PERGAMON Computers and Mathematics with Applications 39 (2000) 85–94

An International Journal
computers & mathematics with applications

www.elsevier.nl/locate/camwa

Accelerated Monotone Iterative Methods for a Boundary Value Problem of Second-Order Discrete Equations

YUAN-MING WANG
Department of Mathematics, East China Normal University
Shanghai 200062, P.R. China

(Received July 1999; accepted August 1999)

Communicated by R. P. Agarwal

Abstract—An accelerated monotone iterative method for a boundary value problem of second-order discrete equations is presented. This method leads to an existence-comparison theorem as well as a computational algorithm for the solutions. The monotone property of the iterations gives improved upper and lower bounds of the solution in each iteration, and the rate of convergence of the iterations is either quadratic or nearly quadratic depending on the property of the nonlinear function. Some numerical results are presented to illustrate the monotone convergence of the iterative sequences and the rate of convergence of the iterations.

Keywords—Boundary value problem of second-order discrete equations, Upper and lower solutions, Accelerated monotone iterative method, Monotone convergence.

1. INTRODUCTION

In studying some problems arising in solid state physics, chemical reactions, population dynamics, and other topics, we often meet discrete boundary value problems. Besides, they are also natural consequences of the discretization of differential boundary value problems. There are numerous works which are devoted to the discrete boundary value problems (see [1–9]). Let $N \geq 2$ be a positive integer, $I_1^{N-1} = \{1, 2, \ldots, N-1\}$ and $I_0^N = I_1^{N-1} \cup \{0, N\}$. For the function $u : I_0^N \longrightarrow \mathbf{R}$, we define

$$\delta^2 u(t) = u(t-1) - 2u(t) + u(t+1), \qquad t \in I_1^{N-1},$$
$$P_N u(t) = \frac{1}{12N^2}\left(u(t-1) + 10u(t) + u(t+1)\right), \qquad t \in I_1^{N-1}.$$

In this paper, we consider the following discrete boundary value problem:

$$-\delta^2 u(t) + P_N f\left(\frac{t}{N}, u(t)\right) = 0, \qquad t \in I_1^{N-1}, \qquad u(0) = \alpha, \qquad u(N) = \beta, \tag{1.1}$$

0898-1221/00/$ - see front matter Typeset by AMS-TEX
PII: S0898-1221(99)00335-1

where the given function $f(\cdot, u)$, which in general is nonlinear in u, is assumed to be continuous differentiable in u, and $\alpha, \beta \in \mathbf{R}$ are known constants. This problem comes from the discretization of the following continuous boundary value problem: $y''(x) = f(x, y)$, $0 < x < 1$; $y(0) = \alpha$, $y(1) = \beta$, by the fourth-order Numerov's method (see [10–12]). If $f(\cdot, u)$ is nonlinear in u, problem (1.1) requires some kind of iterative scheme for the computation of numerical solutions. By the method of upper and lower solutions, the author in [12] proposed a monotone iteration for problem (1.1). This monotone iteration leads not only to the existence and uniqueness of a solution but the process of iteration gives also a computational algorithm for numerical solutions. However, the iteration process used in the above work is of Picard type, and the rate of convergence of the iterations is of linear order. To increase the rate of convergence while maintaining the monotone property of the iteration, we propose an accelerated monotone iterative scheme. An advantage of this scheme is that it leads to a monotone sequence which converges either quadratically or nearly quadratically with only the usual differentiability requirement on the function $f(\cdot, u)$. On the other hand, since the initial iteration in the monotone iterative scheme is either an upper solution or a lower solution, which can be constructed directly from the equation without any knowledge of the solution, this method eliminates the search for the initial iteration as is often needed in the Newton's method. This elimination gives a practical advantage in the computation of numerical solutions.

The outline of this paper is as follows. In Section 2, we give an accelerated iterative scheme for the construction of monotone sequences by the method of upper and lower solutions. This iterative scheme is reduced to the Newton's iterative scheme if $f(\cdot, u)$ possesses a concavity or convexity property between upper and lower solutions. The proof of the quadratic rate of convergence of the iterations is given in Section 3, where some explicit estimates for the rate of convergence are given. Finally, in Section 4, numerical results are presented and the rate of convergence of the monotone iterations are compared with that by the Picard's method.

2. ACCELERATED MONOTONE ITERATIONS

Without further mention, we assume that all the inequalities involving vectors are componentwise. Define $\mathcal{S} = \{u(t) \mid u(t) : I_0^N \longrightarrow \mathbf{R}^n\}$. To obtain sequences which converge to solutions of (1.1), we need to a pair of upper and lower solutions which are defined as follows.

DEFINITION 2.1. *A function $\overline{u}(t) \in \mathcal{S}$ is called an upper solution of problem (1.1) if*

$$-\delta^2\overline{u}(t) + P_N f\left(\frac{t}{N}, \overline{u}(t)\right) \geq 0, \qquad t \in I_1^{N-1}, \tag{2.1}$$

$$\overline{u}(0) \geq \alpha, \qquad \overline{u}(N) \geq \beta, \tag{2.2}$$

and $\underline{u}(t) \in \mathcal{S}$ is called a lower solution if

$$-\delta^2\underline{u}(t) + P_N f\left(\frac{t}{N}, \underline{u}(t)\right) \leq 0, \qquad t \in I_1^{N-1}, \tag{2.3}$$

$$\underline{u}(0) \leq \alpha, \qquad \underline{u}(N) \leq \beta. \tag{2.4}$$

The pair $\overline{u}(t)$, $\underline{u}(t)$ are said to be ordered if $\overline{u}(t) \geq \underline{u}(t)$ for all $t \in I_0^N$.

