A geometric model of tube categories

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ABSTRACT

We give a geometric model for a tube category in terms of homotopy classes of oriented arcs in an annulus with marked points on its boundary. In particular, we interpret the dimensions of extension groups of degree 1 between indecomposable objects in terms of negative geometric intersection numbers between corresponding arcs, giving a geometric interpretation of the description of an extension group in the cluster category of a tube as a symmetrized version of the extension group in the tube. We show that a similar result holds for finite dimensional representations of the linearly oriented quiver of type A_∞^∞.

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1. Introduction

Let k be an algebraically closed field. A *tube category* $\mathcal{T}_n$ of rank n over k can be defined as the category of finite dimensional nilpotent representations of an oriented n-cycle over k. We introduce a geometric model of such a category using an annulus with n marked points on the outer boundary and none on the inner boundary. The indecomposable objects of $\mathcal{T}_n$ are in bijection with certain oriented arcs in the annulus, i.e. those which are oriented anticlockwise and which are not homotopic

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to a boundary arc between successive marked points; we refer to them as admissible arcs. The AR-quiver of the tube can be realised via the geometry of the arcs. We then show that:

Theorem 1.1. *Let α and β be admissible arcs in an annulus with n marked points on its outer boundary, and let A and B be the corresponding indecomposable objects in a tube $\mathcal{T}_n$ of rank n. Then the dimension of $\mathrm{Ext}^1_{\mathcal{T}_n}(A, B)$ coincides with the negative geometric intersection number of the pair α, β.*

Cluster categories were introduced in [BMRRT] (and in [CCS] for type A) in order to model the cluster algebras [FZ1] of Fomin and Zelevinsky. They have since been used widely in representation theory (see e.g. [K2] for a survey). Since a tube is skeletally small it follows from Keller's theorem [K1, §9.9] that the cluster category $\mathcal{C}_{\mathcal{T}_n}$ associated to a tube (which we call a cluster tube) is triangulated (see also [BKL1, §5] and [BMV, §2]). We show that the geometric intersection number of α, β is equal to the sum of the negative geometric intersection number and the positive geometric intersection number. This total number coincides with the dimension of $\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{T}_n}}(A, B)$. We thus obtain a geometric interpretation of the fact that

$$\mathrm{Ext}_{\mathcal{C}_{\mathcal{T}_n}}(A, B) \cong \mathrm{Ext}_{\mathcal{T}_n}(A, B) \oplus D\,\mathrm{Ext}_{\mathcal{T}_n}(B, A),$$

(which can be proved as in [BMRRT, Prop. 1.7]) for indecomposable objects A and B in $\mathcal{T}_n$, where $D = \mathrm{Hom}_k(-, k)$. We also show that similar results hold for the category of finite dimensional representations of the linear orientation of the quiver A_∞^∞, a hereditary category whose AR-quiver is of type $\mathbb{Z}A_\infty$.

We note that geometric models of cluster categories and of the generalised cluster categories of Amiot [A] use unoriented arcs in marked surfaces: see [BZ,CCS,HJ,S]. Such models were used for cluster algebras in [FZ2,FST]. Our perspective here is that by using oriented arcs we can model the classical categories $\mathcal{T}_n$ arising in representation theory. The 2-Calabi–Yau Ext^1-symmetry of the cluster category of $\mathcal{T}_n$ then arises by forgetting the orientation on the arcs. We note that oriented arcs have also been used in [BM] to obtain a geometric model for the root category [H] of type A (the quotient of the bounded derived category by the square of the shift).

We remark that [G] and [BZ, 3.18] give a construction of the AR-quiver of a cluster tube (in the latter case, as a subcategory of a generalised cluster category in a more general situation) using unoriented arcs in an annulus. It follows from [BZ, 5.3] that if the geometric intersection number of two non-homotopic admissible arcs in the annulus is non-zero then the Ext^1-group between the corresponding objects in the cluster tube is non-zero, and it follows from [BZ, 5.1] that the geometric intersection number of an arc with itself vanishes (see Remark 3.2) if and only if the Ext^1-group of the corresponding object with itself in the cluster tube vanishes. Thus, in the case of the cluster tube, we obtain a strengthening of these results.

In Section 2 we recall tube categories and indicate how the AR-quiver of the tube can be constructed geometrically using admissible arcs in the annulus. In Section 3 we compute geometric intersection numbers between admissible arcs in the annulus by lifting them to the universal covering space, and prove the main result. In Section 4 we give an interpretation of our model using doubled arcs which, in particular, gives a connection with a model for the rigid part of the cluster tube given in [BMV, §3]. In Section 5 we consider the case of the category of finite dimensional representations of the linear orientation of A_∞^∞.

Since completing this work, we learnt of interesting independent work in the diploma thesis [W] which, in particular, gives a bijection between string modules over a quiver of type $\widetilde{A}_n$ (i.e. a quiver whose underlying graph is a cycle, but is not an oriented cycle itself) and certain oriented arcs in the annulus (regarded as a cylinder). A geometric interpretation of the AR-quiver is given and it is shown that the negative geometric intersection number of two arcs coincides with the dimension of the extension group of the corresponding modules. Since tubes can be obtained as subcategories of the module categories of such quivers, this gives an alternative proof of Theorem 1.1.

Our approach includes a topological explanation of how geometric intersection numbers in the annulus and its universal cover are related (see Section 3). In addition, it links up interpretations of geometric intersection numbers with and without orientation, thus strengthening results of [BZ] in

this case. We generalise the results obtained here to the $\mathbb{Z}A_\infty$ case using the universal covering of the annulus. We also give an explanation for the doubled arc model for exceptional objects in the tube appearing in [BMV].

2. The AR-quiver of a tube

2.1. Categories of tube shape

We refer to [ARS] for background on the theory of finite dimensional algebras and their representations. For more information on tubes, see [Ri,SS]. A stable translation quiver Γ without multiple arrows is said to be of *tube type* if it is of the form $\Gamma = \mathbb{Z}A_\infty/n$ for some $n > 0$, cf. [Ri, §3]. In that case we say that the tube Γ has rank n and we call the additive hull of the mesh category of Γ a *tube of rank n*, denoting it $\mathcal{T}_n$. If C_n is a cyclically oriented quiver with n vertices then the category of finite dimensional nilpotent representations of kC_n is equivalent to $\mathcal{T}_n$ (this follows from [Ri, 3.6(6)]; see also [RvdB, III.1.1]).

The category $\mathcal{T}_n$ is Hom-finite, abelian, hereditary and uniserial [Ri, §3]. We have the duality $\operatorname{Ext}^1_{\mathcal{T}_n}(X, Y) \cong D \operatorname{Hom}_{\mathcal{T}_n}(Y, \tau X)$ in $\mathcal{T}_n$, where τ denotes the AR-translate. We can form its cluster category as in [BMRRT, §1], obtaining the *cluster tube* $\mathcal{C}_{\mathcal{T}_n}$ of rank n as the orbit category $D^b(\mathcal{T}_n)/\tau^{-1}[1]$. As noted above, Keller's theorem [K1, §9.9] applies, so the cluster tube is triangulated. This category has also been studied in [BKL1,BKL2,BMV]. Its AR-quiver is the same as the AR-quiver of $\mathcal{T}_n$ (arguing as in [BMRRT, Prop. 1.3]). To abbreviate we write $\Gamma_{\mathcal{T}_n} := \Gamma_{\mathrm{AR}}(\mathcal{T}_n)$ for the AR-quiver of $\mathcal{T}_n$.

