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Generic noncommutative surfaces ☆

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Abstract

We study a class of noncommutative surfaces, and their higher dimensional analogs, which come from generic subalgebras of twisted homogeneous coordinate rings of projective space. Such rings provide answers to several open questions in noncommutative projective geometry. Specifically, these rings R are the first known graded algebras over a field k which are noetherian but not strongly noetherian: in other words, $R \otimes_k B$ is not noetherian for some choice of commutative noetherian extension ring B. This answers a question of Artin, Small, and Zhang. The rings R are also maximal orders, but they do not satisfy all of the χ conditions of Artin and Zhang. In particular, they satisfy the χ_1 condition but not χ_i for $i \geqslant 2$, answering a question of Stafford and Zhang and a question of Stafford and Van den Bergh. Finally, we show that the noncommutative scheme R-proj has finite global dimension.

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1. Introduction

In algebraic geometry, the correspondence between projective schemes over a field k and $\mathbb{N}$-graded k-algebras is classical. In the last decade or so, many of the ideas of projective geometry have been successfully generalized to the setting of noncommutative graded rings. This new subject is known as noncommutative projective geometry, and while of theoretical interest in its own right, it has also provided the

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solutions to many purely ring-theoretic questions. For example, the graded domains of dimension two, which correspond to noncommutative curves, have now been completely classified [2]. The noncommutative analogs of the projective plane $\mathbb{P}^2$ have been identified and classified as well. The generic noncommutative projective plane is called the Sklyanin algebra; despite its simple presentation by generators and relations, before the geometric approach of [3,4] was developed it was not even known that this algebra was noetherian.

The classification theory of graded algebras of dimension three, or noncommutative projective surfaces, has also progressed substantially in recent years; for a survey of some of these results see [23]. In this paper, we study a class of algebras of dimension at least three and show that they provide counterexamples to a number of open questions in the literature. In particular, these give new examples of noncommutative surfaces with much different behavior than any of those studied previously. Moreover, the construction of these algebras is quite simple and general.

Definition 1.1. Let $S = \bigoplus_{i \geqslant 0} S_i$ be a generic Zhang twist (see Section 3) of a polynomial ring in $(t+1)$ variables over an algebraically closed field k for some $t \geqslant 2$. Then let R be the subalgebra of S generated by any generic codimension-one subspace of S_1.

To give a very explicit example of the surface case, for $t = 2$ we may take

$$S = k\{x, y, z | xz = pzx, yz = qzy, xy = pq^{-1}yx\}$$

for any scalars $p, q \in k$ which are algebraically independent over the prime subfield of k. Then let R be the subalgebra of S generated over k by any two linearly independent elements $r_1, r_2 \in S_1$ such that the k-span of r_1 and r_2 does not contain x, y, or z.

For the rest of this introduction, we assume that all algebras $A = \bigoplus_{i=0}^{\infty} A_i$ are $\mathbb{N}$-graded and finitely generated by A_1 as an algebra over $A_0 = k$, where k is an algebraically closed field. A *point module* over A is a cyclic $\mathbb{N}$-graded left module M, generated in degree 0, such that $\dim_k M_i = 1$ for all $i \geqslant 0$. In case A is commutative, the isomorphism classes of point modules over A are in natural correspondence with the closed points of the scheme proj A. More generally, for many specific examples of noncommutative graded rings one may show explicitly that the set of point modules is parameterized by a commutative scheme, and the geometry of this scheme often gives important information about the ring itself; for example see [3,4]. This is a very useful technique, and so it is natural to wonder in what generality the point modules for a ring will form a nice geometric object. Let a k-algebra A be called *strongly* (*left*) *noetherian* if $A \otimes_k B$ is a left noetherian ring for all commutative noetherian k-algebras B. A recent theorem of Artin and Zhang (see Theorem 6.2 below) shows that the point modules for any strongly noetherian k-algebra are naturally parameterized by a commutative projective scheme over k.

The strong noetherian property holds for many standard examples of noncommutative rings, including all finitely generated commutative k-algebras, all twisted homogeneous coordinate rings of projective k-schemes, and the AS-regular algebras of dimension 3 [1, Section 4]. On the other hand, Resco and Small [19] have given an example of a noetherian finitely generated algebra over a field which is not strongly noetherian. This algebra is not graded, however, nor is the base field algebraically closed, and so the example falls outside the paradigm of noncommutative projective geometry. Artin, Small and Zhang have asked if perhaps every finitely generated $\mathbb{N}$-graded noetherian k-algebra is strongly noetherian. We will prove the following theorem which answers this question in the negative.

Theorem 1.2 (Theorem 6.8). *The ring R of Definition* 1.1 i*s a connected $\mathbb{N}$-graded k-algebra, finitely generated in degree* 1, *which is noetherian but not strongly noetherian.*

We offer two different proofs that R is not strongly noetherian. First, we will classify the set of point modules for R, from which we can see that R fails to satisfy Artin and Zhang's theorem (Theorem 6.2). For the second proof, we construct an explicit commutative noetherian ring B such that $R \otimes_k B$ is not noetherian. The ring B that works is an infinite affine blowup of affine space, which was defined in [1] and is an interesting construction in itself.

In [5], Artin and Zhang describe a categorical approach to noncommutative geometry. Let A-gr be the category of all noetherian graded left A-modules, and let A-tors be the subcategory of all modules with finite k-dimension. If A is commutative, then part of Serre's theorem states that the category $\operatorname{coh} X$ of coherent sheaves on the commutative projective scheme $X = \operatorname{proj} A$ is equivalent to the quotient category $A\text{-qgr} = A\text{-gr}/A\text{-tors}$ [13, Exercise II.5.9]. Using this as motivation, for any graded algebra A the *noncommutative projective scheme* A-proj is defined to be the pair $(A\text{-qgr}, \pi(A))$, where $\pi(A)$ is the image of A in A-qgr and plays the role of the structure sheaf. One defines cohomology groups for $\mathscr{M} \in A$-qgr by setting $\mathrm{H}^i(\mathscr{M}) = \operatorname{Ext}^i_{A\text{-qgr}}(\pi(A), \mathscr{M})$.

Some homological conditions on the ring A called the χ conditions arise naturally in this approach. Let ${}_Ak = A/\bigoplus_{n=1}^{\infty} A_n$; then we say that A satisfies χ_i if $\dim_k \underline{\operatorname{Ext}}^j_A(k, M) < \infty$ for all $0 \leqslant j \leqslant i$ and all $M \in A$-gr. We say that A satisfies χ if A satisfies χ_i for all $i \geqslant 0$. The most important of these conditions is χ_1: if A satisfies χ_1 then a noncommutative version of Serre's theorem holds (see Theorem 10.3 below), which shows that the ring A is determined in large degree by its associated scheme A-proj together with the natural shift functor.

The χ conditions hold trivially for commutative rings, but Stafford and Zhang [24, Theorem 2.3] gave an example of a noetherian ring T of dimension 2 for which χ fails. The ring T fails χ_i for all $i \geqslant 0$, and so it does not satisfy the noncommutative Serre's theorem. The algebra R of Definition 1.1 is a more interesting example of the failure of χ, since χ_1 holds and the noncommutative Serre's theorem is valid for R. The following theorem thus answers a question of Stafford and Zhang from [24, Section 4].

Theorem 1.3 (Theorem 10.6). *R is a noetherian connected finitely $\mathbb{N}$-graded k-algebra, finitely generated in degree 1, for which χ_1 holds but χ_i fails for all $i \geqslant 2$.* $\square$

One consequence we will draw is that the category R-qgr is necessarily quite different from the standard examples of noncommutative schemes, which come from rings satisfying χ.

Theorem 1.4 (Theorem 10.11). *The category R-qgr is not equivalent to the category $\operatorname{coh} X$ of coherent sheaves on X for any commutative projective scheme X. More generally, $R\text{-qgr} \nsim A\text{-qgr}$ for any graded ring A which satisfies χ_2.*

A *maximal order* is the noncommutative analog of a commutative integrally closed domain; see Section 9 for the formal definition. In [23, p. 194], the authors ask whether the χ conditions perhaps must hold for all maximal orders, one reason being that all of the noetherian examples in [24] for which χ fails are equivalent orders to maximal orders which do satisfy χ. We show to the contrary the following result.

Theorem 1.5 (Theorems 9.5 and 10.6). *R is a connected, finitely $\mathbb{N}$-graded maximal order for which χ_i fails for $i \geqslant 2$.*

For a graded ring A, the *global dimension* of A-qgr is the supremum of all n such that $\operatorname{Ext}^n_{A\text{-qgr}}(\mathscr{M}, \mathscr{N}) \neq 0$ for some $\mathscr{M}, \mathscr{N} \in A$-qgr. Similarly, we define the *cohomological dimension* of A-proj to be the supremum of all n such that $\mathrm{H}^n(\mathscr{F}) \neq 0$ for some $\mathscr{F} \in A$-qgr. Despite the failure of χ for R, the following theorem shows that R-qgr is not too badly behaved.

Theorem 1.6 (Theorem 11.7). *R-qgr has global dimension at most $t+1$ and cohomological dimension at most t, where $t = \mathrm{GK}(R) - 1$.*

The starting point for our study was the work of David Jordan on rings generated by two Eulerian derivatives [14]. These rings are more or less the algebras R of Definition 1.1 for $t = 2$. Jordan had classified the point modules for such algebras and showed that the strong noetherian property must fail, but did not show these algebras were noetherian. Our results imply in fact that rings generated by a generic finite set of Eulerian derivatives are noetherian, answering a question in [14]—see Section 13 for more details.

The results in this paper form part of the author's Ph.D. Thesis [20], and in some cases extra details may be found there. In collaboration with Stafford and Keeler we have recently developed a geometric approach to the study of the rings R which shows that they may be thought of as a kind of peculiar noncommutative blowing up of projective space; details will be given in a further paper.

2. Basic definitions

In this section, we fix some terminology concerning noncommutative graded rings. The reader may wish to skim this section and refer back to it later when necessary.

