20th European Conference on Fracture (ECF20)

The Theory of Critical Distances to model the short- to long-crack transition in geological materials subjected to Mode I static loading

Elisabeth T. Bowman\textsuperscript{a} and Luca Susmel\textsuperscript{a,}\textsuperscript{*}

\textsuperscript{a}Department of Civil and Structural Engineering, The University of Sheffield, Mappin Street, Sheffield, S1 3JD, UK

Abstract

The Theory of Critical Distances postulates that, in cracked materials subjected to static loading, breakage takes place when a distance dependent effective stress exceeds the material tensile strength. Such an effective stress is equal to the stress calculated either at a certain distance from the notch tip (Point Method), averaged over a line (Line Method), over an area (Area Method) or, finally, averaged in a finite volume (Volume Method). The necessary characteristic length is a material property which can directly be determined by combining the material ultimate tensile strength with the plane strain fracture toughness. In the present investigation, by re-analysing a large number of experimental results taken from the literature, it is shown that the Theory of Critical Distances is successful in modelling the transition from the short- to the long-crack regime in geological materials subjected to Mode I static loading.

© 2014 Elsevier Ltd. Open access under CC BY-NC-ND license.
Selection and peer-review under responsibility of the Norwegian University of Science and Technology (NTNU), Department of Structural Engineering

Keywords: Theory of Critical Distance, geological materials, short-/long-cracks

1. Introduction

In many different engineering problems of practical interest, such as, for instance, rock fragmentation, rock cutting, fracking, and earthquake engineering, the mechanical behaviour of cracked geological materials is often modelled by use of Linear Elastic Fracture Mechanics (LEFM) concepts.

\textsuperscript{*}Corresponding author. Tel.: +44 (0) 114 222 5073; fax: +44 (0) 114 222 5700.
E-mail address: l.susmel@sheffield.ac.uk

2211-8128 © 2014 Elsevier Ltd. Open access under CC BY-NC-ND license.
Selection and peer-review under responsibility of the Norwegian University of Science and Technology (NTNU), Department of Structural Engineering
doi:10.1016/j.mspro.2014.06.093
In this setting, Stress Intensity Factor (SIFs) are the stress quantities used to concisely describe the entire stress fields in the vicinity of the crack tips. Mode I SIFs, $K_I$, can be directly estimated according to the following well-known definition (Anderson, 1995):

$$K_I = F \sigma_g \sqrt{\pi a}$$

In Eq. (1) $a$ is the crack length, $\sigma_g$ is the nominal gross stress, and $F$ is the so-called shape factor which depends on the geometrical/loading configuration characterising the specific problem being addressed. LEFM postulates that, in cracked bodies subjected to Mode I static loading, failure occurs when $K_I = K_c$. Accordingly, Eq. (1) can easily be re-arranged to directly estimate the magnitude of the nominal gross stress, $\sigma_{th}$, resulting in static breakage, i.e., resulting in $K_I = K_c$:

$$\sigma_{th} = \frac{K_c}{\sqrt{\pi F^2 a}}$$

By so doing, the above threshold condition can be plotted in a Kitagawa-Takahashi (1976) like diagram (Fig. 1), where the ordinate is the failure nominal gross stress, $\sigma_{th}$, whilst the abscissa is the equivalent crack length calculated as $F^2 a$ (Usami, 1985). The most important peculiarity of this schematisation is that experimental results generated by testing cracked specimens having different values for the shape factor, $F$, can all be brought back to the reference case of a central through-thickness crack in an infinite plate loaded in tension (for which $F$ is invariably equal to unity). As suggested not only by Eq. (2), but also by the log-log chart of Fig. 1, as the equivalent crack length decreases, the corresponding nominal threshold stress, $\sigma_{th}$, increases monotonically, eventually becoming larger than the material tensile strength, $\sigma_t$. This implies that, theoretically speaking, the LEFM concepts should be used to model the mechanical behaviour of cracked materials as long as the equivalent crack length is larger than $a_0$, that is:

![Fig. 1. Kitagawa-Takahashi diagram.](image_url)
Nevertheless, since *natura non facit saltus*, much experimental evidence (Usami et al., 1986) suggests that there is a gradual transition from the short- to the long-crack regime (Fig. 1). Accordingly, the use of the LEFM concepts is seen to result in accurate estimates solely in those situations in which the equivalent crack length is larger than about 10a0 (Taylor, 2007). As far as engineering problems of practical interest are concerned, attention must then be paid not to use the LEFM concepts out of their range of validity, since this would result in non-conservative estimates of strength. In this complex scenario, the aim of the present paper is to investigate whether the Theory of Critical Distances (TCD) is successful in modelling, under Mode I static loading, the transition from the short- to the long-crack regime in geological materials.

