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Journal of Computational and Applied Mathematics 132 (2001) 357–369

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

Necessary conditions for weighted mean convergence of Lagrange interpolation for exponential weights

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Received 28 May 1999; received in revised form 20 February 2000

Abstract

Given a continuous real-valued function f which vanishes outside a fixed finite interval, we establish necessary conditions for weighted mean convergence of Lagrange interpolation for a general class of even weights w which are of exponential decay on the real line or at the endpoints of $(-1,1)$. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 41A10; 42C05

Keywords: Erdős weight; Freud weight; Lagrange interpolation; Orthogonal polynomial; Pollaczek weight; Weighted approximation

1. Introduction and statement of results

Let I denote either the open interval $(-1,1)$ or $\mathbb{R}$ and let $w: I \to (0,\infty)$ be an even continuous weight function with all power moments

$$\int_I x^n w^2(x)\,\mathrm{d}x, \quad n \geqslant 0$$

finite. Then we may define orthonormal polynomials

$$p_n(x) := p_n(w^2,x) = \gamma_n x^n + \cdots, \quad \gamma_n = \gamma_n(w^2) > 0, \quad x \in \mathbb{R}$$

satisfying

$$\int_I p_n(w^2,x)p_m(w^2,x)w^2(x)\,\mathrm{d}x = \begin{cases} 0, & n \neq m, \\ 1, & n = m \end{cases}$$

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PII: S0377-0427(00)00439-8

and with zeros denoted by

$$-\infty < x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} < \infty.$$

For each $n \geqslant 1$ and for the given weight w, we define interpolatory matrices

$$U_n := \{x_{j,n}:\ 1 \leqslant j \leqslant n\} \tag{1.1}$$

and

$$V_{n+2} = U_n \cup \{y_0\} \cup \{-y_0\}, \tag{1.2}$$

where y_0 maximizes $\|Pw\|_{L_\infty(\mathbb{R})}$ over every $P \in \Pi_n$. Let $f : I \to \mathbb{R}$ be a given continuous function and denote by $L_n[f, U_n]$ and $L_n[f, V_{n+2}]$ the Lagrange interpolation polynomials of degree $n-1$ and $n+1$ interpolating f at the points in U_n and V_{n+2}, respectively.

In this article, we establish a necessary condition for weighted mean convergence of Lagrange interpolation in L_p $(0 < p < \infty)$, for continuous functions f which vanish outside finite fixed intervals $J \subset I$ and for even weights on $\mathbb{R}$ that are both of polynomial and of faster than polynomial decay at infinity as well as even weights on $(-1, 1)$ that are of exponential decay near ± 1. Our class of functions is the smallest for which convergence questions in weighted L_p spaces are meaningful and so our main result is the least we can expect to achieve convergence simultaneously for all three classes of weights considered, for both interpolation schemes (1.1) and (1.2) and for every $0 < p < \infty$.

To formulate our main result, we need a suitable class of admissible weights and to this end, let us agree that $I+$ will denote either $(0, \infty)$ if I is $\mathbb{R}$ and $(0, 1)$ if I is $(-1, 1)$. Our class of weights w will then be assumed to be admissible in the sense of the following definition:

1.1. Class of admissible weights

Definition 1.1. Let $w_Q = \exp(-Q)$ where $Q : I \to \mathbb{R}$ is even and continuous:

(a) Assume that Q'' is continuous in $I+$ and Q'', $Q' \geqslant 0$ in $I+$.

(b) The function

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \quad x \in I+$$

satisfies for large enough x or x close enough to ± 1

$$T(x) \sim \frac{xQ'(x)}{Q(x)}.$$

Moreover, T satisfies either:

(b1) There exist $A > 1$ and $B > 1$ such that

$$A \leqslant T(x) \leqslant B, \quad x \in I+.$$

(b2) T is increasing in $I+$ with $\lim_{x \to 0+} T(x) > 1$. If $I = \mathbb{R}$,

$$\lim_{|x| \to \infty} T(x) = \infty$$

and if $I=(-1,1)$, for x close enough to ± 1,

$$T(x)\geqslant \frac{A}{1-x^2}$$

for some $A>2$.

Then w shall be called an admissible weight and we shall write $w\in\mathscr{A}$.
Canonical examples of the class $\mathscr{A}$ are

(a)

$$w_\alpha(x):=\exp(-|x|^\alpha),\quad \alpha>1,\ x\in\mathbb{R}. \tag{1.3}$$

(b)

$$w_{k,\beta}(x):=\exp(-\exp_k(|x|^\beta)),\quad \beta>0,\ k\geqslant 1,\ x\in\mathbb{R}. \tag{1.4}$$

(c)

$$w_{k,\gamma}(x):=\exp(-\exp_k(1-x^2)^{-\gamma}),\ \gamma>0,\ k\geqslant 0,\ x\in(-1,1), \tag{1.5}$$

where $\exp_k$ denotes the kth iterated exponential.