It is obvious that every solution of problem (1.1) is an upper solution as well as a lower solution. Given $u(t)$, $v(t)$, and $w(t)$ in $\mathcal{S}$, we say that $w \in [u, v]$ if $u(t) \leq w(t) \leq v(t)$, for all $t \in I_0^N$. Let M be a given constant and set

$$\mathcal{N}(M) = \begin{cases} \dfrac{M}{12N^2}, & M \geq 0, \\ -\dfrac{5 + \cos(\pi/N)}{24N^2 \sin^2(\pi/2N)} M, & M < 0. \end{cases} \tag{2.5}$$

We have the following positive lemma (see [12]).

LEMMA 2.1. *Let $u(t) \in \mathcal{S}$ such that*

$$-\delta^2 u(t) + P_N(M(t)u(t)) \geq 0, \qquad t \in I_1^{N-1},$$
$$u(0) \geq 0, \qquad u(N) \geq 0,$$

where $M(t) \in \mathcal{S}$. Set $\underline{M} = \min_{t\in I_1^{N-1}} M(t)$ and $\overline{M} = \max_{t\in I_0^N} M(t)$. If $\max(\mathcal{N}(\underline{M}), \mathcal{N}(\overline{M})) < 1$, then $u(t) \geq 0$ for all $t \in I_0^N$.

By Lemma 2.1, we have that for all $M(t), g(t) \in \mathcal{S}$, the linear problem

$$-\delta^2 u(t) + P_N(M(t)u(t)) = g(t), \qquad t \in I_1^{N-1},$$
$$u(0) = \alpha, \qquad u(N) = \beta$$

is uniquely solvable in $\mathcal{S}$ provided $\max(\mathcal{N}(\underline{M}), \mathcal{N}(\overline{M})) < 1$, where $\underline{M} = \min_{t\in I_1^{N-1}} M(t)$ and $\overline{M} = \max_{t\in I_0^N} M(t)$.

Let $\overline{u}(t)$ and $\underline{u}(t)$ be a pair of ordered upper and lower solutions of problem (1.1). To compute the solution of problem (1.1), we use the following iterative scheme:

$$\begin{aligned} &-\delta^2 u^{(m+1)}(t) + P_N\left(M^{(m)}(t)u^{(m+1)}(t)\right) \\ &= P_N\left(M^{(m)}(t)u^{(m)}(t)\right) - P_N f\left(\frac{t}{N}, u^{(m)}(t)\right), \qquad t \in I_1^{N-1}, \\ &u^{(m+1)}(0) = \alpha, \quad u^{(m+1)}(N) = \beta, \qquad (m = 0,1,2,\dots), \end{aligned} \tag{2.6}$$

where $u^{(0)}(t)$ is either $\overline{u}(t)$ or $\underline{u}(t)$, and

$$M^{(m)}(t) = \max\left\{f_u\left(\frac{t}{N}, u(t)\right) : \underline{u}^{(m)}(t) \leq u(t) \leq \overline{u}^{(m)}(t)\right\}, \qquad t \in I_0^N. \tag{2.7}$$

The functions $\overline{u}^{(m)}(t)$, $\underline{u}^{(m)}(t)$ in the definition of $M^{(m)}(t)$ are obtained from (2.6) with $u^{(0)} = \overline{u}(t)$ and $u^{(0)} = \underline{u}(t)$, respectively. It is clear from (2.7) that

$$f\left(\frac{t}{N}, u(t)\right) - f\left(\frac{t}{N}, v(t)\right) \leq M^{(m)}(t)\left(u(t) - v(t)\right), \qquad t \in I_0^N, \tag{2.8}$$

whenever $\underline{u}^{(m)}(t) \leq v(t) \leq u(t) \leq \overline{u}^{(m)}(t)$, $t \in I_0^N$. Moreover, if $f(\cdot, u)$ is a C^2-function then

$$M^{(m)}(t) = f_u\left(\frac{t}{N}, \overline{u}^{(m)}(t)\right), \qquad \text{when } f_{uu}\left(\frac{t}{N}, u(t)\right) \geq 0 \text{ in } \left[\underline{u}^{(m)}, \overline{u}^{(m)}\right]$$

and

$$M^{(m)}(t) = f_u\left(\frac{t}{N}, \underline{u}^{(m)}(t)\right), \qquad \text{when } f_{uu}\left(\frac{t}{N}, u(t)\right) \leq 0 \text{ in } \left[\underline{u}^{(m)}, \overline{u}^{(m)}\right].$$

Hence, if $f_u(\cdot, u)$ is either monotone nondecreasing or monotone nonincreasing in u, then the iteration process (2.6) is reduced to the Newton's form

$$\begin{aligned} &-\delta^2 u^{(m+1)}(t) + P_N\left(f_u\left(\frac{t}{N}, u^{(m)}(t)\right) u^{(m+1)}(t)\right) \\ &= P_N\left(f_u\left(\frac{t}{N}, u^{(m)}(t)\right) u^{(m)}(t)\right) - P_N f\left(\frac{t}{N}, u^{(m)}(t)\right), \qquad t \in I_1^{N-1}, \\ &u^{(m+1)}(0) = \alpha, \quad u^{(m+1)}(N) = \beta, \qquad (m = 0,1,2,\dots) \end{aligned} \tag{2.9}$$

(e.g., see [13]). To show that the sequence given by (2.6) is well defined for an arbitrary C^1-function $f(\cdot, u)$ it is crucial that the sequences $\{\overline{u}^{(m)}(t)\}$, $\{\underline{u}^{(m)}(t)\}$ possess the property $\overline{u}^{(m)}(t) \geq \underline{u}^{(m)}(t)$ for every $m = 1, 2, \dots$ and all $t \in I_0^N$. This is shown in the following lemma.

