ELSEVIER

Discrete Applied Mathematics 100 (2000) 183–202

DISCRETE APPLIED MATHEMATICS

and clique-independent sets

Venkatesan Guruswami *, C. Pandu Rangan

Department of Computer Science and Engineering, Indian Institute of Technology, Madras-600036, India

Received 15 February 1996; revised 22 March 1999; accepted 5 April 1999

Abstract

A minimum clique-transversal set $MCT(G)$ of a graph $G=(V,E)$ is a set $S \subseteq V$ of minimum cardinality that meets all maximal cliques in G. A maximum clique-independent set $MCI(G)$ of G is a set of maximum number of pairwise vertex-disjoint maximal cliques. We prove that the problem of finding an $MCT(G)$ and an $MCI(G)$ is **NP**-hard for cocomparability, planar, line and total graphs. As an interesting corollary we obtain that the problem of finding a minimum number of elements of a poset to meet all maximal antichains of the poset remains **NP**-hard even if the poset has height two, thereby generalizing a result of Duffas et al. (J. Combin. Theory Ser. A 58 (1991) 158–164). We present a polynomial algorithm for the above problems on Helly circular-arc graphs which is the first such algorithm for a class of graphs that is not *clique-perfect*. We also present polynomial algorithms for the weighted version of the clique-transversal problem on strongly chordal graphs, chordal graphs of bounded clique size, and cographs. The algorithms presented run in linear time for strongly chordal graphs and cographs. These mark the first attempts at the weighted version of the problem.

Keywords: Graph algorithm; **NP**-hardness; Clique-transversal set; Clique-independent set; Line graph; Total graph; Poset

1. Introduction

All graphs we deal with are finite, undirected and simple. As usual for a graph G, $\alpha(G)$, $\omega(G)$, $\delta(G)$ and $\Delta(G)$ denote the size of a maximum independent set, the size of a largest clique, the minimum degree and the maximum degree, respectively. A *clique* is a set of pairwise adjacent vertices. A *maximal clique* is a clique that

* Corresponding author. MIT Laboratory for Computer Science, NE43-370, 545 Technology Square, Cambridge, MA 02139, USA.

E-mail addresses: venkat@theory.1cs.mit.edu (V. Guruswami), rangan@iitm.ernet.in (C.P. Rangan)

PII: S0166-218X(99)00159-6

is maximal under inclusion. A set of vertices that meets all maximal cliques of G is called a *clique-transversal set* of G or $CT(G)$. A minimum clique-transversal set $MCT(G)$ is a $CT(G)$ of minimum cardinality. As defined in [20], the *clique-transversal number* $\tau_C(G)$ of G is the size of an $MCT(G)$. A *clique-independent set* $CI(G)$ is a collection of pairwise vertex-disjoint maximal cliques. The *clique-independence number* $\alpha_C(G)$ is the size of a maximum clique-independent set $MCI(G)$ [5]. We have the obvious min–max duality inequality $\tau_C(G) \geqslant \alpha_C(G)$ for any graph G. We define a graph G to be *clique-perfect* if $\tau_C(H) = \alpha_C(H)$ for every induced subgraph H of G.

An extensive study on clique-transversal sets is done from the theoretic point of view in [1,18,20]. The aim of this paper is to investigate the problems of determining $\tau_C(G)$ and $\alpha_C(G)$ of a graph G from the algorithmic point of view. In [5], a linear time algorithm to find $\tau_C(G)$ and $\alpha_C(G)$ of a strongly chordal graph G was presented (given a strong-elimination ordering) and it was proved that the problems are **NP**-hard on split graphs. It was also proved that strongly chordal graphs are clique-perfect. In [2], a polynomial algorithm for the above problems on a comparability graph was presented by reducing the problems to some Menger-type questions and it was shown that comparability graphs are clique-perfect.

In this paper, we prove that determining $\tau_C(G)$ and $\alpha_C(G)$ is **NP**-hard for complements of bipartite graphs, planar graphs and line graphs. We obtain as a corollary that the problem of finding a minimum number of elements of a poset to meet all maximal antichains of a poset (defined later in the paper) is **NP**-hard, even for height-two posets. We then propose an $O(n^2)$ algorithm to find an $MWCT(G)$ – a minimum weighted $CT(G)$ – of a strongly chordal graph with a positive weight attached to each vertex, a linear algorithm for the same problem on interval graphs and a polynomial algorithm for the same on chordal graphs of bounded clique-size. To the best of our knowledge these are the first attempts on the weighted version. We then propose an $O(n^2)$ time algorithm for finding an $MCI(G)$ and an $MCT(G)$ of a Helly circular-arc graph (HC-graph) G. HC-graphs are in general not even perfect (in the usual sense) and in particular they are not clique-perfect. So, this is the first polynomial time solution of the above problems on a class of graphs that is not clique-perfect.

The *line graph* of a graph G, denoted $L(G)$, is defined as follows: The vertices of $L(G)$ correspond to edges of G and two vertices in $L(G)$ are adjacent whenever the corresponding edges of G are. The *total graph* of G, denoted $T(G)$, has vertex set $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent whenever they are neighbors in G. Clearly $T(G)$ has both G and $L(G)$ as induced subgraphs, see [14] for some interesting properties of line and total graphs.

For a graph G, its *vertex-clique incidence matrix* (or just *clique-matrix*) $M(G)$ is a 0–1 valued matrix defined as follows. Let $v_1, v_2, \ldots, v_n$ be the vertices of G and $C_1, C_2, \ldots, C_p$ be the maximal cliques of G. Then $M(G)$ has p rows and n columns and is defined by: For $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant n$, $M(G)_{ij} = 1$ if and only if $v_j \in C_i$. For all graph theoretic terms not defined explicitly here, refer [14].

2. NP-hardness results

The clique-transversal and the clique-independent set problems are known to be **NP**-hard on split graphs. We now obtain **NP**-hardness results on cocomparability graphs, total graphs, planar graphs and line graphs. The decision versions of the problems that are proved to be **NP**-complete are formally specified below:

Problem: CLIQUE-TRANSVERSAL SET

Instance: A graph $G=(V,E)$ and an integer $K\geqslant 1$.

Question: Is there a set of K or fewer vertices that *meets* all maximal cliques of G?

Problem: CLIQUE-INDEPENDENT SET

Instance: A graph $G=(V,E)$ and an integer $K\geqslant 1$.

Question: Is there a collection of K or more pairwise vertex-disjoint maximal cliques?

It is clear that CLIQUE-INDEPENDENT SET is in **NP**. However, CLIQUE-TRANSVERSAL SET is not known to be in **NP**, for a graph may have an exponential number of maximal cliques and verifying whether a given set indeed meets all maximal cliques cannot be done in polynomial time in any obvious fashion.

2.1. Complements of bipartite graphs

Theorem 2.1. *CLIQUE-INDEPENDENT SET is* **NP**-*complete on complements of bipartite graphs.*

Proof. The reduction is from the independent set problem on total graphs of cubic graphs. The **NP**-hardness of this problem follows from the fact that the edge-domination problem is **NP**-hard on planar cubic graphs [16,21] and that a minimum edge dominating set of a graph G corresponds to a maximum independent set of $T(G)$, the total graph of G [21]. Let $T(G)$ be the total graph of a *cubic* graph $G=(V,E)$. We reduce the problem of finding $\alpha(T(G))$ to the problem of finding $\alpha_C(H)$ of a suitably defined complement H of a bipartite graph. Let $H=(X,Y,E_1)$ where $X\cap Y=\emptyset$, X,Y are cliques and $E_1\subseteq X\times Y$ represents edges with one end in X and the other in Y.

To construct H, take $X=E$ and $Y=V$ and $E_1=\{(e,x),(e,y)\colon e=(x,y)\in E\}$. Informally, V and E correspond to the two cliques of H, and a vertex e in $X(=E)$ is adjacent in H to precisely the two vertices in $Y(=V)$ on which e is incident in G. Since G is simple, two vertices in X are adjacent to at most one common vertex in Y. Hence, no clique in H can contain more than one vertex in both X as well as Y.