2.2. Arcs in the annulus and its universal cover

Let $\mathbb{A}(n) = \mathbb{A}(n, 0)$ be an annulus with n marked points on the outer boundary. Our goal is to use oriented arcs in $\mathbb{A}(n)$ as a geometric model for the tube and the cluster tube as introduced in Section 2.1.

Remark 2.1. In [FST, Example 6.9], the authors consider annuli of the form $\mathbb{A}(r, s)$ with $r, s \geqslant 1$. These have r marked points on the outer boundary, s marked points on the inner boundary and unoriented arcs between these marked points. More generally, the authors study triangulated surfaces as a geometric model for cluster algebras. In particular, the annuli $\mathbb{A}(r, s)$ give rise to cluster algebras associated to quivers of type $\widetilde{A}_{r+s-1}$ with r arrows in one direction and s arrows in the other direction. Thus it makes sense to use the annulus $\mathbb{A}(n)$, with n marked points on one boundary component only, to model $\mathcal{T}_n$, whose objects can be considered as (nilpotent) representations of a cyclic quiver with n arrows all in the same direction.

We consider smooth oriented arcs in surfaces with boundary. Recall that two such arcs α, β which start at the same point and finish at the same point are said to be *homotopy equivalent* if there is a homotopy between them fixing the starting point and the finishing point. We write $\alpha \sim \beta$ to denote this.

Let us now consider oriented arcs between marked points of the boundary of $\mathbb{A}(n)$. We start by labelling the marked points of $\mathbb{A}(n)$ with $0, 1, \ldots, n-1$, in an anti-clockwise direction. We will always take arcs up to homotopy fixing the starting and ending points. For simplicity, we will assume that the marked points are equally spaced around the outer boundary of $\mathbb{A}(n)$. An oriented arc from a marked point i to a marked point l, for $i, l \in \{0, 1, \ldots, n-1\}$, is homotopically equivalent to the boundary arc which starts at i, goes around the boundary k times (for some $k \geqslant 0$) and then ends in l; we shall denote this arc by $i_k l$.

It will be most convenient to describe oriented arcs and their intersections using the universal cover of $\mathbb{A}(n)$. To be precise, observe that we may identify the annulus $\mathbb{A}(n)$ with a (bounded) cylinder $\mathrm{Cyl}(n)$ of height 1 with n marked points on the lower boundary: we regard the cylinder as a rectangle in $\mathbb{R}^2$ with the vertical sides identified, cf. Fig. 1. As in the annulus, we label the marked points on the lower boundary of $\mathrm{Cyl}(n)$ from left to right and assume that arcs are oriented anti-clockwise (when viewed from the top of the cylinder), i.e. left to right in the rectangle in $\mathbb{R}^2$.
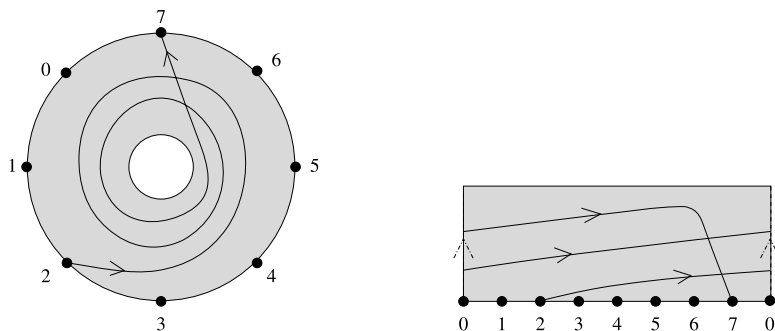
Fig. 1. $\alpha = \pi_8([2, 23])$ in the annulus $\mathbb{A}(8)$ and in the cylinder Cyl(8).

In a second step, we move from the cylinder Cyl(n) to its universal covering space $\mathbb{U} = (\mathbb{U}, \pi_n)$, with $\mathbb{U} = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant y \leqslant 1\}$, an infinite strip in the plane. We adopt the orientation on $\mathbb{U}$ inherited from its embedding in the plane. The covering map $\pi_n : \mathbb{U} \to$ Cyl(n) is induced from wrapping $\mathbb{U}$ around Cyl(n), i.e. the effect of π_n on $(x, y) \in \mathbb{U}$ is to take the first coordinate modulo n:

$$\pi_n : \mathbb{U} \to \text{Cyl}(n), \qquad (x, y) \longmapsto (x \bmod n, y),$$

so $(\lambda n, y)$ is mapped to $(0, y)$ for any $\lambda \in \mathbb{Z}$. We choose an orientation on Cyl(n) such that π_n is orientation-preserving. Via the identification of Cyl(n) with $\mathbb{A}(n)$, this induces an orientation on $\mathbb{A}(n)$. We will also write π_n to indicate the covering map from $\mathbb{U}$ to $\mathbb{A}(n)$ directly. We take the integers $(r, 0) \in \mathbb{U}$, $r \in \mathbb{Z}$, as marked points on the lower boundary of $\mathbb{U}$.

We define $\sigma : \mathbb{U} \to \mathbb{U}$ to be the translation which adds n to the x-coordinate of a point in $\mathbb{U}$, with inverse σ^{-1} subtracting n from the x-coordinate of a point, so that $\pi_n(\sigma^r(u)) = \pi_n(u)$ for all $u \in \mathbb{U}$ and for all $r \in \mathbb{Z}$. We write $G := \langle \sigma \rangle$ for the group generated by σ which thus acts naturally on $\mathbb{U}$.

Observe that for any $s \in \mathbb{R}$, the image of $\mathbb{U}_s = \{(x, y) \in \mathbb{U} \mid s \leqslant x < s + n\}$ under π_n is the cylinder Cyl(n).

It is clear that the oriented arc in the cylinder Cyl(n) corresponding to $i_k l$ starts at a marked point i and then wraps around the cylinder k times before ending at a marked point l. We now introduce an alternative notation for arcs in $\mathbb{U}$ and $\mathbb{A}(n)$. We call the arcs that we would like to focus on *admissible*.

Definition 2.2. Let i, j be integers with $j > i + 1$. An *admissible arc* in $\mathbb{U}$ is an oriented arc in $\mathbb{U}$ starting at the marked point $i = (i, 0)$ and ending in $j = (j, 0)$; we denote it by $[i, j]$. An oriented arc in $\mathbb{A}(n)$ is said to be *admissible* if it is of the form $\pi_n([i, j])$ where $[i, j]$ is admissible in $\mathbb{U}$.

Note that arcs in $\mathbb{A}(n)$ of the form $\pi_n([i, i + 1])$, which are homotopic to the part of the boundary between adjacent marked points, are not admissible, and nor are arcs of the form $\pi_n([i, j])$ with $i \geqslant j$.

Definition 2.3. Let $[i, j]$ be an admissible arc in $\mathbb{U}$. Then we define the *combinatorial length* of $[i, j]$ to be the integer $j - i$.