We make the convention throughout that 0 is a natural number, so that $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$. The rings A we study in this paper are all $\mathbb{N}$-graded algebras $A = \bigoplus_{i=0}^{\infty} A_i$ over an algebraically closed field k. We assume throughout that all algebras A are *connected*, that is that $A_0 = k$, and finitely generated as an algebra by A_1. Let A-Gr be the abelian category whose objects are the $\mathbb{Z}$-graded left A-modules $M = \bigoplus_{i=-\infty}^{\infty} M_i$, and where the morphisms $\mathrm{Hom}_{A\text{-Gr}}(M, N)$ are the module homomorphisms ϕ satisfying $\phi(M_n) \subseteq N_n$ for all n. Let A-gr be the full subcategory of A-Gr consisting of the noetherian objects. For $M \in A$-Gr and $n \in \mathbb{Z}$, the shift of M by n, denoted $M[n]$, is the module with the same ungraded module structure as M but with the grading shifted so that $(M[n])_m = M_{n+m}$. Then for $M, N \in A$-Gr we may define

$$\underline{\mathrm{Hom}}_A(M, N) = \bigoplus_{i=-\infty}^{\infty} \mathrm{Hom}_{A\text{-Gr}}(M, N[i]).$$

The group $\underline{\mathrm{Hom}}(M, N)$ is a $\mathbb{Z}$-graded vector space and we also write $\underline{\mathrm{Hom}}(M, N)_i$ for the ith graded piece $\mathrm{Hom}(M, N[i])$. Under mild hypotheses, for example if M is finitely generated, the group $\underline{\mathrm{Hom}}(M, N)$ may be identified with the set of ungraded module homomorphisms from M to N. It is a standard result that the category A-Gr has enough injectives and so we may define right derived functors $\mathrm{Ext}^i_{A\text{-Gr}}(M, -)$ of $\mathrm{Hom}_{A\text{-Gr}}(M, -)$ for any M. The definition of $\underline{\mathrm{Hom}}$ generalizes to

$$\underline{\mathrm{Ext}}^i_A(M, N) = \bigoplus_{i=-\infty}^{\infty} \mathrm{Ext}^i_{A\text{-Gr}}(M, N[i]).$$

See [5, Section 3] for a discussion of the basic properties of $\underline{\mathrm{Ext}}$.

For a module $M \in A$-Gr, a *tail* of M is any submodule of the form $M_{\geqslant n} = \bigoplus_{i=n}^{\infty} M_i$. A *subfactor* of M is any module of the form N/N' for graded submodules $N' \subseteq N$ of M. For the purposes of this paper, $M \in A$-Gr is called *torsion* if for all $m \in M$ there exists some $n \geqslant 0$ such that $(A_{\geqslant n})m = 0$. Note that if $M \in A$-gr, then M is torsion if and only if $\dim_k M < \infty$. We say that $M \in A$-Gr is *left bounded* if $M_i = 0$ for $i \ll 0$, and *right bounded* if $M_i = 0$ for $i \gg 0$. M is *bounded* if it is both left and right bounded. A (*finite*) *filtration* of $M \in A$-Gr is a sequence of graded submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$; we call the modules M_i/M_{i-1} for $0 < i \leqslant n$ the *factors* of the filtration.

A *point module* over A is a graded module M such that M is cyclic, generated in degree 0, and $\dim_k M_n = 1$ for all $n \geqslant 0$. Note that a tail of a point module is a shift of some other point module. A *point ideal* is a left ideal I of A such that A/I is a point module, or equivalently such that $\dim_k I_n = \dim_k A_n - 1$ for all $n \geqslant 0$. Since A is generated in degree 1, the point ideals of A are in one-to-one correspondence with isomorphism classes of point modules over A.

We will use Gelfand–Kirillov dimension, or GK-dimension for short, as our dimension function for modules; the basic reference for its properties is [15]. Given $M \in A$-gr, the *Hilbert function* of M is the function $H(n) = \dim_k M_n$ for $n \in \mathbb{Z}$. If A is a finitely generated k-algebra and ${}_A M$ is a finitely generated module, then $\mathrm{GK}(M)$ depends only on the Hilbert function of M [15, 6.1]. In particular, if M has a *Hilbert polynomial*, that is $\dim_k M_n = f(n)$ for $n \gg 0$ and some polynomial $f \in \mathbb{Q}[n]$, then $\mathrm{GK}(M) = \deg f + 1$ (with the convention $\deg(0) = -1$). We say $M \in A$-Gr is (*graded*) *critical* if $\mathrm{GK}(M/N) < \mathrm{GK}(M)$ for all nonzero graded submodules N of M. If A is an $\mathbb{N}$-graded noetherian ring, then the GK-dimension for A-modules is *exact*: in other words, given any exact sequence $0 \to M' \to M \to M'' \to 0$ in A-gr, one has $\mathrm{GK}(M) = \max(\mathrm{GK}(M'), \mathrm{GK}(M''))$ [17, 4.9].

3. Zhang twists

Let k be an algebraically closed field. Fix from now on a commutative polynomial ring $U = k[x_0, \dots, x_t]$ in $t+1$ indeterminates, graded as usual with $\deg x_i = 1$ for all i, and the corresponding projective space $\mathrm{Proj}\, U = \mathbb{P}^t$. By a *point* of $\mathbb{P}^t$ we always mean a closed point. The main results of this paper require that $t \geqslant 2$, so we assume this throughout.

Let the symbol $\circ$ indicate the multiplication operation in U. For any graded automorphism ϕ of U we may define a new graded ring $(S, \bigstar)$, where S has the same underlying vector space as U and $f \bigstar g = \phi^n(f) \circ g$ for $f \in U_m$, $g \in U_n$. This is just a special case of the (left) twisting construction studied by Zhang [31].

Let $\mathfrak{m}_d$ stand for the ideal in U of a point $d \in \mathbb{P}^t$. For a homogeneous element $f \in U$, we use the notation $f \in \mathfrak{m}_d$ and $f(d) = 0$ interchangeably to indicate that f vanishes at d. Corresponding to the automorphism ϕ of U is an automorphism φ of $\mathbb{P}^t$ which satisfies $\mathfrak{m}_{\varphi(d)} = \phi^{-1}(\mathfrak{m}_d)$ for all $d \in \mathbb{P}^t$. Automorphisms ϕ_1, ϕ_2 of U give the same automorphism φ of $\mathbb{P}^t$ if and only if $\phi_1 = a\phi_2$ for some $a \in k^\times$ [13, II.7.1.1]. Moreover, automorphisms of U which are scalar multiples give isomorphic twisted algebras S [31, 5.13]. Thus a particular twist S of U is determined up to isomorphism by φ and we write $S = S(\varphi)$. We remark that an alternative description of S may be given using twisted homogeneous coordinate rings [23, Section 3]. In this language, $S = B(\mathbb{P}^t, \mathcal{O}(1), \varphi^{-1})$ where $\mathcal{O}(1)$ is the twisting sheaf of Serre on $\mathbb{P}^t$.

Since S and its subalgebras are our main interest in this paper, our notational convention from now on (except in the appendix) will be to let juxtaposition indicate multiplication in S and to use $\circ$ when multiplication in U is intended. However, we let exponents retain their old commutative meaning, so that if $\mathfrak{m}_d$ is a point ideal of U then $\mathfrak{m}_d^2$ is short for $\mathfrak{m}_d \circ \mathfrak{m}_d$, the ideal of polynomials vanishing twice at d.

Many important properties pass from a graded ring to any Zhang twist. In particular, it is easy to see that S is a noetherian domain of GK dimension $t+1$, since these properties are obvious for U [31, Propositions 5.1, 5.2 and 5.7]. Suppose that $M \in U$-Gr, and let $\circ$ indicate the action of U on M. Then M obtains an S-structure using the rule $sx = \phi^n(s) \circ x$ for $s \in S_m$, $x \in M_n$. This defines a functor θ :

U-Gr $\to$ S-Gr which is an equivalence of categories [31, 3.1]. For any graded ideal I of U, $\theta(I)$ is a graded left ideal of S. We often simply identify the underlying k-spaces of these ideals and call both I.

Since the equivalence of categories preserves Hilbert functions and the property of being cyclic, it is clear that the point modules over S are the modules of the form $\theta(U/\mathfrak{m}_d) = S/\mathfrak{m}_d$ for $d \in \mathbb{P}^t$. We will use the following notation:

Notation 3.1. Given a point $d \in \mathbb{P}^t$, let $P(d)$ be the left point module $\theta(U/\mathfrak{m}_d) = S/\mathfrak{m}_d$ of S.

We may also describe point modules over S by their *point sequences*. If M is a point module over S, then the annihilator of M_n in $S_1 = U_1$ is some codimension 1 subspace which corresponds to a point $d_n \in \mathbb{P}^t$. The point sequence of M is defined to be the sequence $(d_0, d_1, d_2, \dots)$ of points of $\mathbb{P}^t$. Clearly, two S-point modules are isomorphic if and only if they have the same point sequence.

Lemma 3.2. *Let d be an arbitrary point of $\mathbb{P}^t$.*

(1) *$P(d)$ has point sequence $(d, \varphi(d), \varphi^2(d), \dots)$.*
(2) *$(P(d))_{\geqslant n} \cong P(\varphi^n(d))[-n]$ as S-modules.*

Proof. (1) By definition $P(d) = S/\mathfrak{m}_d$. If $f \in S_1$, then $fS_n = \phi^n(f) \circ U_n \subseteq \mathfrak{m}_d$ if and only if $\phi^n(f) \in \mathfrak{m}_d$, in other words $f \in \phi^{-n}(\mathfrak{m}_d) = \mathfrak{m}_{\varphi^n(d)}$.

(2) By part (1), $(P(d))_{\geqslant n}$ is the shift by $(-n)$ of the point module with point sequence $(\varphi^n(d), \varphi^{n+1}(d), \dots)$. $\square$

Finally, we record the following simple facts which we shall use frequently.

Lemma 3.3. *Let $M \in S$-gr.*

(1) *If M is cyclic 1-critical, then $M \cong P(d)[i]$ for some $d \in \mathbb{P}^t$ and $i \in \mathbb{Z}$.*
(2) *M has a finite filtration with factors which are graded cyclic critical S-modules.*

Proof. (1) The equivalence of categories U-Gr $\sim$ S-Gr preserves the GK-dimension of finitely generated modules, since it preserves Hilbert functions, and so it also preserves the property of being GK-critical. It is standard that the cyclic 1-critical graded U-modules are just the point modules over U and their shifts. Under the equivalence of categories, the corresponding S-modules are the S-point modules and their shifts.

(2) Each module $N \in U$-gr has a finite filtration composed of graded cyclic critical U-modules, so the same holds for S-modules by the equivalence of categories. $\square$

4. The algebras $R(\varphi, c)$

Let $S = S(\varphi)$ for some $\varphi \in \operatorname{Aut} \mathbb{P}^t$. For any codimension-1 subspace V of $S_1 = U_1$, we let $R = k\langle V \rangle \subseteq S$ be the subalgebra of S generated by V. The vector subspace V of U_1 corresponds to a unique point $c \in \mathbb{P}^t$. Then R is determined up to isomorphism by the geometric data (φ, c) and we write $R = R(\varphi, c)$. We emphasize again that we always assume that $t \geqslant 2$ from now on; for smaller t the ring R is not very interesting.

We shall see that the basic properties of $R(\varphi, c)$ depend closely on properties of the iterates of the point c under φ. It is convenient to let $c_i = \varphi^{-i}(c)$ for all $i \in \mathbb{Z}$. Then the ideal of the point c_i is $\phi^i(\mathfrak{m}_c)$. In case c has finite order under φ, that is $\varphi^n(c) = c$ for some $n > 0$, the algebra $R(\varphi, c)$ behaves quite differently from the case where c has infinite order. For example, if $\varphi^n = 1$ for some $n \geqslant 0$ then it is easy to see that S and hence R are *PI* rings. The finite order case turns out to have none of the interesting properties of the infinite order case, and so in this paper we will only consider the case where c has infinite order under φ.