Note that, if $G\neq K_1$ or K_2, the only maximal cliques of H are V, E, $C_e=\{e,x,y\}$ where $e=(x,y)\in E$, and $C_v=\{v\}\cup\{e\colon e \text{ is adjacent to } v\}$ where $v\in V$. Hence, unless $G=K_1$ or K_2, we have

$$\alpha_C(H)=\alpha(T(G))$$

since the maximum clique-independent set of H will not include the maximal cliques V or E, and $C_r\cap C_s=\emptyset$ if and only if r and s are two non-adjacent vertices of $T(G)$.

The result now follows from the fact that H can be constructed from G in polynomial time. $\square$

Remark. Along the lines of the above proof, one can also prove that if G is a triangle-free graph with $\delta(G)\geqslant 2$, then $\alpha(T(G))=\alpha_C(T(G))$. Now, the edge-domination problem is **NP**-hard on bipartite graphs G [21] and we may assume that the bipartite graph G has $\delta(G)\geqslant 2$ by using the following attachment at each vertex v of G: add new vertices a_v, x_v, y_v, z_v and w_v and the edges (v, a_v), (a_v, x_v), (x_v, y_v), (y_v, z_v), (z_v, w_v) and (w_v, x_v). It is easy to see that these transformations increase the size of a minimum edge dominating set of G by exactly $2|V(G)|$, Hence, the independent set problem is **NP**-hard on total graphs of bipartite graphs with $\delta\geqslant 2$, and thus, CLIQUE-INDEPENDENT SET is **NP**-complete on the class of total graphs.

We now consider the clique-transversal problem on complements of bipartite graphs. It is known that CLIQUE-TRANSVERSAL SET is **NP**-hard on complements of comparability graphs. This follows from the result of [9] on posets. For a poset $(P,\leqslant)$, define $S\subseteq P$ to be an *antichain* if no two elements of S are comparable. A *maximal antichain* is an antichain that is maximal under inclusion. The height of a poset is the size of a maximum chain (a set of pairwise comparable elements) in the poset.

It is **NP**-hard to decide whether a poset has a fiber of size at most k, where a fiber is defined to be any set that meets all maximal antichains of the poset [9]. However, the poset constructed in the reduction of [9] has height $k+1$ and hence the associated comparability graph will have clique number at least $k+1$. We now prove a stronger hardness result for posets of height two, so that the associated comparability graph has clique number 2, i.e. it is in fact bipartite.

Theorem 2.2. *CLIQUE-TRANSVERSAL SET is* **NP**-*hard on complements of bipartite graphs.*

Proof. The reduction is from the minimum vertex-cover problem on simple graphs G with $\delta(G)\geqslant 2$. Let $G=(V',E')$ be a graph with $\delta(G)\geqslant 2$ with $V'=\{1,2,\dots,n\}$, $E'=\{e_j=(f_j,g_j)\,|\,1\leqslant j\leqslant m\}$. Construct a graph $H=(X,Y,E)$ with $E\subseteq X\times Y$ and X,Y,E have the same meanings as in the previous theorem. Set

$$X=E'\cup\left(\bigcup_{i=1}^{n}\{p_i,q_i\,|\,1\leqslant i\leqslant n\}\right),$$

$$Y=V'\cup\{h_i\,|\,1\leqslant i\leqslant n\},$$

$$E=\{(e_j,f_j),(e_j,g_j)\,|\,1\leqslant j\leqslant m\}\cup\{(p_i,i),(p_i,h_i),(q_i,h_i)\,|\,1\leqslant i\leqslant n\}.$$

Since G is simple, by our construction, two vertices in X are adjacent to at most one common vertex in Y, so no clique in H can have more than one vertex in both

X and Y. The only maximal cliques (other than Y) containing h_i are $\{p_i, h_i, i\}$ and $\{p_i, q_i, h_i\}$ and the only maximal clique (other than X) containing q_i is $\{p_i, q_i, h_i\}$.

We first prove, if $\gamma(G)$ denotes the size of the minimum vertex-cover of G, $\tau_C(H) \leqslant \gamma(G)+n$. Indeed let $\gamma(G)=k$ and $S=\{i_1,\dots,i_k\}$ be a minimum vertex-cover of G. Then $T=\{i_1,\dots,i_k,p_1,p_2,\dots,p_n\}$ is a clique-transversal set of H because it meets X,Y, the cliques comprising i and all vertices in X adjacent to i for $1\leqslant i\leqslant n$, the maximal cliques $\{e_i,f_i,g_i\}$ for $1\leqslant i\leqslant m$ and $\{p_i,q_i,h_i\}$, $\{p_i,h_i,i\}$ for $1\leqslant i\leqslant n$ and these are the only maximal cliques in H.

Conversely, let $T=\{x_1,x_2,\dots,x_r\}$ be a minimum clique-transversal set of H so that $\tau_C(H)=r$. If any x_j $(1\leqslant j\leqslant r)$ is equal to h_i or q_i, replace it with p_i to get T' which will meet all the maximal cliques with the exception of (possibly) Y. Since, to meet $\{p_i,q_i,h_i\}$ at least one of p_i,q_i,h_i must belong to T, we must have $\{p_1,p_2,\dots,p_n\}\subseteq T'$. Let $T'=\{y_1,y_2,\dots,y_r\}$, if any $y_i=e_j$ for some $1\leqslant j\leqslant m$, replace e_j with one of f_j,g_j to get T''. Now $T''\cap E'=\emptyset$. Also T'' is a clique-transversal set of H because if f_j is substituted for e_j, the only maximal clique causing problems is the clique with g_j and all vertices in Y adjacent to g_j, however since $p_{g_j}\in T''$, T'' meets this clique as well. Also clearly $|T''|=|T|=\tau_C(H)$. Let $T''=\{p_1,p_2,\dots,p_n,i_1,i_2,\dots,i_s\}$ with $n+s=r$. Now for T'' to meet all cliques of the form $\{e_j,f_j,g_j\}$ for $1\leqslant j\leqslant m$, $\{i_1,i_2,\dots,i_s\}$ must be a vertex-cover of G, so $s\geqslant\gamma(G)$ which implies $|T''|=\tau_C(H)=r=n+s\geqslant n+\gamma(G)$.

Thus, $\tau_C(H)=n+\gamma(G)$, and the result (**NP**-hardness) now follows since H can be constructed from G in polynomial time. □

Since every bipartite graph is a comparability graph [13], we get:

Corollary 2.1. *The clique-transversal and the clique-independent set problems are* **NP**-*hard on cocomparability graphs.*

Since posets naturally map onto comparability graphs through the Hasse-diagram concept, we get the following interesting result, which strengthens the result in [9]:

Corollary 2.2. *For a poset* $(P,\leqslant)$ *it is* **NP**-*hard to determine a set of minimum cardinality that meets all maximal antichains of the poset even if the height of the poset is two.*

Remark. Using ideas similar to those of the proof of Theorem 2.2, one can prove that CLIQUE-TRANSVERSAL SET is **NP**-hard on the class of total graphs. The reduction is from the vertex-cover problem on triangle-free graphs, which is **NP**-complete [23]. Given a triangle-free graph G on n vertices, construct G' by attaching a new P_4 at each vertex of G, note that $|V(G')|=4n$ and G' is also triangle free. Now consider the total graph $T(G')$ of G. It can be proved that $\tau_C(T(G'))=2n+\gamma(G)$ using techniques similar to those used in proving Theorem 2.2. It follows that finding $\tau_C(H)$ for a total graph H is **NP**-hard.