Remark 2.4. 1) If α is an admissible arc in $\mathbb{A}(n)$ with starting point i, $0 \leqslant i < n$, then it has a unique lift $\widetilde{\alpha}$ in $\mathbb{U}$ which starts at $i = (i, 0)$. We call this lift the *canonical lift of α*. Observe that all other liftings of α can be obtained by iteratively applying σ or σ^{-1} to $\widetilde{\alpha}$. So we get $\pi_n^{-1}(\alpha) = G\widetilde{\alpha}$. See Fig. 2 for an example.

2) Observe that the winding number of the arc $\pi_n([i, j])$ around the inner boundary of $\mathbb{A}(n)$ is given by $k = \lfloor \frac{j-i}{n} \rfloor$.
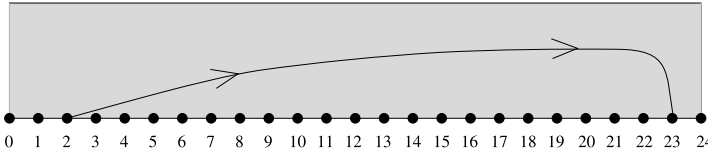
Fig. 2. The universal cover $\mathbb{U}$ with a lift of the arc $\alpha = \pi_8([2, 23])$.
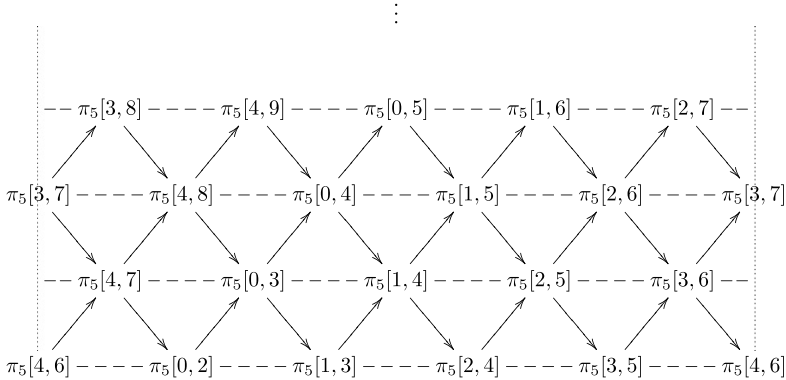

Fig. 3. The translation quiver $\Gamma(\mathbb{A}(5))$.

2.3. A quiver of arcs

We can associate a stable translation quiver to $\mathbb{A}(n)$, called $\Gamma(\mathbb{A}(n)) = \Gamma_{\mathrm{arc}}(\mathbb{A}(n))$. The vertices of $\Gamma(\mathbb{A}(n))$ are the admissible arcs in $\mathbb{A}(n)$. The arrows of $\Gamma(\mathbb{A}(n))$ are as follows. Let α and β be admissible arcs in $\mathbb{A}(n)$. Then there is an arrow from α to β if and only if there are admissible arcs $[a, b]$ and $[c, d]$ in $\mathbb{U}$ with $\pi_n([a, b]) = \alpha$, $\pi_n([c, d]) = \beta$ and either $c = a + 1$ and $d = b$, or $c = a$ and $d = b + 1$.

The translation τ on $\Gamma(\mathbb{A}(n))$ is induced by the map $i \mapsto i - 1$ (taken modulo n) for i a marked point of $\mathbb{A}(n)$. This clearly defines a structure of stable translation quiver on the set of admissible arcs in $\mathbb{A}(n)$. The case $n = 5$ is shown in Fig. 3 as an example.

2.4. Isomorphism of translation quivers

Here we show that the quivers $\Gamma(\mathbb{A}(n))$ and $\Gamma(\mathcal{T}_n)$ are isomorphic and thus get the geometric model for tubes. This isomorphism also appears in the Master's thesis [G] of B. Gehrig and in [BZ, §3.4] (in these two cases, using unoriented arcs), and in [W, 4.18] (using oriented arcs).

To describe the AR-quiver $\Gamma(\mathcal{T}_n)$, recall that $\mathcal{T}_n$ is uniserial. We denote the simple objects by $S_0, S_1, \ldots, S_{n-1}$. Then, if M is an indecomposable object in $\mathcal{T}_n$, it has a unique composition series of the form $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$, for some $k \geqslant 1$, where $i_1, i_2, \ldots, i_k \in \{0, 1, \ldots, n-1\}$ and $i_{j+1} = \overline{i_j + 1}$ for all j, where $\bar{a}$ denotes the reduction of a modulo n. If $a, b \in \mathbb{Z}$ and $a + 1 < b$, we denote the unique indecomposable object of $\mathcal{T}_n$ with composition series $S_{\overline{a+1}}, S_{\overline{a+2}}, \ldots, S_{\overline{b-1}}$ by $M[a, b]$. Note that every indecomposable object arises uniquely in this way up to a shift of both entries by n, i.e. $M[a, b] \cong M[a + n, b + n]$; thus we obtain a unique parametrization of the indecomposable objects of $\mathcal{T}_n$ (up to isomorphism) by restricting a to be in the set $\{0, 1, 2, \ldots, n-1\}$.

The AR-quiver of $\mathcal{T}_n$ is then easily described. It has vertices given by the indecomposable objects $M[a, b]$ of $\mathcal{T}_n$, for $0 \leqslant a \leqslant n - 1$ and $a + 1 < b$ (note that $M[a, a + 2] \cong S_{\overline{a+1}}$, for $a = 0, 1, \ldots, n - 1$). For any such a, b, the inclusion $M[a, b] \to M[a, b + 1]$ and the surjection $M[a, b] \to M[a + 1, b]$ (if $a + 2 < b$) correspond to arrows in the AR-quiver (using the above isomorphism if necessary); all arrows in the AR-quiver arise in this way. The AR-translate τ takes $M[a, b]$ to $M[a - 1, b - 1]$.

It is then straightforward to check that we have the following.

Lemma 2.5. *The quivers $\Gamma(\mathbb{A}(n))$ and $\Gamma_{\mathcal{T}_n}$ are isomorphic as stable translation quivers. On vertices, the isomorphism is given by the map*:

$$\varphi : \big\{ \text{admissible arcs in } \mathbb{A}(n) \big\} \longrightarrow \text{ind}(\mathcal{T}_n)$$

$$\pi_n\big([a, b]\big) \longmapsto M[a, b]$$

where $a \in \{0, 1, \ldots, n-1\}$, $b \in \mathbb{Z}$ and $a + 1 < b$.

We remark that analogous results in the cluster category case have been obtained in type A [CCS, Thm. 5.1], type D [S, §5], for a surface with marked points on the boundary [BZ, §1], and in type A_∞ [HJ] (see Section 5). Note that here we are not considering the cluster category, but as we observed in Section 2.1, the AR-quivers of $\mathcal{T}_n$ and $\mathcal{C}_{\mathcal{T}_n}$ are isomorphic. We remark that in [S], the puncture (and corresponding tagged arcs) allows for the possibility of 3 middle terms in the Auslander–Reiten triangles; this does not happen here, where there are only 1 or 2 middle terms in the Auslander–Reiten sequences. For a geometric interpretation of all non-split short exact sequences in $\mathcal{T}_n$, we refer to [BBM].

