard problem. Consider the "knapsack feasibility problem" which is a well known NP-hard problem: Given $n + 1$ positive integers $a_1, \ldots, a_n$; $b$ is there a feasible solution to the problem:

$$
\text{aizi=b,} \quad i=1, \ldots, n?
$$

This problem is equivalent to the following (in general indefinite) quadratic problem:

$$
\text{global min} \quad f(x) = \left(\sum_{i=1}^{n} a_i x_i - b\right)^2 + x^T(e - x),
$$

subject to $0 \leq x_i \leq 1$, $i = 1, \ldots, n$, \hfill (2.2)

where $e = (1, \ldots, 1) \in \mathbb{R}^n$. Note that for any feasible point, $f(x) \geq 0$. Hence, the knapsack feasibility problem has a solution if and only if the global minimum of the corresponding indefinite quadratic function is zero. This shows that the general problem (1.1) is NP-hard.

In [63] it is also shown that the problem of checking local optimality for a feasible point of (1.1) and the problem of checking if a local minimum is strict are also NP-hard. In addition, it is shown that the problem of checking whether $f(x)$ is locally strictly convex is also NP-hard. This is important, since second-order conditions for local minima of (1.1) require that the objective function be locally strictly convex. It is therefore evident that the general indefinite quadratic problem is indeed a difficult problem to solve.

2.2. Characterization of solution points

In some restricted classes of functions such as strict quasi-convex the local minimizers are also global. For problems with a concave objective function every global (local) minimum occurs at some vertex of the feasible domain $P$. The same is true for linear complementarity problems [60] and problems with a quasi-concave objective function [5,44]. For the general indefinite quadratic problem, the solution occurs on a boundary point, not necessarily a vertex.

**Theorem 2.1.** The optimal solution of problem (1.1) occurs at some boundary point of the feasible domain $P$. 
PROOF. Let \( x^* \) be a global minimum and assume that \( x^* \) is an interior point of \( P \). Let \( \lambda \) be a negative eigenvalue of \( Q \) and let \( u \) be its corresponding eigenvector. Since \( x^* \) is an interior point of \( P \) and \( \nabla f(x^*) = Qx^* + c = 0 \) or \( Qx^* = -c \), there exists \( \varepsilon > 0 \) such that \( y = x^* + \varepsilon u \in P \). Then
\[
f(x^* + \varepsilon u) = f(x^*) + \frac{1}{2}\varepsilon^2 \lambda u^T u,
\]
which implies that \( f(x^* + \varepsilon u) < f(x^*) \), a contradiction.

EXAMPLE. Consider the problem
\[
\text{global min } f(x) = x_1^2 - x_2^2,
\]
where
\[
P = \{ z : -1 \leq z_1 \leq 1, -1 \leq z_2 \leq 1 \}.
\]
Note that \( f(v) = 0 \) for all vertices of \( P \). However, \( f(0,1) = f(0,-1) = -1 \) is the global optimum.

THEOREM 2.2. If \( x^* \) solves (1.1), then \( x^* \) is also optimal for the linear program
\[
\min_{x \in P} \nabla f(x^*)^T x.
\]

PROOF. For any feasible point \( y \) and any \( \lambda \in (0,1) \), the point \( z = x^* + \lambda(y - x^*) \) is also feasible and therefore \( f(z) - f(x^*) \geq 0 \), or \( \lambda(c + Qx^*)^T(y - x^*) + \lambda^2(y - x^*)^T Q(y - x^*)/2 \geq 0 \). Since \( \lambda \neq 0 \), it follows that \( (c + Qx^*)^T(y - x^*) \geq -\lambda(y - x^*)^T Q(y - x^*)/2 \). For \( \lambda \) arbitrarily close to 0, we have that \( (c + Qx^*)^T(y - x^*) \geq 0 \), that is, \( (\nabla f(x^*))^T y \geq (\nabla f(x^*))^T x^* \) for all feasible points \( y \).

The above theorem is useful when the feasible domain of (1.1) is the rectangle \( P = \{ z : a_i \leq z_i \leq b_i, i = 1, \ldots, n \} \). In that case, the range of the gradient in \( P \) can be used to reduce the dimension of the problem by fixing variables. Suppose that \( m_i \leq \partial f(x)/\partial x_i \leq M_i \), for all \( x \in [a, b] \). Then, it is clear that
\[
\begin{align*}
(i) \text{ If } m_i > 0, & \text{ then } x^*_i = a_i. \\
(ii) \text{ If } M_i < 0, & \text{ then } x^*_i = b_i.
\end{align*}
\]

There are very few results that characterize the global solution points of constrained indefinite quadratic problems. Suppose in problem (1.1) the polyhedron \( P \) is given defined by a set of linear inequalities:
\[
P = \{ x \in \mathbb{R}^n : Ax \leq b \},
\]
where \( A \) is an \( m \times n \) matrix and \( b \in \mathbb{R}^m \). The following theorem \([28]\) uses active constraints to characterize the solution of (1.1). A constraint of the form \( a^T x \leq d \) is said to be active at the point \( x^* \) if \( a^T x^* = d \).