2.2. Planar graphs

We now turn towards planar graphs. Two parameters related to $\alpha_C(G)$ and $\tau_C(G)$ are the neighborhood-covering number $\rho_N(G)$ and the neighborhood-independence number $\alpha_N(G)$ (see [17]). Let $G=(V,E)$. The *neighborhood* $N_G(v)$ (or simply $N(v)$), stands for the set $\{u: (v,u)\in E\}$. The *closed neighborhood* $N_G[v]$ (or simply $N[v]$), is given by $N(v)\cup\{v\}$. A *neighborhood-covering set* (NC-set) C is a set of vertices such that $E=\bigcup\{E[v]: v\in C\}$, where $E[v]$ is the set of edges in the subgraph induced by $N[v]$. $\rho_N(G)$ is the size of a minimum NC-set. A *neighborhood-independent set* (NI-set) of G is a set of edges in which no two distinct edges belong to the same $E[v]$ for any $v\in V$. $\alpha_N(G)$ is the size of a maximum NI-set. Clearly $\rho_N(G)\geqslant\alpha_N(G)$. The status of the problem of determining $\rho_N(G)$ and $\alpha_N(G)$ for a planar graph G was left open in [5].

We now proceed to settle the algorithmic complexity of determining $\alpha_C(G)$, $\tau_C(G)$, $\rho_N(G)$ and $\alpha_N(G)$ for a planar graph G.

Theorem 2.3. *The decision version of the neighborhood covering problem as well as CLIQUE-TRANSVERSAL SET are* **NP**-*complete for a planar graph G with $\Delta=3$.*

Proof. The decision version of the neighborhood-covering problem is clearly in **NP**. CLIQUE-TRANSVERSAL SET is in **NP** for a planar graph G since a planar graph has a polynomial number of maximal cliques. We now prove **NP**-hardness.

Let H be an arbitrary planar cubic graph. We reduce the problem of determining the size, $\gamma(H)$, of a minimum vertex-cover of H, which is known to be **NP**-hard [16], to the problem of finding $\rho_N(G)$ and $\tau_C(G)$ for a planar graph G with $\Delta(G)=3$.

Form G by inserting two new vertices in each edge of H. Formally let $H=(V_1,E_1)$ with $V_1=\{1,2,\dots,n\}$ and $E_1=\{e_i=(f_i,g_i)\mid 1\leqslant i\leqslant m\}$. In $G=(V,E)$, u_i,v_i are inserted on e_i for $1\leqslant i\leqslant m$. Formally $V=V_1\cup\{u_i,v_i\mid 1\leqslant i\leqslant m\}$ and $E=\bigcup_{i=1}^{m}\{(f_i,u_i),(u_i,v_i),(v_i,g_i)\}$.

Clearly G has no triangle, in fact girth$(G)\geqslant 9$. So, $\rho_N(G)=\tau_C(G)=\gamma(G)$. If S is a minimum vertex-cover of H, then by adding exactly one vertex from $\{u_i,v_i\}$ for $1\leqslant i\leqslant m$ to S one can easily obtain a vertex-cover of G. So $\gamma(G)\leqslant\gamma(H)+m$.

Conversely, let T be a minimum vertex-cover of G. Clearly, $T\cap\{u_i,v_i\}\neq\emptyset$ for $1\leqslant i\leqslant m$. Also if both $u_i,v_i\in T$ for some i, then we can replace u_i by f_i in T. So we may assume $|T\cap(V-V_1)|=m$. Let $S=T\cap V_1$. Since T meets all edges of G, S must meet all edges of H, which implies $|S|\geqslant\gamma(H)$. Thus, $|T|=\gamma(G)\geqslant\gamma(H)+m$.

Hence we have $\rho_N(G)=\tau_C(G)=\gamma(G)=\gamma(H)+m$. The conclusion now follows since G can be constructed in polynomial time from H. $\square$

We now turn to the related problems of determining $\alpha_N(G)$ and $\alpha_C(G)$ for a planar graph G. We first state some useful lemmas:

Lemma 2.1. *Let G be a triangle-free graph. Then, any maximal clique of the line graph of G, L(G), is the set of edges of G incident to some vertex of G.*

Lemma 2.2. *Let G be as in Lemma* 2.1. *Then, two maximal cliques in L(G) intersect if and only if their corresponding vertices (in G) are adjacent in G.*

Theorem 2.4. *The determination of $\alpha_N(G)$ and $\alpha_C(G)$ for a planar graph with $\Delta=3$ is* **NP**-*hard.*

Proof. The reduction is from the independent set problem on planar cubic graphs which is **NP**-hard [16]. Let $H=(V,E)$ be an arbitrary planar cubic graph, $|E|=m, |V|=n$. Form G as in Theorem 2.3, then $\Delta(G)=3$ and G is triangle free. Let $K=L(G)$ be the line graph of G; from Lemmas 2.1 and 2.2 and the arguments in the proof of Theorem 2.3 it follows that $\alpha_C(K)=\alpha(G)=m+\alpha(H)$.

It is clear that K is planar, because from the way it is constructed, K can be obtained from H by replacing each vertex of H by a triangle and each edge (u,v) of H by a path of length 2 connecting a vertex in the triangle corresponding to u to a vertex in the triangle corresponding to v.

Since, replacing each edge of an NI-set by a maximal clique containing it yields a clique-independent set (CI-set), we have $\alpha_C(K)\geqslant\alpha_N(K)$. Conversely let $T=\{C_1,C_2,\ldots,C_p\}$ be a clique-independent set of K. Define $S=\{e_1,e_2,\ldots,e_p\}$ where e_i is an edge of the graph K whose endpoints are in C_i for $1\leqslant i\leqslant p$. We claim S is an NI-set. Indeed, if $e_j,e_k\in S$, $(j<k)$, then e_j,e_k are independent as T is a CI-set. Let $e_j=(r,s)$ and $e_k=(x,y)$. If possible, let $\{e_j,e_k\}\subset E[w]$. Since $\Delta(K)=3$, $w\in\{r,s,x,y\}$, say $w=r$. So we must have $C_k=\{r,x,y\}$ as $\omega(K)=3$ and an edge of K is in at most one triangle of K (by the construction of K). This implies that $C_j\cap C_k\neq\emptyset$, a contradiction. Thus $\alpha_N(K)\geqslant\alpha_C(K)$.

Hence $\alpha_C(K)=\alpha_N(K)=m+\alpha(H)$. The result now follows since K is planar with $\Delta(K)=3$ and can be obtained from H in polynomial time. □

Corollary 2.3. *The determination of $\alpha_N(G)$ and $\alpha_C(G)$ for a line graph G with $\Delta(G)=3$ is* **NP**-*hard.*

Proof. Follows from the reduction of Theorem 2.4. □

2.3. Line graphs

From Corollary 2.3, we obtain the **NP**-completeness of CLIQUE-INDEPENDENT SET on line graphs. We now turn towards the clique-transversal problem. For a graph $G=(V,E)$ denote by $t(G)$ the minimum number of edges in E such that every triangle in G has at least one of those edges. It is shown in [23] that determination of $t(G)$ is **NP**-complete, but we now prove a stronger result when $\Delta(G)$ is bounded.

Lemma 2.3. *The problem of determining $t(G)$ for a graph G with $\Delta(G)=6$ and $\delta(G)\geqslant 2$ is* **NP**-*hard.*

Proof. The reduction is from the vertex-cover problem on cubic graphs. Let $H=(V,E)$ be a cubic graph, with $V=\{1,2,\dots,n\}$, $E=\{e_j=(g_j,h_j)\mid 1\leqslant j\leqslant m\}$.

Construct $G=(V',E')$ as follows: Initially take n independent edges $f_i=(u_i,u_i')$ for $1\leqslant i\leqslant n$ as part of G (these edges correspond to vertices of H). Then for each $e_j\in E$, introduce three vertices a_j,b_j,c_j that induce a K_3 in G and join a_j (resp. b_j) to both ends of f_{g_j} (resp. f_{h_j}). Also join c_j to one of the ends of both f_{g_j} and f_{h_j}. Note that $V'=\{u_i,u_i'\mid 1\leqslant i\leqslant n\}\cup\{a_j,b_j,c_j\mid 1\leqslant j\leqslant m\}$, $|V'|=2n+3m$ and that $|E'|=9m+n$. Also, clearly $\delta(G)\geqslant 2$.