LEMMA 2.2. *Let $\overline{u}(t)$ and $\underline{u}(t)$ be a pair of ordered upper and lower solution and set*

$$M(t) = \max\left\{f_u\left(\frac{t}{N}, u(t)\right) : \underline{u}(t) \le u(t) \le \overline{u}(t)\right\}, \qquad t \in I_0^N,$$

$$M'(t) = \min\left\{f_u\left(\frac{t}{N}, u(t)\right) : \underline{u}(t) \le u(t) \le \overline{u}(t)\right\}, \qquad t \in I_0^N,$$

$$\overline{M} = \max_{t\in I_0^N} M(t), \qquad \underline{M} = \min_{t\in I_1^{N-1}} M(t), \qquad \underline{M}' = \min_{t\in I_0^N} M'(t).$$

If $\underline{M}' > -8$ and $\max(\mathcal{N}(\underline{M}), \mathcal{N}(\overline{M})) < 1$, then sequences $\{\overline{u}^{(m)}(t)\}$, $\{\underline{u}^{(m)}(t)\}$, and $\{M^{(m)}(t)\}$ given by (2.6) and (2.7) with $\overline{u}^0(t) = \overline{u}(t)$ and $\underline{u}^{(0)}(t) = \underline{u}(t)$ are all well defined and possess the property

$$\underline{u}(t) \le \underline{u}^{(m)}(t) \le \underline{u}^{(m+1)}(t) \le \overline{u}^{(m+1)}(t) \le \overline{u}^{(m)}(t) \le \overline{u}(t), \qquad t \in I_0^N, \quad m = 1, 2, \dots. \tag{2.10}$$

PROOF. Since $M^{(0)}(t) = M(t)$ and $\max(\mathcal{N}(\underline{M}), \mathcal{N}(\overline{M})) < 1$, we have from Lemma 2.1 that the first iterations $\overline{u}^{(1)}(t)$ and $\underline{u}^{(1)}(t)$ are well defined. By (2.6), (2.8), and Definition 1.1, we have that for all $t \in I_1^{N-1}$,

$$\begin{aligned} &-\delta^2\left(\overline{u}^{(1)}(t) - \underline{u}^{(1)}(t)\right) + P_N\left(M^{(0)}(t)\left(\overline{u}^{(1)}(t) - \underline{u}^{(1)}(t)\right)\right) \\ &= P_N\left(M^{(0)}(t)\left(\overline{u}^{(0)}(t) - \underline{u}^{(0)}(t)\right)\right) - P_N\left(f\left(\frac{t}{N}, \overline{u}^{(0)}(t)\right) - f\left(\frac{t}{N}, \underline{u}^{(0)}(t)\right)\right) \ge 0, \end{aligned}$$

$$-\delta^2\left(\underline{u}^{(1)}(t) - \underline{u}^{(0)}(t)\right) + P_N\left(M^{(0)}(t)\left(\underline{u}^{(1)}(t) - \underline{u}^{(0)}(t)\right)\right) \ge 0,$$

$$-\delta^2\left(\overline{u}^{(0)}(t) - \overline{u}^{(1)}(t)\right) + P_N\left(M^{(0)}(t)\left(\overline{u}^{(0)}(t) - \overline{u}^{(1)}(t)\right)\right) \ge 0.$$

Since $\max(\mathcal{N}(\overline{M}), \mathcal{N}(\underline{M})) < 1$, Lemma 2.1 implies that

$$\underline{u}^{(0)}(t) \le \underline{u}^{(1)}(t) \le \overline{u}^{(1)}(t) \le \overline{u}^{(0)}(t), \qquad t \in I_0^N.$$

Therefore, $M^{(1)}(t)$ is well defined. Assume, by induction, that $\underline{u}^{(m)}(t)$, $\overline{u}^{(m)}(t)$, and $M^{(m)}(t)$ are all well defined and

$$\underline{u}(t) \le \underline{u}^{(m-1)}(t) \le \underline{u}^{(m)}(t) \le \overline{u}^{(m)}(t) \le \overline{u}^{(m-1)}(t) \le \overline{u}(t), \qquad t \in I_0^N,$$

for some $m \ge 1$. Let

$$\underline{M}^{(m)} = \min_{t\in I_1^{N-1}} M^{(m)}(t), \qquad \overline{M}^{(m)} = \max_{t\in I_0^N} M^{(m)}(t).$$

Since $\underline{M}' > -8$ and $M'(t) \le M^{(m)}(t) \le M(t)$, for all $t \in I_0^N$, we have that $-8 < \underline{M}^{(m)} \le \underline{M}$ and $-8 < \overline{M}^{(m)} \le \overline{M}$. We show that $\max(\mathcal{N}(\underline{M}^{(m)}), \mathcal{N}(\overline{M}^{(m)})) < 1$. There are three cases.

(i) $-8 < \overline{M}^{(m)} \le 0$. It is easy to check that $(5 + \cos(\pi/N))/(24N^2 \sin^2(\pi/(2N))) \le 1/8$, and so we have $\max(\mathcal{N}(\underline{M}^{(m)}), \mathcal{N}(\overline{M}^{(m)})) < 1$.