Standing Hypothesis 4.1. Assume that $(\varphi, c) \in (\operatorname{Aut} \mathbb{P}^t) \times \mathbb{P}^t$ is given such that c has infinite order under φ, or equivalently the points $\{c_i\}_{i \in \mathbb{Z}}$ are distinct.

We note some relationships among the various $R(\varphi, c)$. In particular, part (1) of the next lemma will allow us to transfer our left sided results to the right.

Lemma 4.2. *Let* $(\varphi, c) \in (\operatorname{Aut} \mathbb{P}^t) \times \mathbb{P}^t$, *and let* ψ *be any automorphism of* $\mathbb{P}^t$. *Then*

(1) $R(\varphi, c)^{op} \cong R(\varphi^{-1}, \varphi(c))$.
(2) $R(\varphi, c) \cong R(\psi\varphi\psi^{-1}, \psi(c))$.

Proof. (1) Set $S = S(\varphi)$ and $S' = S(\varphi^{-1})$, identifying the underlying spaces of each with that of U. Let ϕ be an automorphism of U corresponding to φ. Then it is straightforward to check that the vector space map defined on the graded pieces of U by sending $f \in U_m$ to $\phi^{-m}(f) \in U_m$ is a graded algebra isomorphism from S^{op} to S'. The isomorphism maps $(\mathfrak{m}_c)_1$ to $(\mathfrak{m}_{\varphi(c)})_1$ and so it restricts to an isomorphism $R(\varphi, c)^{op} \cong R(\varphi^{-1}, \varphi(c))$.

(2) Similarly, let σ be an automorphism of U corresponding to ψ. It is easy to check that the vector space map of U defined by $f \mapsto \sigma^{-1}(f)$ is an isomorphism of $S(\varphi)$ onto $S(\psi\varphi\psi^{-1})$ which maps $(\mathfrak{m}_c)_1$ to $\sigma^{-1}(\mathfrak{m}_c)_1 = (\mathfrak{m}_{\psi(c)})_1$, and so restricts to an isomorphism $R(\varphi, c) \cong R(\psi\varphi\psi^{-1}, \psi(c))$. $\square$

In the next theorem we will prove an important characterization of the elements of $R = R(\varphi, c)$ which is foundational for all that follows. First we need the following lemma; the proofs of it and several other technical commutative results which appear later in the paper may be found in the appendix.

Lemma 4.3 (Lemma A.8). *Let $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$ be the ideals of U corresponding to distinct points $d_1, \dots, d_n$ in $\mathbb{P}^t$. Then $(\mathfrak{m}_1 \circ \mathfrak{m}_2 \circ \cdots \circ \mathfrak{m}_n)_{\geqslant n} = (\bigcap_{i=1}^n \mathfrak{m}_i)_{\geqslant n}$.*

Theorem 4.4. *Let $R = R(\varphi, c)$. Then for all $n \geqslant 0$,*

$$R_n = \{ f \in U_n \mid f(c_i) = 0 \text{ for } 0 \leqslant i \leqslant n-1 \}.$$

Proof. By definition $R = k\langle V \rangle \subseteq S$, where $V = (\mathfrak{m}_c)_1$ considered as a subset of U. For $n = 0$ the statement of the theorem is $R_0 = U_0 = k$, which is clearly correct, so assume that $n \geqslant 1$. Then

$$R_n = V^n = \phi^{n-1}(V) \circ \phi^{n-2}(V) \circ \cdots \circ \phi(V) \circ V.$$

Now $\phi^i(V) = (\mathfrak{m}_{c_i})_1$, and the points c_i are distinct by Hypothesis 4.1. Thus by Lemma 4.3 we get that

$$\begin{aligned} R_n &= (\mathfrak{m}_{c_{n-1}})_1 \circ \cdots \circ (\mathfrak{m}_{c_1})_1 \circ (\mathfrak{m}_{c_0})_1 = [(\mathfrak{m}_{c_{n-1}}) \circ \cdots \circ (\mathfrak{m}_{c_1}) \circ (\mathfrak{m}_{c_0})]_n \\ &= [(\mathfrak{m}_{c_{n-1}}) \cap \cdots \cap (\mathfrak{m}_{c_1}) \cap (\mathfrak{m}_{c_0})]_n. \end{aligned}$$

The statement of the theorem in degree n follows. $\square$

The theorem has a number of easy consequences. The first will be a simple calculation of the Hilbert function of R, which depends on the following fundamental commutative result which is proved in the appendix.

Lemma 4.5 (Lemma A.9). *Let $d_1, d_2, \dots, d_n$ be distinct points in $\mathbb{P}^t$, for some $n \geqslant 1$, and let $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$ be the corresponding graded ideals of U. Let $e_i > 0$ for all $1 \leqslant i \leqslant n$. Set $J = \bigcap_{i=1}^n \mathfrak{m}_i^{e_i}$. Then $\dim_k J_m = \binom{m+t}{t} - \sum_i \binom{e_i+t-1}{t}$ for all $m \geqslant (\sum e_i) - 1$. In particular, if $J = \bigcap_{i=1}^n \mathfrak{m}_i$ then $\dim_k J_m = \binom{m+t}{t} - n$ for $m \geqslant n-1$.*

Lemma 4.6. *Let $R = R(\varphi, c)$. Then $\dim_k R_n = \binom{n+t}{t} - n$ for all $n \geqslant 0$. In particular, $\mathrm{GK}(R) = t+1$.*

Proof. The Hilbert function of R follows from Theorem 4.4 and Lemma 4.5. Since we always assume that $t \geqslant 2$, it is clear that the Hilbert polynomial of R has degree t and so $\mathrm{GK}(R) = t+1$. $\square$

Lemma 4.7. *The rings $R = R(\varphi, c)$ and $S = S(\varphi)$ have the same graded quotient ring D and Goldie quotient ring Q. The inclusion $R \hookrightarrow S$ is a essential extension of left (or right) R-modules.*

Proof. Since both R and S are domains of finite GK-dimension, they both have graded quotient rings and Goldie quotient rings [18, C.I.1.6], [15, 4.12]. Clearly, the graded quotient ring D' of R is contained in the graded quotient ring D of S. Since we assume always that $t \geqslant 2$, we may choose a nonzero polynomial $g \in S_1$ with $g \in \mathfrak{m}_{c_0} \cap \mathfrak{m}_{c_1}$. Then Theorem 4.4 implies that $g \in R_1$ and $S_1 g \subseteq R_2$. Thus $S_1 \subseteq R_2(R_1)^{-1} \subseteq D'$ and consequently $D' = D$. Then Q, the Goldie quotient ring of the domain D, is also the Goldie quotient ring for both R and S. The last statement of the proposition is now clear. $\square$

5. The noetherian property for R

Let $S = S(\varphi)$ and $R = R(\varphi, c)$. In this section we will characterize those choices of φ and c satisfying Hypothesis 4.1 for which the ring R is noetherian. To do this, we will first analyze the structure of the factor module ${}_R(S/R)$ in detail, and then use this information to understand contractions and extensions of left ideals between R and S.

The following notation will be convenient in this section.

Notation 5.1. (1) $A_n = \{0, 1, \ldots, n-1\}$ for $n > 0$ and $A_n = \emptyset$ for $n \leqslant 0$.
(2) For $B \subseteq \mathbb{Z}$, set $B + m = \{b + m \mid b \in B\}$.

Definition 5.2. Let $B \subseteq \mathbb{N}$. We define a left R-module $T^B \subseteq S$ by specifying its graded pieces as follows:

$$(T^B)_n = \{f \in S_n \mid f(c_i) = 0 \text{ for } i \in A_n \backslash B\}.$$

We then define the left R-module $M^B = T^B/R \subseteq (S/R)$.

We should check that T^B really is closed under left multiplication by R. If $g \in R_m$ and $f \in (T^B)_n$, then $gf = \phi^n(g) \circ f$. Now $g(c_i) = 0$ for $i \in A_m$ by Theorem 4.4 and $f(c_i) = 0$ for $i \in A_n \backslash B$ by definition. Thus $[\phi^n(g) \circ f](c_i) = 0$ for $i \in (A_m + n) \cup (A_n \backslash B) \supseteq (A_{n+m} \backslash B)$, and so $gf \in (T^B)_{n+m}$ as required. Also, by Theorem 4.4 the extreme cases are $R = T^{\emptyset}$ and $S = T^{\mathbb{N}}$. In particular, $R \subseteq T^B$ always holds, and so M^B is well defined.

Lemma 5.3. *The Hilbert function of M^B is given by*

$$\dim_k (M^B)_n = |A_n \cap B|.$$

Proof. Immediate from Lemma 4.5. $\square$

In the special case of Definition 5.2 where B is a singleton set, M^B is just a shifted R-point module, as follows.

Lemma 5.4. *Let $j \in \mathbb{N}$. Then $M = M^{\{j\}}$ is an R-point module shifted by $j+1$. In fact, $M \cong {}_R(P(c_{-1}))[-j-1]$.*

Proof. By Lemma 5.3 the Hilbert function of M is

$$\dim_k M_n = \begin{cases} 0, & 0 \leqslant n \leqslant j, \\ 1, & j+1 \leqslant n \end{cases}$$

so that M does have the Hilbert function of a point module shifted by $j+1$. For convenience of notation set $m = j+1$, and let us calculate $\mathrm{ann}_R(M_m)$. Now $fM_m = 0$ for $f \in R_n$ if and only if $f(T^{\{j\}})_m \subseteq R_{m+n}$. Since $M_m \neq 0$, we may choose $g \in (T^{\{j\}})_m$ such that $g \notin R$; then $g(c_j) \neq 0$. Also, because $\dim_k M_m = 1$ it is clear that $(T^{\{j\}})_m = R_m + kg$, and so $f(T^{\{j\}})_m \subseteq R$ if and only if $fg \in R$. Now $fg = \phi^m(f) \circ g$, and so by Theorem 4.4 we have that $fg \in R$ if and only if $\phi^m(f)(c_j) = 0$, equivalently $f(c_{-1}) = 0$, since $m = j+1$. In conclusion, $\mathrm{ann}_R(M_m) = \mathfrak{m}_{c_{-1}} \cap R$.

Thus we have an injection of R-modules given by right multiplication by g:

$$\psi : (R/(\mathfrak{m}_{c_{-1}} \cap R))[-m] \xrightarrow{g} T^{\{j\}}/R = M.$$

By Lemma 4.5, $R/(\mathfrak{m}_{c_{-1}} \cap R)$ has the Hilbert function of a point module and so both sides have the same Hilbert function. Thus ψ is actually an isomorphism. In particular, M is cyclic and so is a shifted R-point module.

We also have the injection $R/(\mathfrak{m}_{c_{-1}} \cap R) \to S/(\mathfrak{m}_{c_{-1}}) = P(c_{-1})$, and since both sides have the Hilbert function of a point module this is also an isomorphism of R-modules. So $M \cong {}_R(P(c_{-1}))[-j-1]$. $\square$

We may now understand the structure of ${}_R(S/R)$ completely.

