Journal of Pure and Applied Algebra 217 (2013) 348–363

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Tensor products of strongly graded vertex algebras and their modules

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ARTICLE INFO

Article history:
Received 8 January 2012
Received in revised form 28 May 2012
Available online 31 July 2012
Communicated by C.A. Weibel

MSC: 17B69; 06B15

ABSTRACT

We study strongly graded vertex algebras and their strongly graded modules, which are conformal vertex algebras and their modules with a second, compatible grading by an abelian group satisfying certain grading restriction conditions. We consider a tensor product of strongly graded vertex algebras and its tensor product strongly graded modules. We prove that a tensor product of strongly graded irreducible modules for a tensor product of strongly graded vertex algebras is irreducible, and that such irreducible modules, up to equivalence, exhaust certain naturally defined strongly graded irreducible modules for a tensor product of strongly graded vertex algebras. We also prove that certain naturally defined strongly graded modules for the tensor product strongly graded vertex algebra are completely reducible if and only if every strongly graded module for each of the tensor product factors is completely reducible. These results generalize the corresponding known results for vertex operator algebras and their modules.

1. Introduction

We prove that a tensor product of strongly graded irreducible modules for a tensor product of strongly graded vertex algebras is irreducible, and that conversely, such irreducible modules, up to equivalence, exhaust certain naturally defined strongly graded irreducible modules for a tensor product of strongly graded vertex algebras. (These terms are defined below.) As a consequence, we determine all the strongly graded irreducible modules for the tensor product of the moonshine module vertex operator algebra $V^\natural$ with a vertex algebra associated with a self-dual even lattice, in particular, the two-dimensional Lorentzian lattice.

The moonshine conjecture of Conway and Norton in [2] included the conjecture that there should exist an infinite-dimensional representation V of the (not yet constructed) Fischer–Griess Monster sporadic finite simple group $\mathbb{M}$ such that the McKay–Thompson series T_g for $g \in \mathbb{M}$ acting on V should have coefficients that are equal to the coefficients of the q-series expansions of certain modular functions. In particular, this conjecture incorporated the McKay–Thompson conjecture, which asserted that there should exist a (suitably nontrivial) $\mathbb{Z}$-graded $\mathbb{M}$-module $V = \coprod_{i\geq -1} V_{-i}$ with graded dimension equal to the elliptic modular function $j(\tau) - 744 = \sum_{i\geq -1} c(i)q^i$, where we write q for $e^{2\pi i\tau}$, τ in the upper half-plane. Such an $\mathbb{M}$-module, the "moonshine module", denoted by $V^\natural$, was constructed in [8], and in fact, the construction of [8] gave a vertex operator algebra structure on $V^\natural$ equipped with an action of $\mathbb{M}$. In [8], the authors also gave an explicit formula for the McKay–Thompson series of any element of the centralizer of an involution of type 2**B** of $\mathbb{M}$; the case of the identity element of $\mathbb{M}$ proved the McKay–Thompson conjecture.

Borcherds then showed in [1] that the rest of the McKay–Thompson series for the elements of $\mathbb{M}$ acting on $V^\natural$ are the expected modular functions. He obtained recursion formulas for the coefficients of McKay–Thompson series for $V^\natural$ from the Euler–Poincaré identity for certain homology groups associated with a special Lie algebra, the "monster Lie algebra", which he constructed using the tensor product of the moonshine module vertex operator algebra $V^\natural$ and a natural vertex algebra

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doi:10.1016/j.jpaa.2012.06.021

associated with the two-dimensional Lorentzian lattice. The importance of this tensor product vertex algebra motivates the present paper.

The difference between the terminology "vertex operator algebra", as defined in [8], and "vertex algebra", as defined in [1], is that a vertex operator algebra amounts to a vertex algebra with a conformal vector such that the eigenspaces of the operator $L(0)$ are all finite dimensional with (integral) eigenvalues that are truncated from below (cf. [17]). In [14], the authors use a notion of "conformal vertex algebra", which is a vertex algebra with a conformal vector and with an $L(0)$-eigenspace decomposition, and a notion of "strongly graded conformal vertex algebra", which is a conformal vertex algebra with a second, compatible grading by an abelian group satisfying certain grading restriction conditions.

In a series of papers [10–13,9], the authors developed a tensor product theory for modules for a vertex operator algebra under suitable conditions. A structure called "vertex tensor category structure", which is much richer than braided tensor category structure, has thereby been established for many important categories of modules for classes of vertex operator algebras (see [10]). It is expected that a vertex tensor category together with certain additional structures determines uniquely (up to isomorphism) a vertex operator algebra such that the vertex tensor category constructed from a suitable category of modules for it is equivalent (in the sense of vertex tensor categories) to the original vertex tensor category. In [14], this tensor product theory is generalized to a larger family of categories of "strongly graded modules" for a conformal vertex algebra, under suitably relaxed conditions. We want to investigate the vertex tensor category in the sense of [10], but in the setting of [14], associated with the tensor product of the moonshine module vertex operator algebra $V^\natural$ and the vertex algebra associated with the two-dimensional Lorentzian lattice. The first step in thinking about this is to determine the irreducible modules for this algebra.

For the vertex operator algebra case, it is proved in [7] that a tensor product module $W_1 \otimes \cdots \otimes W_p$ for a tensor product vertex algebra $V_1 \otimes \cdots \otimes V_p$ (where W_i is a V_i-module) is irreducible if and only if each W_i is irreducible. The proof uses a version of Schur's Lemma and also the density theorem [15]. It is also proved in [7] that these irreducible modules W are (up to equivalence) exactly all the irreducible modules for the tensor product algebra $V_1 \otimes \cdots \otimes V_p$. The proof uses the fact that each homogeneous subspace of W is finite dimensional. In this paper, we generalize the arguments in [7] to prove similar, more general results for strongly graded modules for strongly graded conformal vertex algebras.

For the strongly graded conformal vertex algebra case, the homogeneous subspaces of a strongly graded module are no longer finite dimensional. However, by using the fact that each doubly homogeneous subspace (homogeneous with respect to both gradings) of a strongly graded conformal vertex algebra is finite dimensional, we prove a suitable version of Schur's Lemma for strongly graded modules under the assumption that the abelian group that gives the second grading of the strongly graded algebra is countable.

To avoid unwanted flexibility in the second grading such as a shifting of the grading by an element of the abelian group, we suppose that the grading abelian groups A for a strongly graded conformal vertex algebra and $\tilde{A}$ (which includes A as a subgroup) for its strongly graded modules are always determined by a vector space, which we typically call $\mathfrak{h}$, consisting of operators induced by V. We call this kind of strongly graded conformal vertex algebra a "strongly $(\mathfrak{h}, A)$-graded conformal vertex algebra" and its strongly graded modules "strongly $(\mathfrak{h}, \tilde{A})$-graded modules." Important examples of strongly $(\mathfrak{h}, A)$-graded conformal vertex algebras and their strongly $(\mathfrak{h}, \tilde{A})$-graded modules are the vertex algebras associated with nondegenerate even lattices and their modules.

For strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded modules W_i for strongly $(\mathfrak{h}_i, A_i)$-graded conformal vertex algebras V_i, we construct a tensor product strongly $(\oplus_{i=1}^p \mathfrak{h}_i, \oplus_{i=1}^p \tilde{A}_i)$-graded module $W_1 \otimes \cdots \otimes W_p$ for the tensor product strongly graded conformal vertex algebra $V_1 \otimes \cdots \otimes V_p$. Then we prove that this tensor product module $W_1 \otimes \cdots \otimes W_p$ is irreducible if and only if each W_i is irreducible, under the assumption that each grading abelian group A_i for V_i is a countable group.

To determine all the irreducible strongly graded modules (up to equivalence) for the tensor product strongly graded conformal vertex algebra $V_1 \otimes \cdots \otimes V_p$, the main difficulty is that we need to deal with the second grading by the abelian groups. For the strongly $(\oplus_{i=1}^p \mathfrak{h}_i, \tilde{A})$-graded modules W for the tensor product strongly $(\oplus_{i=1}^p \mathfrak{h}_i, \oplus_{i=1}^p A_i)$-graded vertex algebra $V_1 \otimes \cdots \otimes V_p$, we assume there is a decomposition $\tilde{A} = \tilde{A_1} \oplus \cdots \oplus \tilde{A_p}$, such that W is an $(\mathfrak{h}_i, \tilde{A}_i)$-graded module (that is, a strongly graded module except for the grading restriction conditions) when viewed as a V_i-module. We call this kind of strongly $(\oplus_{i=1}^p \mathfrak{h}_i, \tilde{A})$-graded module a strongly $((\mathfrak{h}_1, \tilde{A_1}), \ldots, (\mathfrak{h}_p, \tilde{A_p}))$-graded module. In the main theorem, we prove that if such a module is irreducible, then it is a tensor product of strongly graded irreducible modules. Then, as a corollary of the main theorem, we classify the strongly graded modules for the tensor product strongly graded conformal vertex algebra $V^\natural \otimes V_L$, where L is an even lattice, and in particular, where L is the (self-dual) two-dimensional Lorentzian lattice.

It is proved in [6] that every module for the tensor product vertex operator algebra $V_1 \otimes \cdots \otimes V_p$ is completely reducible if and only if every module for each vertex operator algebra V_i is completely reducible. We also generalize the argument in [6] to prove a similar result for tensor product strongly $(\mathfrak{h}, A)$-graded conformal vertex algebras.