(ii) $\overline{M}^{(m)} > 0$ and $\underline{M}^{(m)} \ge 0$. In this case, $\mathcal{N}(\overline{M}^{(m)}) \le \mathcal{N}(\overline{M})$ and $\mathcal{N}(\underline{M}^{(m)}) \le \mathcal{N}(\underline{M})$. Therefore, $\max(\mathcal{N}(\overline{M}^{(m)}), \mathcal{N}(\underline{M}^{(m)})) \le \max(\mathcal{N}(\overline{M}), \mathcal{N}(\underline{M})) < 1$.

(iii) $\overline{M}^{(m)} > 0$ and $-8 < \underline{M}^{(m)} < 0$. In this case, $\mathcal{N}(\underline{M}^{(m)}) < 1$ because of $(5 + \cos(\pi/N))/(24N^2 \sin^2(\pi/(2N))) \le 1/8$ and $\mathcal{N}(\overline{M}^{(m)}) < \mathcal{N}(\overline{M}) < 1$. Then we have $\max(\mathcal{N}(\underline{M}^{(m)}), \mathcal{N}(\overline{M}^{(m)})) < 1$.

Hence by Lemma 2.1, $\overline{u}^{(m+1)}(t)$ and $\underline{u}^{(m+1)}(t)$ are well defined. Moreover, the iteration process (2.6) implies that

$$\begin{aligned}&-\delta^2\left(\underline{u}^{(m+1)}(t)-\underline{u}^{(m)}(t)\right)+P_N\left(M^{(m)}(t)\left(\underline{u}^{(m+1)}(t)-\underline{u}^{(m)}(t)\right)\right)\\&=P_N\left(M^{(m-1)}(t)\left(\underline{u}^{(m)}(t)-\underline{u}^{(m-1)}(t)\right)\right)-P_N\left(f\left(\frac{t}{N},\underline{u}^{(m)}(t)\right)-f\left(\frac{t}{N},\underline{u}^{(m-1)}(t)\right)\right).\end{aligned}$$

Using relation (2.8) yields

$$-\delta^2\left(\underline{u}^{(m+1)}(t)-\underline{u}^{(m)}(t)\right)+P_N\left(M^{(m)}(t)\left(\underline{u}^{(m+1)}(t)-\underline{u}^{(m)}(t)\right)\right)\geq 0,\qquad t\in I_1^{N-1}.$$

By Lemma 2.1, $\underline{u}^{(m)}(t)\leq\underline{u}^{(m+1)}(t)$, for all $t\in I_0^N$. A similar argument gives $\overline{u}^{(m+1)}(t)\leq\overline{u}^{(m)}(t)$ and $\overline{u}^{(m+1)}(t)\geq\underline{u}^{(m+1)}(t)$, for all $t\in I_0^N$. Furthermore, $M^{(m+1)}(t)$ is well defined. The conclusion of the lemma follows from the principle of induction.

The following theorem gives an existence-uniqueness result as well as a computational algorithm for (1.1).

THEOREM 2.1. *Let the hypothesis in Lemma 2.2 hold. Then the sequences $\{\overline{u}^{(m)}(t)\}$, $\{\underline{u}^{(m)}(t)\}$ given by (2.6) with $\overline{u}^{(0)}(t)=\overline{u}(t)$ and $\underline{u}^{(0)}(t)=\underline{u}(t)$, converge monotonically from above and below to the solutions $\overline{u}^*(t)$ and $\underline{u}^*(t)$ of (1.1), respectively. Moreover, for all $m=1,2,\dots,$*

$$\underline{u}(t)\leq\underline{u}^{(m)}(t)\leq\underline{u}^{(m+1)}(t)\leq\underline{u}^*(t)\leq\overline{u}^*(t)\leq\overline{u}^{(m+1)}(t)\leq\overline{u}^{(m)}(t)\leq\overline{u}(t),\qquad t\in I_0^N,\tag{2.11}$$

and for any one solution $u^*(t)$ of (1.1) in $[\underline{u},\overline{u}]$, we have $u^*\in[\underline{u}^*,\overline{u}^*]$. If $\mathcal{N}(\underline{M}')<1$, in addition, then $\overline{u}^*(t)\equiv\underline{u}^*(t)$ and is the unique solution of (1.1) in $[\underline{u},\overline{u}]$.

PROOF. In view of (2.10), there exist limits $\underline{u}^*(t)$ and $\overline{u}^*(t)$ such that

$$\lim_{m\to\infty}\overline{u}^{(m)}(t)=\overline{u}^*(t),\qquad\lim_{m\to\infty}\underline{u}^{(m)}(t)=\underline{u}^*(t),\qquad t\in I_0^N$$

and (2.11) holds. Letting $m\to\infty$ in (2.6) shows that both $\overline{u}^*(t)$ and $\underline{u}^*(t)$ are the solutions of (1.1). Let $u^*(t)$ be any other solution of (1.1) in $[\underline{u},\overline{u}]$. Assume that

$$\underline{u}^{(m)}(t)\leq u^*(t)\leq\overline{u}^{(m)}(t),\qquad t\in I_0^N,\tag{2.12}$$

for some $m\geq 0$. Then (2.6) and (1.1) imply that

$$\begin{aligned}&-\delta^2\left(\overline{u}^{(m+1)}(t)-u^*(t)\right)+P_N\left(M^{(m)}(t)\left(\overline{u}^{(m+1)}(t)-u^*(t)\right)\right)\\&=P_N\left(M^{(m)}(t)\left(\overline{u}^{(m)}(t)-u^*(t)\right)\right)-P_N\left(f\left(\frac{t}{N},\overline{u}^{(m)}(t)\right)-f\left(\frac{t}{N},u^*(t)\right)\right),\qquad t\in I_1^{N-1}.\end{aligned}$$