2. Fundamentals of the Theory of Critical Distances

The TCD postulates that failure occurs when a distance dependent effective stress, σ_{eff}, exceeds the material tensile strength, σ_t. Accordingly, the threshold condition under Mode I static loading can be expressed as follows:

\[ σ_{eff} = σ_t \]  

(4)

Examination of the state of the art suggests that effective stress σ_{eff} can be calculated in different ways, including the Point, Line, Area, and Volume Methods (Taylor, 1999; Taylor, 2007). Observing that the use of the above formalisations of the TCD results in similar estimates (Taylor, 2007), for the sake of conciseness, in what follows solely the Point (PM) and Line Methods (LM) will be considered. According to the PM and LM then, σ_{eff} takes on the following values (Taylor, 2007):

\[ σ_{eff} = σ_y(θ = 0, r = L/2) \]  

(5)

\[ σ_{eff} = \frac{1}{2L} \int_{0}^{L} σ_y(θ = 0, r)dr \]  

(6)
the meaning of the adopted symbols being explained in Fig. 2a. In the definitions for $\sigma_{\text{eff}}$ reported above, critical distance $L$ is a material property which can be determined through the plane strain fracture toughness, $K_{\text{IC}}$, and the tensile strength, $\sigma_t$, as follows (Schmidt, 1980; Taylor, 2007):

$$L = \frac{1}{\pi} \left( \frac{K_{\text{IC}}}{\sigma_t} \right)^2$$

(7)

In order to directly use the above formalisations of the TCD to model the transition from the short- to the long-crack regime, consider a uniaxially loaded infinite plate containing a central through-thickness crack of semi-length $a$ (Fig. 2a). According to Westergaard (1939), the linear-elastic stress field along the crack bisector (i.e., $\theta=0^\circ$ in Fig. 2a) can be estimated via the following relationship:

$$\sigma_y(\theta=0, r) = \frac{\sigma_g}{\sqrt{1 - \left( \frac{a}{a+r} \right)^2}}$$

(8)

If stress $\sigma_t$ is calculated through Eq. (8) at a distance $r$ from the crack tip equal to $L/2$ and the failure condition is expressed according to Eq. (4), the PM can directly be re-written to model the transition from the short- to the long-crack regime as (Taylor, 1999):

$$\sigma_{th} = \sigma_t \sqrt{1 - \left( \frac{a}{a+L/2} \right)^2}$$

(9)

In a similar way, by averaging $\sigma_y$ - estimated according to Eq. (8) - over a straight line of length $2L$, the LM effective stress takes on the following form:

$$\sigma_{\text{eff}} = \frac{1}{2L} \int_0^{2L} \frac{\sigma_g}{\sqrt{1 - \left( \frac{a}{a+r} \right)^2}} dr = \sigma_g \sqrt{\left( \frac{a+L}{L} \right)}$$

(10)

so that, by making use of failure condition (4), the transition from the short- to the long-crack regime can explicitly be modelled as (Taylor, 1999):

$$\sigma_{th} = \sigma_t \sqrt{\frac{L}{a+L}}$$

(11)

The normalised Kitagawa-Takashi diagram of Fig. 2b shows that both the PM, Eq. (9), and the LM, Eq. (11), are equally capable of matching, on the left-hand side, the plain material static strength and, on the right-hand side, the cracked plate nominal strength estimated according to LEFM. To conclude, it is worth observing that, in the transition region, the LM is seen to be slightly more conservative than the PM (see Fig. 2b).