This paper is organized as follows. In Section 2, we introduce the definitions and some basic properties of strongly graded vertex algebras and their strongly graded modules. Then we construct a tensor product of strongly graded vertex algebras and its tensor product strongly graded modules in Section 3. In Section 4, we introduce the definition of strongly $(\mathfrak{h}, A)$-graded vertex algebra and strongly $(\mathfrak{h}, \tilde{A})$-graded module. In Section 5, we prove the main theorem, which classifies the irreducible strongly $((\mathfrak{h}_1, \tilde{A_1}), \ldots, (\mathfrak{h}_p, \tilde{A_p}))$-graded $V_1 \otimes \cdots \otimes V_p$-modules. Then we use the main theorem to determine all the strongly graded modules for $V^\natural \otimes V_L$. In Section 6, we consider strongly graded conformal vertex algebras whose strongly graded modules are all completely reducible and prove that every strongly $((\mathfrak{h}_1, \tilde{A_1}), \ldots, (\mathfrak{h}_p, \tilde{A_p}))$-graded module for the tensor product strongly graded algebra $V_1 \otimes \cdots \otimes V_p$ is completely reducible if and only if every strongly $(\mathfrak{h}_i, \tilde{A_i})$-graded module for each V_i is completely reducible.

2. Strongly graded vertex algebras and their modules

We recall the following four definitions from [14].

Definition 2.1. A *conformal vertex algebra* is a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)} \tag{2.1}$$

(for $v \in V_{(n)}$, we say the *weight* of v is n and we write $\mathrm{wt}\, v = n$) equipped with a linear map $V \otimes V \to V[[x, x^{-1}]]$, or equivalently,

$$\begin{aligned} V &\to (\mathrm{End}\, V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \mathrm{End}\, V), \end{aligned} \tag{2.2}$$

$Y(v, x)$ denoting the *vertex operator associated with* v, and equipped also with two distinguished vectors $\mathbf{1} \in V_{(0)}$ (the *vacuum vector*) and $\omega \in V_{(2)}$ (the *conformal vector*), satisfying the following conditions for $u, v \in V$: the *lower truncation condition*:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large} \tag{2.3}$$

(or equivalently, $Y(u, x)v \in V((x))$); the *vacuum property*:

$$Y(\mathbf{1}, x) = 1_V; \tag{2.4}$$

the *creation property*:

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x\to 0} Y(v, x)\mathbf{1} = v \tag{2.5}$$

(that is, $Y(v, x)\mathbf{1}$ involves only nonnegative integral powers of x and the constant term is v); the *Jacobi identity* (the main axiom):

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2) \tag{2.6}$$

(note that when each expression in (2.6) is applied to any element of V, the coefficient of each monomial in the formal variables is a finite sum; on the right-hand side, the notation $Y(\cdot, x_2)$ is understood to be extended in the obvious way to $V[[x_0, x_0^{-1}]]$); the *Virasoro algebra relations*:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}c \tag{2.7}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e., } Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}, \tag{2.8}$$

$$c \in \mathbb{C} \tag{2.9}$$

(the *central charge* or *rank* of V);

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x) \tag{2.10}$$

(the $L(-1)$-*derivative property*); and

$$L(0)v = nv = (\mathrm{wt}\, v)v \quad \text{for } n \in \mathbb{Z} \quad \text{and} \quad v \in V_{(n)}. \tag{2.11}$$

This completes the definition of the notion of conformal vertex algebra. We will denote such a conformal vertex algebra by $(V, Y, \mathbf{1}, \omega)$.

Definition 2.2. Given a conformal vertex algebra $(V, Y, \mathbf{1}, \omega)$, a *module* for V is a $\mathbb{C}$-graded vector space

$$W = \coprod_{n\in\mathbb{C}} W_{(n)} \tag{2.12}$$

(graded by *weights*) equipped with a linear map $V \otimes W \to W[[x, x^{-1}]]$, or equivalently,

$$\begin{aligned} V &\to (\mathrm{End}\, W)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \mathrm{End}\, W) \end{aligned} \tag{2.13}$$

(note that the sum is over $\mathbb{Z}$, not $\mathbb{C}$), $Y(v, x)$ denoting the *vertex operator on W associated with v*, such that all the defining properties of a conformal vertex algebra that make sense hold. That is, the following conditions are satisfied: the lower truncation condition: for $v \in V$ and $w \in W$,

$$v_n w = 0 \quad \text{for } n \text{ sufficiently large} \tag{2.14}$$

(or equivalently, $Y(v, x)w \in W((x))$); the vacuum property:

$$Y(\mathbf{1}, x) = 1_W; \tag{2.15}$$

the Jacobi identity for vertex operators on W: for $u, v \in V$,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2) \tag{2.16}$$

(note that on the right-hand side, $Y(u, x_0)$ is the operator on V associated with u); the Virasoro algebra relations on W with scalar c equal to the central charge of V:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}c \tag{2.17}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e., } Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}; \tag{2.18}$$

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x) \tag{2.19}$$

(the $L(-1)$-derivative property); and

$$(L(0) - n)w = 0 \quad \text{for } n \in \mathbb{C} \quad \text{and} \quad w \in W_{(n)}, \tag{2.20}$$

where $n = \mathrm{wt}\, w$.

This completes the definition of the notion of module for a conformal vertex algebra.

Definition 2.3. Let A be an abelian group. A conformal vertex algebra

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)}$$

is said to be *strongly graded with respect to A* (or *strongly A-graded*, or just *strongly graded* if the abelian group A is understood) if it is equipped with a second gradation, by A,

$$V = \coprod_{\alpha\in A} V^{(\alpha)},$$

such that the following conditions are satisfied: the two gradations are compatible, that is,

$$V^{(\alpha)} = \coprod_{n\in\mathbb{Z}} V^{(\alpha)}_{(n)} \quad (\text{where } V^{(\alpha)}_{(n)} = V_{(n)} \cap V^{(\alpha)}) \quad \text{for any } \alpha \in A;$$

for any $\alpha, \beta \in A$ and $n \in \mathbb{Z}$,

$$V^{(\alpha)}_{(n)} = 0 \quad \text{for } n \text{ sufficiently negative}; \tag{2.21}$$

$$\dim V^{(\alpha)}_{(n)} < \infty; \tag{2.22}$$

$$\mathbf{1} \in V^{(0)}_{(0)}; \tag{2.23}$$

$$\omega \in V^{(0)}_{(2)}; \tag{2.24}$$

$$v_l V^{(\beta)} \subset V^{(\alpha+\beta)} \quad \text{for any } v \in V^{(\alpha)},\ l \in \mathbb{Z}. \tag{2.25}$$

This completes the definition of the notion of strongly A-graded conformal vertex algebra.

For modules for a strongly graded algebra we will also have a second grading by an abelian group, and it is natural to allow this group to be larger than the second grading group A for the algebra. (Note that this already occurs for the *first* grading group, which is $\mathbb{Z}$ for algebras and $\mathbb{C}$ for modules.)

Definition 2.4. Let A be an abelian group and V a strongly A-graded conformal vertex algebra. Let $\tilde{A}$ be an abelian group containing A as a subgroup. A V-module

$$W = \coprod_{n\in\mathbb{C}} W_{(n)}$$

is said to be *strongly graded with respect to* $\tilde{A}$ (or *strongly* $\tilde{A}$*-graded*, or just *strongly graded* if the abelian group $\tilde{A}$ is understood) if it is equipped with a second gradation, by $\tilde{A}$,

$$W = \coprod_{\beta\in\tilde{A}} W^{(\beta)}, \tag{2.26}$$

such that the following conditions are satisfied: the two gradations are compatible, that is, for any $\beta \in \tilde{A}$,

$$W^{(\beta)} = \coprod_{n\in\mathbb{C}} W^{(\beta)}_{(n)} \quad (\text{where } W^{(\beta)}_{(n)} = W_{(n)} \cap W^{(\beta)})$$

for any $\alpha \in A$, $\beta \in \tilde{A}$ and $n \in \mathbb{C}$,

$$W^{(\beta)}_{(n+k)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative;} \tag{2.27}$$

$$\dim W^{(\beta)}_{(n)} < \infty \tag{2.28}$$

$$v_l W^{(\beta)} \subset W^{(\alpha+\beta)} \quad \text{for any } v \in V^{(\alpha)},\ l \in \mathbb{Z}. \tag{2.29}$$

This completes the definition of the notion of strongly $\tilde{A}$-graded module for a strongly A-graded conformal vertex algebra.

Remark 2.5. It is always possible that there are different gradings on W by $\tilde{A}$, such as by shifting by an element in $\tilde{A}$. However, in this paper, we shall fix one particular $\tilde{A}$-grading on W.

In order to study strongly graded V-modules for tensor product algebras, we shall need the following generalization:

Definition 2.6. In the setting of Definition 2.4 (the definition of "strongly graded module"), a V-module (not necessarily strongly graded, of course) is *doubly graded with respect to* $\tilde{A}$ if it satisfies all the conditions in Definition 2.4 except perhaps for (2.27) and (2.28).