Relation (2.8) ensures that

$$P_N\left(M^{(m)}(t)\left(\overline{u}^{(m)}(t)-u^*(t)\right)\right)-P_N\left(f\left(\frac{t}{N},u^{(m)}(t)\right)-f\left(\frac{t}{N},u^*(t)\right)\right)\geq 0,\qquad t\in I_1^{N-1}.$$

It follows from Lemma 2.1 that $\overline{u}^{(m+1)}(t)\geq u^*(t)$, for all $t\in I_0^N$. Using the similar argument gives $\underline{u}^{(m+1)}(t)\leq u^*(t)$, for all $t\in I_0^N$. By the induction principle, the monotone property (2.12) holds for all $m\geq 0$. Letting $m\to\infty$ in (2.12), we get $\underline{u}^*(t)\leq u^*(t)\leq\overline{u}^*(t)$ which gives $u^*\in[\underline{u}^*,\overline{u}^*]$. The proof of the uniqueness of the solution can be found in [4].

When $f_u(\cdot,u)$ is monotone nondecreasing or monotone nonincreasing in u, iteration (2.6) is reduced to the Newton's iteration (2.9). As a consequence of Theorem 2.1, we have the following conclusion.

COROLLARY 2.1. *Let the hypothesis in Lemma 2.2 hold. Assume that $f(\cdot,u)$ is a C^2-function of u. Then the sequence $\{\overline{u}^{(m)}(t)\}$ given by (2.9) with $\overline{u}^{(0)}(t)=\overline{u}(t)$ converges monotonically from above to a solution $\overline{u}^*(t)$ of (1.1) in $[\underline{u},\overline{u}]$ if $f_{uu}((t/N),u(t))\geq 0$ for $t\in I_0^N$ and $u\in[\underline{u},\overline{u}]$. Similarly, the sequence $\{\underline{u}^{(m)}(t)\}$ given by (2.9) with $\underline{u}^{(0)}(t)=u(t)$ converges monotonically from below to a solution $\underline{u}^*(t)$ of (1.1) in $[\underline{u},\overline{u}]$ if $f_{uu}((t/N),u(t))\leq 0$ for $t\in I_0^N$ and $u\in[\underline{u},\overline{u}]$.*

3. RATE OF CONVERGENCE

In this section, we show the quadratic and nearly quadratic rate of convergence of the sequences given by (2.6) and (2.9). We first introduce the concept of monotone matrix. An $n \times n$ real matrix $A = (A_{i,j})$ is called a monotone matrix if $AZ \geq 0$ implies $Z \geq 0$ for any vector $Z \in \mathbf{R}^n$. A necessary and sufficient condition for the monotonicity of an $n \times n$ real matrix A is the existence of the inverse $A^{-1} \geq 0$ (see [14,15]). Define the symmetric tridiagonal matrices $A = (A_{i,j})$ and $B = (B_{i,j})$ as

$$\begin{aligned} A_{i,i} &= 2, & B_{i,i} &= \frac{5}{6}, & i &= 1,2,\dots,N-1, \\ A_{i,i-1} &= -1, & B_{i,i-1} &= \frac{1}{12}, & i &= 2,3,\dots,N-1, \\ A_{i,i+1} &= -1, & B_{i,i+1} &= \frac{1}{12}, & i &= 1,2,\dots,N-2. \end{aligned}$$

By Lemma 2.1, we have the following result.

LEMMA 3.1. *Let $D = \operatorname{diag}(M(1), M(2), \dots, M(N-1))$. Then the matrix $A + (1/N^2)BD$ is monotone provided $\max(\mathcal{N}(\underline{M}), \mathcal{N}(\overline{M})) < 1$, where $\underline{M} = \min_{t \in I_1^{N-1}} M(t)$ and $\overline{M} = \max_{t\in I_1^{N-1}} M(t)$.*

The following theorem gives an estimate for the rate of convergence of the sequences from (2.6).

THEOREM 3.1. *Let the hypothesis in Lemma 2.2 hold. Assume that $f(\cdot, u)$ is a C^2-function of u. Let also $\{\overline{u}^{(m)}(t)\}$, $\{\underline{u}^{(m)}(t)\}$ be the sequences given by (2.6), and $\overline{u}^*(t)$ and $\underline{u}^*(t)$ be the limits of them, respectively. Then there exists constant ρ, independent of m, such that*

$$\begin{gathered} \max_{t\in I_0^N} \left|\overline{u}^{(m+1)}(t) - \overline{u}^*(t)\right| \leq \rho \max_{t\in I_0^N} \left|\overline{u}^{(m)}(t) - \overline{u}^*(t)\right| \cdot \max_{t\in I_0^N} \left|\overline{u}^{(m)}(t) - \underline{u}^{(m)}(t)\right|, \\ \max_{t\in I_0^N} \left|\underline{u}^{(m+1)}(t) - \underline{u}^*(t)\right| \leq \rho \max_{t\in I_0^N} \left|\underline{u}^{(m)}(t) - \underline{u}^*(t)\right| \cdot \max_{t\in I_0^N} \left|\overline{u}^{(m)}(t) - \underline{u}^{(m)}(t)\right|, \\ (m = 1,2,\dots) \end{gathered} \tag{3.1}$$

and if $\underline{u}^*(t) \equiv \overline{u}^*(t)$, then

$$\begin{gathered} \max_{t\in I_0^N} \left|\overline{u}^{(m+1)}(t) - \overline{u}^*(t)\right| + \max_{t\in I_0^N} \left|\underline{u}^{(m+1)}(t) - \underline{u}^*(t)\right| \\ \leq \rho \left(\max_{t\in I_0^N} \left|\overline{u}^{(m)}(t) - \overline{u}^*(t)\right| + \max_{t\in I_0^N} \left|\underline{u}^{(m)}(t) - \underline{u}^*(t)\right| \right)^2, \qquad (m = 1,2,\dots). \end{gathered} \tag{3.2}$$