The weights listed above are, respectively, examples of Freud, Erdős and generalized Pollaczek weights. Freud weights are characterized by their smooth polynomial decay at infinity and Erdős weights by their faster than smooth polynomial decay at infinity. Generalized Pollaczek weights decay strongly near ± 1 as exponentials and are of faster decay than classical Jacobi weights. They violate the well-known Szegő condition for orthogonal polynomials, [9, Chapter 5, p. 208].

Following is our main result:

Theorem 1.2. *Let* $w_Q\in\mathscr{A}$, $w\geqslant 0\in L_1(\mathbb{R})$ *and* $0<p<\infty$ *be given*:

(a) *If for every continuous function* f *vanishing outside a fixed finite interval* $J\subset I$

$$\lim_{n\to\infty}\int_I [|f(x)-L_n(f,U_n)(x)|^p w(x)]\,\mathrm{d}x=0 \tag{1.6}$$

holds, *then we have*

$$\int_I [w_Q^{-1}(x)/(1+|x|)]^p w(x)\,\mathrm{d}x<\infty. \tag{1.7}$$

(b) *Moreover if for every continuous function* f *vanishing outside a fixed finite interval*

$$\lim_{n\to\infty}\int_I [|f(x)-L_{n+2}(f,V_{n+2})(x)|^p w(x)]\,\mathrm{d}x=0 \tag{1.8}$$

holds, *then we have* (1.7).

Our theorem shows that if for a certain $w\geqslant 0\in L_1(\mathbb{R})$, (1.7) fails, then there exists a continuous function f which vanishes outside a finite fixed interval for which there is no convergence in (1.6) and (1.8). As our class of functions is the smallest class for which convergence questions in weighted

L_p are meaningful, (1.7) is the least we should expect to conclude that we have convergence in general for every p. Our main emphasis in this paper is to derive a necessary condition for mean convergence of Lagrange interpolation which works for as small a class of functions as possible and simultaneously for Freud, Erdős and exponential-type weights on $\mathbb{R}$ and $(-1,1)$. At the same time, our main result applies to both interpolation arrays U_n and V_{n+2} and so makes more precise and general earlier necessary conditions obtained by Matjila, Lubinsky and the authors. There is a vast literature dealing with necessary and sufficient conditions for mean convergence of Lagrange interpolation for even Freud, Erdős, and generalized Pollaczek weights for larger classes of functions and for these results, we refer the reader to [1–9,12,14–22,25–27] and the many references cited therein. For those who are not familiar, the array V_{n+2} has recently been shown to yield typically better sufficient results for mean and uniform convergence of weighted Lagrange interpolation and for these results we refer the reader to [26,2,3,16,15,6,4] and the references cited therein.

We now show that under certain conditions, we may modify the bound in (1.7). This result is contained in:

Corollary 1.3. *Let $w_Q \in \mathscr{A}$ under assumption* (b2). *Let $w \geqslant 0 \in L_1(I)$ and assume it satisfies the following condition*:

Let $0 < A < B < \infty$ and suppose that uniformly for $n \geqslant 1$ and $1 \leqslant j \leqslant n$,

$$A \leqslant w(x)/w(x_{jn}) \leqslant B \quad x \in [x_{j+1,n}, x_{jn}]. \tag{1.9}$$

Then for every continuous function f vanishing outside a fixed finite interval $J \subset I$ for which (1.6) *holds, we have*

$$\int_{x_{nn}}^{x_{1n}} (|1-|x|/a_n| + \delta_n)^{-p/4} [w_Q^{-1}(x)/(1+|x|)]^p w(x)\, \mathrm{d}x < \infty, \tag{1.10}$$

where

$$\delta_n := (nT(a_n))^{-2/3}$$

and a_n is the well-known Mhaskar–Rakhmanov–Saff number for w_Q^2, see [24, *Chapter* 2].

The remainder of this paper is organized as follows. In Section 2, we provide numerous examples from the literature as to how our main results may be understood while in Section 3, we present our proofs.

2. Examples

In this section, we will illustrate our main results with numerous examples from the literature where for typically larger classes of functions necessary and sufficient conditions for mean convergence are obtained. Because the class of functions that we consider is much smaller than those considered below, a completely new proof of Theorem 1.2 is needed for the various weights considered.

Throughout, for any two sequences b_n and c_n of nonzero real numbers, we shall write $b_n \lesssim (\gtrsim) c_n$ if there exists a positive constant C, independent of n, such that

$$b_n \leqslant (\geqslant) C_1 c_n, \ n \to \infty$$

and $b_n \sim c_n$ if

$$b_n \lesssim c_n \quad \text{and} \quad b_n \gtrsim c_n.$$

Similar notation will be used for functions and sequences of functions.