Example 2.7. Note that the notion of conformal vertex algebra strongly graded with respect to the trivial group is exactly the notion of vertex operator algebra. Let V be a vertex operator algebra, viewed (equivalently) as a conformal vertex algebra strongly graded with respect to the trivial group. Then the V-modules that are strongly graded with respect to the trivial group (in the sense of Definition 2.4) are exactly the ($\mathbb{C}$-graded) modules for V as a vertex operator algebra, with the grading restrictions as follows: For $n \in \mathbb{C}$,

$$W_{(n+k)} = 0 \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative} \tag{2.30}$$

and

$$\dim W_{(n)} < \infty. \tag{2.31}$$

Example 2.8. An important source of examples of strongly graded conformal vertex algebras and modules comes from the vertex algebras and modules associated with even lattices. We recall the following construction from [8]. Let L be an even lattice, i.e., a finite-rank free abelian group equipped with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$, not necessarily positive definite, such that $\langle\alpha,\alpha\rangle \in 2\mathbb{Z}$ for all $\alpha \in L$. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. Then $\mathfrak{h}$ is a vector space with a nonsingular bilinear form $\langle\cdot,\cdot\rangle$, extended from L. We form a Heisenberg algebra

$$\widehat{\mathfrak{h}}_{\mathbb{Z}} = \coprod_{n\in\mathbb{Z},\ n\neq 0} \mathfrak{h}\otimes t^n \oplus \mathbb{C}c.$$

Let $(\widehat{L},\bar{\ })$ be a central extension of L by a finite cyclic group $\langle\kappa \mid \kappa^s = 1\rangle$. Fix a primitive sth root of unity, say ω, and define the faithful character

$$\chi : \langle\kappa\rangle \to \mathbb{C}^*$$

by the condition

$$\chi(\kappa) = \omega.$$

Denote by $\mathbb{C}_\chi$ the one-dimensional space $\mathbb{C}$ viewed as a $\langle\kappa\rangle$-module on which $\langle\kappa\rangle$ acts according to χ:

$$\kappa \cdot 1 = \omega,$$

and denote by $\mathbb{C}\{L\}$ the induced $\widehat{L}$-module

$$\mathbb{C}\{L\} = \mathrm{Ind}_{\langle\kappa\rangle}^{\widehat{L}} \mathbb{C}_\chi = \mathbb{C}[\widehat{L}] \otimes_{\mathbb{C}[\langle\kappa\rangle]} \mathbb{C}_\chi.$$

Then

$$V_L = S(\widehat{\mathfrak{h}}_{\mathbb{Z}}^-) \otimes \mathbb{C}\{L\}$$

has a natural structure of conformal vertex algebra; see [1] and Chapter 8 of [8]. For $\alpha \in L$, choose an $a \in \widehat{L}$ such that $\bar{a} = \alpha$. Define

$$\iota(a) = a \otimes 1 \in \mathbb{C}\{L\}$$

and

$$V_L^{(\alpha)} = \mathrm{span}\,\{h_1(-n_1)\cdots h_k(-n_k) \otimes \iota(a)\},$$

where $h_1, \ldots, h_k \in \mathfrak{h}$, $n_1, \ldots, n_k > 0$, and where $h(n)$ is the operator associated with $h \otimes t^n$ via the $\hat{\mathfrak{h}}_{\mathbb{Z}}$-module structure of V_L. Then V_L is equipped with a natural second grading given by L itself. Also for $n \in \mathbb{Z}$, we have

$$(V_L)_{(n)}^{(\alpha)} = \mathrm{span}\,\left\{h_1(-n_1)\cdots h_k(-n_k) \otimes \iota(a) \mid \bar{a} = \alpha, \sum_{i=1}^{k} n_i + \frac{1}{2}\langle\alpha, \alpha\rangle = n\right\},$$

making V_L a strongly L-graded conformal vertex algebra in the sense of Definition 2.3. When the form $\langle\cdot,\cdot\rangle$ on L is also positive definite, then V_L is a vertex operator algebra, that is, as in Example 2.7, V_L is a strongly A-graded conformal vertex algebra for A the trivial group. In general, a conformal vertex algebra may be strongly graded for several choices of A.

Any sublattice M of the "dual lattice" L° of L containing L gives rise to a strongly M-graded module for the strongly L-graded conformal vertex algebra (see Chapter 8 of [8]; cf. [17]). In fact, any irreducible V_L-module is equivalent to a V_L-module of the form $V_{L+\beta} \subset V_{L^\circ}$ for some $\beta \in L^\circ$ and any V_L-module W is equivalent to a direct sum of irreducible V_L-modules, i.e.,

$$W = \coprod_{\gamma_i \in L^\circ,\ i=1,\ldots,n} V_{\gamma_i + L},$$

where γ_i's are arbitrary elements of L°, and $n \in \mathbb{N}$ (see [3,5]; cf. [17]). In general, a module for a strongly graded vertex algebra may be strongly graded for several choices of $\tilde{A}$.

Notation 2.9. In the remainder of this section, without further assumption, we will let A be an abelian group and V be a strongly A-graded conformal vertex algebra. Also, we will let $\tilde{A}$ be an abelian group containing A and W be a doubly graded V-module with respect to $\tilde{A}$. When we need W to be strongly graded, we will say it explicitly.

Definition 2.10. The subspaces $V_{(n)}^{(\alpha)}$ for $n \in \mathbb{Z}$, $\alpha \in A$ in Definition 2.6 are called the *doubly homogeneous subspaces* of V. The elements in $V_{(n)}^{(\alpha)}$ are called *doubly homogeneous* elements. Similar definitions can be used for $W_{(n)}^{(\beta)}$ in the module W.

Notation 2.11. Let v be a doubly homogeneous element of V. Let wt v_n, $n \in \mathbb{Z}$, refer to the weight of v_n as an operator acting on W, and let A-wt v_n refer to the A-weight of v_n on W.

Lemma 2.12. *Let $v \in V_{(n)}^{(\alpha)}$, for $n \in \mathbb{Z}$, $\alpha \in A$. Then for $m \in \mathbb{Z}$,* wt $v_m = n - m - 1$ *and A-*wt $v_m = \alpha$.

Proof. The first equation is standard from the theory of graded conformal vertex algebras and the second follows easily from the definitions. □

Definition 2.13. The algebra $A(V; W)$ *associated with V and W* is defined to be the algebra of operators on W induced by V, i.e., the algebra generated by the set

$$\{v_n \mid v \in V,\ n \in \mathbb{Z}\}.$$

For a subspace V' of V, we use $A(V'; W)$ to denote the subalgebra of $A(V; W)$ generated by the set

$$\{v_n \mid v \in V',\ n \in \mathbb{Z}\}.$$

For a subspace W' of W, we use $A(V; W')$ to denote the subalgebra of $A(V; W)$ preserving W'. Similarly for V' and W', we use $A(V'; W')$ to denote the subalgebra of $A(V; W)$ generated by the operators on W' induced by V'.

Remark 2.14. When W' is a submodule of W, there are two possible definitions for $A(V; W')$ in Definition 2.13. One is as an algebra associated with V and W', the other is as a subalgebra of $A(V; W)$. But it does not matter because they are both algebras of operators on W' generated by the set

$$\{v_n \mid v \in V,\ n \in \mathbb{Z}\}.$$

Similar comments hold for V' a subalgebra of V.

The following lemma follows easily from Lemma 2.12:

Lemma 2.15. *The algebra $A(V; W)$ is doubly graded by $\mathbb{Z}$ and A. Moreover for $n \in \mathbb{Z}$,*

$$A(V; W)_{(n)} = \text{span}\left\{(v_1)_{j_1} \cdots (v_m)_{j_m} \;\middle|\; \sum_{i=1}^{m} \text{wt } (v_i)_{j_i} = n, \text{where } m \in \mathbb{N},\ v_i \in V,\ j_i \in \mathbb{Z},\ \text{for } i = 1, \ldots, m\right\}$$

and for $\alpha \in A$,

$$A(V; W)^{(\alpha)} = \text{span}\left\{(v_1)_{j_1} \cdots (v_m)_{j_m} \;\middle|\; \sum_{i=1}^{m} A\text{-wt } (v_i)_{j_i} = \alpha, \text{where } m \in \mathbb{N},\ v_i \in V,\ j_i \in \mathbb{Z},\ \text{for } i = 1, \ldots, m\right\}.$$

Proposition 2.16. *Let W be an irreducible doubly graded V-module with respect to $\tilde{A}$. Then we have the following results:*

(a) *Each weight subspace $W_{(h)}$ ($h \in \mathbb{C}$) is irreducible under the algebra $A(V; W_{(h)})$.*
(b) *Each $\tilde{A}$-homogeneous subspace $W^{(\beta)}$ ($\beta \in \tilde{A}$) is irreducible under the algebra $A(V; W^{(\beta)})$.*
(c) *Each doubly homogeneous subspace $W^{(\beta)}_{(h)}$ ($h \in \mathbb{C}$, $\beta \in \tilde{A}$) is irreducible under the algebra $A(V; W^{(\beta)}_{(h)})$.*

Proof. We only prove statement (a), the proofs of statements (b) and (c) being similar. If $W_{(h)}$ is not irreducible, we can find a nontrivial proper submodule U of $W_{(h)}$ under the algebra $A(V; W_{(h)})$. This submodule cannot generate all W under the action by the algebra $A(V; W)$, since by Lemma 2.15,

$$A(V; W)U = \coprod_{n\in\mathbb{Z}} A(V; W)_{(n)} U \subset U \oplus \coprod_{m\in\mathbb{Z}, m\neq h} W_{(m)}.$$

This contradicts the irreducibility of W. $\square$

Remark 2.17. A V-module W decomposes into submodules corresponding to the congruence classes of its weights modulo $\mathbb{Z}$: For $\mu \in \mathbb{C}/\mathbb{Z}$, let

$$W_{(\mu)} = \coprod_{\bar{n}=\mu} W_{(n)}, \tag{2.32}$$

where $\bar{n}$ denotes the equivalence class of $n \in \mathbb{C}$ in $\mathbb{C}/\mathbb{Z}$. Then

$$W = \coprod_{\mu\in\mathbb{C}/\mathbb{Z}} W_{(\mu)} \tag{2.33}$$

and each $W_{(\mu)}$ is a V-submodule of W. Thus if a module W is indecomposable (in particular, if it is irreducible), then all complex numbers n for which $W_{(n)} \neq 0$ are congruent modulo $\mathbb{Z}$ to each other.