PROOF. Consider the sequence $\{\overline{u}^{(m)}(t)\}$. By (1.1) and (2.6),

$$\begin{gathered} -\delta^2 \left(\overline{u}^{(m+1)}(t) - \overline{u}^*(t)\right) + P_N \left(M^{(m)}(t) \left(\overline{u}^{(m+1)}(t) - \overline{u}^*(t)\right)\right) \\ = P_N \left(M^{(m)}(t) \left(\overline{u}^{(m)}(t) - \overline{u}^*(t)\right)\right) - P_N \left(f\left(\frac{t}{N}, \overline{u}^{(m)}(t)\right) - f\left(\frac{t}{N}, \overline{u}^*(t)\right)\right), \\ t \in I_1^{N-1}, \\ \overline{u}^{(m+1)}(t) = \overline{u}^*(t), \qquad t = 0, N. \end{gathered}$$

By (2.7) and the mean-value theorem, there exist $\xi^{(m)}(t)$ in $[\underline{u}^{(m)}, \overline{u}^{(m)}]$ and $\eta^{(m)}(t)$ in $[\overline{u}^*, \overline{u}^{(m)}]$ such that

$$\begin{aligned} M^{(m)}(t) &= f_u\left(\frac{t}{N}, \xi^{(m)}(t)\right), & t &\in I_0^N, \\ f\left(\frac{t}{N}, \overline{u}^{(m)}(t)\right) - f\left(\frac{t}{N}, \overline{u}^*(t)\right) &= f_u\left(\frac{t}{N}, \eta^{(m)}(t)\right)\left(\overline{u}^{(m)}(t) - \overline{u}^*(t)\right), & t &\in I_0^N, \end{aligned} \tag{3.3}$$

respectively. Since

$$f_u\left(\frac{t}{N},\xi^{(m)}(t)\right)-f_u\left(\frac{t}{N},\eta^{(m)}(t)\right)=f_{uu}\left(\frac{t}{N},\theta^{(m)}(t)\right)\left(\xi^{(m)}(t)-\eta^{(m)}(t)\right),$$

for some intermediate value $\theta^{(m)}(t)$ between $\xi^{(m)}(t)$ and $\eta^{(m)}(t)$, we have

$$\begin{aligned}&P_N\left(M^{(m)}(t)\left(\overline{u}^{(m)}(t)-\overline{u}^*(t)\right)\right)-P_N\left(f\left(\frac{t}{N},\overline{u}^{(m)}(t)\right)-f\left(\frac{t}{N},\overline{u}^*(t)\right)\right)\\&\quad=P_N\left(f_{uu}\left(\frac{t}{N},\theta^{(m)}(t)\right)\left(\xi^{(m)}(t)-\eta^{(m)}(t)\right)\left(\overline{u}^{(m)}(t)-\overline{u}^*(t)\right)\right),\qquad t\in I_1^{N-1}.\end{aligned}$$

It follows from $|\xi^{(m)}(t)-\eta^{(m)}(t)|\le|\overline{u}^{(m)}(t)-\underline{u}^{(m)}(t)|\le\max_{t\in I_0^N}|\overline{u}^{(m)}(t)-\underline{u}^{(m)}(t)|$ that

$$\begin{aligned}&-\delta^2\left(\overline{u}^{(m+1)}(t)-\overline{u}^*(t)\right)+P_N\left(M^{(m)}(t)\left(\overline{u}^{(m+1)}(t)-\overline{u}^*(t)\right)\right)\\&\qquad\le\sigma\max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\underline{u}^{(m)}(t)\right|\cdot P_N\left(\left|\overline{u}^{(m)}(t)-\overline{u}^*(t)\right|\right),\qquad t\in I_1^{N-1},\\&\qquad\overline{u}^{(m+1)}(t)=\overline{u}^*(t),\qquad t=0,N,\end{aligned}$$

where $\sigma=\max_{t\in I_0^N}\sigma(t)$ and $\sigma(t)=\max\{f_{uu}(t/N,u(t)):\underline{u}(t)\le u(t)\le\overline{u}(t)\}$. Let

$$D^m=\operatorname{diag}\left(M^{(m)}(1),\dots,M^{(m)}(N-1)\right),$$
$$\overline{U}^{(m)}=\left(\overline{u}^{(m)}(1),\dots,\overline{u}^{(m)}(N-1)\right)^{\top},\qquad\overline{U}^*=\left(\overline{u}^*(1),\dots,\overline{u}^*(N-1)\right)^{\top}.$$

Then we have that for all $m\ge1$,

$$\left(A+\frac{1}{N^2}BD^{(m)}\right)\left(\overline{U}^{(m+1)}-\overline{U}^*\right)\le\frac{\sigma}{N^2}\max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\underline{u}^{(m)}(t)\right|B\left|\overline{U}^{(m)}-\overline{U}^*\right|.$$