Since H is 3-regular it is evident that $\Delta(G)\leqslant 7$. Also for $1\leqslant i\leqslant n$ one can ensure that, of the three c_j's adjacent to an end of any f_i, two of them are adjacent to one end (of f_i) and the third c_j is adjacent to the other end (of f_i). This will make sure that $\Delta(G)=6$.

If S is a set of $t(G)$ edges meeting all triangles in G, then one can choose S such that for each j ($1\leqslant j\leqslant m$) S uses at least one of the two edges f_{g_j},f_{h_j} and exactly two edges involving a_j,b_j and c_j. Indeed, this can be seen as follows: Let d_j (resp. e_j) denote the end of f_{g_j} (resp. f_{h_j}) to which c_j is adjacent. To meet the triangle $\{a_j,b_j,c_j\}$, there is no advantage in choosing the edge (a_j,b_j), and so assume for definiteness that (a_j,c_j) is chosen. Then one can choose the edge f_{g_j} in order to meet the triangle formed by the ends of f_{g_j} and a_j and choose the edge (b_j,e_j) to meet the triangle formed by b_j and the ends of f_{h_j} as well as the triangle $\{b_j,c_j,e_j\}$.

Conversely, it is easy to see that any such set S must use at least two edges involving a_j,b_j and c_j for $1\leqslant j\leqslant m$ irrespective of the number of edges in S from $\{f_{g_j},f_{h_j}\}$. Thus we may assume that S contains $2m$ edges from $E'-\{f_i\mid 1\leqslant i\leqslant n\}$ and a set $U=\{f_{i_1},f_{i_2},\dots,f_{i_k}\}$ of k edges from $\{f_i\mid 1\leqslant i\leqslant n\}$. Clearly, S meets all triangles in G iff for each $j, 1\leqslant j\leqslant m, U$ has at least one of f_{g_j},f_{h_j} i.e, iff $\{i_1,i_2,\dots,i_k\}$ forms a vertex-cover of H. Hence $t(G)=2m+\gamma(H)$. Since G can be constructed from H in polynomial time, the result follows. □

We now prove the central theorem of this subsection.

Theorem 2.5. *Let $L(H)$ be the line graph of H where $\Delta(H)=7$. Then CLIQUE-TRANSVERSAL SET is* **NP**-*complete when restricted to such graphs H.*

Proof. Presence in **NP** follows from the fact that a line graph has a polynomial number of maximal cliques. The reduction is from the problem of finding $t(G)$ for a graph $G=(V,E)$ with $\delta(G)\geqslant 2$ and $\Delta(G)=6$.

Let $V=\{v_1,v_2,\dots,v_n\}$. Form $H=(V',E')$ as follows: Initially H will contain G as an induced subgraph, and for each $i, 1\leqslant i\leqslant n$, add vertices w_i,u_{i1},u_{i2},u_{i3} and u_{i4}; let $\{u_{ij}\mid 1\leqslant j\leqslant 4\}$ induce a K_4 in H and join v_i to w_i and w_i to u_{i1}. We observe the following:

- $d_H(v) \geqslant 3$ for all $v \in V$ since $\delta(G) \geqslant 2$,
- $d_H(u_{ij}) \geqslant 3$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant 4$,
- for $1 \leqslant i \leqslant n$, $N_H(w_i) = \{v_i, u_{i1}\}$ and $(v_i, u_{i1}) \notin E'$,
- $\Delta(H) = 7$ since $\Delta(G) = 6$.

It follows that the set of edges incident at any vertex in H would form a maximal clique in $L(H)$, the line graph of H. Also, the edges of any triangle of H will form a maximal clique of size 3 in $L(H)$. Hence we conclude that $\tau_C(L(H)) = tec(H)$ where $tec(H)$ denotes the size of a minimum edge-cover of H that includes an edge from each triangle in H.

Claim. $tec(H) = 3n + t(G)$.

Proof. For each i, $1 \leqslant i \leqslant n$, in order to meet the triangles in the K_4 formed by u_{ij} $(1 \leqslant j \leqslant 4)$, exactly two edges (assume $p_i = (u_{i1}, u_{i4})$ and $q_i = (u_{i2}, u_{i3})$ for definiteness) are chosen. Thus to cover w_i, we may choose $r_i = (v_i, w_i)$ (since u_{i1} is already covered by p_i). The edges p_i, q_i, r_i $(1 \leqslant i \leqslant n)$ would cover all the vertices in H and the optimal way to meet the remaining triangles in H would be to meet the triangles in G optimally by using $t(G)$ edges from E. It follows that $tec(H) = 3n + t(G)$ and claim is proved.

Hence $\tau_C(L(H)) = 3n + t(G)$, and the result now follows from Lemma 2.3 and the fact that $L(H)$ can be constructed from G in polynomial time. □

3. Minimum weighted clique-transversal sets

We now turn to the weighted version of the clique-transversal problem where vertices of the graph have positive weights attached to them and we seek to find a clique-transversal set which has a minimum total weight.

3.1. Strongly chordal graphs

A graph is an *interval graph* if it is the intersection graph of a family of intervals on the real line [13]. A graph is *chordal* if it contains no cycle of length greater than three as an induced subgraph. An *s-sun* is a chordal graph with a Hamiltonian cycle $x_1, y_1, x_2, y_2, \ldots, x_s, y_s, x_1$ such that each y_i is of degree two [6]. A *strongly chordal graph* is a chordal graph with no s-sun as an induced subgraph, for all $s \geqslant 3$. It is proven in [10] that a graph is strongly chordal if and only if its vertices have a *strong elimination order* $v_1, v_2, \ldots, v_n$; i.e. for each $i, 1 \leqslant i \leqslant n, N_i[v_j] \subseteq N_i[v_k]$ whenever $v_j, v_k \in N_i[v_i]$ and $j < k$ (here $N_i[x]$ stands for the closed neighborhood of x in the subgraph G_i of G induced by $\{v_i, v_{i+1}, \ldots, v_n\}$).

Let $G = (V, E, w)$ be a strongly chordal graph with a strong elimination order $v_1, v_2, \ldots, v_n$ and with a positive weight $w(v)$ attached to each vertex $v \in V$. Construct a weighted graph, called the vertex-clique incidence graph $H=(V',E',w')$ of G, as follows:

The vertex set of H is $V\cup C$ where $C=\{C_1,C_2,\dots,C_p\}$ is the set of maximal cliques of G. In H, two vertices of V are adjacent iff they are adjacent in G, C is an independent set and $C_i\in C$ is adjacent to $v_j\in V$ iff $v_j\in C_i$ in G. The weight function w' is such that

$$w'(v)=\begin{cases} w(v) & \text{if } v\in V,\\ \infty & \text{otherwise, i.e. } v\in C.\end{cases}$$

Lemma 3.1. *If $G=(V,E)$ is strongly chordal, the graph H constructed above is also a strongly chordal graph.*

Proof. Let $v_1,v_2,\dots,v_n$ be a strong elimination ordering of the vertices in G. Since G is chordal, if $C_1,C_2,\dots,C_p$ are the maximal cliques of G, we may assume that $C_j=N_{i_j}[v_{i_j}]$ for $1\leqslant j\leqslant p$, for some $1\leqslant i_1<i_2<\cdots<i_p\leqslant n$. Consider the ordering π of the vertices of H obtained by first taking the ordering $v_1,v_2,\dots,v_n$ and inserting the "vertex" C_j just before v_{i_j} for each $j, 1\leqslant j\leqslant p$.

We claim π is a strong elimination ordering of H. To prove this, note that, for $1\leqslant r\leqslant n$, the vertices that occur later than v_r in π (including the vertex v_r itself) are $S_r=\{v_r,v_{r+1},\dots,v_n,C_{q_r},C_{q_r+1},\dots,C_p\}$ where $q_r=\min\{t\colon 1\leqslant t\leqslant p, i_t>r\}$, and, for $1\leqslant s\leqslant p$, the vertices that occur later than C_s in π are $T_s=\{v\colon v\in C_s\}$.