2.1. Freud weights

Theorem 1.2(a) was first proved by Nevai in [21, Theorem 2] for the Hermite weight $w_Q = \exp(-x^2/2)$ and its present form for a related class of Freud weights is due to Sakai in [25, (1.2) and (1.3)]. Let us define $w(x) := w_Q^p(x)(1+|x|)^{-\Delta p}$ for $\Delta + 1 > 1/p$. Then it is easy to see that we have (1.7). Indeed for a larger class of functions and for w as above, Lubinsky and Matjila and Matjila in [17, Theorem 1.3; 19, Theorem 1.1] have shown the following:

Let $w_Q \in \mathscr{A}$, $0 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$ and $\hat{\alpha} := \min\{1, \alpha\}$. Then for

$$\lim_{n\to\infty} \|(f(x) - L_n(f, U_n)(x))w_Q(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})} = 0$$

to hold for every continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} |f(x)|w_Q(x)(1+|x|)^{\alpha} = 0,$$

if $p \leqslant 4$, it is necessary that

$$\hat{\alpha} + \Delta > \frac{1}{p},$$

if $p > 4$ and $\alpha \neq 1$, it is necessary that

$$a_n^{1/p-(\hat{\alpha}+\Delta)} n^{(1-4/p)/6} = O(1), \quad n \to \infty$$

and if $p > 4$ and $\alpha = 1$, it is necessary that

$$a_n^{1/p-(\hat{\alpha}+\Delta)} n^{(1-4/p)/6} = O(1/\log n), \quad n \to \infty.$$

Moreover, in [16, Theorem 1.4], it was shown that for the same class of functions as above, $\hat{\alpha} + \Delta > 1/p$ is necessary for mean convergence for every $1 < p < \infty$ if U_n is replaced by V_{n+2}. Clearly, if $\hat{\alpha} + \Delta > 1/p$, then $1 + \Delta > 1/p$.

2.2. Erdős weights

For larger classes of functions, the following results of Damelin and Lubinsky and Damelin, Jung and Kwon, are given in [7, Theorem 1.3; 8, Theorems 1.3–1.4; 5, Theorem 1.1; 6, Theorems 1.2–1.3; 4, Corollaries 2.3–2.4]. To illustrate these results let us choose $0 < p < \infty$, $\Delta, \beta \in \mathbb{R}$, $\kappa, \alpha > 0$, $\hat{\alpha} := \{1, \alpha\}$ and $w_Q \in \mathscr{A}$ satisfying (b2) and for every $\varepsilon > 0$,

$$T(x) \lesssim O(Q^{\varepsilon}(x)). \tag{2.1}$$

(a) Functions that decay as logarithms:

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that

$$\lim_{|x|\to\infty} |f(x)|w_Q(x)(\log|x|)^{1+\kappa} = 0. \tag{2.2}$$

For

$$\lim_{n\to\infty} \|(f - L_n(f, U_n))(x)w_Q(x)(1 + Q(x))^{-\Delta}\|_{L_p(\mathbb{R})} = 0$$

to hold, it is necessary that

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}.$$

Moreover if we assume for every $\varepsilon > 0$, *the stronger condition*

$$T(x) \lesssim \left(\log Q'(x)\right)^{1+\varepsilon}$$

holds instead of (2.1), *then for*

$$\lim_{n\to\infty} \|(f - L_{n+2}(f, V_{n+2}))w_Q(x)(1 + |x|)^{-\beta}(\log(2 + Q(x)))^{-1}\|_{L_p(\mathbb{R})} = 0$$

to hold it is necessary that $\beta \geqslant 1/p$ *and for*

$$\lim_{n\to\infty} \|(f - L_{n+2}(f, V_{n+2}))w_Q(x)(1 + |x|)^{-1/p}(\log(2 + Q(x)))^{-\beta}\|_{L_p(\mathbb{R})} = 0$$

to hold it is necessary that $\beta \geqslant 1$.

(b) Functions that decay as polynomials:

Let $f : \mathbb{R} \to \mathbb{R}$ *be continuous and suppose that*

$$\lim_{|x|\to\infty} |f(x)|w_Q(x)(1 + |x|)^{\alpha} = 0. \tag{2.3}$$

For

$$\lim_{n\to\infty} \|(f - L_{n+2}(f, V_{n+2}))w(1 + |x|)^{-\Delta}\|_{L_p(\mathbb{R})} = 0$$

to hold it is necessary that

$$\hat{\alpha} + \Delta > 1/p. \tag{2.4}$$

Moreover if $0 < p < 4$, *then for*

$$\lim_{n\to\infty} \|(f - L_n(f, U_n))w(1 + |x|)^{-\Delta}\|_{L_p(\mathbb{R})} = 0$$

to hold it is necessary that (2.4) *holds.*

Let us define $w(x) := (w_Q(x)(1+Q(x))^{-\Delta})^p$ for $\Delta > \max\{0, \frac{2}{3}(\frac{1}{4} - 1/p)\}$. Then w satisfies condition (1.9). Moreover, using Lemma 2.3(a), (2.24), (2.16) and Lemma 2.4 of [7] together with (2.1), we see that there exists $\varepsilon > 0$ such that for any $0 < \eta < 1$,