THEOREM 2.3. If the matrix \( Q \) has \( s \) negative eigenvalues, then at least \( s \) of the constraints are active at any local (global) solution point.

If the matrix \( Q \) is negative definite, this theorem implies the classical result that any local (global) minimum of a concave function occurs at an extreme point of the feasible domain \( P \). It also implies that when \( Q \) has at least one negative eigenvalue, then at least one constraint is active, and therefore the global minimizer lies on the boundary of \( P \). Furthermore, it can be shown that if \( Q \) is indefinite and the global minimum occurs at an interior point of a face of \( P \), then \( Q \) must have exactly one negative eigenvalue. These results have been applied to generate test problems of indefinite quadratic problems with a known (nonvertex) solution.

Since \( \min_{x \in P} f(x) = -\max_{x \in P} (-f(x)) \), we may consider the problem of globally maximizing or minimizing an indefinite quadratic form.
3. BENDERS DECOMPOSITION METHOD

Kough [39] proposed an algorithm for the indefinite quadratic problem that uses a generalized Benders cut procedure that was developed by Geoffrion [25] (see also [9]). The problem considered is in the following (polar) form:

\[
\begin{align*}
\max & \quad x^T x - y^T y, \\
\text{s.t.} & \quad Ax + By + c \geq 0, \quad z \in \mathbb{R}^n, \quad y \in \mathbb{R}^t
\end{align*}
\]

and \(A, B,\) are \(m \times n\) and \(m \times t\) matrices respectively and \(c \in \mathbb{R}^m\). The separability of the objective function into \(x, y\) variables (convex and concave part) suggests a natural decomposition for the Benders cut problem. If \(x_0\) is a fixed feasible point we define the following convex problem \(P(x_0)\):

\[
\begin{align*}
\max & \quad -y^T y, \\
\text{s.t.} & \quad By \geq -(Ax_0 + c).
\end{align*}
\]

Let \(y_0\) be the optimal solution to the above problem. Using problem \(P(x_0)\) we define the function

\[v(x_0) = x^T x - y_0^T y_0.\]

Then problem (3.1) is equivalent to the next problem \(P_v(x)\):

\[
\begin{align*}
\max & \quad v(x), \\
\text{s.t.} & \quad x \in \mathbb{R} = \{x : Ax + By + c \geq 0 \text{ for some } y\}.
\end{align*}
\]

Here \(R\) is the projection of \(\{x : Ax + By + c \geq 0\}\) into the \(x\)-space. When the Benders cut method is applied to (3.1) it generates approximations \(v_k(x)\) of \(v(x)\) and \(R_k\) of \(R\) satisfying \(v_k(x) \geq v_{k+1}(x) \geq v(x)\) for all \(x\) and \(R_{k+1} \supseteq R_k \supseteq R\) for \(k_2 \geq k_1\). At the \(k\)th step the approximate problem

\[
\begin{align*}
\max & \quad v_k(x), \\
\text{s.t.} & \quad x \in \mathbb{R}_k,
\end{align*}
\]

is solved to obtain an approximate optimal \(x_k\). If \(x_k \in R\), a Benders feasibility cut is generated in order to obtain \(R_{k+1}\). Otherwise a Benders cut is generated to obtain \(v_{k+1}(x)\). This algorithm will process \(\varepsilon\)-finite convergence [25]. Furthermore, Kough developed exact cuts and proposed a modified finite algorithm. A generalized Benders decomposition approach that can be used to solve indefinite quadratic problems is also discussed in [22].