Definition 2.18. Let W_1 and W_2 be doubly graded V-modules with respect to $\tilde{A}$. A *module homomorphism* from W_1 to W_2 is a linear map ψ such that

$$\psi(Y(v, x)w) = Y(v, x)\psi(w) \quad \text{for } v \in V,\ w \in W_1,$$

and such that ψ preserves the grading by $\tilde{A}$. An *isomorphism* is a bijective homomorphism. An *endomorphism* is a homomorphism from W to itself, we denote the endomorphism ring by $\text{End}^{\tilde{A}}_V(W)$.

Remark 2.19. Suppose V, W_1, W_2, ψ are as in Definition 2.18. Then ψ is compatible with both gradings:

$$\psi((W_1)^{(\beta)}_{(h)}) \subset (W_2)^{(\beta)}_{(h)}, \quad h \in \mathbb{C},$$

because ψ commutes with $L(0)$ (see Section 4.5 of [17]), and because ψ preserves the grading by $\tilde{A}$.

Remark 2.20. The endomorphism ring $\text{End}^{\tilde{A}}_V(W)$ is a subring of the commuting ring

$$\text{End}_V(W) := \{\text{linear maps } \psi : W \to W \mid \psi(Y(v, x)w) = Y(v, x)\psi(w), \text{for } v \in V, w \in W\}.$$

Proposition 2.21. *Suppose W is an irreducible strongly $\tilde{A}$-graded V-module. Then* $\text{End}^{\tilde{A}}_V(W) = \mathbb{C}$.

Proof. For any $\lambda \in \mathbb{C}$, $\psi \in \mathrm{End}_V^{\tilde{A}}(W)$, let W_λ^ψ be the λ-eigenspace of ψ. Then W_λ^ψ is a V-submodule of W. Because W is irreducible, $W_\lambda^\psi = 0$ or W. We still need to show $W_\lambda^\psi \neq 0$, for some $\lambda \in \mathbb{C}$.

Choose $h \in \mathbb{C}$, $\beta \in \tilde{A}$ such that $W_{(h)}^{(\beta)} \neq 0$. Then by Remark 2.19, ψ preserves $W_{(h)}^{(\beta)}$. Since $\dim W_{(h)}^{(\beta)} < \infty$ and we are working over $\mathbb{C}$, ψ has an eigenvector in $W_{(h)}^{(\beta)}$. Therefore $W_\lambda^\psi \neq 0$ for some $\lambda \in \mathbb{C}$. $\square$

Proposition 2.22. *Suppose A is a countable abelian group. Then* $\mathrm{End}_V(W) = \mathbb{C}$.

Proof. From Definition 2.3, $V_{(n)} = \coprod_{\alpha\in A} V_{(n)}^{(\alpha)}$, where each doubly homogeneous subspace $V_{(n)}^{(\alpha)}$ has finite dimension. Since A is a countable group, there are countably many such doubly homogeneous subspaces $V_{(n)}^{(\alpha)}$, and hence V has countable dimension. Since W is irreducible, from Proposition 4.5.6 of [17], we know

$$W = \mathrm{span}\{v_n w \mid v \in V, n \in \mathbb{Z}\},$$

for any nonzero element w in W. Since V has countable dimension, so does W. Then the result follows from Dixmier's Lemma, which says that if S is an irreducible set of operators on a vector space W of countable dimension over $\mathbb{C}$, then the commuting ring of S on W consists of the scalars (cf. Lemma 2.2 in [16], and [18], p. 11), where we take S to be $A(V; W)$. $\square$

3. Tensor product of strongly graded vertex algebras and their modules

In this section, we are going to introduce the notion of tensor product of finitely many strongly graded conformal vertex algebras and their modules.

Let $A_1, \ldots, A_p$ be abelian groups, and let $V_1, \ldots, V_p$ be strongly $A_1, \ldots, A_p$-graded conformal vertex algebras with conformal vectors $\omega^1, \ldots, \omega^p$, respectively.

Let

$$A = A_1 \oplus \cdots \oplus A_p.$$

Then the vector space

$$V = V_1 \otimes \cdots \otimes V_p$$

becomes a strongly A-graded conformal vertex algebra, which we shall call the *tensor product strongly A-graded conformal vertex algebra*, with the following structure:

$$Y(v^{(1)} \otimes \cdots \otimes v^{(p)}, x) = Y(v^{(1)}, x) \otimes \cdots \otimes Y(v^{(p)}, x)$$

for $v^{(i)} \in V_i$ and the vacuum vector is

$$\mathbf{1} = \mathbf{1} \otimes \cdots \otimes \mathbf{1}.$$

(Here we use the notation $\mathbf{1}$ for the vacuum vectors of V and each V_i.) The conformal vector is

$$\omega = \omega^1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \omega^p.$$

Then

$$L(n) = L_1(n) \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes L_p(n)$$

or $n \in \mathbb{Z}$. (Here we use the notation $L_i(n)$ for the operators on V_i associated with ω^i, $i = 1, \ldots, p$.) The A-grading of V is given by

$$V = \coprod_{\alpha\in A} V^{(\alpha)},$$

with

$$V^{(\alpha)} = V_1^{(\alpha_1)} \otimes \cdots \otimes V_p^{(\alpha_p)},$$

where $\alpha_i \in A_i$, $i = 1, \ldots, p$, are such that $\alpha_1 + \cdots + \alpha_p = \alpha$. The $\mathbb{Z}$-grading of V is given by

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)},$$

where

$$V_{(n)} = \coprod_{n_1+\cdots+n_p=n} (V_1)_{(n_1)} \otimes \cdots \otimes (V_p)_{(n_p)}.$$

(It follows that the $\mathbb{Z}$-grading is given by $L(0)$ defined above.)

Proposition 3.1. *The tensor product of finitely many strongly graded conformal vertex algebras is a strongly graded conformal vertex algebra whose central charge is the sum of the central charges of the tensor factors.*

Proof. The grading restrictions (2.21) and (2.22) clearly hold. The Jacobi identity follows from the weak commutativity and weak associativity properties, as in Section 3.4 of [17]. □

Notation 3.2. For each $i = 1, \dots, p$, we identify V_i with the subspace $1 \otimes \cdots \otimes 1 \otimes V_i \otimes 1 \otimes \cdots \otimes 1$ of V. The strongly graded conformal vertex algebra V_i is a vertex subalgebra of V. However, it is not a conformal vertex subalgebra of V because the conformal vector of V and V_i do not match.

Remark 3.3. From the definition of tensor product strongly graded conformal vertex algebra, we see that

$$Y((1 \otimes \cdots \otimes 1 \otimes v^{(i)} \otimes 1 \otimes \cdots \otimes 1, x) = 1_{V_1} \otimes \cdots \otimes 1_{V_{i-1}} \otimes Y(v^{(i)}, x) \otimes 1_{V_{i+1}} \otimes \cdots \otimes 1_{V_p},$$

for $v^{(i)} \in V_i$. In particular, we have

$$[Y(V_i, x_1), Y(V_j, x_2)] = 0,$$

for $i, j = 1, \dots, p$ and $i \neq j$.

Lemma 3.4. *For all $n \in \mathbb{Z}$, $(v^{(1)} \otimes \cdots \otimes v^{(p)})_n$ can be expressed as a linear combination, finite on any given vector, of operators of the form $(v^{(1)} \otimes 1 \otimes \cdots \otimes 1)_{i_1} \cdots (1 \otimes \cdots \otimes 1 \otimes v^{(p)})_{i_p}$.*

Proof. We prove the result as in [7] by induction. When $p = 2$, taking Res_{x_1} and the constant term in x_0 of the Jacobi identity, we find that

$$\begin{aligned} Y(v^{(1)} \otimes v^{(2)}, x_2) &= \mathrm{Res}_{x_0} x_0^{-1} Y(Y(v^{(1)} \otimes 1, x_0)(1 \otimes v^{(2)}), x_2) \\ &= \mathrm{Res}_{x_1} (x_1 - x_2)^{-1} Y(v^{(1)} \otimes 1, x_1) Y(1 \otimes v^{(2)}, x_2) \\ &\quad - \mathrm{Res}_{x_1} (-x_2 + x_1)^{-1} Y(1 \otimes v^{(2)}, x_2) Y(v^{(1)} \otimes 1, x_1), \end{aligned}$$

so that for all $n \in \mathbb{Z}$, $(v^{(1)} \otimes v^{(2)})_n$ can be expressed as a linear combination, finite on any given vector, of operators of the form $(v^{(1)} \otimes 1)_{n_1} (1 \otimes v^{(2)})_{n_2}$. (Note that we do not need operators of the form $(1 \otimes v^{(2)})_{n_2} (v^{(1)} \otimes 1)_{n_1}$ because of the Remark 3.3.)