Proposition 5.5. *The modules $\{M^{\{j\}}\}_{j \in \mathbb{N}}$ are independent submodules of S/R. Also, for $B \subseteq \mathbb{N}$,*

$$M^B = \bigoplus_{j \in B} M^{\{j\}}.$$

Proof. We first show the independence of the $M^{\{j\}}$. It is enough to work with homogeneous elements; fix $n \geqslant 0$ and let $\sum_{j \in \mathbb{N}} f_j = 0$ for some $f_j \in (M^{\{j\}})_n$. Let $f_j = g_j + R$ for some elements $g_j \in (T^{\{j\}})_n \subseteq S_n$. Thus $\sum g_j \in R_n$. Suppose that some $f_j \neq 0$, and let $k = \min\{j \mid f_j \neq 0\}$. Now $(M^{\{j\}})_{\leqslant j} = 0$ for all j by Lemma 5.3, and so we must have $k \leqslant n-1$. Then $k \in A_n \backslash \{j\}$ for any $j \neq k$, so that $g_j(c_k) = 0$ for all $j \neq k$, by the definition of $T^{\{j\}}$. Since $\sum g_j \in R_n$, Theorem 4.4 implies that $(\sum g_j)(c_k) = 0$ also holds and so $g_k(c_k) = 0$. But then $g_k \in (T^{\{k\}})_n \cap (\mathfrak{m}_{c_k}) = R_n$, and thus $f_k = 0$, a contradiction. We conclude that all $f_j = 0$, and so the $M^{\{j\}}$ are independent.

For the second statement of the proposition, by Lemma 5.3 the Hilbert function of M^B is $\dim_k(M^B)_n = |A_n \cap B|$, while the Hilbert function of $\bigoplus_{j\in B} M^{\{j\}}$ is

$$\dim_k \left[\bigoplus_{j\in B} M^{\{j\}} \right]_n = \#\{j\in B \mid j\leqslant n-1\} = |A_n \cap B|.$$

Thus the Hilbert functions are the same on both sides of our claimed equality. Since $\sum_{j\in B} M^{\{j\}} \subseteq M^B$ is clear and we know that the $M^{\{j\}}$ are independent by the first part of the proposition, the equality follows. $\square$

Corollary 5.6. (1) *Given $B\subseteq \mathbb{N}$, M^B is a noetherian R-module if and only if the set B has finite cardinality.*

(2) ${}_R(S/R) \cong \bigoplus_{j=0}^{\infty} {}_R P(c_{-1})[-j-1]$. *In particular, ${}_R(S/R)$ is not finitely generated.*

Proof. Condition (1) is clear since a point module is noetherian. Condition (2) follows by taking $B = \mathbb{N}$ in the proposition and using also Lemma 5.4. $\square$

Next, we analyze the noetherian property for some special types of R-modules which may be realized as subfactors of S/R.

Proposition 5.7. *For $f\in R_n$, let $N = (Sf \cap R)/Rf \in R$-Gr. Set $D = \{i\in\mathbb{N} \mid f(c_i) = 0\}$ and $B = (D-n)\cap\mathbb{N}$. Then*

(1) $N \cong M^B[-n]$

(2) *N is noetherian if and only if $|D| < \infty$.*

Proof. First, if we set $T = \{g\in S \mid gf\in R\}$ and $M = T/R$, then $N\cong M[-n]$. So it is enough for (1) to show that $T = T^B$.

Let $g\in S_m$ be arbitrary. Note that $A_n \subseteq D$ since $f\in R_n$. Then

$$\begin{aligned} gf &= \phi^n(g)\circ f \in R \\ &\Leftrightarrow [\phi^n(g)\circ f](c_i) = 0 \quad \text{for all } i\in A_{n+m} \\ &\Leftrightarrow \phi^n(g)(c_i) = 0 \quad \text{for all } i\in A_{n+m}\backslash D \\ &\Leftrightarrow g(c_i) = 0 \quad \text{for all } i\in (A_{n+m}\backslash D) - n \\ &\Leftrightarrow g(c_i) = 0 \quad \text{for all } i\in A_m\backslash (D-n) \quad (\text{since } A_n\subseteq D) \\ &\Leftrightarrow g\in T^B \quad \text{by Definition 5.2.} \end{aligned}$$

Thus $T = T^B$ and (1) holds.

For (2), note that D has finite cardinality if and only if B does, and apply Corollary 5.6(1). $\square$

Proposition 5.8. *For $f \in R_n$, let $M = S/(R + Sf) \in R\text{-Gr}$. Set $D = \{i \in \mathbb{N} \mid f(c_i) = 0\}$. Then*

(1) $M \cong M^D$,
(2) *M is noetherian if and only if $|D| < \infty$.*

Proof. Set $B = \mathbb{N} \backslash D$. We will show that $R + Sf = T^B$. Then we will have that $S/(R + Sf) = S/T^B \cong (M^{\mathbb{N}}/M^B) \cong M^D$ by Proposition 5.5.

Suppose that $h \in (R + Sf)_m$; then $h = g_1 + g_2 f = g_1 + \phi^n(g_2) \circ f$ for some $g_1 \in R_m$ and $g_2 \in S_{m-n}$. Now $g_1(c_i) = 0$ for $i \in A_m$, and $f(c_i) = 0$ for $i \in D$, so that $h(c_i) = 0$ for $i \in A_m \cap D$. So we have $(R + Sf) \subseteq T^B$. Note that $(R + Sf)_m = R_m = (T^B)_m$ for $m < n$.

Now, we can calculate the Hilbert function of $R + Sf$. By Propositions 5.7(1) and Lemma 5.3, we have that $\dim_k((Sf \cap R)/Rf)_m = |A_{m-n} \cap (D - n)| = |A_m \cap D| - n$ for $m \geqslant n$, since $A_n \subseteq D$. Since the Hilbert functions of R, Sf, and Rf are all known, one may calculate that $\dim_k(R + Sf)_m = \binom{m+2}{2} - |A_m \cap D|$ for $m \geqslant n$, which is equal to $\dim_k(T^B)_m$. Thus $R + Sf = T^B$.

Part (2) is then immediate from Corollary 5.6(1). $\square$

Given a left ideal I of R, we may extend to a left ideal SI of S, and then contract back down to get the left ideal $SI \cap R$ of R. The factor $(SI \cap R)/I$ is built up out of the 2 types of modules we considered in Propositions 5.7 and 5.8.

Lemma 5.9. *Let I be a finitely generated nonzero graded left ideal of R, and set $M = (SI \cap R)/I$. Then M has a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$ such that each factor M_{i+1}/M_i is isomorphic with shift to a subfactor of either $(Ss_i \cap R)/Rs_i$ or $S/(R + Ss_i)$ for some nonzero homogeneous $s_i \in R$.*

Proof. Let $I = \sum_{i=1}^{n} Rr_i$ for some homogeneous $r_i \in R$. If $n = 1$ the result is obvious, so assume that $n \geqslant 2$.

Set $J = \sum_{i=1}^{n-1} Rr_i$. By induction on n, $(SJ \cap R)/J$ and hence also its surjective image $(SJ \cap R) + I/I$ have filtrations of the required type. It is enough then to show that

$$N = (SI \cap R)/((SJ \cap R) + I) = (SI \cap R)/((SJ + Rr_n) \cap R)$$

has the required filtration. But N injects into $L = SI/(SJ + Rr_n)$. Now R is an Ore domain by Lemma 4.7, so we may choose a homogeneous element $0 \neq r \in R$ such that $rr_n \in J$. Then L is a surjective image (with shift) of $S/(R + Sr)$, so N is a shift of a subfactor of $S/(R + Sr)$. $\square$

In certain circumstances the noetherian property passes to subrings. The following lemma is just a slight variant of a number of similar results in the literature (for example, see [1, Lemma 4.2]).

Lemma 5.10. *Let $A \hookrightarrow B$ be any extension of $\mathbb{N}$-graded rings. Suppose that B is left noetherian, and that $(BI \cap A)/I$ is a noetherian left A-module for all finitely generated homogeneous left ideals I of A. Then A is left noetherian.*

Proof. It is enough to prove that A is graded left noetherian, that is that all homogeneous left ideals are finitely generated. Let I be a homogeneous left ideal of A. Then BI is a homogeneous left ideal of B, which is finitely generated since B is noetherian, and so we may pick a finite set of homogeneous generators $r_1, r_2, \dots, r_n \in I$ such that $BI = \sum_{i=1}^{n} Br_i$. Let $J = \sum_{i=1}^{n} Ar_i$. Then $BI = BJ$, and since J is finitely generated over A we may apply the hypothesis to conclude that $(BJ \cap A)/J = (BI \cap A)/J$ is a noetherian A-module. The submodule I/J of $(BI \cap A)/J$ is then noetherian over A, in particular finitely generated over A. Finally, since J is finitely generated over A, so is I. $\square$

We note the definition of an unusual geometric condition on a set of points of a variety, which appeared in [1, p. 582].

Definition 5.11. Let $\mathscr{C}$ be an infinite set of (closed) points of a variety X. We say $\mathscr{C}$ is *critically dense* in X if every proper Zariski-closed subset $Y \subsetneq X$ contains only finitely many points of $\mathscr{C}$.

We may now prove our main result characterizing the noetherian property for R.

Theorem 5.12. *Let $R = R(\varphi, c)$ for some $(\varphi, c) \in (\operatorname{Aut} \mathbb{P}^t) \times \mathbb{P}^t$ such that Hypothesis* 4.1 *holds. As always, set $c_i = \varphi^{-i}(c)$. Then*

(1) *$R(\varphi, c)$ is left noetherian if and only the set $\{c_i\}_{i \geqslant 0}$ is critically dense in $\mathbb{P}^t$.*
(2) *$R(\varphi, c)$ is right noetherian if and only the set $\{c_i\}_{i \leqslant -1}$ is critically dense* in $\mathbb{P}^t$.
(3) *$R(\varphi, c)$ is noetherian if and only the set $\{c_i\}_{i \in \mathbb{Z}}$ is critically dense in $\mathbb{P}^t$.*

Proof. (1) Set $\mathscr{C} = \{c_i\}_{i \geqslant 0}$ and suppose that $\mathscr{C}$ is critically dense. Then for any nonzero homogeneous polynomial $f \in R$, the set $D = \{i \in \mathbb{N} \mid f(c_i) = 0\}$ has finite cardinality, so by Propositions 5.7 and 5.8 the left R-modules $(Sf \cap R)/Rf$ and $S/(R + Sf)$ are noetherian. By Lemma 5.9, for any finitely generated homogeneous left ideal I of R, the left R-module $(SI \cap R)/I$ is noetherian. By Lemma 5.10, R is a left noetherian ring.