Consider the *higher neighborhood* $N_r[\pi]=N_H[v_r]\cap S_r$ of the vertex v_r in the ordering π. We have $N_r[\pi]\subseteq\{v_r,v_{r+1},\dots,v_n\}$, since if $(v_r,C_j)\in E(H)$, then $v_r\in C_j$, which implies $i_j\leqslant r$ (by definition of i_j as the smallest index of a vertex in C_j) and hence $j<q_r$ and $C_j\notin N_r[\pi]$. Let $v_j,v_k\in N_r[\pi]$ with $r\leqslant j<k\leqslant n$. We will prove that $N_H[v_j]\cap S_r\subseteq N_H[v_k]\cap S_r$. Since $v_1,v_2,\dots,v_n$ was a strong elimination ordering of G, $N_i[v_j]\subseteq N_i[v_k]$, which implies

$$N_H[v_j]\cap\{v_r,v_{r+1},\dots,v_n\}\subseteq N_H[v_k]\cap\{v_r,v_{r+1},\dots,v_n\}.$$

Now suppose $C_s\in N_H[v_j]$, for some $q_r\leqslant s\leqslant p$. Then, since $r<i_s\leqslant j<k$, $(v_r,v_j)\in E,(v_r,v_k)\in E$, and $(v_{i_s},v_j)\in E$, we have, by the strong elimination ordering of $v_1,v_2,\dots,v_n$, that $(v_k,v_{i_s})\in E$, which implies $v_k\in C_{i_s}$, or equivalently, $C_s\in N_H[v_k]$. Hence,

$$N_H[v_j]\cap\{C_{q_r},C_{q_r+1},\dots,C_p\}\subseteq N_H[v_k]\cap\{C_{q_r},C_{q_r+1},\dots,C_p\}.$$

Combining the above two inclusions, we have for $1\leqslant r\leqslant n$, and for $r\leqslant j<k\leqslant n$,

$$N_H[v_j]\cap S_r\subseteq N_H[v_k]\cap S_r. \tag{1}$$

Similarly one can prove that, for $1\leqslant s\leqslant p$, if $v_j,v_k\in T_s$ for some $j,k,i_s\leqslant j<k\leqslant n$, then,

$$N_H[v_j]\cap F_s\subseteq N_H[v_k]\cap F_s, \tag{2}$$

where F_s is the set of vertices occurring later than C_s (including C_s) in the ordering π.

It is now easy to see that (1) and (2) imply that π is a strong elimination ordering of H. $\square$

Lemma 3.2. *A minimum weighted clique-transversal set of G corresponds to a minimum weighted vertex-dominating set of H.*

Proof. Follows easily from the way H is constructed and the fact that no C_i can be in a minimum weighted vertex-dominating set of H since it is assigned infinite weight. □

Lemma 3.3. *The vertex-clique incidence matrix of a chordal graph has* $\mathrm{O}(n+m)$ *entries equal to* 1.

Proof. Let G be a chordal graph and let $v_1, v_2, \ldots, v_n$ be a perfect elimination ordering of the vertices of G so that for each $i, 1 \leqslant i \leqslant n$, $N_i[v_i]$ is a clique. Let $C_1, C_2, \ldots, C_p$ be the maximal cliques of G. Clearly each C_j, $1 \leqslant j \leqslant p$ is of the form $N_{i_j}[v_{i_j}]$ for some i_j, $1 \leqslant i_j \leqslant n$. Moreover the i_j's, $1 \leqslant j \leqslant p$, are all distinct. Thus $p \leqslant n$ and the number N of 1's in the matrix is given by

$$N = \sum_{j=1}^{p} |C_j| = \sum_{j=1}^{p} |N_{i_j}[v_{i_j}]| \leqslant m + p \leqslant m + n. \qquad \square$$

Theorem 3.1. *A minimum weighted clique-transversal set of a strongly chordal graph G can be obtained in* $\mathrm{O}(n+m)$ *time, given a strong elimination ordering of the vertices of G.*

Proof. The $\mathrm{O}(n)$ maximal cliques of G can be obtained in linear time. Construct the vertex-clique incidence graph H of G as detailed above. By Lemma 3.3, the vertex-clique incidence matrix of G has only $\mathrm{O}(n+m)$ entries equal to 1, so we find that H has $\mathrm{O}(n+m)$ edges and $\mathrm{O}(n)$ vertices. Also, by Lemma 3.1, H is strongly chordal, and a strong elimination ordering of its vertices can be obtained from a given one for G in $\mathrm{O}(n+m)$ time. The result now follows from Lemma 3.2 and the linear algorithm for finding a minimum weighted dominating set of a strongly chordal graph given its strong elimination ordering (see [11,15]). □

Since interval graphs form a subclass of strongly chordal graphs, we have the following corollary:

Corollary 3.1. *A minimum weighted clique-transversal set of an interval graph can be obtained in linear time.*

3.2. Chordal graphs of bounded clique size

Let $G = (V, E)$ be a chordal graph with $\omega(G) = k$ where k is a constant. We shall now prove that the minimum weighted clique-transversal set problem can be solved in polynomial time for chordal graphs whose maximum clique size is k for some constant k (this was also noted by Chang et al. [4] in their paper on generalized clique transversal on k-trees). A *maximum k-colorable subgraph* of a graph G is defined as an induced subgraph H of G such that $\chi(H) \leqslant k$ and H has a maximum number of vertices.

Theorem 3.2 (Yannakakis and Gavril [22]). *The maximum weighted k-colorable subgraph problem in chordal graphs is polynomial-time solvable when k is fixed.*

Lemma 3.4 (Corneil and Fonlupt [7]). *Let $G=(V,E)$ be a chordal graph in which all maximal cliques are of size k. Then, $W \subseteq V$ is a clique-transversal set of G if and only if $V-W$ induces a $(k-1)$-colorable subgraph of G.*

Theorem 3.3. *A minimum weighted clique-transversal set of a chordal graph with bounded clique size, say $\omega(G)=k$ for some constant k, can be found in polynomial time.*

Proof. Let $G=(V,E,w)$ be a chordal graph with $\omega(G)=k$ and a weight $w(v)$ attached to each $v \in V$. Let $C_1, C_2, \dots, C_p$ be the maximal cliques of G. Form H as follows: For each C_i $(1 \leqslant i \leqslant p)$ with $|C_i| < k$ add $k-|C_i|$ *new* vertices and join each of them to all vertices of C_i. Also assign very large weights M to the newly introduced vertices. Clearly H is a chordal graph with all maximal cliques of size k. From the way weights are given to the new vertices, it follows that any $MWCT(H) \subseteq V$. For some $W \subseteq V(H)$, let $H[W]$ be a maximum weight $(k-1)$-colorable subgraph of H. Then by Lemma 3.4 it follows that $V(H)-W \subseteq V$ is a minimum weight clique-transversal set of G. The result now follows since such a W can be obtained in polynomial time by Theorem 3.2. □

A graph is said to be a *k-tree* if it can be obtained, starting with a clique of size k, by repeated addition of a vertex and making it adjacent to an existing k-clique. It is easy to see that if G is a k-tree, then G is chordal and $\omega(G) \leqslant k+1$, so using Theorem 3.3 we can conclude:

Corollary 3.2 (Chang et al. [4]). *A minimum weighted clique-transversal set of a k-tree G can be obtained in polynomial time when k is fixed.*

3.3. Cographs

Cographs are graphs with no induced P_4. Cographs may also be defined recursively as follows:
- The graph consisting of an isolated vertex is a cograph.
- If G_1 and G_2 are cographs, then so is their union $G_1 \cup G_2$ and their join $G_1 + G_2$.