For general p, taking Res_{x_1} and the constant term in x_0 of the Jacobi identity, we have

$$\begin{aligned} Y(v^{(1)} \otimes \cdots \otimes v^{(p)}, x_2) &= \mathrm{Res}_{x_0} x_0^{-1} Y(Y(v^{(1)} \otimes \cdots \otimes v^{(p-1)} \otimes 1, x_0)(1 \otimes \cdots \otimes 1 \otimes v^{(p)}), x_2) \\ &= \mathrm{Res}_{x_1} (x_1 - x_2)^{-1} Y(v^{(1)} \otimes \cdots \otimes v^{(p-1)} \otimes 1, x_1) Y(1 \otimes \cdots \otimes 1 \otimes v^{(p)}, x_2) \\ &\quad - \mathrm{Res}_{x_1} (-x_2 + x_1)^{-1} Y(1 \otimes \cdots \otimes 1 \otimes v^{(p)}, x_2) Y(v^{(1)} \otimes \cdots \otimes v^{(p-1)} \otimes 1, x_1). \end{aligned}$$

It follows that $(v^{(1)} \otimes \cdots \otimes v^{(p)})_n$ is a linear combination of the operators $(v^{(1)} \otimes \cdots \otimes v^{(p-1)} \otimes 1)_{n_1} \cdot (1 \otimes \cdots \otimes 1 \otimes v^{(p)})_{n_2}$. Thus the lemma holds by the induction hypothesis. □

Now we define the notion of tensor product module for tensor product strongly $A = A_1 \oplus \cdots \oplus A_p$-graded conformal vertex algebra $V = V_1 \otimes \cdots \otimes V_p$ with the notions above. Let $\tilde{A}_1, \dots, \tilde{A}_p$ be abelian groups containing $A_1, \dots, A_p$ as subgroups, respectively, and let $W_1, \dots, W_p$ be strongly $\tilde{A}_1, \dots, \tilde{A}_p$-graded modules for $V_1, \dots, V_p$, respectively.

Let

$$\tilde{A} = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_p.$$

Then we can construct the *tensor product strongly $\tilde{A}$-graded module*

$$W = W_1 \otimes \cdots \otimes W_p$$

for the tensor product strongly A-graded algebra V by means of the definition

$$Y(v^{(1)} \otimes \cdots \otimes v^{(p)}, x) = Y(v^{(1)}, x) \otimes \cdots \otimes Y(v^{(p)}, x) \quad \text{for } v^{(i)} \in V_i,\ i = 1, \dots, p,$$

$$L(n) = L_1(n) \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes L_p(n) \quad \text{for } n \in \mathbb{Z}.$$

(Here we use the notation $L_i(n)$ for the operators associated with ω^i on W_i, $i = 1, \dots, p$.) The $\tilde{A}$-grading of W is defined as

$$W = \coprod_{\beta \in \tilde{A}} W^{(\beta)},$$

with

$$W^{(\beta)} = W_1^{(\beta_1)} \otimes \cdots \otimes W_p^{(\beta_p)},$$

where $\beta_i \in \tilde{A}_i$, $i = 1, \dots, p$, are such that $\beta_1 + \cdots + \beta_p = \beta$. The $\mathbb{C}$-grading of W is defined as

$$W = \coprod_{n \in \mathbb{C}} W_{(n)},$$

where

$$W_{(n)} = \sum_{n_1+\cdots+n_p=n} (W_1)_{(n_1)} \otimes \cdots \otimes (W_p)_{(n_p)}.$$

It follows that the $\mathbb{C}$-grading is given by the operator $L(0)$ on W defined above. It is clear that the algebra V is also a module for itself.

Proposition 3.5. *The structure W constructed above is a strongly $\tilde{A}$-graded module for the tensor product strongly A-graded conformal vertex algebra V.*

Assumption 3.6. In the remainder of this paper, we always assume that A, and that each A_i $(i = 1, \ldots, p)$ is a countable abelian group.

Using Proposition 2.22, we now prove the following.

Theorem 3.7. *Let $W = W_1 \otimes \cdots \otimes W_p$ be a strongly $\tilde{A} = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_p$-graded V-module, with the notations as above. Then W is irreducible if and only if each W_i is irreducible.*

Proof. The "only if" part is trivial. For the "if" part, for simplicity of notation, we take $p = 2$ without losing any essential content. Take a nonzero submodule $W \subset W_1 \otimes W_2$, let $w_1^{(1)}, \ldots, w_n^{(1)} \in W_1$ and $w_1^{(2)}, \ldots, w_n^{(2)} \in W_2$ be linearly independent such that $\Sigma_{j=1}^n a_j(w_j^{(1)} \otimes w_j^{(2)}) \in W$, where each $a_j \neq 0$. Take any $w^{(1)} \in W_1$, $w^{(2)} \in W_2$. By Proposition 2.22, the commuting ring consists of the scalars for W_1 and W_2. Thus by the density theorem (see for example Section 5.8 of [15]), there are $b_1 \in A(V_1; W_1 \otimes W_2)$, $b_2 \in A(V_2; W_1 \otimes W_2)$ such that

$$\begin{aligned} b_1 \cdot w_1^{(1)} &= w^{(1)}, \qquad b_1 \cdot w_i^{(1)} = 0, \quad \text{for } i = 2, \ldots, n. \\ b_2 \cdot w_1^{(2)} &= w^{(2)}, \qquad b_2 \cdot w_i^{(2)} = 0, \quad \text{for } i = 2, \ldots, n. \end{aligned}$$

Then

$$(b_1 b_2) \cdot \Sigma_{j=1}^n a_j(w_j^{(1)} \otimes w_j^{(2)}) = a_1(w^{(1)} \otimes w^{(2)}) \in W.$$

Hence $w^{(1)} \otimes w^{(2)} \in W$, and so $W = W_1 \otimes W_2$. $\square$

4. Strongly $(\mathfrak{h}, A)$-graded vertex algebras and their strongly $(\mathfrak{h}, \tilde{A})$-graded modules

For some strongly A-graded vertex algebras V, there is a vector space $\mathfrak{h}$ consisting of mutually commuting operators induced by V such that the A-grading of V is given by $\mathfrak{h}$ in the following way: for $\alpha \in A$, $V^{(\alpha)}$ is the weight space of $\mathfrak{h}$ of weight α. Here is an example:

Example 4.1. Consider the strongly L-graded conformal vertex algebra V_L in Example 2.8. For $h \in \mathfrak{h}$, there is an operator $h(0)$ on V_L such that

$$h(0) \cdot V_L^{(\alpha)} = \langle h, \alpha \rangle V_L^{(\alpha)}.$$

We identify $\mathfrak{h}$ with the set of operators

$$\{h(0) \;=\; (h(-1) \cdot \mathbf{1})_0 \mid h \in \mathfrak{h}\}$$

(see Chapter 8 of [8]). Since the symmetric bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate, $V_L^{(\alpha)}$ is characterized as the weight space of $\mathfrak{h}$ of weight α.

Consider the tensor algebra $T(V[t, t^{-1}])$ over the vector space $V[t, t^{-1}]$. Then any V-module W, in particular, V itself, can be regarded as a $T(V[t, t^{-1}])$-module uniquely determined by the condition that for $v \in V$, $n \in \mathbb{Z}$, $v \otimes t^n$ acts on W as v_n. In the following definitions, we consider a particular subspace of $T(V[t, t^{-1}])$ acting on V and W.

Definition 4.2. A strongly A-graded vertex algebra equipped with a vector subspace

$$\mathfrak{h} \subset T(V[t, t^{-1}])$$

is called *strongly $(\mathfrak{h}, A)$-graded* if there is a nondegenerate pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{h} \times A &\longrightarrow \mathbb{C} \\ (h, \alpha) &\longmapsto \langle h, \alpha \rangle \end{aligned}$$

linear in the first variable and additive in the second variable, such that $\mathfrak{h}$ acts commutatively on V and

$$V^{(\alpha)} = \{v \in V \mid h \cdot v = \langle h, \alpha \rangle v, \text{ for all } h \in \mathfrak{h}\}.$$

By Definition 4.2, the strongly graded conformal vertex algebra V_L in Example 4.1 is strongly $(\mathfrak{h}, L)$-graded, where $\mathfrak{h}$ is the set of operators $\{(h(-1)\cdot \mathbf{1})_0 \mid h \in L\otimes_{\mathbb{Z}} \mathbb{C}\}$.