Conversely, if $\mathscr{C}$ fails to be critically dense, then we may choose a nonzero homogeneous polynomial $h \in S$ which vanishes at infinitely many points of $\mathscr{C}$. Since by Lemma 4.7 we know that $R \hookrightarrow S$ is an essential extension of R-modules, there exists a homogeneous $g \in R$ such that $0 \neq f = gh \in R$. Then f also vanishes at infinitely many points of $\mathscr{C}$, and so by Proposition 5.7, the left R-module $(Sf \cap R)/Rf$ is not noetherian. Since this module is a subfactor of R, we conclude that R is not a left noetherian ring.

(2) Using Lemma 4.2(1), this part follows immediately from part (1).

(3) This follows from the fact that for any infinite sets $\mathscr{C}_1, \mathscr{C}_2 \subseteq \mathbb{P}^t$, $\mathscr{C}_1 \cup \mathscr{C}_2$ is critically dense if and only if both $\mathscr{C}_1$ and $\mathscr{C}_2$ are. $\square$

In Section 12 we will examine the critical density condition appearing in Theorem 5.12 more closely. In particular, we shall prove that there exist many choices of φ and c for which $R(\varphi, c)$ is noetherian:

Proposition 5.13 (See Theorem 12.3 below). *Let φ be the automorphism of $\mathbb{P}^t$ defined by $(a_0 : a_1 : \cdots : a_t) \mapsto (a_0 : p_1 a_1 : p_2 a_2 : \cdots : p_t a_t)$, and let c be the point $(1 : 1 : \cdots : 1) \in \mathbb{P}^t$. If the scalars $\{p_1, p_2, \dots, p_t\}$ are algebraically independent over the prime subfield of k, then $\{\varphi^i(c)\}_{i \in \mathbb{Z}}$ is critically dense and $R(\varphi, c)$ is noetherian.* $\square$

The noetherian case is our main interest, so in the remainder of the paper (except Section 12) we will assume the following hypothesis.

Standing Hypothesis 5.14. Let $c_i = \varphi^{-i}(c)$. Assume that φ and c are chosen such that the point set $\{c_i\}_{i \in \mathbb{Z}}$ is critically dense in $\mathbb{P}^t$, so that $R(\varphi, c)$ is noetherian. We will refer to this as the *critical density condition*.

Below, we will frequently use the following exact sequence to study an arbitrary cyclic left R-module R/I:

$$0 \to (SI \cap R)/I \to R/I \to S/SI \to S/(R + SI) \to 0. \tag{5.15}$$

We note for future reference what the results of this section tell us about the terms of this sequence.

Lemma 5.16. *Assume the critical density condition, and let $0 \neq I$ be a graded left ideal of R.*

1. *As left R-modules, $(SI \cap R)/I$ and $S/(R + SI)$ have finite filtrations with factors which are either torsion or a tail of the shifted R-point module ${}_R(P(c_{-1}))[-i]$ for some $i \geqslant 0$. In particular, $S/(R + SI)$ is a noetherian left R-module.*
2. *${}_R(S/J)$ is a noetherian module for all nonzero left ideals J of S.*

Proof. (1) Let $0 \neq f \in R$ be arbitrary. Since $\{c_i\}_{i \in \mathbb{Z}}$ is a critically dense set, $f(c_i) = 0$ holds for only finitely many $i \in \mathbb{Z}$. Then by the results 5.4–5.8, the left R-modules $(Sf \cap R)/Rf$ and $S/(R + Sf)$ are isomorphic to finite direct sums of shifted point modules of the form ${}_R(P(c_{-1}))[-i]$ for various $i \geqslant 0$. Now using Lemma 5.9, it is clear that $(SI \cap R)/I$ has a filtration of the right kind. Similarly, $S/(R + SI)$ is a homomorphic image of $S/(R + Sf)$ for any $0 \neq f \in I$, so it also has the required filtration and is clearly noetherian.

(2) It is immediate from the exact sequence (5.15) for $I = Rr$ and part (1) that ${}_R(S/Sr)$ is noetherian for any homogeneous $0 \neq r \in R$. It is enough to show that ${}_R(S/Sx)$ is noetherian for an arbitrary homogeneous $0 \neq x \in S$. There is some

nonzero homogeneous $y \in R$ such that $yx \in R$, by Lemma 4.7. Then since (S/Syx) is a noetherian R-module, so is S/Sx. $\square$

6. Point modules and the strong noetherian property

Let $S = S(\varphi)$ and $R = R(\varphi, c)$ for (φ, c) satisfying the critical density condition, so that R is noetherian. Recall the definition of the strong noetherian property:

Definition 6.1. A k-algebra A is called *strongly* (*left*) *noetherian* if $A \otimes_k B$ is a left noetherian ring for all commutative noetherian k-algebras B.

Artin and Zhang showed that the point modules for a strongly noetherian algebra have a nice geometric structure. The following is a special case of their theorem.

Theorem 6.2 (Artin and Zhang [6, Corollaries E4.11, E4.12]). *Let A be a connected $\mathbb{N}$-graded strongly noetherian algebra over an algebraically closed field k.*

(1) *The point modules over A are naturally parameterized by a commutative projective scheme over k.*
(2) *There is some $d \geqslant 0$ such that every point module M for A is uniquely determined by its truncation $M/M_{\geqslant d}$.* $\square$

Using an explicit presentation for the ring, Jordan [14] classified the point modules for the algebra R in a special case. In this section, we classify the point modules for the rings $R(\varphi, c)$ in general using a different method which does not rely on relations and get a similar result. The classification will show that part (2) of Theorem 6.2 fails for R, and so R cannot be strongly noetherian.

In this section we will make frequent use of the criterion for R-membership given in Theorem 4.4 without comment. Also, recall that $\circ$ indicates multiplication in the polynomial ring U, and juxtaposition indicates multiplication in S (or R). Some of the results in this section will rely on the following technical commutative lemma which is proved in the appendix.

Lemma 6.3 (Lemma A.10). *Let the points $d_1, d_2, \ldots, d_n, d_{n+1} \in \mathbb{P}^t$ be distinct, and assume that the points $d_1, \ldots, d_n$ do not all lie on a line. Let $\mathfrak{m}_i \subseteq U$ be the homogeneous ideal corresponding to d_i.*

(1) $(\bigcap_{i=1}^{n} \mathfrak{m}_i)_{n-1} \circ (\mathfrak{m}_{n+1})_1 = (\bigcap_{i=1}^{n+1} \mathfrak{m}_i)_n$.
(2) $(\bigcap_{i=1}^{n} \mathfrak{m}_i)_{n-1} \circ (\mathfrak{m}_1)_1 = (\bigcap_{i=2}^{n} \mathfrak{m}_i \cap \mathfrak{m}_1^2)_n$.
(3) $(\bigcap_{i=2}^{n} \mathfrak{m}_i \cap \mathfrak{m}_1^2)_n \circ (\mathfrak{m}_{n+1})_1 = (\bigcap_{i=2}^{n+1} \mathfrak{m}_i \cap \mathfrak{m}_1^2)_{n+1}$.
(4) *Let $b_1, b_2 \in \mathbb{P}^t$, with corresponding ideals $\mathfrak{n}_1, \mathfrak{n}_2$, be such that $b_j \neq d_i$ for $j = 1, 2$ and $1 \leqslant i \leqslant n$. Then $(\bigcap_{i=1}^{n} \mathfrak{m}_i \cap \mathfrak{n}_1)_n = (\bigcap_{i=1}^{n} \mathfrak{m}_i \cap \mathfrak{n}_2)_n$ implies $b_1 = b_2$.*

We have already seen that the point modules over S are easily classified up to isomorphism—they are simply the $\{P(d) \mid d \in \mathbb{P}^t\}$ (recall Notation 3.1). There is a close relationship between the point modules over the rings S and R, as we begin to see in the next proposition.

Proposition 6.4. (1) *Let M be a point module over R. Then $M_{\geqslant n} \cong {}_R(P_{\geqslant n})$ for some S-point module P and some $n \geqslant 0$.*

(2) *If ${}_RP(d_1)_{\geqslant n} \cong {}_RP(d_2)_{\geqslant n}$ for $d_1, d_2 \in \mathbb{P}^t$ and some $n \geqslant 0$, then $d_1 = d_2$.*

Proof. (1) We have $M = R/I$ for a unique point ideal I of R. We will use the exact sequence (5.15); there are two cases.

Suppose first that $(SI \cap R)/I = 0$. Then we have an injection $R/I \to S/SI$. By Lemma 5.16(1) we know that $\mathrm{GK}_R(S/(R + SI)) \leqslant 1$, and clearly $\mathrm{GK}_R(R/I) = 1$, so that $\mathrm{GK}_R(S/SI) = 1$ since GK-dimension is exact for modules over the graded noetherian ring R. By Lemma 5.16(2), ${}_R(S/SI)$ is finitely generated, and so $\mathrm{GK}_S(S/SI) = \mathrm{GK}_R(S/SI) = 1$ since for finitely generated modules the GK-dimension depends only on the Hilbert function. Now may choose a filtration of S/SI composed of cyclic critical S-modules, where the factors must be shifts of S-point modules and ${}_Sk$ (Lemma 3.3). Since M is a R-submodule of S/SI, this forces some tail of M to agree with a tail of an S-point module.

Suppose instead that $N = (SI \cap R)/I \neq 0$. Then N is a nonzero submodule of the point module M, so it is equal to a tail of M. By Lemma 5.16(1), some tail of N, and thus a tail of M, must be isomorphic as an R-module to a tail of some $P(c_{-1})[-i] \cong P(c_{-1-i})_{\geqslant i}$ (using also Lemma 3.2).

(2) By Lemma 3.2, we have for any $d \in \mathbb{P}^t$ that $P(d)_{\geqslant n} \cong P(\varphi^n(d))[-n]$ as S-modules. Thus we may reduce to the case where $n = 0$.

Since $\mathrm{ann}_S\, P(d_i)_0 = \mathfrak{m}_{d_i}$, we must have $\mathfrak{m}_{d_1} \cap R = \mathfrak{m}_{d_2} \cap R$. In degree m this means

$$(\mathfrak{m}_{c_0} \cap \cdots \cap \mathfrak{m}_{c_{m-1}} \cap \mathfrak{m}_{d_1})_m = (\mathfrak{m}_{c_0} \cap \cdots \cap \mathfrak{m}_{c_{m-1}} \cap \mathfrak{m}_{d_2})_m. \tag{6.5}$$

Suppose first that $d_1, d_2 \notin \{c_i\}_{i \in \mathbb{N}}$. Since the point set $\{c_i\}_{i \in \mathbb{Z}}$ is critically dense, it follows that for $m \gg 0$ the points $\{c_i\}_{i=0}^{m-1}$ do not all lie on a line. Then by Lemma 6.3(4), Eq. (6.5) for $m \gg 0$ implies that $d_1 = d_2$.

Otherwise we may assume, without loss of generality, that $d_1 = c_j$ for some $j \geqslant 0$ and that $d_2 \notin \{c_i\}_{i=0}^{j-1}$. Then the equation (6.5) for $m = j + 1$ violates Lemma 4.5 unless $d_1 = c_j = d_2$. $\square$

We may now classify the point modules over the ring $R(\varphi, c)$.