4. CONCAVE PROGRAMMING METHODS

Using an affine transformation (e.g., [61]) we can transform a quadratic problem to one with a separable objective function. Assuming that the quadratic problem is indefinite, the problem can be assumed to have the following separable form:

\[
\begin{align*}
\min & \quad f(x) - g(y), \\
\text{s.t.} & \quad Ax + By + c \leq 0, \\
& \quad x \in X, \quad y \in Y,
\end{align*}
\]

where \(X, Y\) are polyhedral sets in \(\mathbb{R}^{n_1}, \mathbb{R}^{m_2}\), respectively. In this form the objective function has been written as the difference of convex functions. Tuy [81,82] used this formulation to express (4.1) as an equivalent global concave minimization problem (introducing a new variable \(t\))

\[
\begin{align*}
\min & \quad t - g(y), \\
\text{s.t.} & \quad f(x) \leq t, \\
& \quad Ax + By + c \leq 0, \\
& \quad x \in X, \quad y \in Y.
\end{align*}
\]
The objective function $t - g(y)$ is concave, and the constraint set is convex since $f(x)$ is a convex function. Tuy uses concave programming techniques adopted for this particular structure to develop an algorithm for the problem (4.2) when the original function is quadratic indefinite. In this case Tuy considers $f(x) = x^T x$ and $g(y) = y^T y$.

A transformation that preserves the eigenstructure of the quadratic form was used in [58,69] to obtain an equivalent separable problem. Suppose the objective function is given by $\phi(x) = c^T x - \frac{1}{2} x^T Q x$, where the symmetric matrix $Q_{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the equivalent separable problem has the form

$$\min_{x \in \Omega} f(x) = f_1(x) + f_2(x), \quad (4.3)$$

where $\Omega = \{ x : Ax \leq b, 0 \leq x_i \leq \beta_i, i = 1, \ldots, n \} \subseteq \mathbb{R}^n$ and the objective function is partitioned to concave and convex parts:

$$f_1(x) = \sum_{i=1}^{k} \theta_i(x_i) = \sum_{i=1}^{k} \left( c_i x_i - \frac{1}{2} \lambda_i x_i^2 \right),$$

the concave part corresponding to $\lambda_i > 0 \ i = 1, \ldots, k$

$$f_2(x) = \sum_{i=k+1}^{n} \theta_i(x_i) = \sum_{i=k+1}^{n} \left( c_i x_i - \frac{1}{2} \lambda_i x_i^2 \right),$$

the convex part corresponds to $\lambda_i \leq 0 \ i = k + 1, \ldots, n$.

Initially a linear approximation of the concave part is obtained, that is $\gamma_i(x_i) = (c_i - \frac{1}{2} \lambda_i \beta_i) x_i \leq \theta_i(x_i), \ i = 1, \ldots, k$, and the following convex problem is solved

$$\min_{x \in \Omega} \gamma(x) = \sum_{i=1}^{k} \gamma_i(x_i) + f_2(x). \quad (4.4)$$

**Theorem 4.1.** $|f(x) - \gamma(x)| \leq \frac{1}{8} \sum_{i=1}^{k} \lambda_i \beta_i^2$.

**Proof.** It is easy to verify that

$$|\theta_i(x_i) - \gamma_i(x_i)| \leq \frac{1}{2} \lambda_i |x_i(x_i - \beta_i)|, \ i = 1, \ldots, k.$$

Then the maximum of $|x_i(x_i - \beta_i)|$ for $0 \leq x_i \leq \beta_i$ occurs at the midpoint $x_i = \beta_i/2$. Therefore $|\theta_i(x_i) - \gamma_i(x_i)| \leq \frac{1}{8} \lambda_i \beta_i^2$ and the theorem is proved.

If $\hat{x}$ is the solution of the approximate problem (4.4) and $f^*$ the global optimum, we have that

$$0 \leq f(\hat{x}) - f^* \leq \frac{1}{8} \sum_{i=1}^{k} \lambda_i \beta_i^2,$$

$$\gamma(\hat{x}) \leq f^* \leq f(\hat{x}).$$

A more practical error bound has been obtained using an appropriate scaling factor $\Delta f_1$, which measures the range of $f_1(x)$ over the rectangle $R = \{ x_i : 0 \leq x_i \leq \beta_i, i = 1, \ldots, k \}$. This bound is given by

$$\frac{f(\hat{x}) - f^*}{\Delta f_1} \leq \sigma, \quad \text{where } 1/4 \leq \sigma \leq 1.$$

It is important to notice here that this is a worst-case bound. Preliminary computational results indicate that the actual error is on average much smaller.
A branch and bound bisection technique is used next to obtain improved upper and lower bounds for the global optimum. In the worst case, the method uses piecewise linear approximations of the separable convex and concave parts. The piecewise linear program can be formulated as a linear zero-one mixed integer program. The introduction of the zero-one integer variables is required only for the concave part, since minimization of a piecewise linear convex problem is equivalent to a linear problem. By appropriately closing the piecewise approximation an $\epsilon$-approximate solution is guaranteed (for any given tolerance $\epsilon > 0$). The complete analysis of this approach can be found in [51,58,69]. Concave programming techniques proposed in [35] can also be extended for the solution of indefinite quadratic problems. Similar approaches are also discussed in [64].