For a strongly $(\mathfrak{h}, A)$-graded vertex algebra V, a natural module category is the category of strongly $\tilde{A}$-graded V-modules W with an action of $\mathfrak{h}$, such that the $\tilde{A}$-grading on W is given by weight spaces of $\mathfrak{h}$. Here is an example:

Example 4.3. As in Example 2.8, any sublattice M of L° containing L gives rise to a strongly M-graded V_L-module V_M. Take $\mathfrak{h} = L\otimes_{\mathbb{Z}}\mathbb{C}$ and identify $\mathfrak{h}$ as the set of operators $\{(h(-1)\cdot\mathbf{1})_0 \mid h\in\mathfrak{h}\}$ as in Example 4.1. Then for $\beta\in M$,

$$V_M^{(\beta)} = \{w\in V_M \mid h\cdot w = \langle h,\beta\rangle w,\ \text{for all } h\in\mathfrak{h}\}$$

so that we have examples of the following:

Definition 4.4. A strongly $\tilde{A}$-graded module for a strongly $(\mathfrak{h}, A)$-graded vertex algebra is said to be *strongly $(\mathfrak{h}, \tilde{A})$-graded* if there is a nondegenerate pairing

$$\begin{aligned}\langle\cdot,\cdot\rangle : \mathfrak{h}\times\tilde{A} &\longrightarrow \mathbb{C}\\ (h,\beta) &\longmapsto \langle h,\beta\rangle\end{aligned}$$

linear in the first variable and additive in the second variable, such that the operators in $\mathfrak{h}$ act commutatively on W and

$$W^{(\beta)} = \{w\in W \mid h\cdot w = \langle h,\beta\rangle w,\ \text{for all } h\in\mathfrak{h}\}.$$

Remark 4.5. Submodules and quotient modules of strongly $(\mathfrak{h}, \tilde{A})$-graded conformal modules are also strongly $(\mathfrak{h}, \tilde{A})$-graded modules. Irreducible strongly $(\mathfrak{h}, \tilde{A})$-graded modules are strongly $(\mathfrak{h}, \tilde{A})$-graded modules without nontrivial submodules. Strongly $(\mathfrak{h}, \tilde{A})$-graded module homomorphisms are strongly $\tilde{A}$-graded module homomorphisms which commute with the actions of $\mathfrak{h}$.

The following propositions are natural analogues of Propositions 3.1 and 3.5.

Proposition 4.6. *Let $V_1,\ldots,V_p$ be strongly $(\mathfrak{h}_1, A_1),\ldots,(\mathfrak{h}_p, A_p)$-graded conformal vertex algebras, respectively. Let $A = A_1\oplus\cdots\oplus A_p$, $\mathfrak{h} = \mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_p$, and let $\langle\cdot,\cdot\rangle_i$ denote the pairing between $\mathfrak{h}_i$ and A_i, for $i = 1,\ldots,p$. Then the tensor product algebra $V = V_1\otimes\cdots\otimes V_p$ becomes a strongly $(\mathfrak{h}, A)$-graded conformal vertex algebra, where the nondegenerate pairing is given by*

$$\begin{aligned}\langle\cdot,\cdot\rangle : \mathfrak{h}\times A &\longrightarrow \mathbb{C}\\ (h,\alpha) &\longmapsto \sum_{i=1}^{p}\langle h_i,\alpha_i\rangle_i,\end{aligned}$$

where $h = h_1+\cdots+h_p$, $\alpha = \alpha_1+\cdots+\alpha_p$, for $h_i\in\mathfrak{h}_i$, $\alpha_i\in A_i$, $i = 1,\ldots,p$, and

$$V^{(\alpha)} = V_1^{(\alpha_1)}\otimes\cdots\otimes V_p^{(\alpha_p)} = \{v\in V_1\otimes\cdots\otimes V_p \mid h\cdot v = \langle h,\alpha\rangle v,\ \text{for all } h\in\mathfrak{h}\}.$$

Proof. It is easy to see that the pairing defined above is nondegenerate, and $V^{(\alpha)}$ is characterized uniquely as the eigenspace of $\mathfrak{h}$. $\square$

Proposition 4.7. *Let $W_1,\ldots,W_p$ be strongly $(\mathfrak{h}_1, \tilde{A}_1),\ldots,(\mathfrak{h}_p, \tilde{A}_p)$-graded conformal modules for strongly $(\mathfrak{h}_1, A_1),\ldots,(\mathfrak{h}_p, A_p)$-graded conformal vertex algebras $V_1,\ldots,V_p$, respectively. Let $\tilde{A} = \tilde{A}_1\oplus\cdots\oplus\tilde{A}_p$, $\mathfrak{h} = \mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_p$, and let $\langle\cdot,\cdot\rangle_i$ denote the pairing between $\mathfrak{h}_i$ and $\tilde{A}_i$, for $i = 1,\ldots,p$. Then the tensor product module $W = W_1\otimes\cdots\otimes W_p$ becomes a strongly $(\mathfrak{h}, \tilde{A})$-graded module for the strongly graded vertex algebra V, where the nondegenerate pairing is given by*

$$\begin{aligned}\langle\cdot,\cdot\rangle : \mathfrak{h}\times\tilde{A} &\longrightarrow \mathbb{C}\\ (h,\beta) &\longmapsto \sum_{i=1}^{p}\langle h_i,\beta_i\rangle_i,\end{aligned}$$

where $h = h_1+\cdots+h_p$, $\beta = \beta_1+\cdots+\beta_p$, for $h_i\in\mathfrak{h}_i$, $\beta_i\in\tilde{A}_i$, $i = 1,\ldots,p$, and

$$W^{(\beta)} = W_1^{(\beta_1)}\otimes\cdots\otimes W_p^{(\beta_p)} = \{w\in W_1\otimes\cdots\otimes W_p \mid h\cdot w = \langle h,\beta\rangle w,\ \text{for all } h\in\mathfrak{h}\}.\quad\square$$

The following proposition is an analogue and consequence of Theorem 3.7.

Theorem 4.8. *Let $W = W_1\otimes\cdots\otimes W_p$ be a strongly $(\mathfrak{h}, \tilde{A})$-graded module constructed in Proposition 4.7. Then W is irreducible if and only if each W_i is irreducible.*

5. Irreducible modules for tensor product strongly graded algebra

Our goal is to determine all the strongly $(\mathfrak{h}, \tilde{A})$-graded irreducible modules for the tensor product strongly $(\mathfrak{h}, A)$-graded conformal vertex algebra constructed in Proposition 4.6. To do this, we need to define a more specific kind of strongly $(\mathfrak{h}, \tilde{A})$-graded modules as follows:

Definition 5.1. Let $V_1, \ldots, V_p, V$ be strongly $(\mathfrak{h}_1, A_1), \ldots, (\mathfrak{h}_p, A_p), (\mathfrak{h}, A)$-graded conformal vertex algebras, respectively, as in the setting of Proposition 4.6. Let W be a strongly $(\mathfrak{h}, \tilde{A})$-graded V-module, where $\tilde{A}$ is an abelian group containing A as a subgroup, so that in particular, for $\beta \in \tilde{A}$,

$$W^{(\beta)} = \{w \in W \mid h \cdot w = \langle h, \beta\rangle w, \text{ for all } h \in \mathfrak{h}\}.$$

Assume that there exists an abelian subgroup $\tilde{A}_i$ of $\tilde{A}$ containing A_i as a subgroup for each $i = 1, \ldots, p$ such that

$$\tilde{A} = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_p,$$
$$\langle \mathfrak{h}_i, \tilde{A}_j\rangle = 0 \quad \text{if } i \neq j$$

and such that W is a doubly graded V_i-module with respect to $\tilde{A}_i$ and the $\tilde{A}_i$-grading is given by $\mathfrak{h}_i$ in the following way: For $\beta_i \in \tilde{A}_i$,

$$W^{(\beta_i)} = \{w \in W \mid h_i \cdot w = \langle h_i, \beta_i\rangle w, \text{ for all } h_i \in \mathfrak{h}_i\}.$$

Then W is called a *strongly* $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$*-graded* V*-module*.

Remark 5.2. Submodules and quotient modules of strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded V-modules are also strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded modules. Irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded modules are strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded modules without nontrivial submodules. Strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded module homomorphisms are strongly $(\mathfrak{h}, \tilde{A})$-graded V-module homomorphisms.

Example 5.3. The strongly $(\mathfrak{h}, \tilde{A})$-graded tensor product module $W_1 \otimes \cdots \otimes W_p$ constructed in Proposition 4.7 is a strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded $V_1 \otimes \cdots \otimes V_p$-module.

From Example 2.8, we can see that any V_L-module is a strongly L°-graded module. Based on this fact, it is easy to show that the following example satisfies the conditions in Definition 5.1.

Example 5.4. Let $V^\natural$ be the moonshine module constructed in [8], which is a strongly $(\langle 0\rangle, \langle 0\rangle)$-graded conformal vertex algebra as in Example 2.7; let V_L be the conformal vertex algebra associated with the even two-dimensional unimodular Lorentzian lattice L, which is a strongly $(\mathfrak{h}, L)$-graded conformal vertex algebra as constructed in Example 2.8. Then any strongly $(\mathfrak{h}, L)$-graded module for $V^\natural \otimes V_L$ is strongly $((\langle 0\rangle, \langle 0\rangle), (\mathfrak{h}, L))$-graded (note that L is a self-dual lattice, i.e., $L^\circ = L$).

Notation 5.5. For $\beta_1 \in \tilde{A}_1, \ldots, \beta_p \in \tilde{A}_p$, we let $W^{(\beta_1,\ldots,\beta_p)}$ denote the following common weight space of $\mathfrak{h}_1, \ldots, \mathfrak{h}_p$, i.e.,

$$W^{(\beta_1,\ldots,\beta_p)} := \{w \in W \mid h_i \cdot w = \langle h_i, \beta_i\rangle w, \text{ for all } h_i \in \mathfrak{h}_i,\ i = 1, \ldots, p\}.$$

Next we assume W to be a strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded $V_1 \otimes \cdots \otimes V_p$-module, with the notation as in Definition 5.1.