$$\begin{aligned}
&\int_{x_{nn}}^{x_{1n}} (|1 - |x|/a_n| + \delta_n)^{-p/4}[w_Q^{-1}(x)/(1 + |x|)]^p w(x)\,\mathrm{d}x \\
&\quad\lesssim \int_{0\leqslant|x|\leqslant a_{\eta n}} + \int_{a_{\eta n}\leqslant|x|<x_{1n}} (|1 - |x|/a_n| + \delta_n)^{-p/4}[(1 + Q(x))^{-\Delta}/(1 + |x|)]^p\,\mathrm{d}x \\
&\quad\lesssim \int_{0\leqslant|x|\leqslant a_{\eta n}} [T^{1/4}(x)(1 + Q(x))^{-\Delta}/(1 + |x|)]^p\,\mathrm{d}x
\end{aligned}$$

$$+ Q^{-\Delta p}(a_n)/a_n^p \int_{a_{\eta n} \leqslant |x| < x_{1n}} (|1 - |x|/a_n| + \delta_n)^{-p/4}\,\mathrm{d}x$$

$$\lesssim \int_{0 \leqslant |x| \leqslant a_{\eta n}} [1/(1 + |x|)^2]\,\mathrm{d}x + a_n^{-p} Q^{-\Delta p}(a_n) a_n \delta_n^{\min\{0,1-p/4\}} \log n$$

$$\lesssim 1 + n^{-(\Delta - \max\{0, 2/3(1/4 - 1/p)\})p + \varepsilon} \lesssim 1.$$

Thus we have (1.10) and so (1.7) follows. In the case where we use the extended Lagrange interpolation polynomial $L_{n+2}[V_{n+2}]$, we may define

$$w(x) := (w_Q(x)(1 + |x|)^{-\beta}(\log(2 + Q(x))^{-1})^p, \quad \beta \geqslant 1/p,$$

or

$$w_1(x) := (w_Q(x)(1 + |x|)^{-1/p}(\log(2 + Q(x))^{-\beta})^p, \quad \beta \geqslant 1,$$

respectively. Then using Lemma 2.3(a) in [7], we may deduce that

$$\int_I [w_Q^{-1}(x)/(1 + |x|)]^p w(x)\,\mathrm{d}x$$

$$= \int_I [(\log(2 + Q(x)))^{-1}/(1 + |x|)^{1+\beta}]^p$$

$$\lesssim \int_I [(\log(2 + Q(x)))^{-1}/(1 + |x|)^{1+1/p}]^p$$

$$\lesssim \int_I [(\log(2 + |x|))^{-p}/(1 + |x|)^{p+1}] \lesssim 1$$

and proceed similarly for w_1. Thus we have (1.7) for both cases.

2.3. Exponential weights on $(-1, 1)$

Let $w_Q \in \mathscr{A}$, $4 < p < \infty$ and $\Delta \in \mathbb{R}$. In [14, Theorem 1.5], Lubinsky established the following result:

For

$$\lim_{n \to \infty} \|(f - L_n(f, U_n)) w_Q (1 + Q^{2/3} T)^{-\Delta}\|_{L_p(-1,1)} = 0$$

to hold for every continuous function

$$f : (-1, 1) \to \mathbb{R}$$

vanishing outside $[-\frac{1}{2}, \frac{1}{2}]$ *it is necessary that*

$$\Delta \geqslant \tfrac{1}{4} - 1/p.$$

Motivated by this result, let us set $w(x) := w_Q^p(x)(1+Q(x)^{2/3}T(x))^{-\Delta p}$ for $\Delta \geqslant 1/4 - 1/p$. Then w satisfies condition (1.9). Moreover, given $0 < \eta < 1$,

$$\int_{x_{nn}}^{x_{1n}} (|1-|x|/a_n| + \delta_n)^{-p/4}[w_Q^{-1}(x)/(1+|x|)]^p w(x)\,\mathrm{d}x$$

$$= \int_{x_{nn}}^{x_{1n}} (|1-|x|/a_n| + \delta_n)^{-p/4}(1+Q(x)^{2/3}T(x))^{-\Delta p}\,\mathrm{d}x$$

$$\lesssim \int_{x_{nn}}^{x_{1n}} (|1-|x|/a_n| + \delta_n)^{-p/4}(1+Q(x)^{2/3}T(x))^{-(p/4-1)}\,\mathrm{d}x$$

$$\lesssim \int_{|x|\leqslant a_{\eta n}} + \int_{a_{\eta n}\leqslant |x|\leqslant x_{1n}} (|1-|x|/a_n| + \delta_n)^{-p/4}(Q(x)^{2/3}T(x))^{-(p/4-1)}\,\mathrm{d}x.$$