Theorem 6.6. *Assume the critical density condition* (*Hypothesis* 5.14).

(1) *For any point $d \in \mathbb{P}^t \backslash \{c_i\}_{i \geqslant 0}$, the S-point module $P(d)$ is an R-point module, with point ideal $(R \cap \mathfrak{m}_d)$.*

(2) *For each* $i \geqslant 0$, *the* S*-module* $P(c_i)_{\geqslant i+1}$ *is a shifted* R*-point module. There is a* $\mathbb{P}^{t-1}$*-parameterized family of nonisomorphic* R*-point modules* $\{P(c_i, e) \mid e \in \mathbb{P}^{t-1}\}$ *with* $P(c_i, e)_{\geqslant i+1} \cong {}_R P(c_i)_{\geqslant i+1}$ *and* $P(c_i, e)_{\leqslant i} \cong {}_R P(c_i)_{\leqslant i}$ for any $e \in \mathbb{P}^{t-1}$. *These are exactly the point modules whose point ideals contain the left ideal* $(R \cap \mathfrak{m}_{c_i}^2)$ *of* R.

(3) *All of the point modules given in parts* (1) *and* (2) *above are nonisomorphic, and every point module over* $R(\varphi, c)$ *is isomorphic to one of these.*

Proof. Suppose that $d \in \mathbb{P}^t$, so $P(d) = S/\mathfrak{m}_d$ by definition. For any $i \geqslant 0$,

$$R_1(P(d))_i = 0 \;\Leftrightarrow\; R_1 S_i \subseteq \mathfrak{m}_d \;\Leftrightarrow\; (\mathfrak{m}_{c_i})_1 \circ U_i \subseteq \mathfrak{m}_d \;\Leftrightarrow\; d = c_i. \tag{6.7}$$

(1) Let $d \notin \{c_i\}_{i \geqslant 0}$. In this case it is clear from (6.7) that $P(d)$ is already an R-point module. Also, the corresponding point ideal is $\operatorname{ann}_R P(d)_0 = R \cap \mathfrak{m}_d$.

(2) Fix some $i \geqslant 0$. From (6.7) it is clear that $M = {}_R P(c_i) = M_{\leqslant i} \oplus M_{\geqslant i+1}$ where $M_{\leqslant i}$ is the torsion submodule of M and $M_{\geqslant i+1}$ is a shifted R-point module.

We define a left ideal $J = J^{(i)}$ of R by setting $J_{\leqslant i} = (R \cap \mathfrak{m}_{c_i})_{\leqslant i}$ and $J_{\geqslant i+1} = (R \cap \mathfrak{m}_{c_i}^2)_{\geqslant i+1}$. To check that J really is a left ideal of R, one calculates

$$R_1 J_i = \phi^i(\mathfrak{m}_{c_0})_1 \circ J_i = (\mathfrak{m}_{c_i})_1 \circ (R_i \cap \mathfrak{m}_{c_i})_i \subseteq R_{i+1} \cap \mathfrak{m}_{c_i}^2 = J_{i+1}.$$

We will now classify the point ideals of R which contain J. By Lemma 4.5, the Hilbert function of R/J must be

$$\dim_k (R/J)_n = \begin{cases} 1, & n \leqslant i, \\ t, & n \geqslant i+1. \end{cases}$$

Then using Lemma 4.5 again, the natural injection

$$(R/J)_{\geqslant i+1} = \frac{(\mathfrak{m}_{c_0} \cap \mathfrak{m}_{c_1} \cap \cdots \cap \mathfrak{m}_{c_i})_{\geqslant i+1}}{(\mathfrak{m}_{c_0} \cap \mathfrak{m}_{c_1} \cap \cdots \cap \mathfrak{m}_{c_i}^2)_{\geqslant i+1}} \hookrightarrow (\mathfrak{m}_{c_i}/\mathfrak{m}_{c_i}^2)_{\geqslant i+1}$$

is an isomorphism of left R-modules, since the Hilbert functions on both sides are the same.

As a module over the polynomial ring U, we have an isomorphism

$$(\mathfrak{m}_{c_i}/\mathfrak{m}_{c_i}^2)_{\geqslant i+1} \cong \bigoplus_{j=1}^{t} (U/\mathfrak{m}_{c_i})_{\geqslant i+1}$$

which by the equivalence of categories $U\text{-Gr} \sim S\text{-Gr}$ translates to an S-isomorphism as follows:

$${}_S(\mathfrak{m}_{c_i}/\mathfrak{m}_{c_i}^2)_{\geqslant i+1} \cong \bigoplus_{j=1}^{t} P(c_i)_{\geqslant i+1}.$$

By part (1), $P(c_i)_{\geqslant i+1} \cong P(c_{-1})[-i-1]$ is a shifted R-point module, so we conclude that $M = (R/J)_{\geqslant i+1}$ is a direct sum of t isomorphic shifted R-point modules. Then every choice of a codimension-one vector subspace $V = L/(J_{i+1})$ of $(R/J)_{i+1}$ generates a different R-submodule N of M with $M/N \cong_R P(c_i)_{\geqslant i+1}$, and then $J + RL$ is a point ideal for R. Clearly any point ideal containing J must arise in this way, and the set of codimension-one subspaces of $(R/J)_{i+1}$ is parameterized by $\mathbb{P}^{t-1}$. Thus, the set of point ideals of R which contain J is naturally parameterized by a copy of $\mathbb{P}^{t-1}$.

For each $e \in \mathbb{P}^{t-1}$, we have a corresponding point ideal I containing J and we set $P(c_i, e) = R/I$. Then $P(c_i, e)_{\geqslant i+1} \cong_R P(c_i)_{\geqslant i+1}$ and $P(c_i, e)_{\leqslant i} \cong (R/J)_{\leqslant i} \cong_R P(c_i)_{\leqslant i}$.

Finally, note that all of the point ideals constructed above contain $(R \cap \mathfrak{m}_{c_i}^2)$. Conversely, if I is any point ideal which contains $(R \cap \mathfrak{m}_{c_i}^2)$, then I contains J, since $J/(R \cap \mathfrak{m}_{c_i}^2)$ is torsion and I, being a point ideal, is closed under extensions inside R by torsion modules. Then I is one of the point ideals we already constructed. This finishes the proof of part (2).

(3) Note that for fixed i the $P(c_i, e)$ are nonisomorphic for distinct e by construction; then it follows easily from Proposition 6.4(2) that all of the point modules we have constructed in parts (1) and (2) are non-isomorphic.

Suppose that M is an R-point module. Let $\mathscr{P}$ be the set of all R-modules isomorphic to a shift of one of the R-point modules constructed in parts (1) and (2) above. By Proposition 6.4, $M_{\geqslant n} \cong_R P(d)_{\geqslant n}$ for some $n \geqslant 0$ and $d \in \mathbb{P}^t$. For $m \gg 0$, note that $\varphi^m(d) \notin \{c_i\}_{i \geqslant 0}$ and so $M_{\geqslant m+n} \in \mathscr{P}$ by part (1) above. Thus to finish the proof of (3) it is enough by induction to show that given any R-point module N, if $N_{\geqslant 1} \in \mathscr{P}$ then $N \in \mathscr{P}$.

Let N be an R-point module such that $N_{\geqslant 1} \in \mathscr{P}$. Let $I = \operatorname{ann}_R N_0$ be the point ideal of N. There are a number of cases.

Case 1: Suppose first that $N_{\geqslant 1} \cong_R P(d)[-1]$ for some $d \notin \{c_i\}_{i \geqslant -1}$. Then $(R \cap \mathfrak{m}_d)R_1 \subseteq I$, in other words

$$(\mathfrak{m}_{\varphi^{-1}(d)} \cap \mathfrak{m}_{c_1} \cap \mathfrak{m}_{c_2} \cap \cdots \cap \mathfrak{m}_{c_n})_n \circ (\mathfrak{m}_{c_0})_1 \subseteq I_{n+1}$$

for each $n \geqslant 0$. By the critical density condition, for $n \gg 0$ the points $\{c_1, \ldots, c_n\}$ will not lie on a line, so that Lemma 6.3(1) applies and gives

$$(\mathfrak{m}_{\varphi^{-1}(d)} \cap \mathfrak{m}_{c_0} \cap \mathfrak{m}_{c_1} \cap \mathfrak{m}_{c_2} \cap \cdots \cap \mathfrak{m}_{c_n})_{n+1} \subseteq I_{n+1}.$$

In other words, $(R \cap \mathfrak{m}_{\varphi^{-1}(d)})_{\geqslant m} \subseteq I$ for $m \gg 0$. Note that $\varphi^{-1}(d) \notin \{c_i\}_{i \geqslant 0}$, so that $(R \cap \mathfrak{m}_{\varphi^{-1}(d)})$ is one of the point ideals appearing in part (1) above. Since I is a point ideal and is thus closed under extensions inside R by torsion modules, $(R \cap \mathfrak{m}_{\varphi^{-1}(d)}) \subseteq I$ and so comparing Hilbert functions, $(R \cap \mathfrak{m}_{\varphi^{-1}(d)}) = I$. Thus $N \cong_R (P(\varphi^{-1}(d)) \in \mathscr{P}$.

Case 2: If $N_{\geqslant 1} \cong_R P(c_{-1})$, then $(R \cap \mathfrak{m}_{c_{-1}})R_1 \subseteq I$; using Lemma 6.3(2) and a similar argument to that in case 1, this implies that $(R \cap \mathfrak{m}_{c_0}^2)_{\geqslant m} \subseteq I$ for $m \gg 0$. Then since I is a point ideal, $(R \cap \mathfrak{m}_{c_0}^2) \subseteq I$. By part (2) above this forces $N \cong P(c_0, e)$ for some $e \in \mathbb{P}^{t-1}$, and so $N \in \mathscr{P}$.

Case 3: Let $N_{\geqslant 1} \cong P(c_i, e)[-1]$ for some $i \geqslant 0$ and $e \in \mathbb{P}^{t-1}$. Then $(R \cap \mathfrak{m}_{c_i}^2)R_1 \subseteq I$. Now the same argument as in the other cases, except using Lemma 6.3(3), will show that $N \cong P(c_{i+1}, e') \in \mathscr{P}$ for some $e' \in \mathbb{P}^{t-1}$. $\square$

The failure of the strong noetherian property for $R = R(\varphi, c)$ now follows immediately from Theorem 6.6(2). This proves Theorem 1.2 from the introduction.

Theorem 6.8. *Assume the critical density condition. Then $R = R(\varphi, c)$ is a connected graded noetherian algebra, finitely generated in degree* 1, *which is noetherian but not strongly noetherian.*

Proof. We only need to prove that R is not strongly noetherian. For each $i \geqslant 0$, Theorem 6.6(2) provides a whole $\mathbb{P}^{t-1}$ of point modules $P(c_i, e)$ which have isomorphic truncations $P(c_i, e)_{\leqslant i}$. By Theorem 6.2(2), R cannot be a strongly noetherian k-algebra. $\square$

We remark that the point modules over R still appear to have an interesting geometric structure. By Theorem 6.6, there is a single point module corresponding to each point $d \in \mathbb{P}^t \backslash \{c_i\}_{i \geqslant 0}$ and a $\mathbb{P}^{t-1}$-parameterized family of exceptional point modules corresponding to each point c_i with $i \geqslant 0$. Since blowing up $\mathbb{P}^t$ at a point in some sense replaces that point by a copy of $\mathbb{P}^{t-1}$, the intuitive picture of the geometry of the point modules for R is an infinite blowup of projective space at a countable point set.