Proposition 5.6. *Suppose that W is irreducible. Then for $\beta_1 \in \tilde{A}_1, \ldots, \beta_p \in \tilde{A}_p$, $W^{(\beta_1,\ldots,\beta_p)}$ is irreducible under the algebra of operators $A(V_1 \otimes \cdots \otimes V_p; W^{(\beta_1,\ldots,\beta_p)})$.*

Proof. The proof is similar to the proof of Proposition 2.16. □

Lemma 5.7. *For $\beta \in \tilde{A}$, we have*

$$W^{(\beta)} = W^{(\beta_1,\ldots,\beta_p)},$$

where $\beta = \beta_1 + \cdots + \beta_p$.

Proof. This is a consequence of Definition 5.1. □

Theorem 5.8. *Let W be a strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded irreducible $V_1 \otimes \cdots \otimes V_p$-module, with the notions as in Definition 5.1. Then W is a tensor product of irreducible strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded V_i-modules, for $i = 1, \ldots, p$.*

Proof. For simplicity of notation, we take $p = 2$, as above. Since W is irreducible, by Remark 2.17, $W = \coprod_{\bar{n}=\mu} W_{(n)}$ for some $\mu \in \mathbb{C}/\mathbb{Z}$, where $\bar{n}$ denotes the equivalent class of $n \in \mathbb{C}$ in $\mathbb{C}/\mathbb{Z}$. Choose $\beta \in \tilde{A}$ such that $W^{(\beta)} \neq 0$. Then there exists $n_0 \in \mathbb{C}$ such that $W^{(\beta)}_{(n_0)}$ is the lowest weight space of $W^{(\beta)}$. Since $W^{(\beta)}_{(n_0)}$ is finite dimensional and we are working over $\mathbb{C}$, there exists a simultaneous eigenvector $w_0 \in W^{(\beta)}_{(n_0)}$ for the commuting operators $L_i(0)$ and the operators in $\mathfrak{h}_i$, $i = 1, 2$. Denote by $n_1, n_2 \in \mathbb{Z}$ the corresponding eigenvalues for $L_1(0)$, $L_2(0)$. Then we have $n_0 = n_1 + n_2$. Denote by $\beta_1 \in \tilde{A}_1$, $\beta_2 \in \tilde{A}_2$ the corresponding weights for $\mathfrak{h}_1$, $\mathfrak{h}_2$. By Lemma 5.7, we have $W^{(\beta)} = W^{(\beta_1,\beta_2)}$, and $\beta = \beta_1 + \beta_2$.

Now the $L(-1)$-derivative condition and the $L(0)$-bracket formula imply that

$$[L_1(0), Y(v^{(1)} \otimes 1, x)] = Y(L_1(0)(v^{(1)} \otimes 1), x) + x\frac{d}{dx}Y(v^{(1)} \otimes 1, x)$$

for $v^{(1)} \in V_1$. Thus for doubly homogeneous vector $v^{(1)}$ and $n \in \mathbb{Z}$,

$$\mathrm{wt}_1(v^{(1)} \otimes 1)_n = \mathrm{wt}_1(v^{(1)} \otimes 1) - n - 1,$$

where wt_1 refers to $L_1(0)$-eigenvalue on both $V_1 \otimes V_2$ and the space of operators on W. In particular, $(v^{(1)} \otimes 1)_n$ permutes $L_1(0)$-eigenspaces. Moreover, since $(1 \otimes v^{(2)})_n$, for $v^{(2)} \in V_2$, commutes with $L_1(0)$, it preserves $L_1(0)$-eigenspaces. Of course, similar statements hold for $L_2(0)$, $\mathfrak{h}_1(0)$, $\mathfrak{h}_2(0)$.

By Lemma 5.6, $W^{(\beta_1,\beta_2)}$ is irreducible under the algebra of the operators $A(V_1 \otimes V_2; W^{(\beta_1,\beta_2)})$. Then $W^{(\beta_1,\beta_2)}$ is generated by w_0 by the irreducibility, and is spanned by elements of the form

$$(v_1^{(1)} \otimes 1)_{m_1} \cdots (v_k^{(1)} \otimes 1)_{m_k} (1 \otimes v_1^{(2)})_{n_1} \cdots (1 \otimes v_l^{(2)})_{n_l} w_0$$

where $v_i^{(1)} \in V_1$ and $v_j^{(2)} \in V_2$, $v_i^{(1)}$, $v_j^{(2)}$ are doubly homogeneous, and the A-weights of $\sum_{i=1}^m v_i^{(1)}$ and $\sum_{j=1}^n v_j^{(2)}$ are 0.

Hence $W^{(\beta_1,\beta_2)}$ is the direct sum of its simultaneous eigenspaces for $L_i(0)$ and $\mathfrak{h}_i$, for $i = 1, 2$, and the $L_1(0)$, $L_2(0)$-eigenvalues are bounded below by n_1, n_2, respectively. It follows that the lowest weight space $W^{(\beta_1,\beta_2)}_{(n_0)}$ is filled up by the simultaneous eigenspace for the operators $L_i(0)$ with eigenvalues n_i. To be more precise, we use $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$ to denote the subspace $W^{(\beta_1,\beta_2)}_{(n_0)}$. By a similar argument as in Proposition 5.6, $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$ is irreducible under the algebra of operators $A(V_1 \otimes V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$.

By the density theorem, the algebra $A(V_1 \otimes V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ fills up $\mathrm{End}\, W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$. Because $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ and $A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ are commuting algebras of operators and $A(V_1 \otimes V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ is generated by $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ and $A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$, we see that

$$\mathrm{End}\, W^{(\beta_1,\beta_2)}_{(n_1,n_2)} = A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}).$$

Choose an irreducible $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$-submodule M_1 of $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$. Then $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ acts faithfully on M_1 since any element of $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ which annihilates M_1 annihilates $A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) \cdot M_1 = A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) \cdot M_1 = (\mathrm{End}\, W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) M_1 = W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$. Thus $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ restricts faithfully to $\mathrm{End}\, M_1$ and hence is isomorphic to a full matrix algebra. Similarly, $A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ is isomorphic to a full matrix algebra. It follows that

$$\mathrm{End}\, W^{(\beta_1,\beta_2)}_{(n_1,n_2)} = A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) \otimes A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}).$$

Then $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$ has the structure

$$W^{(\beta_1,\beta_2)}_{(n_1,n_2)} = M_1 \otimes M_2$$

as an irreducible $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) \otimes A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$-module. Here, as an irreducible $A(V_i; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$-submodule of $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$, M_i has $\tilde{A}_i$-grading β_i induced by $\mathfrak{h}_i$, and has $\mathbb{C}$-grading n_i induced by $L_i(0)$, respectively, for $i = 1, 2$.

Let

$$w^0 = y_1 \otimes y_2$$

(where $y_i \in M_i$, for $i = 1, 2$) be a nonzero decomposable tensor in $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$. Let W_i be the doubly graded V_i-submodule of W generated by w^0. Then the module W_1 has a strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded V_1-module structure such that

$$W_1 = \coprod_{n \in \mathbb{C},\ \gamma \in \tilde{A}} (W_1)^{(\gamma)}_{(n)},$$

where

$$(W_1)^{(\gamma)}_{(n)} = \mathrm{span}\{(v_1^{(1)} \otimes \mathbf{1})_{s_1} \cdots (v_p^{(1)} \otimes \mathbf{1})_{s_p} w^0 | \mathrm{wt}\, v_1^{(1)} - s_1 - 1 + \cdots + \mathrm{wt}\, v_p^{(1)} - s_p - 1 = n - n_1,$$
$$A\text{-wt}\, v_1^{(1)} + \cdots + A\text{-wt}\, v_p^{(1)} = \gamma - \beta_1,\ v_1^{(1)}, \ldots, v_p^{(1)} \in V_1,\ s_1, \ldots, s_p \in \mathbb{Z}\}.$$

This module we constructed satisfies the grading restrictions (2.27) and (2.28) in Definition 2.4, which follows from the fact that W is a strongly graded $V_1 \otimes V_2$-module and each doubly homogeneous subspace of W_1 lies in the doubly homogeneous subspace of W. Also, $W_1^{(\gamma)}$ is the weight space of $\mathfrak{h}_1$ with weight γ, hence by Definition 4.4, W_1 is a strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded V_1-module.

We claim that W_1 is V_1-irreducible (and similarly for W_2). In fact, consideration of the abelian group grading shows that any nonzero V_1-submodule of W_1 not intersecting $W^{(\beta_1,\beta_2)}$ will give rise to a nonzero $V_1 \otimes V_2$-submodule of W not intersecting $W^{(\beta_1,\beta_2)}$. Thus any nonzero V_1-submodule of W_1 must intersect $W^{(\beta_1,\beta_2)}$. Then consideration of the weight shows that the (β_1, β_2)-subspace of any nonzero V_1-submodule of W_1 not intersecting $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$ would give rise to a nonzero $A(V_1 \otimes V_2; W^{(\beta_1,\beta_2)})$-submodule of $W^{(\beta_1,\beta_2)}$ not intersecting $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$. Thus any nonzero V_1-submodule of W_1 must intersect $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$. But the irreducible $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$-module $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) \cdot w^0$ is the full intersection of W_1 and $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$, so that the V_1-submodule must contain w^0 and hence be all of W_1. This proves the V_1-irreducibility of W_1.