We also note that by a result of Ringel [Ri, §3] (see also [SS, X.2.6]), $\mathcal{T}_n$ is known to be standard, i.e. the subcategory ind($\mathcal{T}_n$) is equivalent to the mesh category of $\Gamma_{\mathcal{T}_n}$. Thus Lemma 2.5 can be regarded as giving a geometric realisation of the category $\mathcal{T}_n$ (i.e. as the mesh category of $\Gamma(\mathbb{A}(n))$), which is analogous to [CCS, Thm. 5.2], [S, Thm. 4.3]. As mentioned in Section 2.1, the AR-quiver of $\mathcal{C}_{\mathcal{T}_n}$ is isomorphic to that of $\mathcal{T}_n$. It follows that a $\mathcal{C}_{\mathcal{T}_n}$ is not standard (as it is not equivalent to $\mathcal{T}_n$) and we do not obtain a similar description of the cluster tube.

3. Geometric intersection numbers as dimensions of extension groups

In this section we show that negative geometric intersection numbers between admissible arcs in $\mathbb{A}(n)$ can be interpreted as dimensions of extension groups between the corresponding objects in $\mathcal{T}_n$ via the isomorphism in Lemma 2.5. To do this we need to compute these geometric intersection numbers; we do it by showing that they can be computed in the universal cover $\mathbb{U}$ of $\mathbb{A}(n)$. We were unable to find the exact statement we needed in the literature so include here a full proof.

We have the following (see e.g. [C, 5.4]).

Theorem 3.1 (Monodromy Theorem). *Let $\pi : E \to X$ be a covering space of a surface X and let γ, δ be paths in E with common starting point. Then $\gamma \sim \delta$ if and only if $\pi(\gamma) \sim \pi(\delta)$. It follows that, in this situation, γ and δ have a common finishing point.*

Two arcs α and β in a surface X are said to be in *general position* provided they intersect each other only transversely and have no points of intersection of multiplicity greater than two (other than possibly their end-points). We denote this by writing $\alpha \pitchfork \beta$. The *geometric intersection number* of two arcs α and β, denoted $I_X(\alpha, \beta)$ is defined by:

$$I_X(\alpha, \beta) = \min_{\substack{\alpha' \sim \alpha, \beta' \sim \beta \\ \alpha' \pitchfork \beta'}} \big| \alpha' \cap \beta' \big|,$$

where $\alpha' \cap \beta'$ denotes the intersection of α' and β', excluding their end-points. This is finite (see e.g. [GP, §3]). We call elements of $\alpha' \cap \beta'$ *crossings*. We will usually just refer to $I_X(\alpha, \beta)$ as the *intersection number* of α, β. If $I_X(\alpha, \beta) = 0$, we say that α and β are *noncrossing*.

If X is oriented, a crossing between two oriented arcs α, β in X in general position is called *positive* if the orientation arising from the pair of tangents to the arcs at the crossing point is compatible with the orientation of X; it is *negative* otherwise. Fig. 4 illustrates the two kinds of crossing for arcs in the plane (or $\mathbb{U}$). We remark that the sign of a crossing in $\mathbb{A}(n)$ is the same for either the orientation
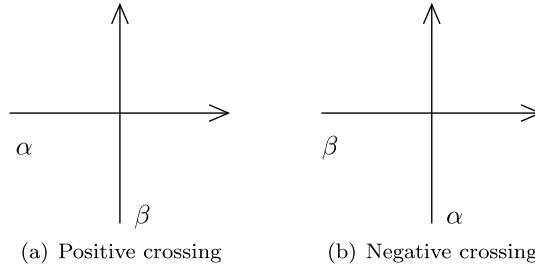
(a) Positive crossing   (b) Negative crossing

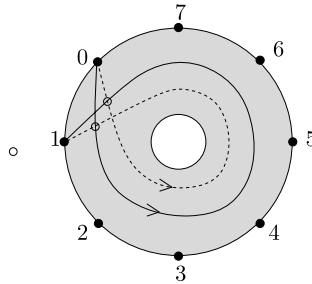**Fig. 4.** The two types of crossing between arcs α and β.

Fig. 5. $I_{\mathbb{A}(8)}(\pi_8([0,9]), \pi_8([0,9])) = 2$.

we gave $\mathbb{A}(n)$ in Section 2.2 (i.e. compatible with that on $\mathrm{Cyl}(n)$) or the orientation inherited from its embedding into the plane.

We then define (for any pair of arcs in X)

$$I_X^+(\alpha, \beta) = \min_{\substack{\alpha' \sim \alpha, \beta' \sim \beta \\ \alpha' \pitchfork \beta'}} |\alpha' \cap^+ \beta'|,$$

where $\alpha' \cap^+ \beta'$ denotes the set of positive crossings between α' and β'. The number $I_X^-(\alpha, \beta)$ is defined similarly.

Remark 3.2. If $\alpha = \beta$, then $I_X(\alpha, \alpha)$ is the minimum number of crossings between a pair of arcs α', β' in general position satisfying $\alpha' \sim \alpha$ and $\beta' \sim \alpha$. For example, consider the arc $\alpha = \pi_8([0,9])$ in $\mathbb{A}(8)$. Fig. 5 depicts two arcs α', β' homotopic to $\pi_8([0,9])$ in general position, with minimum number of intersections. The intersections are indicated by small circles. We have $I_{\mathbb{A}(8)}(\pi_8([0,9]), \pi_8([0,9])) = 2$.

Recall that a *lift* of an arc α in $\mathbb{A}(n)$ (to $\mathbb{U}$) is an arc $\widetilde{\alpha}$ in $\mathbb{U}$ such that $\pi_n(\widetilde{\alpha}) = \alpha$.

Lemma 3.3. *Let $\widetilde{\alpha}, \widetilde{\beta}$ be lifts of oriented arcs α, β in $\mathbb{A}(n)$ respectively. Then*

$$I_{\mathbb{A}(n)}(\alpha, \beta) = \min_{\substack{\gamma \sim \widetilde{\alpha}, \delta \sim \widetilde{\beta} \\ G\gamma \pitchfork G\delta}} |\gamma \cap G\delta|.$$

Furthermore, $I_{\mathbb{A}(n)}^+(\alpha, \beta)$ (respectively, $I_{\mathbb{A}(n)}^-(\alpha, \beta)$) can be computed in the same way except that on the right-hand side, only positive (respectively, negative) crossings are counted.

Proof. We note that the right-hand side of the above is always finite as γ, δ are compact. We first prove the following claim:

Claim. *Let γ and δ be lifts in $\mathbb{U}$ of α' and β', respectively. Then the following are equivalent.*

(a) $\gamma \sim \widetilde{\alpha}$, $\delta \sim \widetilde{\beta}$, and $G\gamma \pitchfork G\delta$.
(b) $\alpha' \sim \alpha$, $\beta' \sim \beta$, and $\alpha' \pitchfork \beta'$.

Proof of claim: (a) implies (b). Given γ, δ as in (a), the fact that $\alpha' \sim \alpha$ and $\beta' \sim \beta$ follows from Theorem 3.1. If α' and β' intersected non-transversely, two lifts $\sigma^a(\gamma)$ and $\sigma^b(\delta)$ of α' and β' would also intersect non-transversely, a contradiction. And if α' and β' intersected in a point of multiplicity greater than 2 then, taking the pre-image of an admissible neighbourhood of that point, we'd have that $G\gamma$ and $G\delta$ intersected in a point of multiplicity greater than 2, a contradiction. Hence (b) holds.