Now choose l such that

$$2^{l-1} \leqslant \eta n \leqslant 2^l.$$

Then using [14, Lemma 2.2], we deduce that there exists a constant $\varepsilon > 0$ such that

$$\int_{|x|\leqslant a_{\eta n}} (|1-|x|/a_n| + \delta_n)^{-p/4}(Q(x)^{2/3}T(x))^{-(p/4-1)}\,\mathrm{d}x$$

$$\lesssim \sum_{k=0}^{l} \int_{a_{2^k}}^{a_{2^{k+1}}} (|1-|x|/a_n| + \delta_n)^{-p/4}(Q(x)^{2/3}T(x))^{-(p/4-1)}\,\mathrm{d}x$$

$$\lesssim \sum_{k=0}^{l} Q^{-2/3(p/4-1)}(a_{2^k}) \lesssim \sum_{k=0}^{l} 2^{-\varepsilon k(p/4-1)} \lesssim 1$$

and

$$\int_{a_{\eta n}\leqslant |x|\leqslant x_{1n}} (|1-|x|/a_n| + \delta_n)^{-p/4}(Q(x)^{2/3}T(x))^{-(p/4-1)}\,\mathrm{d}x$$

$$\lesssim (Q(a_n)^{2/3}T(a_n))^{-(p/4-1)} \int_{a_{\eta n}\leqslant |x|\leqslant x_{1n}} (|1-|x|/a_n| + \delta_n)^{-p/4}$$

$$\lesssim a_n(Q(a_n)^{2/3}T(a_n))^{-(p/4-1)}(\delta_n)^{-p/4+1}$$

$$\sim (nT(a_n))^{-2/3(p/4-1)}(\delta_n)^{-p/4+1}$$

$$\sim (\delta_n)^{(p/4-1)}(\delta_n)^{-p/4+1}$$

$$\sim 1.$$

It follows that

$$\int_{x_{nn}}^{x_{1n}} (|1-|x|/a_n| + \delta_n)^{-p/4}[w_Q^{-1}(x)/(1+|x|)]^p w(x)\,\mathrm{d}x$$

$$\lesssim \int_{|x|\leqslant a_{\eta n}} + \int_{a_{\eta n}\leqslant |x|\leqslant x_{1n}} (|1-|x|/a_n| + \delta_n)^{-p/4}(Q(x)^{2/3}T(x))^{-(p/4-1)}\,\mathrm{d}x$$

$$\lesssim 1,$$

which gives (1.10) and hence (1.7).

Concerning the array V_{n+2}, Lubinsky has shown in [15, Theorem 1.9] that for $1 < p < \infty$ and every Riemann integrable f with fw bounded

$$\lim_{n\to\infty} \|(f - L_n(f, V_{n+2}))w_Q(1-t^2)^d\|_{L_p(-1,1)} = 0$$

whenever $d > -1/p$. An easy calculation then shows that $w(x) := w_Q^p(1-x^2)^{dp}$ gives (1.7) when $d > -1/p$ and so for a smaller class of functions, Theorem 1.2 gives a necessary condition for the above theorem of Lubinsky to hold.

3. The proofs of Theorem 1.2 and Corollary 1.3

In this section, we give our proofs.

Proof of Theorem 1.2. We shall prove Theorem 1.1(b) for Theorem 1.2(a) is similar. We shall assume first that (b2) holds and that $I = \mathbb{R}$. Using ideas from [21,25], we let $C_0(-2,-1)$ be the space of continuous functions on $\mathbb{R}$ with support in $[-2,-1]$. Since for this space, (1.8) holds for the linear functional $L_{n+2}(;, V_{n+2})$, we may apply the generalized uniform boundedness theorem, see [23], and conclude that for all $f \in C_0(-2,-1)$, we have

$$\int_{-\infty}^{\infty} |L_{n+2}(f, V_{n+2})(x)|^p w(x)\,\mathrm{d}x \lesssim \max_{-2\leqslant x\leqslant -1}|f(x)|^p. \tag{3.1}$$

Let $\{p_n\}_{n=1}$ be the orthonormal polynomials with respect to the admissible weight w_Q^2 and for each $n = 1,2,3,\ldots$, let us consider a sequence of functions $\{g_n\} \in C_0(-2,-1)$ satisfying

$$\max_{-2\leqslant x\leqslant -1} |g_n(x)| = 1$$

and

$$g_n(x_{kn}) = \mathrm{sign}(\tilde{p}_n'(x_{kn})), \quad x_{kn} \in (-2,-1),$$

where $\tilde{p}_n(x) = p_n(x)(x-y_0)(x+y_0)$. Thus we learn that for each n and for every $x \in \mathbb{R}$

$$L_{n+2}(g_n; V_{n+2})(x) = \tilde{p}_n(x) \sum_{-2\leqslant x_{kn}\leqslant -1} |\tilde{p}_n'(x_{kn})|^{-1}(x - x_{kn})^{-1}.$$