5. BILINEAR PROGRAMMING APPROACH

The bilinear programming problem belongs to the class of nonconvex optimization problems and in its general form can be stated as follows:

\[
\text{global min } \ f(z, y), \tag{5.1}
\]
\[ z \in X, y \in Y \]

where $X, Y$ are polyhedral sets in $R^n$ and $R^m$, respectively and $f(z, y)$ is a continuous bilinear function. Here we consider the case where $f(z, y) = c^Tz + d^Ty + z^TQy$ and the polyhedral sets are given by

\[
X = \{ z \in R^n : A_1z = b_1, z \geq 0 \},
\]
\[
Y = \{ y \in R^m : A_2y = b_2, y \geq 0 \},
\]

where $Q, A_1, A_2$ are matrices of dimensions $n \times m, k \times n, l \times m$ respectively, and $c \in R^n, d \in R^m, b_1 \in R^k, b_2 \in R^l$. Then this bilinear problem can be expressed as an (indefinite) quadratic problem of the special structure

\[
\text{global min } \ a^Tz + \frac{1}{2}z^TMz, \tag{5.2}
\]
\[ z \in Z = \{ z \in R^{n+m} : Az = b, z \geq 0 \},
\]

where

\[
z = \begin{bmatrix} x \\ y \end{bmatrix}, \quad a = \begin{bmatrix} c \\ d \end{bmatrix}, \quad M = \begin{bmatrix} 0 & Q \\ Q^T & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.
\]

Consider now the quadratic problem

\[
\min_{z \in P} \phi(z) = c^Tx + x^TQx, \tag{5.3}
\]

where $P$ is a bounded polyhedron in $R^n$, $Q$ is an $n \times n$ symmetric matrix, and $z, c \in R^n$. This problem can be formulated as a jointly constrained bilinear programming problem of the form

\[
\text{min } B(x, y) = x^Ty, \tag{5.4}
\]
\[ x \in P, \quad y - Qx = c.
\]

A branch and bound procedure that has been developed originally by Al-Khayyal and Falk [1,2] can be used to solve (5.4). This approach is based on the use of convex envelopes [37] of $B(x, y)$ over hyperrectangles. Given the rectangle $R = \{(x, y) : l \leq x \leq L, m \leq y \leq M \}$, define $R_i = \{(x_i, y_i) : l_i \leq x_i \leq L_i, m_i \leq y_i \leq M_i \}$ so that $R = R_1 \times \ldots \times R_n$. The convex envelope of the function $B_i = x_iy_i$ over $R_i$ is given by

\[
E_i(x_i, y_i) = \max\{m_i x_i + l_i y_i - l_i m_i, M_i x_i + L_i y_i - L_i M_i \},
\]
and therefore the convex envelope of $B(x, y) = x^T y$ over $R$ is given by
\[ E(x, y) = \sum_{i=1}^{n} E_i(x_i, y_i). \]

Other bilinear programming algorithms can be used to solve problem (5.4). Gallo and Ulkucu [23] developed algorithms that are based in Tuy's cone-splitting method [80] for linearly constrained concave minimization problems. Another algorithm that guarantees finite convergence, based again on Tuy's method, has been proposed by Vaish and Shetty [83,84] and later improved by Sherali and Shetty [74,75]. The equivalence of the general concave problem and bilinear programming has been proved by [79]. Different approaches for bilinear programming algorithms can be found in [3,17,18,38].

6. PRODUCT OF LINEAR FORMS

A special class of indefinite quadratic problems has the form
\[ \text{global } \min_{x \in P} f(x) = l_1(x)l_2(x), \]
where $P$ is a bounded polyhedron in $\mathbb{R}^n$, $l_1(x) = a^T x + \alpha$, $l_2(x) = b^T x + \beta$ where $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.

Note that if $a, b$ are linearly dependent vectors, then the above problem is reduced to the solution of at most two linear programs. Therefore we can always assume that $a, b$ are linearly independent vectors. For quadratic forms that can be expressed as a product of linear functions, see also [71].

One of the first algorithms for the solution of (6.1) was proposed in [77,78] for the special case where $f(x)$ is quasi-concave.