The following interesting characterization of cographs in terms of clique-transversal sets appears in [18]:

Theorem 3.4. *A connected graph $G=(V,E)$ is a cograph if and only if every minimal clique-transversal set of G is a maximal independent set.*

It can be proved (for instance by induction) that any maximal independent set of a cograph is also a clique-transversal set, and hence, using Theorem 3.4, one can reduce the problem of finding the minimum weighted clique-transversal set of a cograph G to the problem of computing the minimum weighted maximal independent set of $G-\{\text{isolated vertices in } G\}$, which can be done in linear time using standard cograph techniques. We may hence conclude:

Theorem 3.5. *A minimum weighted clique-transversal set of a cograph can be computed in linear time.*

We now also present a direct approach to the clique-transversal problem on cographs. Let $MWCT(G)$ stand for any minimum weight clique-transversal set of a graph G with positive weights attached to its vertices.

Lemma 3.5. *If $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ with $V_1 \cap V_2 = \emptyset$ and $G = G_1 + G_2$, then the maximal cliques C of G are precisely those of the form $C = C_1 \cup C_2$ where C_i is a maximal clique of G_i for $i = 1,2$.*

Proof. Clearly if $C = C_1 \cup C_2$ with C_i a maximal clique of G_i $(i = 1,2)$, then C is a maximal clique of $G = G_1 + G_2$. Conversely, if C is a maximal clique of G, define $C_i = C \cap V_i$ for $i = 1,2$; then $C = C_1 \cup C_2$. If say C_1 is not maximal in G_1, i.e $C_1 \subset C_1' \subseteq V_1$ where C_1' induces a clique, then we have $C = C_1 \cup C_2 \subset C_1' \cup C_2 \subseteq V$ with $C_1' \cup C_2$ inducing a clique in G, a contradiction since C is a maximal clique of G. $\square$

Let the weight of a set be the sum of weights of its elements. For a family of sets $F = \{S_1, S_2, \ldots, S_k\}$ define $\text{Min}(S_1, \ldots, S_k)$ to be the set in F with smallest weight. We now establish recurrence equations for $MWCT(G)$.

Lemma 3.6. *Let $G_i = (V_i, E_i)$ for $1 \leqslant i \leqslant k$ where $V_i \cap V_j = \emptyset$ for $1 \leqslant i < j \leqslant k$. Then*
(i) $MWCT(G_1 \cup G_2 \cup \cdots \cup G_k) = \bigcup_{i=1}^{k} MWCT(G_i)$,
(ii) $MWCT(G_1 + G_2 + \cdots + G_k) = \text{Min}(MWCT(G_1), \ldots, MWCT(G_k))$.

Proof. Part (i) is obvious. Part (ii) follows from Lemma 3.5 since all weights are positive. $\square$

The recurrence equations of Lemma 3.6, together with "standard" cograph techniques, yield Theorem 3.5.

4. Helly circular-arc graphs

In this section we consider another class of graphs for which the maximal cliques have a nice structure, viz. Helly circular-arc graphs. A family $\{T_i\}_{i\in I}$ of subsets of a set T is said to satisfy the Helly property if $J\subseteq I$ and $T_i\cap T_j\neq\emptyset$ for all $i,j\in J$ imply $\bigcap_{j\in J}T_j\neq\emptyset$. A graph is a *Helly circular-arc graph* if it is the intersection graph of a family of arcs on a circle that satisfies the Helly property [13]. Helly circular-arc graphs (HC-graphs) form a superclass of interval graphs. In this section, we describe polynomial algorithms for the clique-transversal and clique-independent set problems on HC-graphs.

4.1. Preliminaries

Lemma 4.1. *If C is a maximal clique of a Helly circular-arc graph G, then $G[V-C]$ is an interval graph.*

Proof. The arcs corresponding to vertices in C intersect pairwise and hence due to Helly property, they have a point P on the circle in common. Owing to the maximality of C, arcs corresponding to vertices in $V-C$ do not contain P, or equivalently $G[V-C]$ is an interval graph. □

Lemma 4.2. *An HC-graph G on n vertices has at most n maximal cliques.*

Proof. Let $\{A_j=(l(A_j),r(A_j))\colon 1\leqslant j\leqslant n\}$ denote the circular arcs corresponding to the n vertices of G where A_j is the arc joining the points $l(A_j)$ and $r(A_j)$ and oriented in counterclockwise sense from $l(A_j)$ to $r(A_j)$. $r(A_j)$ is called the right endpoint of the arc A_j.

Now, consider a maximal clique C of G. Due to the Helly property, the arcs corresponding to the vertices in C have a common point, say P, on the circle. Now suppose we move along the circle in the counterclockwise direction starting from P till we reach the first point Q on the circle which is the endpoint of one of the arcs $\{A_j\colon 1\leqslant j\leqslant n\}$, say that of A_s. Q must be a right endpoint of A_s for otherwise the clique C will not be maximal under inclusion. Moreover, by nature of its choice, this point Q will be present in all the arcs that correspond to vertices in C.

Thus, it follows that with each maximal clique C one can associate the right endpoint of an arc A_{f_C} where $1\leqslant f_C\leqslant n$ such that C is precisely the set of all arcs that contain $r(A_{f_C})$. Moreover, it is easy to see that the above construction ensures that if $C_1\neq C_2$ are two maximal cliques, then $f_{C_1}\neq f_{C_2}$. It therefore follows that the number of maximal cliques is at most n. □

Lemma 4.3. *Let $S_1,S_2,\ldots,S_k$ be an ordering of some non-empty subsets of a set $X=\{x_1,x_2,\ldots,x_n\}$. Moreover, for $1\leqslant j\leqslant n$ let the S_i's to which x_j belongs occur consecutively in the above ordering. Then the minimum cardinality of a subset $T\subseteq X$*

which satisfies $T \cap S_i \neq \emptyset$ for $1 \leqslant i \leqslant k$ equals the maximum number of pairwise disjoint sets that can be chosen from $S_1, \ldots, S_k$.

Proof. For $1 \leqslant i \leqslant n$, let r_i denote the index of the last set in the ordering $S_1, S_2, \ldots, S_k$ to which x_i belongs. Consider the following greedy algorithm:

(i) Set $i = 1$, $H = \emptyset$, $Y = \emptyset$

(ii) While $i \leqslant k$

- Set $H = H \cup \{S_i\}$.
- Let $x_j \in S_i$ be such that $r_j = \max\{r_p \mid x_p \in S_i\}$.
  Set $Y = Y \cup \{x_j\}$ and $i = r_j + 1$.

When the above algorithm terminates, we clearly have $Y \cap S_i \neq \emptyset$ for $1 \leqslant i \leqslant k$ and that H is set of pairwise disjoint S_j's.

Also we have $|H| = |Y|$. The conclusion now easily follows. □

Remark. Note that the above algorithm runs in $O(|X|)$ time once the r_i's are found.

A (0,1)-valued matrix is said to have *circular* 1'*s property for columns* if its rows can be permuted in such a way that the 1's in each column occur in a circular consecutive order; regard the matrix as wrapped around a cylinder. The following characterization of HC-graphs is due to Gavril:

Theorem 4.1 (Gavril [12]). *An undirected graph G is a HC-graph iff its vertex-clique incidence matrix has circular 1's property for columns.*

4.2. Idea behind the algorithm

Let $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$, be a HC-graph whose maximal cliques have been ordered as $C_1, C_2, \ldots, C_p$ so that the vertex-clique incidence matrix $M(G)$ has circular 1's property for columns.

For $1 \leqslant r,\ s \leqslant p$ the *circular segment* $[r, s]$ is defined as follows:

$$[r,s] = \begin{cases} \{z: r \leqslant z \leqslant s\} & \text{if } r \leqslant s, \\ \{1, 2, \ldots, p\} - \{z: s < z < r\} & \text{if } s < r. \end{cases}$$

Note that $[r,s] \cup [s,r] = \{1, 2, \ldots, p\}$ and $[r,s] \cap [s,r] = \{r,s\}$. Also if T is a proper subset of $\{1, 2, \ldots, p\}$ comprising of consecutive indices modulo p (i.e., indices p and 1 are considered consecutive) then there is a *unique* way to write T as a circular segment $[l_T, r_T]$. Also if $[r,s]$ is a circular segment, denote by $[r,s]^c$ the circular segment $\{1, 2, \ldots, p\} - [r,s]$.