Moreover, using [11, Corollary 1.4(b), p. 205] and the identity $|x_{kn} \pm y_0| \sim a_n$, uniformly for n, we obtain for $-2 < x_{kn} < -1$ and uniformly for n

$$|\tilde{p}_n'(x_{kn})|^{-1} = |p_n'(x_{kn})(x_{kn} - y_0)(x_{kn} + y_0)|^{-1}$$

$$\sim n^{-1}a_n^{-1/2}.$$

Then for $x>0$ and using [11, Corollary 1.3, p. 205], we deduce that

$$\begin{aligned}|L_{n+2}(g_n;V_{n+2})(x)| &\gtrsim \left|\frac{\tilde{p}_n(x)}{x+1}\right| \sum_{-2\leqslant x_{kn}\leqslant -1} n^{-1}a_n^{-1/2}\\ &\sim a_n^{-3/2}\left|\frac{\tilde{p}_n(x)}{x+1}\right| \sum_{-2\leqslant x_{kn}\leqslant -1} x_{kn}-x_{k+1,n}\\ &\sim \left|\frac{a_n^{-3/2}\tilde{p}_n(x)}{x+1}\right|. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we deduce that

$$\begin{aligned}L &:= \limsup_{n\to\infty}\int_0^\infty \left|a_n^{-3/2}\frac{\tilde{p}_n(x)}{1+x}\right|^p w(x)\,\mathrm{d}x\\ &\lesssim \limsup_{n\to\infty}\int_0^\infty |L_{n+2}(g_n;V_{n+2})(x)|^p w(x)\,\mathrm{d}x\\ &\lesssim \max_{-2\leqslant x\leqslant -1}|g_n(x)|^p<\infty.\end{aligned}$$

The proof will be complete if we can show that the integral in (1.7) is bounded by L as L is finite.

To see this, we proceed as follows: Let us define for $\epsilon>0$

$$I_{jn}=I_{jn}(\varepsilon)=[x_{jn}+\varepsilon a_n/n, x_{j-1,n}-\varepsilon a_n/n],\quad j=2,3,\ldots,n$$

and

$$\bar{I}_{jn}=\bar{I}_{jn}(\varepsilon)=[x_{j-1,n}-\varepsilon a_n/n, x_{j-1,n}+\varepsilon a_n/n],\quad j=2,3,\ldots,n.$$

Firstly using [13], we have for $x\in[x_{jn},x_{j-1,n}]$, the Erdös–Turan identity

$$l_{jn}(x)w_Q(x)w_Q^{-1}(x_{jn})+l_{j-1,n}(x)w_Q(x)w_Q^{-1}(x_{j-1,n})\geqslant 1.$$

Applying the triangle inequality to this identity we see that

$$|p_nw_Q(x)|\left(\frac{1}{|p_n'(x_{jn})w_Q(x_{jn})(x-x_{jn})|}+\frac{1}{|p_n'(x_{j-1,n})w_Q(x_{j-1,n})(x-x_{j-1,n})|}\right)\geqslant 1.$$

Next let $0<\delta<1$. Since for

$$x\in I_{jn}(\varepsilon)\cap[0,\delta a_n]$$

$|x-x_{jn}|\geqslant\varepsilon a_n/n$ and $|x-x_{j-1,n}|\geqslant\varepsilon a_n/n$, we have using [11, Corollary 1.4(b), p. 205] that for

$$x\in I_{jn}(\varepsilon)\cap[0,\delta a_n],$$

$$\begin{aligned}1&\leqslant |p_nw_Q(x)|\left(\frac{1}{|p_n'(x_{jn})w_Q(x_{jn})(x-x_{jn})|}+\frac{1}{|p_n'(x_{j-1,n})w_Q(x_{j-1,n})(x-x_{j-1,n})|}\right)\\ &\leqslant \varepsilon\frac{n}{a_n}|p_nw_Q(x)|\left(\frac{1}{|p_n'(x_{jn})w_Q(x_{jn})|}+\frac{1}{|p_n'(x_{j-1,n})w_Q(x_{j-1,n})|}\right)\\ &\sim \varepsilon a_n^{1/2}|p_nw_Q(x)|.\end{aligned}$$

Thus for $x \in I_{jn}(\varepsilon) \cap [0, \delta a_n]$, we learn that

$$|\tilde{p}_n w_Q(x)| = |p_n w_Q(x)||(x - y_0)(x + y_0)| \sim a_n^2 |p_n w_Q(x)| \gtrsim a_n^{3/2}$$

and so consequently

$$\sum_{j=2}^{n} \int_{I_{jn} \cap [0, \delta a_n]} |w_Q^{-1}(x)/(1+x)|^p w(x)\,\mathrm{d}x \lesssim L. \tag{3.3}$$