7. Extending the base ring

Let $S = S(\varphi)$ and $R = R(\varphi, c)$ for (φ, c) satisfying the critical density condition, and let $c_i = \varphi^{-i}(c) \in \mathbb{P}_k^t$ as usual. We now know by Theorem 6.8 that R is not strongly noetherian, but this proof is quite indirect and it is not obvious which choice of extension ring B makes $R \otimes_k B$ non-noetherian. In this section we construct such a noetherian commutative k-algebra B which is even a UFD.

Let B be an arbitrary commutative k-algebra which is a domain. We will use subscripts to indicate extension of the base ring, so for example we write $R_B = R \otimes_k B$. The automorphism ϕ of U extends uniquely to an automorphism of U_B fixing B, which we also call ϕ. We continue to identify the underlying B-module of S_B with that of U_B, and we use juxtaposition for multiplication in S_B and the symbol $\circ$ for multiplication in U_B, as in our current convention (see Section 3). The multiplication of S_B is still given by $fg = \phi^n(f) \circ g$ for $f \in (S_B)_m$, $g \in (S_B)_n$; in other words, S_B is the left Zhang twist of U_B by the twisting system $\{\phi^i\}_{i \in \mathbb{N}}$, just as before.

Let d be a point in $\mathbb{P}_k^t$. Since the homogeneous coordinates for d are defined only up to a scalar multiple in $k^\times$, given $f \in U_B$ the expression $f(d)$ is defined up to a nonzero element of k; we will use this notation only in contexts where the ambiguity does not matter. For example, the condition $f(d) = 0$ makes sense and is equivalent to the condition $f \in \mathfrak{m}_d \circ U_B$, where $\mathfrak{m}_d \subseteq U$ and $\mathfrak{m}_d \circ U_B$ is a graded prime ideal of U_B.

The natural analog of Theorem 4.4 still holds in this setting:

Proposition 7.1. *For all* $n \geqslant 0$, *we have*

$$(R_B)_n = \{f \in (U_B)_n \text{ such that } f(c_i) = 0 \text{ for } 0 \leqslant i \leqslant n-1\}.$$

Proof. As subsets of U_B, using Theorem 4.4 we have

$$(R_B)_n = R_n \otimes B = \left(\bigcap_{i=0}^{n-1} \mathfrak{m}_{c_i} \right)_n \otimes B = \bigcap_{i=0}^{n-1} (\mathfrak{m}_{c_i} \circ U_B)_n$$

and the proposition follows. $\square$

We now give sufficient conditions on B for the ring $R \otimes_k B$ to fail to have the left noetherian property.

Proposition 7.2. *Assume that B is a UFD. Suppose that there exist nonzero homogeneous elements $f, g \in (U_B)_1$ satisfying the following conditions*:

(1) *$f(c_i)$ divides $g(c_i)$ in B for all $i \geqslant 0$.*
(2) *For all $i \gg 0$, $f(c_i)$ is not a unit of B.*
(3) $\gcd(f, g) = 1$ *in* U_B.

Then $R \otimes_k B$ is not a left noetherian ring.

Proof. Note that $U_B \cong B[x_0, x_1, \dots, x_t]$ is a UFD, since B is, so condition (3) makes sense.

For convenience, fix some homogeneous coordinates for the c_i. For each $n \geqslant 0$, we may choose a polynomial $\theta_n \in S_n$ with coefficients in k such that $\theta_n(c_i) = 0$ for $-1 \leqslant i \leqslant n-2$ and $\theta_n(c_{n-1}) \neq 0$. This is possible, for example, by Lemma 4.5. By hypothesis (1), for each $n \geqslant 0$ we may write $\Omega_n = g(c_n)/f(c_n) \in B$. Now let $t_n = \theta_n(\Omega_n f - g) \in (S_B)_{n+1}$ for each $n \geqslant 0$.

Since $\phi(\theta_n)$ vanishes at c_i for $0 \leqslant i \leqslant n-1$ and $[\Omega_n f - g](c_n) = 0$, the element $t_n = \phi(\theta_n) \circ (\Omega_n f - g)$ is in $(R_B)_{n+1}$, by Proposition 7.1. We will show that for $n \gg 0$ we have $t_{n+1} \notin \sum_{i=0}^n (R_B) t_i$, which will imply that R_B is not left noetherian.

Suppose that $t_{n+1} = \sum_{i=0}^n r_i t_i$ for some $r_i \in (R_B)_{n+1-i}$. Writing out the explicit expressions for the t_i, this is

$$\theta_{n+1}(\Omega_{n+1} f - g) = \sum_{i=0}^{n} r_i \theta_i (\Omega_i f - g).$$

Considering these expressions in U_B, after some rearrangement we obtain (since f, g have degree 1)

$$\phi\left[\theta_{n+1}\Omega_{n+1} - \sum_{i=0}^{n} r_i\theta_i\Omega_i\right]\circ f + \phi\left[-\theta_{n+1} + \sum_{i=0}^{n} r_i\theta_i\right]\circ g = 0.$$

Now by hypothesis (3), g must divide the polynomial

$$h = \phi\left[\theta_{n+1}\Omega_{n+1} - \sum_{i=0}^{n} r_i\theta_i\Omega_i\right] = \Omega_{n+1}\phi(\theta_{n+1}) - \sum_{i=0}^{n} \Omega_i\phi^{i+1}(r_i)\circ\phi(\theta_i).$$

We note that $[\phi(\theta_{n+1})](c_{n+1}) \in k^\times$ by the definition of the θ_i, and $[\phi^{i+1}(r_i)](c_{n+1}) = 0$ for $0 \leqslant i \leqslant n$, since $r_i \in (R_B)_{n-i+1}$. Thus evaluating at c_{n+1} we conclude that $h(c_{n+1}) \in \Omega_{n+1}k^\times$ and thus $g(c_{n+1})$ divides Ω_{n+1}. But since $\Omega_{n+1} = g(c_{n+1})/f(c_{n+1})$, this implies that $f(c_{n+1})$ is a unit in B. For all $n \gg 0$, this contradicts hypothesis (2), and so $t_{n+1} \notin \sum_{i=0}^{n}(R_B)t_i$ for $n \gg 0$, as we wished to show. □

Next, we construct a commutative noetherian ring B which satisfies the hypotheses of Proposition 7.2. We shall obtain such a ring as an infinite blowup of affine space, to be defined presently. See [1, Section 1] for more details about this construction.

Let A be a commutative domain, and let X be the affine scheme Spec A. Suppose that d is a closed nonsingular point of X with corresponding maximal ideal $\mathfrak{p} \subseteq A$, and let $z_0, z_1, \ldots, z_r$ be some choice of generators of the ideal $\mathfrak{p}$ such that $z_0 \notin \mathfrak{p}^2$. The *affine blowup* of X at d (with denominator z_0) is $X' = \operatorname{Spec} A'$ where $A' = A[z_1z_0^{-1}, z_2z_0^{-1}, \ldots, z_rz_0^{-1}]$.

Consider the special case where $A = k[y_1, y_2, \ldots, y_t]$ is a polynomial ring, $X = \mathbb{A}^t$, and $d = (a_1, a_2, \ldots, a_t)$. The affine blowup of $\mathbb{A}^t$ at d with the denominator $(y_1 - a_1)$ is $X' = \operatorname{Spec} A'$ for the ring

$$A' = A[(y_2 - a_2)(y_1 - a_1)^{-1}, \ldots, (y_t - a_t)(y_1 - a_1)^{-1}].$$

Note that also $A' = k[y_1, (y_2 - a_2)(y_1 - a_1)^{-1}, \ldots, (y_t - a_t)(y_1 - a_1)^{-1}]$, so A' is itself isomorphic to a polynomial ring in t variables over k and $X' = \mathbb{A}^t$ as well. The blowup map $X' \to X$ is an isomorphism outside of the closed set $\{y_1 = a_1\}$ of X.

Given a sequence of points $\{d_i = (a_{i1}, a_{i2}, \ldots, a_{it})\}_{i \geqslant 0}$ such that $a_{i1} \neq a_{j1}$ for $i \neq j$, we may iterate the blowup construction, producing a union of commutative domains

$$A \subseteq A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots,$$

where each A_i is isomorphic to a polynomial ring in t variables over k. In this case we set $B = \bigcup A_i$ and $Y = \operatorname{Spec} B$, and call Y (or B) the *infinite blowup* of $\mathbb{A}^t$ at the sequence of points $\{d_i\}$. Explicitly, $B = A[\{(y_j - a_{ij})(y_1 - a_{i1})^{-1} \mid 2 \leqslant j \leqslant t,\ i \geqslant 0\}]$.

That there should be some connection between such infinite blowups and the algebras $R(\varphi, c)$ is strongly suggested by the following result (cf. Theorem 5.12).

Theorem 7.3 (Artin et al. [1, Theorem 1.5]). *The infinite blowup B is a noetherian ring if and only if the set of points $\{d_i\}_{i \geqslant 0}$ is a critically dense subset of $\mathbb{A}^t$.*

Now we show the failure of the strong noetherian property for the noetherian rings $R(\varphi, c)$ explicitly. Below, we will identify automorphisms of $\mathbb{P}^t$ with elements of $\mathrm{PGL}_{t+1}(k) = \mathrm{GL}_{t+1}(k)/k^{\times}$ [13, p. 151]; explicitly, we let matrices in $\mathrm{GL}_{t+1}(k)$ act on the left on the homogeneous coordinates $(a_0 : a_1 : \cdots : a_t)$ of $\mathbb{P}^t$, considered as column vectors.

Theorem 7.4. *Let $(\varphi, c) \in (\mathrm{Aut}\, \mathbb{P}^t_k) \times \mathbb{P}^t_k$ satisfy the critical density condition. There is an affine patch $\mathbb{A}^t \subseteq \mathbb{P}^t$ such that $\{c_i\}_{i \in \mathbb{Z}} \subseteq \mathbb{A}^t$. Let B be the infinite blowup of $\mathbb{A}^t$ at the points $\{c_i\}_{i \geqslant 0}$. Then $R = R(\varphi, c)$ is noetherian, but B is a commutative noetherian k-algebra which is a UFD such that $R \otimes_k B$ is not a left noetherian ring.*

Proof. By changing coordinates, we may replace φ by a conjugate without loss of generality, so we may assume that when represented as a matrix φ is lower triangular. Also, we may multiply this matrix by a nonzero scalar without changing the automorphism of $\mathbb{P}^t$ it represents, and so we also assume that the top left entry of the matrix is 1.