For $1 \leqslant j \leqslant n$, let $\mathscr{C}_v(j)$ denote the set of indices of the maximal cliques that contain v_j; and for $1 \leqslant i \leqslant p$, let $INT(i)$ be the set of indices of the maximal cliques that intersect C_i. Due to the circular 1's property of the matrix $M(G)$, it follows that $\mathscr{C}_v(j)$ and $INT(i)$ are circular segments; let $INT(i) = [l_i, r_i]$, $1 \leqslant i \leqslant p$ and $\mathscr{C}_v(j) = [a_j, b_j]$, $1 \leqslant j \leqslant n$. Without loss of generality, we assume that $\mathscr{C}_v(j) \neq \{1, 2, \ldots, p\}$ for $1 \leqslant j \leqslant n$, for otherwise there exists v_j such that v_j belongs to all maximal cliques and this happens if and only if $deg_G(v_j) = n - 1$ which can be readily checked for.

For $1 \leqslant i \leqslant p$, consider the set of cliques Ω_i which is the set of cliques that are vertex-disjoint with C_i; clearly the set of indices of the cliques in Ω_i is $[l_i, r_i]^c$. Denote by $MCT(G_i)$ a set $P_i \subseteq V - C_i$ of minimum cardinality that meets all cliques in Ω_i and by $MCI(G_i)$ a set $Q_i \subseteq \Omega_i$ of maximum number of pairwise vertex-disjoint cliques in Ω_i. By Lemma 4.3, the remark following it and the circular 1's ordering of cliques, it follows that $MCT(G_i)$ and $MCI(G_i)$ can be computed in $\mathrm{O}(n)$ time and that $|MCT(G_i)| = |MCI(G_i)|$.

Let v_{x_i} be the vertex in $C_i \cap C_{l_i}$ belonging to the maximum number of cliques among the cliques $\{C_j\colon j \in INT(i)\}$. Similarly $v_{y_i} \in C_i \cap C_{r_i}$ is defined.

For $1 \leqslant i \leqslant p$, $S_i = MCI(G_i) \cup \{C_i\}$ is a $CI(G)$ and $T_i = MCT(G_i) \cup \{x_i, y_i\}$ is a $CT(G)$ (in fact, $MCT(G_i) \cup \{s_i, t_i\}$ is a $CT(G)$ for any $s_i \in C_i \cap C_{l_i}$ and any $t_i \in C_i \cap C_{r_i}$). It is clear that

$$\alpha_C(G) = \max_{1 \leqslant i \leqslant p} |S_i| = \max_{1 \leqslant i \leqslant p} |MCI(G_i)| + 1. \tag{3}$$

Now, for each i, $1 \leqslant i \leqslant p$,

$$\begin{aligned} \tau_C(G) &\leqslant |MCT(G_i)| + 2 \\ &= |MCI(G_i)| + 2 \text{ (because } |MCT(G_i)| = |MCI(G_i)|) \\ &\leqslant \alpha_C(G) + 1 \text{ (using(3))}. \end{aligned}$$

Combining this with the weak-duality relation $\tau_C(G) \geqslant \alpha_C(G)$, we get

$$\alpha_C(G) \leqslant \tau_C(G) \leqslant \alpha_C(G) + 1.$$

We may thus record the preceding discussion in the following result:

Theorem 4.2. *For a HC-graph G*, $\alpha_C(G) \leqslant \tau_C(G) \leqslant \alpha_C(G) + 1$.

Remark. It is easy to see that C_5, the cycle on five vertices, is an HC-graph with $\alpha_C(G) = 2$ and $\tau_C(G) = 3$, and so the above bound is tight.

The idea behind our algorithm is the following: Find S_i and T_i in the order $i = 1, 2, \ldots, p$ (recall that $|T_i| = |S_i| + 1$). Check if a T_j exists with $|T_j| = |S_i|$ or if a S_j exists with $|S_j| = |T_i|$ for some j, $1 \leqslant j < i$. In either case we have a $CT(G)$ and a $CI(G)$ of equal size and we can output these as the optimal $MCT(G)$ and $MCI(G)$. If this fails, we have $|S_1| = |S_2| = \cdots = |S_p| = \alpha_C(G)$. Then find a set of vertices Z_j that meets all cliques in $\{C_k\colon k \in [a_j, b_j]^c\}$ for $1 \leqslant j \leqslant n$. If any Z_j is such that $|Z_j \cup \{v_j\}| = \alpha_C(G)$, output $Z_j \cup \{v_j\}$ as $MCT(G)$ and S_1 as $MCI(G)$. Otherwise output S_1 and T_1 as $MCI(G)$ and $MCT(G)$, respectively.

4.3. The algorithm

We now proceed to formally specify the algorithm.

Algorithm *HC-Arc-CT-CI*
Input: A Family of Arcs $\mathscr{F}=\{A_j\colon 1\leqslant j\leqslant n\}$ on a circle that satisfy the Helly property.
Output: $MCT(G)$ and $MCI(G)$ where G is the intersection graph of the family $\mathscr{F}$.
Step 0:
Determine and order the maximal cliques and the vertices of G with circular 1's property as $C_1,\ldots,C_p$ and $v_1,v_2,\ldots,v_n$. For $1\leqslant i\leqslant p$, determine l_i,r_i,x_i and y_i. Also for $1\leqslant j\leqslant n$, determine a_j and b_j. (This finishes all the preprocessing.)
Step 1:
For $i=1,2,\ldots,p$ **do**
(i) Find $MCT(G_i)$ and $MCI(G_i)$ by the algorithm of Lemma 4.3 using the ordering $C_{r_i+1},\cdots,C_{l_i-1}$ of the cliques in Ω_i. Set
$S_i=MCI(G_i)\cup\{C_i\}$ and
$T_i=MCT(G_i)\cup\{v_{x_i},v_{y_i}\}$.
(ii) **If** $i=1$ **then** set
$size=|S_1|$
$CT\_save=T_1$ and
$CI\_save=S_1$.
(iii) **If** $i\neq 1$ and $|S_i|=size+1$ **then** set
$MCI(G)=S_i$,
$MCT(G)=CT\_save$; Exit.
(iv) **If** $i\neq 1$ and $|S_i|=size-1$ **then** set
$MCI(G)=CI\_save$,
$MCT(G)=T_i$; Exit.
/$*$ If algorithm has still not terminated, CI_save is actually a maximum clique-independent set and $\alpha_C(G)=size$; Proceed further to ascertain whether $\tau_C(G)$ is actually $\alpha_C(G)+1$. $*$/
Step 2:
For $j=1,2,\ldots,n$ **do**
(i) Run the algorithm of Lemma 4.3 for the ordering $C_{b_j+1},\ldots,C_{a_j-1}$ to get a transversal set Z_j. (Clearly $Z_j\cup\{v_j\}$ is a $CT(G)$.)
(ii) **If** $|Z_j|=size-1$ **then** set:
$MCI(G)=CI\_save$,
$MCT(G)=Z_j\cup\{v_j\}$; Exit.
Step 3: /$*$ Still here, so $\tau_C(G)=\alpha_C(G)+1$ $*$/
Set $MCI(G)=CI\_save$, $MCT(G)=CT\_save$; Exit.

4.4. Proof of correctness

The correctness of the algorithm follows from the following observations:
(i) The $MCT(G)$ output is a $CT(G)$ and the $MCI(G)$ output is a $CI(G)$.
(ii) Clearly if the algorithm reports $\tau_C(G)=\alpha_C(G)$ then the sets output ($MCI(G)$ and $MCT(G)$) are the right ones (this follows from observation (i)).

(iii) Since $|MCT(G_i)| = |MCI(G_i)|$ for $1 \leqslant i \leqslant p$, it follows that if the algorithm outputs $MCT(G)$ and $MCI(G)$ from Steps 1 or 2, then $|MCI(G)| = |MCT(G)|$ and so the sets output are the correct ones by observation (ii).