Taking $-8<\gamma\le\min\{0,\underline{M}'\}$, we have

$$A+\frac{1}{N^2}BD^{(m)}\ge A+\frac{\gamma}{N^2}B.$$

Furthermore, by $\overline{U}^{(m)}\ge\overline{U}^*$,

$$\left(A+\frac{\gamma}{N^2}B\right)\left(\overline{U}^{(m+1)}-\overline{U}^*\right)\le\frac{\sigma}{N^2}\max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\underline{u}^{(m)}(t)\right|B\left|\overline{U}^{(m)}-\overline{U}^*\right|.$$

Since $\mathcal{N}(\gamma)<1$, we have from Lemma 3.1 that

$$\left(A+\frac{\gamma}{N^2}B\right)^{-1}\ge0$$

and therefore,

$$0\le\left(\overline{U}^{(m+1)}-\overline{U}^*\right)\le\frac{\sigma}{N^2}\max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\underline{u}^{(m)}(t)\right|\left(A+\frac{\gamma}{N^2}B\right)^{-1}B\left|\overline{U}^{(m)}-\overline{U}^*\right|.$$

Using the vector norm, we obtain

$$\left\|\overline{U}^{(m+1)}-\overline{U}^*\right\|_\infty\le\frac{\sigma}{N^2}\max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\underline{u}^{(m)}(t)\right|\left\|\left(A+\frac{\gamma}{N^2}B\right)^{-1}\right\|_\infty\cdot\|B\|_\infty\cdot\left\|\overline{U}^{(m)}-\overline{U}^*\right\|_\infty,$$

which gives that

$$\max_{t\in I_0^N}\left|\overline{u}^{(m+1)}(t)-\overline{u}^*(t)\right| \le \rho \max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\underline{u}^{(m)}(t)\right|\cdot \max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\overline{u}^*(t)\right|,$$

where $\rho = (\sigma/N^2)\|(A+(\gamma/N^2)B)^{-1}\|_\infty \cdot \|B\|_\infty$. The proof of the second relation in (3.1) is similar.

An addition of the relations in (3.1) yields

$$\begin{aligned}\max_{t\in I_0^N}&\left|\overline{u}^{(m+1)}(t)-\overline{u}^*(t)\right| + \max_{t\in I_0^N}\left|\underline{u}^{(m+1)}(t)-\underline{u}^*(t)\right|\\ &\le \rho\left(\max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\overline{u}^*(t)\right| + \max_{t\in I_0^N}\left|\underline{u}^{(m)}(t)-\underline{u}^*(t)\right|\right)\cdot \max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\underline{u}^{(m)}(t)\right|.\end{aligned}$$

If $\overline{u}^*(t)=\underline{u}^*(t)$, we have $|\overline{u}^{(m)}(t)-\underline{u}^{(m)}(t)| \le |\overline{u}^{(m)}(t)-\overline{u}^*(t)| + |\underline{u}^*(t)-\underline{u}^{(m)}(t)|$. Therefore, (3.2) holds.

Theorem 3.1 gives a nearly quadratic convergence of the sequences $\{\overline{u}^{(m)}(t)\}$ and $\{\underline{u}^{(m)}(t)\}$, and a quadratic convergence for the sum of these two sequences. The following theorem shows that if $f_u(\cdot,u)$ is monotone nondecreasing or monotone nonincreasing in u, then one of the two sequences converges quadratically to the solution.

THEOREM 3.2. *Let the conditions in Theorem 3.1 hold. Then there exists a constant ρ, independent of m, such that*

$$\max_{t\in I_0^N}\left|\overline{u}^{(m+1)}(t)-\overline{u}^*(t)\right| \le \rho\left(\max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\overline{u}^*(t)\right|\right)^2, \qquad (m=1,2,\dots), \tag{3.4}$$

if $f_{uu}(t/N,u(t)) \ge 0$ for $t\in I_0^N$ and $u\in[\underline{u},\overline{u}]$, and

$$\max_{t\in I_0^N}\left|\underline{u}^{(m+1)}(t)-\underline{u}^*(t)\right| \le \rho\left(\max_{t\in I_0^N}\left|\underline{u}^{(m)}(t)-\underline{u}^*(t)\right|\right)^2, \qquad (m=1,2,\dots), \tag{3.5}$$

if $f_{uu}(t/N,u)\le 0$ for $t\in I_0^N$ and $u\in[\underline{u},\overline{u}]$.

PROOF. Consider the case $f_{uu}(t/N,u(t))\ge 0$. In this case, $M^{(m)}(t) = f_u(t/N,\overline{u}^{(m)}(t))$. This implies $\xi^{(m)}(t)=\overline{u}^{(m)}(t)$, where $\xi^{(m)}(t)$ is the intermediate value appeared in (3.3). Since $\eta^{(m)}(t)$ is in $[\overline{u}^*,\overline{u}^{(m)}]$, we see that

$$\left|\xi^{(m)}(t)-\eta^{(m)}(t)\right| \le \left|\overline{u}^{(m)}(t)-\overline{u}^*(t)\right| \le \max_{t\in I_0^N}\left|\overline{u}^{(m)}(t)-\overline{u}^*(t)\right|.$$

The argument in the proof of Theorem 3.1 shows that (3.4) holds with

$$\rho = \frac{\sigma}{N^2}\left\|\left(A+\frac{\gamma}{N}B\right)^{-1}\right\|_\infty \cdot \|B\|_\infty,$$

where σ and γ is the same as before. The proof for (3.5) is similar.