Conversely, suppose that α' and β' are as in (b). Then $\gamma \sim \widetilde{\alpha}$ and $\delta \sim \widetilde{\beta}$ follow from Theorem 3.1. If $G\gamma$ and $G\delta$ intersected non-transversely, the image of a non-transverse intersection would be a non-transverse intersection of α' and β', a contradiction. Similarly, if $G\gamma$ and $G\delta$ intersected in a point of multiplicity greater than 2, the image would be a point of multiplicity greater than 2 in the intersection of α' and β', a contradiction. Hence (a) holds.

Now in order to prove the lemma we must show that, in the circumstances of the claim, we have

$$|\gamma \cap G\delta| = |\alpha' \cap \beta'|.$$

Let f be π_n restricted to $\gamma \cap G\delta$. It is clear that the image of f is contained in $\alpha' \cap \beta'$. If $x \in \alpha' \cap \beta'$ there is $u \in \mathbb{U}$ such that $\pi_n(u) = x$. Then we have:

$$u \in G\gamma \cap G\delta$$

since $\pi_n\gamma = \alpha'$ and $\pi_n\delta = \beta'$. So $u \in \sigma^a(\gamma) \cap \sigma^b(\delta)$ for some $a, b \in \mathbb{Z}$. Hence $\sigma^{-a}(u) \in \gamma \cap G\delta$ and $\pi_n(\sigma^{-a}(u)) = \pi_n(u) = x$ so f is surjective. If $u \neq u' \in \gamma \cap G\delta$ and $f(u) = f(u')$ then $u, u' \in \gamma$, $u \in \sigma^a(\delta)$ and $u' \in \sigma^b(\delta)$ for some $a, b \in \mathbb{Z}$. But then $\pi_n(u)$ is a point in the intersection of α' and β' of multiplicity greater than 2, a contradiction. Hence f is injective and thus a bijection. We have the seen that the first part of the lemma holds. The other cases can be dealt with in a similar way, since π_n preserves orientation. $\square$

To compute the intersection number of two admissible arcs α and β in $\mathbb{A}(n)$, we will later exhibit explicit arcs in $\mathbb{U}$ homotopic to lifts of α and β giving an upper bound for $I_{\mathbb{A}(n)}(\alpha, \beta)$ using Lemma 3.3. The following result gives a lower bound which we will use to get equality. This lower bound is easily computable as it just involves intersection numbers in $\mathbb{U}$.

Corollary 3.4. *Let $\widetilde{\alpha}, \widetilde{\beta}$ be lifts of oriented arcs α, β in $\mathbb{A}(n)$ respectively. Then*

$$I_{\mathbb{A}(n)}(\alpha, \beta) \geqslant \sum_{m \in \mathbb{Z}} I_{\mathbb{U}}(\widetilde{\alpha}, \sigma^m(\widetilde{\beta})).$$

Similar results hold for $I^+_{\mathbb{A}(n)}(\alpha, \beta)$ and $I^-_{\mathbb{A}(n)}(\alpha, \beta)$.

Proof. Note that the sum on the right-hand side is finite since $\widetilde{\alpha}$ and $\widetilde{\beta}$ are compact, so only finitely many of the intersection numbers are non-zero. Using Lemma 3.3, we have:

$$
\begin{aligned}
I_{\mathbb{A}(n)}(\alpha, \beta) &= \min_{\substack{\gamma \sim \widetilde{\alpha}, \delta \sim \widetilde{\beta} \\ G\gamma \pitchfork G\delta}} |\gamma \cap G\delta| \\
&= \min_{\substack{\gamma \sim \widetilde{\alpha}, \delta \sim \widetilde{\beta} \\ G\gamma \pitchfork G\delta}} \sum_{m \in \mathbb{Z}} |\gamma \cap \sigma^m\delta| \\
&\geqslant \sum_{m \in \mathbb{Z}} \min_{\substack{\gamma \sim \widetilde{\alpha}, \delta \sim \widetilde{\beta} \\ G\gamma \pitchfork G\delta}} |\gamma \cap \sigma^m(\delta)|
\end{aligned}
$$

$$\geqslant \sum_{m\in\mathbb{Z}} \min_{\substack{\gamma\sim\widetilde{\alpha},\,\delta\sim\sigma^m(\widetilde{\beta})\\ \gamma\pitchfork\delta}} |\gamma\cap\delta|$$

$$= \sum_{m\in\mathbb{Z}} I_{\mathbb{U}}\big(\widetilde{\alpha},\sigma^m(\beta)\big),$$

as required.   □

We can now put the above results together to obtain a computation of intersection numbers of admissible arcs in the annulus.

Proposition 3.5. *Let $[a,b]$ and $[c,d]$ be admissible arcs in $\mathbb{U}$ and suppose that $d-c\geqslant b-a$, i.e. the combinatorial length of $[c,d]$ is greater than or equal to the combinatorial length of $[a,b]$. Then:*

(a) $I^+_{\mathbb{A}(n)}\big(\pi_n([a,b]),\pi_n([c,d])\big) = \big|\{m\in\mathbb{Z}\colon a<\sigma^m(c)<b\}\big|$;

(b) $I^-_{\mathbb{A}(n)}\big(\pi_n([a,b]),\pi_n([c,d])\big) = \big|\{m\in\mathbb{Z}\colon a<\sigma^m(d)<b\}\big|$;

(c) $I_{\mathbb{A}(n)}\big(\pi_n([a,b]),\pi_n([c,d])\big) = I^+_{\mathbb{A}(n)}\big(\pi_n([a,b]),\pi_n([c,d])\big) + I^-_{\mathbb{A}(n)}\big(\pi_n([a,b]),\pi_n([c,d])\big)$.

Proof. We note that $[c,d]$ is homotopy equivalent to the oriented arc δ consisting of two line segments, the first, L_1, joining $(c,0)$ to $(\frac{1}{2}(c+d),h)$ and the second, L_2, joining $(\frac{1}{2}(c+d),h)$ to $(d,0)$, where $0<h<1$ is fixed. Note that in order to ensure the arc is smooth, we round it off at the point $(\frac{1}{2}(c+d),h)$; it is clear that we can do this within a ball of radius $\varepsilon>0$ for arbitrarily small ε so it will not affect the rest of the proof.

By translating $[a,b]$ if necessary, we may assume that $c\leqslant a<c+n$. Let M'_1 be the line segment joining $(a,0)$ to the point $(\frac{1}{2}(c+d+n),h)$, i.e. to the mid-point between the top of δ and the top of $\sigma(\delta)$.

Let δ' be the σ-translate of δ whose end-point is $(y,0)$ with $y<b$ maximal. Let M'_2 be the line segment joining $(b,0)$ with $(y+(c-d+n)/2,h)$, i.e. with the mid-point between the top of δ' and the top of $\sigma(\delta')$. Let v be the point where M'_1 and M'_2 meet; then let M_1 be the line joining $(a,0)$ to v and let M_2 be the line joining v to $(b,0)$. Then γ, the oriented arc which starts at $(a,0)$, travels along M_1 to v and then along M_2 to $(b,0)$ is homotopy equivalent to $[a,b]$ (rounded off at v as before). See Fig. 6 for an example.