By interchanging n for $n+1$, we deduce that

$$\sum_{j=2}^{n+1} \int_{I_{j,n+1} \cap [0, \delta a_{n+1}]} |w_Q^{-1}(x)/(1+x)|^p w(x)\,\mathrm{d}x \lesssim L. \tag{3.4}$$

We now claim that for a certain ε

$$\bar{I}_{jn}(\varepsilon) \cap [0, \delta a_n] \subset I_{j,n+1}(\varepsilon) \cap [0, \delta a_n]. \tag{3.5}$$

To see this, observe first that using [11, Corollary 1.4(b), p. 205; 11, Corollary 1.3, (1.24), p. 205], we have for $|x_{jn}| \leqslant \delta a_n$ and uniformly for n the identity

$$|x_{j,n+1} - x_{jn}| \gtrsim \left| \frac{p_n(x_{j,n+1}) w_Q(x_{j,n+1})}{p_n'(x_{jn}) w_Q(x_{jn})} \right| \sim \frac{a_n}{n}.$$

Moreover, using the interlacing properties of the zeros it follows quite easily that indeed

$$|x_{j,n+1} - x_{jn}| \sim |x_{j-1,n} - x_{j,n+1}| \sim \frac{a_n}{n}.$$

Thus (3.5) holds for some ε. Hence, (3.3) becomes

$$\sum_{j=2}^{n} \int_{\bar{I}_{j,n} \cap [0, \delta a_n]} |w_Q^{-1}(x)/(1+x)|^p w(x)\,\mathrm{d}x$$

$$\lesssim \sum_{j=2}^{n+1} \int_{I_{j,n+1} \cap [0, \delta a_{n+1}]} |w_Q^{-1}(x)/(1+x)|^p w(x)\,\mathrm{d}x \lesssim L. \tag{3.6}$$

By (3.4) and (3.6), we have

$$\int_0^{\delta a_n} |w_Q^{-1}(x)/(1+x)|^p w(x)\,\mathrm{d}x \lesssim L.$$

Thus, we deduce that

$$\int_0^{\infty} |w_Q^{-1}(x)/(1+x)|^p w(x)\,\mathrm{d}x \lesssim L$$

as required. The case $x \leqslant 0$ is similar. Suppose next that (b2) holds and that $I = (-1, 1)$. We then proceed as above with some changes. Firstly, in place of (3.1) we conclude that for all $f \in C_0(-\frac{1}{2}, -\frac{1}{4})$, we have

$$\int_{-1}^{1} |L_{n+2}(f, V_{n+2})(x)|^p w(x)\,\mathrm{d}x \lesssim \max_{-1/2 \leqslant x \leqslant -1/4} |f(x) w_Q(x)|^p. \tag{3.7}$$

Then using [10, Theorem 1.2 p. 7; 10, Corollary 1.5(iii) p. 10; 10 Corollary 1.4(ii) p. 9], we conclude that for $0 < x < 1$ and uniformly for large enough n

$$\begin{aligned} |L_{n+2}(g_n;x)| &\gtrsim \left|\frac{\tilde{p}_n(x)}{x+1/4}\right| \sum_{-1/2 \leqslant x_{kn} \leqslant -1/4} n^{-1} \\ &\sim \left|\frac{\tilde{p}_n(x)}{x+1/4}\right| \sum_{-1/2 \leqslant x_{kn} \leqslant -1/4} x_{kn} - x_{k+1,n} \\ &\sim \left|\frac{\tilde{p}_n(x)}{x+1/4}\right| \\ &\sim \left|\frac{\tilde{p}_n(x)}{x+1}\right|. \end{aligned} \tag{3.8}$$

This shows as above that

$$L := \limsup_{n\to\infty} \int_0^1 \left|\frac{\tilde{p}_n(x)}{1+x}\right|^p w(x)\,\mathrm{d}x$$

is finite and so the proof is complete if we can show that the integral in (1.7) is bounded by L. We now proceed exactly as in the case $I = \mathbb{R}$ except we use [2, (3.1);13]. Finally suppose that $I = \mathbb{R}$ and (b1) holds. Then we proceed as in [25] and the above using [26, Lemma 2]. This completes the proof of Theorem 1.2. □

The Proof of Corollary 1.3. Under the assumptions of Corollary 1.3, we may apply the method of Lemma 2.3 in [14] and deduce that

$$\left\|a_n^{1/2} p_n(x) \frac{w^{1/p}(x)}{1+|x|}\right\|^p_{L_p[0,x_{1n})} \gtrsim \left\|\frac{w_Q^{-1}(x)}{1+|x|} w^{1/p}(x)(1-|x|/a_n+\delta_n)^{-1/4}\right\|^p_{L_p[0,x_{1n})}.$$

By the same argument as the proof of Theorem 1.2, we then have

$$\left\|a_n^{1/2}\frac{p_n(x)}{1+x} w^{1/p}(x)\right\|_{L_p[0,\infty)} < \infty.$$

Applying a similar estimate to the case $x < 0$, gives the result. □

Acknowledgements

The first author was partially funded by the Centre for Applicable Analysis and Number Theory. The second author was partially supported by the Korean Research Foundation and the third author was partially supported by KOSEF(98-0701-03-01-5) and the Korea Ministry of Education (BK-21).