3. Validation by experimental data

In order to check the accuracy of the TCD in modelling the transition from the short- to the long-crack regime in geological materials subjected to Mode I static loading, a number of experimental results were selected from the technical literature. Table 1 summarises the tensile strength, $\sigma_t$, the plain fracture toughness, $K_{\text{IC}}$, and the critical distance value, $L$, for the investigated materials, the geometries of the considered samples being shown in Fig. 3.
To consistently post-process all the selected experimental results in accordance with Eq. (1), wherever it was necessary, shape factors were recalculated according to the following definition:

\[ F = \frac{K_I}{\sigma_g \sqrt{\pi \cdot a}} \]  

(12)

Table 1. Summary of the reanalysed experimental results – swc=saturated water content (%); SCS=Semi-Circular Specimen; SECB=Single-Edge Crack Beam; SECRB=Single-Edge Crack Round Bar; SND=Straight Notched Disk (see also Fig. 3).

<table>
<thead>
<tr>
<th>Material</th>
<th>References</th>
<th>( \sigma_t )</th>
<th>( K_{Ic} )</th>
<th>( L )</th>
<th>Specimen’s geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Synthetic Rock (swc=18%)</td>
<td>Lim et al., 1994</td>
<td>0.42</td>
<td>0.062</td>
<td>6.936</td>
<td>SCS</td>
</tr>
<tr>
<td>Synthetic Rock (swc=12%)</td>
<td>Haberfield et al., 1990</td>
<td>0.88</td>
<td>0.092</td>
<td>3.479</td>
<td>SECB</td>
</tr>
<tr>
<td>Synthetic Rock (swc=15.5%)</td>
<td>Haberfield et al., 1990</td>
<td>0.57</td>
<td>0.090</td>
<td>7.936</td>
<td>SECB</td>
</tr>
<tr>
<td>Synthetic Rock (swc=17%)</td>
<td>Haberfield et al., 1990</td>
<td>0.48</td>
<td>0.079</td>
<td>8.622</td>
<td>SECB</td>
</tr>
<tr>
<td>Synthetic Rock (swc=18%)</td>
<td>Haberfield et al., 1990</td>
<td>0.43</td>
<td>0.039</td>
<td>2.618</td>
<td>SECB</td>
</tr>
<tr>
<td>Synthetic Rock (swc=18.5%)</td>
<td>Haberfield et al., 1990</td>
<td>0.40</td>
<td>0.038</td>
<td>2.873</td>
<td>SECB</td>
</tr>
<tr>
<td>Indiana Limestone</td>
<td>Bazan et al., 1993 (Schmidt, 1976)</td>
<td>(5.38)</td>
<td>0.792</td>
<td>6.898</td>
<td>SECB</td>
</tr>
<tr>
<td>Welsh Limestone</td>
<td>Singh, Sun, 1990</td>
<td>8.49</td>
<td>0.990</td>
<td>4.328</td>
<td>SCS</td>
</tr>
<tr>
<td>Indiana Limestone</td>
<td>Schmidt, 1976</td>
<td>5.38</td>
<td>0.870</td>
<td>8.324</td>
<td>SECB</td>
</tr>
<tr>
<td>Stripa Granite</td>
<td>Swan, 1980; Sun et al., 1986</td>
<td>27.60</td>
<td>1.820</td>
<td>1.384</td>
<td>SECRB</td>
</tr>
<tr>
<td>Andesite</td>
<td>Tutluoglu et al., 2011 (Koca et al., 2011)</td>
<td>(3.31)</td>
<td>1.000</td>
<td>29.053</td>
<td>SND</td>
</tr>
</tbody>
</table>

Fig. 3. Geometries of the investigated cracked samples.

The normalised Kitagawa-Takashi diagrams of Fig. 4 summarise the accuracy of both the PM (Fig. 4a) and LM (Fig. 4b) in modelling the transition from the short- to the long-crack regime in geological materials subjected to Mode I static loading: such charts make it evident that, in spite of the physiological scattering always characterising the mechanical behaviour of cracked materials, the TCD is capable of correctly capturing the essence of the investigated phenomenon.
Fig. 4. Accuracy of the PM (a) and LM (b) in modelling the transition from the short- to the long-crack regime in geological materials subjected to Mode I static loading (See also Table 1).

4. Conclusions

- With geological materials, the LEFM concepts are recommended to be used in the presence of cracks having equivalent length F²a larger than 10L, critical distance L being calculated according to Eq. (7).
- The TCD is seen to be successful in modelling the transition from the short- to the long-crack regime under Mode I static loading in geological materials.

References