It is clear that γ intersects $G\delta$ transversely, and hence so does $G\gamma$. Note that M_1 crosses only σ-translates of L_2 and σ-translates of M_2. Similarly, M_2 crosses only σ-translates of L_1 and σ-translates of M_1. The same is true of σ-translates of M_1 and M_2. It follows that there are no points of multiplicity greater than 2 in the intersection of $G\gamma$ and $G\delta$.

Let $\alpha=\pi_n([a,b])$ and $\beta=\pi_n([c,d])$. By Lemma 3.3, we have that $I^+_{\mathbb{A}(n)}(\alpha,\beta)$ is less than or equal to the number of positive crossings between γ and $G\delta$. Using the explicit description of γ and δ above, the latter number equals $\sum_{m\in\mathbb{Z}} I^+_{\mathbb{U}}(\gamma,\sigma^m\delta)$, which equals $|\{m\in\mathbb{Z}\colon a<\sigma^m(c)<b\}|$. By Corollary 3.4, we have

$$I^+_{\mathbb{A}(n)}(\alpha,\beta) \geqslant \sum_{m\in\mathbb{Z}} I^+_{\mathbb{U}}\big([a,b],\sigma^m[c,d]\big).$$

Since $\gamma\sim[a,b]$ and $\delta\sim[c,d]$, it follows that

$$I^+_{\mathbb{A}(n)}(\alpha,\beta) \geqslant \sum_{m\in\mathbb{Z}} I^+_{\mathbb{U}}\big(\gamma,\sigma^m\delta\big).$$

Putting this together with the above, we have equality and that part (a) of the proposition holds.
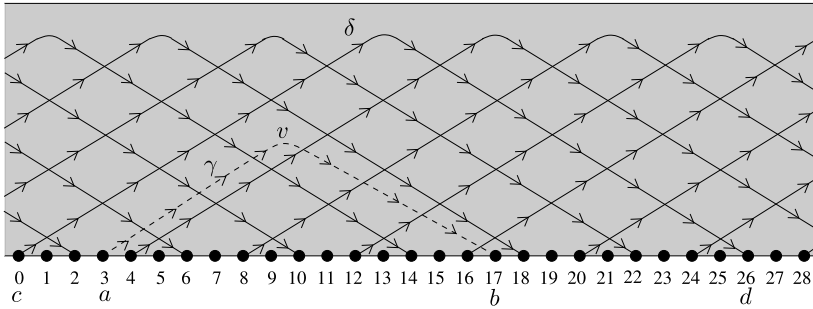
Fig. 6. Computing the intersection numbers of $\pi_4([0, 26])$ and $\pi_4([3, 17])$. We have $I^+_{\mathbb{A}(4)}(\pi_4([0, 26]), \pi_4([3, 17])) = 3$, $I^-_{\mathbb{A}(4)}(\pi_4([0, 26]), \pi_4([3, 17])) = 4$ and $I_{\mathbb{A}(4)}(\pi_4([0, 26]), \pi_4([3, 17])) = 7$.

Parts (b) and (c) of the proposition are proved in a similar way, using the same representatives γ and δ as above. $\quad\square$

For an example of Proposition 3.5 in the case $n = 4$, see Fig. 6.

We note that it follows from the above proof that the inequality in Corollary 3.4 is an equality, i.e. that intersection numbers between admissible arcs in $\mathbb{A}(n)$ can be computed in $\mathbb{U}$.

Corollary 3.6. *Let $\widetilde{\alpha}$, $\widetilde{\beta}$ be lifts of admissible arcs α, β in $\mathbb{A}(n)$ respectively. Then*

$$I_{\mathbb{A}(n)}(\alpha, \beta) = \sum_{m \in \mathbb{Z}} I_{\mathbb{U}}\big(\widetilde{\alpha}, \sigma^m(\widetilde{\beta})\big).$$

Similar results hold for $I^+_{\mathbb{A}(n)}(\alpha, \beta)$ and $I^-_{\mathbb{A}(n)}(\alpha, \beta)$.

Remark 3.7. Let c be a single oriented cycle in $\mathbb{A}(n)$ from 0 to itself, i.e. a generator of the fundamental group at 0. We note that it follows from Proposition 3.5 that, for $r, s > 0$, $I^+_{\mathbb{A}(n)}(c^r, c^s) = I^-_{\mathbb{A}(n)}(c^r, c^s) = \min(r, s) - 1$ and $I_{\mathbb{A}(n)}(c^r, c^s) = 2\min(r, s) - 2$.

Recall the bijection between admissible arcs in $\mathbb{A}(n)$ and indecomposable objects in $\mathcal{T}_n$ from Lemma 2.5, taking $\pi_n([a, b])$ to $M[a, b]$ for an admissible arc $\pi_n([a, b])$. The object $M[a, b]$ has a unique composition series (ordered from the socle upwards), $S_{\overline{a+1}}, S_{\overline{a+2}}, \ldots, S_{\overline{b-1}}$. We now have all we need in order to show that the dimensions of extension groups (of degree 1) in the tube can be interpreted as negative intersection numbers of the corresponding admissible arcs in the annulus, our main result.

Theorem 3.8. *Let $\pi_n([a, b])$ and $\pi_n([c, d])$ be admissible arcs in $\mathbb{A}(n)$. Then*

(a) $\dim \mathrm{Ext}^1_{\mathcal{T}_n}\big(M[a, b], M[c, d]\big) = I^-_{\mathbb{A}(n)}\big(\pi_n([a, b]), \pi_n([c, d])\big);$

(b) $\dim \mathrm{Ext}^1_{\mathcal{T}_n}\big(M[c, d], M[a, b]\big) = I^+_{\mathbb{A}(n)}\big(\pi_n([a, b]), \pi_n([c, d])\big);$

(c) $\dim \mathrm{Ext}^1_{\mathcal{C}_{\mathcal{T}_n}}\big(M[a, b], M[c, d]\big) = I_{\mathbb{A}(n)}\big(\pi_n([a, b]), \pi_n([c, d])\big).$

Proof. We first prove (a). We have

$$\mathrm{Ext}^1_{\mathcal{T}_n}\big(M[a, b], M[c, d]\big) \cong D\mathrm{Hom}_{\mathcal{T}_n}\big(M[c, d], \tau M[a, b]\big)$$
$$\cong D\mathrm{Hom}_{\mathcal{T}_n}\big(M[c, d], M[a - 1, b - 1]\big).$$

Assume first that the combinatorial length of $[c, d]$ is greater than or equal to that of $[a, b]$. Then a homomorphism in $\text{Hom}_{\mathcal{T}_n}(M[c, d], M[a - 1, b - 1])$ must map the top, $S_{\overline{d-1}}$, of $M[c, d]$ onto a composition factor $S_{\overline{d-1}}$ in $M[a - 1, b - 1]$ and is uniquely determined up to a scalar by this choice. So the dimension of the last space above is the number of times $\overline{d-1}$ appears in the sequence $[\bar{a}, \overline{a+1}, \ldots, \overline{b-2}]$, i.e. the number of times $\bar{d}$ appears in the sequence $\overline{a+1}, \overline{a+2}, \ldots, \overline{b-1}$. This is equal to $I_{\mathbb{A}(n)}^{-}(\pi_n([a, b]), \pi_n([c, d]))$ by Proposition 3.5.