References

[1] S. Bonan, Weighted mean convergence of Lagrange interpolation, Ph.D. thesis, Ohio-State University, Columbus, OH, 1982.

[2] S.B. Damelin, The weighted Lebesgue constant of Lagrange interpolation for exponential weights on $[-1,1]$, Acta-Math. (Hung.) 81 (3) (1998) 211–228.
[3] S.B. Damelin, The Lebesgue function and Lebesgue constant of Lagrange interpolation for Erdős weights, J. Approx. Theory 94 (2) (1998) 235–262.
[4] S.B. Damelin, H.S. Jung, K.H. Kwon, Converse Marcinkiewicz–Zygmund inequalities on the real line with applications to mean convergence of Lagrange interpolation, postscript.
[5] S.B. Damelin, H.S. Jung, K. Kwon, A note on mean convergence of Lagrange interpolation for Erdős weights in L_p $(0 < p \leqslant 1)$, postscript.
[6] S.B. Damelin, H.S. Jung, K.H. Kwon, On mean convergence of extended Lagrange interpolation for Erdős weights, postscript.
[7] S.B. Damelin, D.S. Lubinsky, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős weights, Canad. J. Math. 48 (4) (1996) 710–736.
[8] S.B. Damelin, D.S. Lubinsky, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős weights II, Canad. J. Math. 48 (4) (1996) 737–757.
[9] G. Freud, Orthogonal Polynomials, Pergamon Press, Oxford, 1971.
[10] A.L. Levin, D.S. Lubinsky, Christoffel functions and orthogonal polynomials for exponential weights on $[-1,1]$, Memoir Amer. Math. Soc. 535 (1994).
[11] A.L. Levin, D.S. Lubinsky, T.Z. Mthembu, Christoffel functions and orthogonal polynomials for Erdös weights on $(-\infty,\infty)$, Rend. Mat. Roma 14 (1994) 199–289.
[12] D.S. Lubinsky, Converse quadrature sum inequalities for polynomials with Freud weights, Acta Sci. Math. (Szeged) 60 (1995) 525–557.
[13] D.S. Lubinsky, An extension of the Erdös–Turan inequality for the sum of successive fundamental polynomials, Ann. Numer. Math. 2 (1995) 305–309.
[14] D.S. Lubinsky, Mean convergence of Lagrange interpolation for exponential weights on $[-1,1]$, Canad. J. Math. 50 (6) (1998) 1273–1297.
[15] D.S. Lubinsky, On converse Marcinkiewicz–Zygmund inequalities in L_p, $p > 1$, Constr. Approx 15 (1999) 577–610.
[16] D.S. Lubinsky, G. Mastroianni, Mean convergence of extended Lagrange interpolation with Freud weights, Acta Math. Hung. 84 (1–2) (1999) 47–63.
[17] D.S. Lubinsky, D.M. Matjila, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Freud weights, SIAM J. Math. Anal. 26 (1995) 238–262.
[18] D.S. Lubinsky, T.Z. Mthembu, Mean convergence of Lagrange interpolation for Erdős weights, J. Comput. Appl. Math. 47 (1993) 369–390.
[19] D.M. Matjila, Convergence of Lagrange interpolation for Freud weights in weighted $L_p(\mathbb{R})$, $0 < p \leqslant 1$, in: A. Cuyt (Ed.), Nonlinear Numerical Methods and Rational Approximation II, Kluwer Academic Publishers, Dordrecht, 1994, pp. 25–35.
[20] H.N. Mhaskar, Introduction to the Theory of Weighted Polynomial Approximation, Series in Approximations and Decompositions, Vol. 7, World Scientific, Singapore.
[21] P. Nevai, Mean convergence of Lagrange interpolation II, J. Approx. Theory 30 (1980) 263–276.
[22] P. Nevai, Geza Freud, orthogonal polynomials and Christoffel functions, J. Approx. Theory 48 (1986) 3–167.
[23] M.M. Rao, Measure Theory and Integration, Wiley, New York, 1987.
[24] E.B. Saff, V. Totik, Logarithmic Potentials with External Fields, Springer, Heidelberg, 1997.
[25] R. Sakai, Lagrange interpolation based at the zeros of orthonormal polynomials with Freud weights, J. Approx. Theory 92 (1998) 116–127.
[26] J. Szabados, Weighted Lagrange interpolation and Hermite–Fejer interpolation on the real line, J. Inequalities Appl. 1 (1997) 99–123.
[27] J. Szabados, P. Vertesi, Interpolation of Functions, World Scientific, Singapore, 1991.