4. NUMERICAL RESULTS

In this section, we give some numerical results. Consider problem (1.1) with

$$f\left(\frac{t}{N},u\right) = \kappa u^2 - \pi^2\sin\left(\frac{\pi}{N}t\right) - \kappa\sin^2\left(\frac{\pi}{N}t\right),$$

where $\kappa>0$ is given constant. It can be checked that $\overline{u}(t) = ((\pi^2+1)/2)(t/N)(1-(t/N))$ is an upper solution and $\underline{u}(t)\equiv 0$ is a lower solution. Set $N=20$ and $\kappa=2$. We use the

Table 1.

m	$u^{(m)}(2)$	$u^{(m)}(4)$	$u^{(m)}(6)$	$u^{(m)}(8)$	$u^{(m)}(10)$
1	0.314664	0.598534	0.823685	0.968128	1.017873
2	0.309029	0.587809	0.809051	0.951097	1.000043
3	0.309018	0.587788	0.809020	0.951060	1.000004

Table 2.

m	$u^{(m)}(2)$	$u^{(m)}(4)$	$u^{(m)}(6)$	$u^{(m)}(8)$	$u^{(m)}(10)$
1	0.248095	0.469177	0.642157	0.752009	0.789644
2	0.307159	0.584143	0.803841	0.944827	0.993394
3	0.309017	0.587784	0.809016	0.951055	0.999998

Table 3.

κ	Picard	Acceler.
50	12	5
100	13	5
200	13	5
400	13	5
800	14	5

iterative scheme (2.6) to solve this problem and denote by $u^{(m)}(t)$ the m'^{th} value of iteration. Numerical results show that if $u^{(0)}(t) = \overline{u}(t)$, then $u^{(m)}(t)$ is a monotone nonincreasing sequence (see Table 1), while if $u^{(0)}(t) = \underline{u}(t)$, then $u^{(m)}(t)$ is a monotone nondecreasing sequence (see Table 2). The monotonicity in Tables 1 and 2 agrees with that described by Theorem 2.1. In all computations, we also find that the above two sequences tend to the same limit. This coincides with the uniqueness result in Theorem 2.1, because the uniqueness condition of the solution is satisfied in this example.

Next, starting the same initial values $\overline{u}^{(0)}(t) = \overline{u}(t)$ and $\underline{u}^{(0)}(t) = \underline{u}(t)$, we compute the sequences $\overline{u}^{(m)}(t)$ and $\underline{u}^{(m)}(t)$ from iteration (2.6) and the following Picard iteration:

$$\begin{aligned} -\delta^2 u^{(m+1)}(t) &+ P_N\left(M(t)u^{(m+1)}(t)\right) \\ &= P_N\left(M(t)u^{(m)}(t)\right) - P_N f\left(\frac{t}{N}, u^{(m)}(t)\right), \qquad t \in I_1^{N-1}, \\ u^{(m+1)}(0) &= \alpha, \quad u^{(m+1)}(N) = \beta, \qquad (m = 0, 1, 2, \dots), \end{aligned} \tag{4.1}$$

where $M(t) = \max\{f_u(t/N, u(t)) : \underline{u}(t) \le u(t) \le \overline{u}(t)\}$, $t \in I_0^N$ (see [4]). The last two columns of Table 3 give the number of iterations by these two iterations for the different κ, where a tolerance $\epsilon = 10^{-6}$ for $\max_{t \in I_0^N} |\overline{u}^{(m)}(t) - \underline{u}^{(m)}(t)|$ is used.

REFERENCES

1. R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, (1992).
2. R.P. Agarwal, On multi-point boundary value problems for discrete equations, *J. Math. Anal. Appl.* **96**, 520–534, (1983).
3. R.P. Agarwal, On boundary value problems for second order discrete systems, *Applicable Analysis* **20**, 1–17, (1985).
4. R.P. Usmani and R.P. Agarwal, On the numerical solution of two point boundary value problems, *Appl. Math. Comp.* **25**, 247–264, (1988).
6. P.Y.H. Pang and R.P. Agarwal, On discrete boundary value problems arising in transport phenomena, *Appl. Math. Comp.* **60**, 193–213, (1994).
7. P.Y.H. Pang and R.P. Agarwal, Monotone iterative methods for a general class of discrete boundary value problems, *Computers Math. Applic.* **28** (1-3), 243–254, (1994).
8. W. Zhuang, Y. Cheng and S.S. Cheng, Monotone methods for a discrete boundary problem, *Computers Math. Applic.* **32** (12), 41–49, (1996).

9. Y.-M. Wang, Monotone enclosure for a class of discrete boundary value problems without monotone nonlinearities, *Computers Math. Applic.* **35** (6), 51–60, (1998).
10. Y.-M. Wang and B.-Y. Guo, On Numerov scheme for nonlinear two-points boundary value problems, *J. Comp. Math.* **16** (4), 345–366, (1998).
11. R.P. Agarwal, On Numerov's method for solving two point boundary value problems, *Utilitas Math.* **28**, 159–174, (1985).
12. Y.-M. Wang, Monotone methods for a boundary value problem of second-order discrete equation, *Computers Math. Applic.* **36** (6), 77–92, (1998).
13. J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).
14. L. Collatz, *The Numerical Treatment of Differential Equations*, 3rd Edition, Springer-Verlag, Berlin, (1964).
15. R.S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, (1962).