If the combinatorial length of $[c, d]$ is less than or equal to that of $[a, b]$, a homomorphism in $\text{Hom}_{\mathcal{T}_n}(M[c, d], M[a - 1, b - 1])$ must map a composition factor $S_{\bar{a}}$ of $M[c, d]$ onto the socle $S_{\bar{a}}$ of $M[a - 1, b - 1]$, and is uniquely determined up to a scalar by this choice. Hence the dimension of this Hom-space is the number of times $\bar{a}$ appears in the sequence $[\overline{c+1}, \overline{c+2}, \ldots, \overline{d-1}]$, i.e. $I_{\mathbb{A}(n)}^{+}(\pi_n([c, d]), \pi_n([a, b]))$ by Proposition 3.5. This is the same as $I_{\mathbb{A}(n)}^{-}(\pi_n([a, b]), \pi_n([c, d]))$, as required. Part (a) is proved.

The proof of part (b) is similar. Part (c) follows from the fact that:

$$\text{Ext}_{\mathcal{C}_{\mathcal{T}_n}}\big(M[a, b], M[c, d]\big) \cong \text{Ext}_{\mathcal{T}_n}\big(M[a, b], M[c, d]\big) \oplus D\,\text{Ext}_{\mathcal{T}_n}\big(M[c, d], M[a, b]\big) \tag{1}$$

(proved as in [BMRRT, Prop. 1.7]) where $D = \text{Hom}_k(-, k)$ is the linear duality, and Proposition 3.5(c). $\quad\square$

Remark 3.9. In the light of Theorem 3.8, Proposition 3.5(c) can be regarded as a geometric version of Eq. (1) for the cluster tube, i.e. passing to the cluster category (of the tube) corresponds to passing from oriented to unoriented arcs in the annulus.

Using the results [BZ, 5.1,5.3], (in the case of an annulus X with a single marked point on the inner boundary and n marked points on the outer boundary), we get that, in the situation of Theorem 3.8, $I_{\mathbb{A}(n)}(\pi_n([a, b]), \pi_n([c, d])) \neq 0$ implies that $\text{Ext}_{\mathcal{C}_{\mathcal{T}_n}}^1(M[a, b], M[c, d]) \neq 0$ and $\text{Ext}_{\mathcal{C}_{\mathcal{T}_n}}^1(M[c, d], M[a, b]) \neq 0$ (with equivalence in the case where $\pi_n([a, b]) = \pi_n([c, d])$). We use the fact (see [BMV, §2]) that the cluster tube is a thick subcategory of the cluster category associated to X. Thus Theorem 3.8(c) can be regarded as a strengthening of this result in the case of a tube. See also [W, 4.23] for an alternative proof of parts (a) and (b).

We also remark that a geometric interpretation of dimensions of extension groups in cluster categories (i.e. a result analogous to Theorem 3.8(c)) has been obtained in type A [CCS, Rk. 5.3], in type D [S, Thm. 4.3] and in type A_∞ [HJ, Lemma 3.6] (see also Section 5). Note that in the tube or cluster tube, and also in the setting of [BZ], but not in [CCS,S,HJ], extension groups between indecomposable objects can have arbitrarily large dimension; in fact the dimension of the extension group of an indecomposable object with itself can be arbitrarily large. Furthermore, all but finitely many objects in the tube (or cluster tube) have non-zero self-extension groups, corresponding to the fact that the corresponding oriented (respectively, unoriented) arcs have non-zero self-intersection numbers.

Recall that an object M in $\mathcal{T}_n$ is said to be *rigid* if $\text{Ext}_{\mathcal{T}_n}^1(M, M) = 0$, and *maximal rigid* if $\text{Ext}_{\mathcal{T}_n}^1(M \oplus X, M \oplus X) = 0$ implies X lies in $\text{add}\,M$ for any object X in $\mathcal{T}_n$. By Theorem 3.8(c), we see that an object of $\mathcal{T}_n$ is rigid if and only if the admissible arcs corresponding to its indecomposable summands do not cross (themselves or each other), and such an object is maximal rigid if and only if it is a maximal noncrossing collection of admissible arcs. We recall that in [Bu, §2], a general notion of triangulation is considered for marked surfaces (such as $\mathbb{A}(n)$) in which it is allowed for boundary components not to contain marked points. The arcs in the triangulation, which must not be self-crossing, must divide up the surface into triangles (possibly self-folded) or annuli with a single marked point on the outer boundary.

Proposition 3.10. *The map φ from admissible arcs in $\mathbb{A}(n)$ to $\text{ind}(\mathcal{T}_n)$ induces a bijection between triangulations of $\mathbb{A}(n)$ (in the sense of [Bu]) and maximal rigid objects of $\text{ind}(\mathcal{T}_n)$.*

Proof. It is clear that any triangulation of $\mathbb{A}(n)$ must contain an arc dividing $\mathbb{A}(n)$ into a disk and one such annulus (since at least one component of the complement of the arcs in the triangulation must

have the inner boundary component of $\mathbb{A}(n)$ on its boundary). The remaining arcs in the triangulation triangulate the disk. Forgetting the arcs in the triangulation homotopic to arcs along the boundary and orienting all arcs anticlockwise, it is clear we obtain a maximal collection of noncrossing admissible arcs.

Conversely, any maximal collection of noncrossing admissible arcs must contain the arc from some marked point on the outer boundary to itself, looping once around the inner boundary component (if it doesn't, this arc can be added). As above, this divides $\mathbb{A}(n)$ into a disk and an annulus of the above kind. The remaining arcs form a maximal noncrossing collection of admissible arcs in the disk, and thus triangulate it. It follows that, if we ignore the orientations and add all the boundary arcs (arcs between adjacent marked points on the outer boundary, and the arc around the inner boundary), we obtain a triangulation.

It is clear that the above maps are inverses to each other. It follows that we have a bijection between triangulations of $\mathbb{A}(n)$ and maximal noncrossing collections of admissible arcs and we are done. $\quad\square$

4. Doubled arc model

In [BMV, §3], the additive subcategory of a tube of rank n generated by the rigid indecomposable objects is modelled by pairs of unoriented arcs in a regular $2n$-sided polygon. We now indicate how this model is related to the model presented here.

For an integer $n \geqslant 2$, consider the annulus $\mathbb{A}(n)$ as being embedded in the complex plane with its centre at the origin with marked points (on the outer boundary) at the nth roots of unity. Then the map $\psi : z \mapsto z^2$ (followed by a scaling to map the new inner boundary onto the same circle it was to start with) induces a map from $\mathbb{A}(2n)$ to $\mathbb{A}(n)$, mapping the marked points of $\mathbb{A}(2n)$ onto those of $\mathbb{A}(n)$. We see that ψ maps $i, i + n$ onto i for each i. The preimage under ψ of $\pi_n([a, b])$ is $\pi_{2n}([a, b]) \cup \pi_{2n}([a + n, b + n])$, up to homotopy equivalence. This pair of oriented arcs corresponds to a pair of oriented arcs in a regular $2n$-gon as in [BMV, §3] (if the orientation is dropped).