**Theorem 6.1.** If $l_1(x) > 0$, $l_2(x) > 0$ for all $x \in P$ then $f(x)$ is quasi-concave.

A simplex type algorithm is used to obtain local solutions. Note that if $f(x)$ is quasi-convex, then the solution is attained at some vertex of $P$. More conditions on quasi-concavity or quasi-convexity of $f(x)$ can be found in [5,44]. Assuming that $a, b$ are linearly independent and $||a|| = ||b|| = 1$ we observe that $f(x) = \frac{1}{2} x^T Q x + c^T x + C_0$ where $Q = ab^T + ba^T$ has at most 3 distinct eigenvalues:
\[ \lambda = 0 \text{ with multiplicity } n - 2, \]
\[ \lambda = a^T b + 1 \text{ (with corresponding eigenvector } a + b), \]
\[ \lambda = a^T b - 1 \text{ (with corresponding eigenvector } a - b). \]

Using this fact and [44] we can prove the following necessary and sufficient conditions for quasi-convexity and quasi-concavity.

**Theorem 6.2.**

(a) $f(x)$ is quasi-concave in $\mathbb{R}^n_+$ iff $Q \preceq 0$, $c \preceq 0$ and there exists $q \in \mathbb{R}^n$ such that $Qq = c$ with $c^T q \geq 0$.

(b) $f(x)$ is quasi-convex in $\mathbb{R}^n_+$ iff $Q \succeq 0$ and there exists $q \in \mathbb{R}^n$ such that $Qq = c$ with $q^T c \geq 0$.

Another finite algorithm for solving problem (6.1) when $f(x)$ is quasi-concave has been proposed by [27]. This again uses a simplex type technique in which two basic variables are replaced by two nonbasic variables at a time. The problem is solved starting with a basic feasible solution and showing the conditions under which the solution can be improved.

A different algorithm, again for the quasi-concave case, has been developed in [4] by solving a sequence of linear problems to obtain the global optimum. These linear programs have the same constraint set $P$, which makes the approach computationally feasible. Recently [62], a more efficient algorithm has been proposed for problem (6.1), without any restriction on the objective function. The algorithm is based on a method that reduces the original problem to a
two-dimensional problem. The feasible domain of the reduced problem, is the projection of the original feasible domain onto the space of variables corresponding to the nonzero eigenvalues. In addition, it can be shown that if the number of vertices of the projected domain is bounded by a polynomial in \( n \), then problem (6.1) is solved in polynomial time. We close this section by mentioning that problems of (6.1) type have applications in microeconomics [4], and certain aspects of physical chip design [41].

7. CONCLUDING REMARKS

We gave an overview of the most general approaches for the solution of indefinite quadratic problems. Specialized methods exist for the solution of problems with network constraints [21] and a number of heuristics. Mueller [49] developed a method based on the gradient projection method. Although the method is not guaranteed to converge, it was successful on the test problems used. Ritter [68] used a cutting plane algorithm but Zwart [86] constructed a counterexample for which Ritter's method converges to a nonglobal optimum in the limit. Other approaches not discussed here can be found in the references.

Indefinite quadratic programming is an intrinsically hard problem and it is not expected to develop efficient solution methods for the general problem when its size is very large. However, it is possible to devise numerically practical algorithms that obtain good approximate solutions for fairly large number of convex variables and a reasonable number of concave variables (see for example [51,58]).

There is little computational experience with algorithms for indefinite quadratic programs. Many algorithms have been implemented and tested on certain problems. The process of evaluating and comparing such implementations is very difficult since it requires a variety of test problems with a known solution. Except for the special case of concave programming, very few methods have been proposed for generating test problems for global optimization algorithms [61]. A method proposed in [53] generates quadratic test problems for global optimization algorithms. Given a bounded polytope and a vertex, the method constructs a quadratic (indefinite) function whose global minimum is attained at the selected vertex. Using this method, specific structure can be imposed on the problem by specifying the eigenvalues of the corresponding symmetric matrix and other parameters on which the problem difficulty depends. Generation of problems with a nonvertex solution is discussed in [28].

With the advent of supercomputers, further advances in both algorithm development and experimentation are now possible for large-scale global optimization problems. Using vector processing or multiprocessing techniques, supercomputers are an important component in a rapidly growing field of parallel processing. Algorithms for global optimization problems involve a large number of function evaluations, a large number of iterations and a large number of local searches. Concurrent function evaluations are well suited for multiprocessing. Similarly concurrent iterations used for local minimizers can be implemented using multiprocessors. An introduction to this area of parallel computing for global optimization problems can be found in [52,57,60,64,66].

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