(iv) The algorithm always outputs a correct $MCI(G)$; this follows from the comment at the end of Step 1.

(v) The only case that remains is when the algorithm outputs the sets from Step 3. This case is settled by Lemma 4.4.

Lemma 4.4. *If the algorithm outputs S_1 as $MCI(G)$ and S_2 as $MCT(G)$ from Step 3 with $|S_2| - |S_1| = 1$, then for the HC-graph G, $\tau_C(G) = \alpha_C(G) + 1$.*

Proof. Suppose on the contrary, $\tau_C(G) = \alpha_C(G) = t$. Then let $MCI(G) = \{C_{i_1}, \ldots, C_{i_t}\}$ with $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant p$ and $MCT(G) = \{w_1, w_2, \ldots, w_t\}$ with $w_k \in C_{i_k}$ for $1 \leqslant k \leq t$ and let $w_k = v_{j_k}$ for $1 \leqslant k \leqslant t$. Since $\{w_1, w_2, \ldots, w_t\}$ meets all maximal cliques of G, we have: For $1 \leqslant k < t$, $w_k \notin C_{i_{k+1}}$ and $w_t \notin C_{i_1}$, $b_{j_k} + 1 \geqslant a_{j_{k+1}}$ and $b_{j_t} + 1 \geq a_{j_1}$. When Step 2 is performed with $j = j_1$, we would obtain Z_{j_1} with $|Z_{j_1}| = t - 1 = \alpha_C(G) - 1 = size - 1$ since the set $\{v_{j_2}, \ldots, v_{j_t}\} \subseteq V - C_{i_1}$ is a set of size $t - 1$ that meets all the cliques in $\{C_k : k \in [a_{j_1}, b_{j_1}]^c\}$. Hence the algorithm would have output S_1, S_2 with $|S_1| = |S_2|$ from Step 2 itself, a contradiction. □

4.5. Complexity analysis

The vertex-clique incidence matrix M can be obtained in $O(n^2)$ time once the circular-arc representation is given using a method like the one of Lemma 4.2. M has n columns and at most n rows by Lemma 4.2. The ordering of the cliques so that M has circular 1's property can be done in $O(n^2)$ time [3]. Once this is done, the determination of l_i, r_i, a_j, b_j etc can be managed in $O(n^2)$ time in a straightforward manner. Thus Step 0 takes $O(n^2)$ time.

Step 1 takes $O(n^2)$ time, as $p \leqslant n$ and after the cliques have been ordered each iteration takes $O(n)$ time by Lemma 4.3. Using the same argument one notes that Step 2 can be performed in $O(n^2)$ time. Hence we conclude:

Theorem 4.3. *For a Helly circular-arc graph G, the clique-transversal and clique-independent set problems can be solved in $O(n^2)$ time.*

5. Conclusions

The complexity status of the clique-transversal and the clique-independence problems on several classes of graphs were determined in this paper. In particular, a polynomial algorithm was presented for Helly circular-arc graphs. Circular-arc graphs form a superset of HC-graphs and may have an exponential number of maximal cliques. The above problems are open on circular-arc graphs. The paper also marks the first successful attempts at the problems on a class of graphs that are not clique-perfect and of the weighted versions of the problem. The result on cocomparability graphs also yields an interesting result in the theory of posets as noted in Corollary 2.2. In light of the result on cographs, the problem may also be investigated on super-classes of cographs like permutation and distance-hereditary graphs.

Also on the theoretic side, one may attempt to characterize clique-perfect graphs and in particular determine whether all clique-perfect graphs are perfect. Note that for $n \geqslant 2$, $\tau_C(C_{2n+1}) = n+1$, $\alpha_C(C_{2n+1}) = n$ and if G is the complement of C_{2n+1} then $\tau_C(G) = 3$ and $\alpha_C(G) = 2$. Hence it follows that all clique-perfect graphs are Berge (i.e. have no odd hole or odd antihole) and that a triangle-free graph is clique-perfect if and only if it is bipartite. It follows that if the strong perfect graph conjecture holds then all clique-perfect graphs are perfect.

6. Unlinked References

[7,8,19]

Acknowledgements

We thank the anonymous referees for their insightful comments which considerably improved the presentation of the paper, especially that of Section 4. We thank one of the referees for pointing out Refs. [9] and [18] to us. We are grateful to the referees for their remarks which led to improvements in the result of Lemma 4.2 and in the complexity of the algorithm of Section 4.3, and also in the algorithm for the weighted clique-transversal problem on strongly chordal graphs.

References

[1] T. Andreae, M. Schughart, Zs. Tuza, Clique-transversal sets of line graphs and complements of line graphs, Discrete Math. 88 (1991) 11–20.
[2] V. Balachandhran, P. Nagavamsi, C. Pandu Rangan, Clique transversal and clique independence on comparability graphs, Inform. Process. Lett. 58 (1996) 181–184.
[3] K.S. Booth, G.S. Lueker, Testing for the consecutive ones property, interval graphs and graph planarity using P–Q tree algorithms, J. Comput. System Sci. 13 (1976) 335–379.
[4] M.S. Chang, Y.H. Chen, G.J. Chang, J.H. Yan, Algorithmic aspects of the genralized clique-transversal problem on chordal graphs, Discrete Appl. Math. 66 (1996) 189–203.
[5] G.J. Chang, M. Farber, Z. Tuza, Algorithmic aspects of neighbourhood numbers, SIAM J. Discrete Math. 6 (1993) 24–29.
[6] G.J. Chang, G.L. Nemhauser, The k-domination and k-stability problems on sun-free chordal graphs, SIAM J. Algebraic Discrete Methods 5 (1984) 332–345.
[7] D.G. Corneil, J. Fonlupt, The complexity of generalized clique covering, Discrete Appl. Math. 22 (1988/1989) 109–118.
[8] D.G. Corneil, Y. Perl, L.K. Stewart, A linear recognition algorithm for cographs, SIAM J. Comput. 14 (1985) 926–934.

[9] D. Duffus, H.A. Kierstead, W.T. Trotter, Fibres and ordered set coloring, J. Combin. Theory Ser. A 58 (1991) 158–164.
[10] M. Farber, Characterizations of strongly chordal graphs, Discrete Math. 43 (1983) 173–189.
[11] M. Farber, Domination, independent domination, and duality in strongly chordal graphs, Discrete Appl. Math. 7 (1984) 115–130.
[12] F. Gavril, Algorithms on circular-arc graphs, Networks 4 (1974) 357–369.
[13] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Academic Press, New York, 1980.
[14] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.
[15] A.J. Hoffman, A.W.J. Kolen, M. Sakarovitch, Totally-balanced and greedy matrices, SIAM J. Algebraic Discrete Methods 6 (1985) 721–730.
[16] J.D. Horton, K. Kilakos, Minimum edge dominating sets, SIAM J. Discrete Math. 6 (1993) 375–387.
[17] J. Lehel, Zs. Tuza, Neighbourhood perfect graphs, Discrete Math. 61 (1986) 93–101.
[18] Z. Lonc, I. Rival, Chains, antichains and fibres, J. Combin. Theory Ser. A 44 (1987) 207–228.
[19] G. Ramalingam, C. Pandu Rangan, A unified approach to domination problems on interval graphs, Inform. Process. Lett. 27 (1988) 271–274.
[20] Zs. Tuza, Covering all cliques of a graph, Discrete Math. 86 (1990) 117–126.
[21] M. Yannakakis, F. Gavril, Edge dominating sets in graphs, SIAM J. Appl. Math. 38 (1980) 364–372.
[22] M. Yannakakis, F. Gavril, The maximum k-colorable subgraph problem for chordal graphs, Inform. Process. Lett. 24 (1987) 133–137.
[23] M. Yannakakis, Edge-deletion problems, SIAM J. Comput. 10 (1981) 297–309.