Proposition 4.1. *Let* $[a, b]$, $[c, d]$ *be admissible arcs in* $\mathbb{U}$. *Then*

(a) $I^+_{\mathbb{A}(2n)}\big(\psi^{-1}\big(\pi_n([a, b])\big), \psi^{-1}\big(\pi_n([c, d])\big)\big) = 2 I^+_{\mathbb{A}(n)}\big(\pi_n([a, b]), \pi_n([c, d])\big).$

(b) $I^-_{\mathbb{A}(2n)}\big(\psi^{-1}\big(\pi_n([a, b])\big), \psi^{-1}\big(\pi_n([c, d])\big)\big) = 2 I^-_{\mathbb{A}(n)}\big(\pi_n([a, b]), \pi_n([c, d])\big).$

(c) $I_{\mathbb{A}(2n)}\big(\psi^{-1}\big(\pi_n([a, b])\big), \psi^{-1}\big(\pi_n([c, d])\big)\big) = 2 I_{\mathbb{A}(n)}\big(\pi_n([a, b]), \pi_n([c, d])\big).$

Proof. We note that for oriented arcs γ, γ', δ in $\mathbb{U}$ which do not self-intersect, it is easy to see that

$$I^+_{\mathbb{U}}\big(\gamma \cup \gamma', \delta\big) = I^+_{\mathbb{U}}(\gamma, \delta) + I^+_{\mathbb{U}}\big(\gamma', \delta\big).$$

It follows from Corollary 3.6 that $I^+_{\mathbb{A}(n)}$ has the same additivity property. Let $[a, b]$, $[c, d]$ be admissible arcs in $\mathbb{U}$. Then:

$$
\begin{aligned}
I^+_{\mathbb{A}(2n)}\big(\psi^{-1}\big(\pi_n([a, b])\big), \psi^{-1}\big(\pi_n([c, d])\big)\big) = {} & I^+_{\mathbb{A}(2n)}\big(\pi_{2n}([a, b]), \pi_{2n}([c, d])\big) \\
& + I^+_{\mathbb{A}(2n)}\big(\pi_{2n}([a, b]), \pi_{2n}([c + n, d + n])\big) \\
& + I^+_{\mathbb{A}(2n)}\big(\pi_{2n}([a + n, b + n]), \pi_{2n}([c, d])\big) \\
& + I^+_{\mathbb{A}(2n)}\big(\pi_{2n}([a + n, b + n]), \pi_{2n}([c + n, d + n])\big).
\end{aligned}
$$

Hence

$$I^+_{\mathbb{A}(2n)}\big(\psi^{-1}\big(\pi_n\big([a,b]\big)\big), \psi^{-1}\big(\pi_n\big([c,d]\big)\big)\big) = \big|\big\{m \in \mathbb{Z}: a < \sigma^{2m}(c) < b\big\}\big|$$

$$+ \big|\big\{m \in \mathbb{Z}: a < \sigma^{2m+1}(c) < b\big\}\big|$$

$$+ \big|\big\{m \in \mathbb{Z}: \sigma(a) < \sigma^{2m}(c) < \sigma(b)\big\}\big|$$

$$+ \big|\big\{m \in \mathbb{Z}: \sigma(a) < \sigma^{2m+1}(c) < \sigma(b)\big\}\big|,$$

so

$$I^+_{\mathbb{A}(2n)}\big(\psi^{-1}\big(\pi_n\big([a,b]\big)\big), \psi^{-1}\big(\pi_n\big([c,d]\big)\big)\big) = 2I^+_{\mathbb{A}(n)}\big(\pi_n\big([a,b]\big), \pi_n\big([c,d]\big)\big),$$

as required for (a). Part (b) is proved similarly, and part (c) follows. $\quad\square$

Corollary 4.2. *Let* $[a,b]$, $[c,d]$ *be admissible arcs in* $\mathbb{U}$. *Then*

$$2 \dim \mathrm{Ext}^1_{\mathcal{C}_{\mathcal{T}_n}}\big(M[a,b], M[c,d]\big) = I_{\mathbb{A}(2n)}\big(\psi^{-1}\big(\pi_n\big([a,b]\big)\big), \psi^{-1}\big(\pi_n\big([c,d]\big)\big)\big).$$

Proof. This follows from Proposition 4.1(c) and Theorem 3.8. $\quad\square$

We thus recover [BMV, 3.2] in the case where $\pi_n([a,b])$ and $\pi_n([c,d])$ are non self-intersecting, i.e. the corresponding indecomposable objects in $\mathcal{T}_n$ are rigid.

5. AR-quivers of type $\mathbb{Z}A_\infty$

By [RvdB, III.1.1], the category $\mathcal{T}_\infty$ of finite dimensional representations of the quiver A_∞^∞:

$$\cdots - 3 \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \cdots$$

is a connected Hom-finite noetherian hereditary abelian category with almost split sequences whose AR-quiver is of type $\mathbb{Z}A_\infty$. Hence one can form the cluster category $\mathcal{C}_{\mathcal{T}_\infty}$ of $\mathcal{T}_\infty$ as the quotient $D^b(\mathcal{T}_\infty)/(\tau^{-1}[1])$; see [KR1, 2.1]. Since $\mathcal{T}_\infty$ is skeletally small, it follows from [K1, 9.9] that it is triangulated and thus also 2-Calabi-Yau. As pointed out to us by P. Jørgensen, it follows from [KR2, Thm. 2.1] that this category coincides with the category D considered by Holm and Jørgensen in [HJ] (referred to there as a cluster category of type A_∞).

The indecomposable objects in $\mathcal{T}_\infty$ (and thus, also, in $\mathcal{C}_{\mathcal{T}_\infty}$) are, up to isomorphism, of the form $X[a,b]$ where $[a,b]$ is an admissible arc in $\mathbb{U}$. The representation $X[a,b]$ has a unique composition series (ordered from the socle upwards) given by $S_{a+1}, S_{a+2}, \ldots, S_{b-1}$. There is an irreducible map in $\mathcal{T}_\infty$ from $X[a,b]$ to $X[c,d]$, for admissible arcs $[a,b]$ and $[c,d]$ in $\mathbb{U}$, if and only if $c = a$ and $d = b + 1$ or $c = a + 1$ and $d = b$ (compare with Section 2.3). The AR-translate takes $X[a,b]$ to $X[a-1, b-1]$. The irreducible maps and AR-translate have the same description in $\mathcal{T}_\infty$ and in $\mathcal{C}_{\mathcal{T}_\infty}$; see [BMRRT, 1.3].

Theorem 5.1. *Let* $[a,b]$ *and* $[c,d]$ *be admissible arcs in* U. *Then*

(a) $\dim \mathrm{Ext}^1_{\mathcal{T}_\infty}\big(X[a,b], X[c,d]\big) = I^-_\mathbb{U}\big([a,b], [c,d]\big);$

(b) $\dim \mathrm{Ext}^1_{\mathcal{T}_\infty}\big(X[c,d], X[a,b]\big) = I^+_\mathbb{U}\big([a,b], [c,d]\big);$

(c) $\dim \mathrm{Ext}^1_{\mathcal{C}_{\mathcal{T}_\infty}}\big(X[a,b], X[c,d]\big) = I_\mathbb{U}\big([a,b], [c,d]\big).$

Proof. This result is proved in the same way as Theorem 3.8 (except that the reduction modulo n is omitted). Note that the same proof of formula (1) holds in this context. $\quad\square$

Acknowledgments

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