Finally, to show that W is isomorphic to $W_1 \otimes W_2$, consider the abstract tensor product $V_1 \otimes V_2$-module $W_1 \otimes W_2$, where W_i is the strongly $\tilde{A}_i$-graded V_i-module defined above, for $i = 1, 2$. Define a linear map

$$\varphi : W_1 \otimes W_2 \to W$$
$$b_1 \cdot w^0 \otimes b_2 \cdot w^0 \mapsto b_1 b_2 \cdot w^0,$$

where b_i is any operator induced by V_i. Then φ is well defined and is a $V_1 \otimes V_2$-module homomorphism. Since $W_1 \otimes W_2$ is irreducible by Theorem 3.7, φ is a module isomorphism. □

Example 5.9. Let V_{L_i} be the conformal vertex algebra associated with an even lattice L_i as in Example 2.8, where $i = 1, \ldots, p$. Let $V_{L_1} \otimes \cdots \otimes V_{L_p}$ be the tensor product strongly graded vertex algebra of $V_{L_1}, \ldots, V_{L_p}$. By the construction of a lattice vertex algebra in Example 2.8, we have

$$V_{L_1} \otimes \cdots \otimes V_{L_p} = V_{L_1 \oplus \cdots \oplus L_p},$$

and every irreducible $V_{L_1 \oplus \cdots \oplus L_p}$-module is equivalent to a module of the form

$$V_{L_1+\gamma_1 \oplus \cdots \oplus L_p+\gamma_p} = V_{L_1+\gamma_1} \otimes \cdots \otimes V_{L_p+\gamma_p},$$

for some $\gamma_i \in L_i^{\circ}$, $i = 1, \ldots, p$. This example illustrates Theorem 5.8.

Now we can describe our main examples:

Corollary 5.10. *The only irreducible strongly $(\mathfrak{h}, L)$-graded module of $V^\natural \otimes V_L$, where L is the unique even two-dimensional unimodular Lorentzian lattice and $\mathfrak{h} = \{(h(-1) \cdot 1)_0 \mid h \in L \otimes_{\mathbb{Z}} \mathbb{C}\}$, up to equivalence, is itself.*

Proof. Let W be an irreducible strongly $(\mathfrak{h}, L)$-graded module of $V^\natural \otimes V_L$. Then by Example 5.4, W is a strongly $((\langle 0\rangle, \langle 0\rangle), (\mathfrak{h}, L))$-graded module of $V^\natural \otimes V_L$. By Theorem 5.8, it is a tensor product of an irreducible strongly $(\langle 0\rangle, \langle 0\rangle)$-graded $V^\natural$-module with an irreducible strongly $(\mathfrak{h}, L)$-graded V_L-module. By Dong [4], $V^\natural$ is its only irreducible module, up to equivalence. Also, by Dong [3] (cf. [17], Example 2.8), V_L is its only irreducible module because L is self-dual. Therefore

$$W = V^\natural \otimes V_L$$

as claimed. □

Remark 5.11. In Corollary 5.10, the two-dimensional self-dual Lorentzian lattice can of course be generalized to any self-dual nondegenerate even lattice.

6. Complete reducibility

Definition 6.1. Let V be a strongly $(\mathfrak{h}, A)$-graded conformal vertex algebra. Then a strongly $(\mathfrak{h}, \tilde{A})$-graded V-module is called *completely reducible* if it is a direct sum of irreducible strongly $(\mathfrak{h}, \tilde{A})$-graded V-modules.

Notation 6.2. In the remainder of this section, we will always let $A = A_1 \oplus \cdots \oplus A_p$, $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p$, and $V = V_1 \otimes \cdots \otimes V_p$.

Definition 6.3. A strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded module for the tensor product conformal vertex algebra V is called *completely reducible* if it is a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded V-modules.

Theorem 6.4. *Let $V_1, \ldots, V_p$ be strongly $(\mathfrak{h}_1, A_1), \ldots, (\mathfrak{h}_p, A_p)$-graded conformal vertex algebras, respectively, and let V be their tensor product strongly $(\mathfrak{h}, A)$-graded conformal vertex algebra. Then every strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded V-module is completely reducible if and only if every strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded V_i-module is completely reducible.*

Proof. It suffices to prove the result for $n = 2$. Let W be a strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V = V_1 \otimes V_2$-module. Then by Proposition 5.7, we can take $w \in W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$, where $\beta_i \in \tilde{A}_i$, $n_i \in \mathbb{C}$, for $i = 1, 2$.

Let M be the strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-submodule of W generated by w, i.e., M is spanned by elements of the form

$$(v^{(1)}_1 \otimes \mathbf{1})_{s_1} \cdots (v^{(1)}_p \otimes \mathbf{1})_{s_p} (\mathbf{1} \otimes v^{(2)}_1)_{t_1} \cdots (\mathbf{1} \otimes v^{(2)}_q)_{t_q} w$$

where $v^{(1)}_1, \ldots, v^{(1)}_p$ are doubly homogeneous elements in V_1 and $v^{(2)}_1, \ldots, v^{(2)}_q$ are doubly homogeneous elements in V_2, respectively, and $s_1, \ldots, s_p, t_1, \ldots, t_q \in \mathbb{Z}$. Let M_i be the doubly graded V_i-submodule of M generated by w. Then M_i is a strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded V_i-module, respectively, for $i = 1, 2$, in an obvious way as in the proof of Theorem 5.8.

By Proposition 4.7 and Example 5.3, $M_1 \otimes M_2$ is strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded. Moreover, we have a strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module epimorphism from $M_1 \otimes M_2$ to M by sending $b_1 w \otimes b_2 w \mapsto b_1 b_2 w$, where b_i is an operator induced by V_i, for $i = 1,\ 2$. If every strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded V_i-module is completely reducible, then M_i is a direct sum of irreducible strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded V_i-modules and therefore $M_1 \otimes M_2$ is a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-modules (see Theorem 4.8). Then as a quotient module of $M_1 \otimes M_2$, M is also a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-modules, and consequently, W is a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-modules.

Conversely, assume that every strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module W is completely reducible. We first observe that $V_1 \otimes V_2$ is strongly $((\mathfrak{h}_1, A_1), (\mathfrak{h}_2, A_2))$-graded, hence a $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module itself by Proposition 4.6 and Example 5.3, and hence is a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded modules. Let W be an irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module. Then W is a tensor product of an irreducible strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded module for V_1 and an irreducible strongly $(\mathfrak{h}_2, \tilde{A}_2)$-graded module for V_2 by Theorem 5.8. In particular, V_1 has irreducible strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded modules and V_2 has irreducible strongly $(\mathfrak{h}_2, \tilde{A}_2)$-graded modules, respectively.

Let W_1 be a strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded V_1-module and W_2 be an irreducible strongly $(\mathfrak{h}_2, \tilde{A}_2)$-graded V_2-module. Since every strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module is completely reducible, $W_1 \otimes W_2$ is a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded modules:

$$W_1 \otimes W_2 = \coprod_i M_i$$

where each M_i is an irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module. Fix i and let $x^{(i)}_1, \ldots, x^{(i)}_n \in W_1$ and $y^{(i)}_1, \ldots, y^{(i)}_n \in W_2$ be linearly independent doubly homogeneous elements such that $\sum_j c_j x^{(i)}_j \otimes y^{(i)}_j \in M_i$, where $c_j \in \mathbb{C}$, $c_j \neq 0$. By the density theorem (as in the proof of Theorem 3.7), each $x^{(i)}_j \otimes y^{(i)}_j \in M_i$. Let W_{i1} be the doubly graded V_1-submodule of W_1 generated by $x^{(i)}_{j_0}$, for some $j_0 \in \{1, 2, \ldots, n\}$. Then W_{i1} is a strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded V_1-submodule as in the proof of Theorem 5.8. By the irreducibility of M_i, we see that $M_i = W_{i1} \otimes W_2$ and that W_{i1} is an irreducible strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded V_1-submodule of W_1. Therefore, $W_1 \otimes W_2 = (\coprod_i W_{i1}) \otimes W_2$. By the density theorem, for any nonzero $w_2 \in W_2$, $W_1 \otimes w_2 = (\coprod_i W_{i1}) \otimes w_2$. Hence as a V_1-module, $W_1 \cong (\coprod_i W_{i1})$, and thus W_1 is completely reducible. Similarly for V_2. $\square$

Example 6.5. Let V_{L_i} be the conformal vertex algebra associated with an even lattice L_i as in Example 2.8, where $i = 1, \ldots, p$. Let $V_{L_1} \otimes \cdots \otimes V_{L_p}$ be the tensor product strongly graded vertex algebra of $V_{L_1}, \ldots, V_{L_p}$. By the construction of a lattice vertex algebra as in Example 2.8, we have

$$V_{L_1} \otimes \cdots \otimes V_{L_p} = V_{L_1 \oplus \cdots \oplus L_p}.$$

As in Example 2.8, every module for $V_{L_1 \oplus \cdots \oplus L_p}$, hence for $V_{L_1} \otimes \cdots \otimes V_{L_p}$, is completely reducible. This example illustrates Theorem 6.4.

Corollary 6.6. *Every strongly $(\mathfrak{h}, L)$-graded module for the strongly $(\mathfrak{h}, L)$-graded conformal vertex algebra $V^{\natural} \otimes V_L$, where L is the unique even two-dimensional unimodular Lorentzian lattice and $\mathfrak{h} = \{(h(-1) \cdot 1)_0 \mid h \in L \otimes_{\mathbb{Z}} \mathbb{C}\}$, is completely reducible.*

Acknowledgments

I would like to thank Professor James Lepowsky for helpful discussions and suggestions. I would also like to thank Professor Yi-Zhi Huang and Professor Haisheng Li for useful discussions.

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