By assumption the set of points $\{c_i\}_{i \in \mathbb{Z}}$ is critically dense in $\mathbb{P}^t$, and so $R(\varphi, c)$ is noetherian. Let X_0 be the hyperplane $\{x_0 = 0\}$ of $\mathbb{P}^t$. Since φ is upper triangular, $\varphi(X_0) = X_0$, so if some $c_i \in X_0$ then $\{c_i\}_{i \in \mathbb{Z}} \subseteq X_0$ which contradicts the critical density condition. So certainly $\{c_i\}_{i \in \mathbb{Z}} \subseteq \mathbb{A}^t = \mathbb{P}^t \backslash X_0$. Since the top left entry of φ is 1, we may fix homogeneous coordinates for the c_i of the form $c_i = (1 : a_{i1} : a_{i2} : \cdots : a_{it})$. Let $y_i = x_i/x_0$, so that $k[y_1, y_2, \ldots, y_t]$ is the coordinate ring of $\mathbb{A}^t$. In affine coordinates, $c_i = (a_{i1}, a_{i2}, \ldots, a_{it})$.

If $a_{i1} = a_{j1}$ for some $i < j$, then since φ is lower triangular it follows that $a_{i1} = a_{k1}$ for all $k \in (j - i)\mathbb{Z}$. Then the hyperplane $\{a_{i1}x_0 - x_1 = 0\}$ of $\mathbb{P}^t$ contains infinitely many of the c_i, again contradicting the critical density of $\{c_i\}_{i \in \mathbb{Z}}$. So the scalars $\{a_{i1}\}_{i \in \mathbb{Z}}$ are all distinct, and the infinite blowup B of $\mathbb{A}^t$ at the points $\{c_i\}_{i \geqslant 0}$ is well defined. The ring B is generated over $k[y_1, y_2, \ldots, y_t]$ by the elements $\{(y_j - a_{ij})(y_1 - a_{i1})^{-1} \mid 2 \leqslant j \leqslant t,\ i \geqslant 0\}$. Clearly the points $\{c_i\}_{i \geqslant 0}$ must be critically dense subset of $\mathbb{A}^t$, since they are a critically dense subset of $\mathbb{P}^t$. Thus B is noetherian by Theorem 7.3.

The ring B is obtained as a directed union of k-algebras A_i which are each isomorphic to a polynomial ring. In each ring A_i the group of units is just $k^{\times}$, and so this is also the group of units of B. It follows that if $z \in A_i$ is an irreducible element of B, then z is irreducible in A_i. Since B is noetherian, every element of B is a finite product of irreducibles, and the uniqueness of such a decomposition follows by the uniqueness in each UFD A_i. Thus B is a UFD.

Fix the two elements $f = y_1x_0 - x_1$ and $g = y_2x_0 - x_2$ of $U_B \cong B[x_0, x_1, \ldots, x_t]$. Since f and g are homogeneous of degree 1 in the x_i and are not divisible by any non-unit of B, it is clear that f and g are distinct irreducible elements of U_B, and so in particular $\gcd(f,g)=1$. Now $f(c_i)=y_1-a_{i1}$ and $g(c_i)=y_2-a_{i2}$, so $\Omega_i = g(c_i)/f(c_i) = (y_2-a_{i2})(y_1-a_{i1})^{-1} \in B$ and thus $f(c_i)$ divides $g(c_i)$ for all $i \geqslant 0$.

Finally, $f(c_i)=(y_1-a_{i1})$ is not in the group of units $k^\times$ of B. We see that all of the hypotheses of Proposition 7.2 are satisfied, and so $R \otimes_k B$ is not left noetherian. □

8. Special subcategories and homological lemmas

Let $S=S(\varphi)$ and $R=R(\varphi,c)$, and assume the critical density condition (Hypothesis 5.14), in particular that R is noetherian. First, we introduce some notation for the subcategories of S-Gr and R-Gr which are generated by the "distinguished" S-point modules $P(c_i)$.

Definition 8.1. (1) Let S-dist be the full subcategory of S-gr consisting of all S-modules M with a finite S-module filtration whose factors are either torsion or a shift of $P(c_i)$ for some $i \in \mathbb{Z}$.

(2) Let R-dist be the full subcategory of R-gr consisting of all R-modules M having a finite R-module filtration whose factors are either torsion or a shift of the module ${}_RP(c_i)$ for some $i \in \mathbb{Z}$.

Note that by Theorem 6.6(1) and (2), ${}_RP(c_i)$ is finitely generated for any $i \in \mathbb{Z}$ and so part (2) of the definition makes sense. We also define S-Dist to be the smallest full subcategory of S-Gr containing S-dist and closed under direct limits. The subcategory R-Dist of R-Gr is defined similarly. We will use frequently later in this section the fact that $S/R \in R$-Dist, which follows from Corollary 5.6.

If $\mathscr{C}$ is any abelian category, a full subcategory $\mathscr{D}$ of $\mathscr{C}$ is called *Serre* if for any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{C}$, $M \in \mathscr{D}$ if and only if both $M' \in \mathscr{D}$ and $M'' \in \mathscr{D}$. All of the subcategories defined above are clearly Serre. In fact, we may describe S-dist as the smallest Serre subcategory of S-gr which contains all of the $P(c_i)[j]$; a similar description holds for the other categories.

The special categories over the two rings are related as follows.

Lemma 8.2. *Let* $M \in S$-gr. *Then* ${}_RM \in R$-dist *if and only if* ${}_SM \in S$-dist. *Similarly, if* $M \in S$-Gr *then* ${}_RM \in R$-Dist *if and only if* ${}_SM \in S$-Dist.

Proof. If ${}_SM \in S$-dist, then it follows directly from Definition 8.1 that ${}_RM \in R$-dist. Conversely, suppose that ${}_RM \in R$-dist. Clearly $\mathrm{GK}_R(M) \leqslant 1$, so we have $\mathrm{GK}_S(M) \leqslant 1$ since we can measure GK-dimension using the Hilbert function. By Lemma 3.3, M has a finite filtration over S with cyclic critical factors, which must in this case be shifts of ${}_Sk$ and S-point modules. Suppose that a shift of $P(d)$ is one of the factors occurring. Then $N = {}_RP(d) \in R$-dist. By the definition of R-dist, some tail of N is isomorphic to a shift of some ${}_RP(c_i)$ for some $i \in \mathbb{Z}$. Using Lemma 3.2, this forces ${}_RP(d) \cong {}_RP(c_j)$ for some $j \in \mathbb{Z}$ and so by Proposition 6.4(2) we have $d = c_j$. Thus the only point modules which may occur as factors in the S-filtration of M are shifts of the $P(c_j)$ for $j \in \mathbb{Z}$ and so $M \in S$-dist.

The second statement is an easy consequence of the definitions of S-Dist and R-Dist and the first statement. $\square$

In the rest of this section, we gather some definitions and lemmas concerning homological algebra over the rings R and S. Let A be a connected $\mathbb{N}$-graded k-algebra, finitely generated in degree 1, and let $k=(A/A_{\geqslant 1})$. We say that A satisfies χ_i if $\dim_k \underline{\mathrm{Ext}}^j(k,M)<\infty$ for all $M\in A$-gr and all $0\leqslant j\leqslant i$, and that A satisfies χ if A satisfies χ_i for all $i\geqslant 0$. If $M\in A$-gr, the *grade* of M is the number $j(M)=\min\{i\,|\,\underline{\mathrm{Ext}}^i_A(M,A)\neq 0\}$. We say that A is *Cohen–Macaulay* if $j(M)+\mathrm{GK}(M)=\mathrm{GK}(A)$ for all $M\in A$-gr. Finally, A is *Artin–Schelter regular* (or *AS-regular*) if A has finite global dimension d, finite GK-dimension, and satisfies the *Gorenstein condition*: $\underline{\mathrm{Ext}}^i_A({}_Ak,A)=0$ if $i\neq d$, and $\underline{\mathrm{Ext}}^d_A({}_Ak,A)\cong k_A$ (up to some shift of grading).

The ring S obtains many nice homological properties simply because it is a Zhang twist of a commutative polynomial ring.

Lemma 8.3. (1) *S has global dimension $t+1$.*
(2) *S is Cohen–Macaulay.*
(3) *S is Artin–Schelter regular.*
(4) *S satisfies χ.*

Proof. All of these properties are standard for the polynomial ring U. Properties (1)–(3) follow for the Zhang twist S of U by Zhang [31, Propositions 5.7, 5.11]. Then since S is Artin–Schelter regular it satisfies χ [5, Theorem 8.1]. $\square$

Recall from Section 3 that there is an equivalence of categories $\theta: U\text{-Gr}\sim S\text{-Gr}$, and that we identify the graded left ideals of S and the graded ideals of U. Under the equivalence of categories we have $\theta(U/I)\cong S/I$. Also, graded injective objects correspond under the equivalence and so it is immediate that $\mathrm{Ext}^i_{U\text{-Gr}}(M,N)\cong \mathrm{Ext}^i_{S\text{-Gr}}(\theta M,\theta N)$ as vector spaces for all $M,N\in U$-Gr and all $i\geqslant 0$. On the other hand, the relationship between $\underline{\mathrm{Ext}}$ groups over the two rings is more complicated, since the shift functors in S and U do not correspond under the equivalence of categories. We work out this relationship in detail for cyclic modules, which is the only case we will need.

Lemma 8.4. *Let $M=U/I$ and $N=U/J$ for some graded ideals I and J of U. For any $n\in\mathbb{Z}$ we have*

(1) $\theta((U/\phi^{-n}(J))[n])\cong (S/J)[n]$.
(2) $\underline{\mathrm{Ext}}^i_S(S/I,S/J)_n\cong \underline{\mathrm{Ext}}^i_U(U/I,U/\phi^{-n}(J))_n$ *as k-spaces.*

Proof. (1) Obviously, the property of being cyclic is preserved by the equivalence of categories, and so $\theta((U/\phi^{-n}(J))[n])$ is a cyclic S-module generated in degree $-n$. Thus we need only show that the annihilator in S of a generator is the left S-ideal J, but this is immediate from the definition of the equivalence θ.

(2) By definition, $\underline{\mathrm{Ext}}^i_U(U/I, U/\phi^{-n}(J))_n = \mathrm{Ext}^i_{U\text{-Gr}}(U/I, (U/\phi^{-n}(J))[n])$ and $\underline{\mathrm{Ext}}^i_S(S/I, S/J)_n = \mathrm{Ext}^i_{S\text{-Gr}}(S/I, (S/J)[n])$. We know that $\theta(U/I) \cong S/I$, and $\theta((U/\phi^{-n}(J))[n]) \cong (S/J)[n]$ by part (1). We are done since the Ext groups in the graded category correspond under the equivalence of categories. $\square$

The next proposition shows that the critical density of the set $\{c_i\}_{i\in\mathbb{Z}}$, besides characterizing the noetherian property for R, also has implications for the homological properties of the S-point modules $P(c_i)$. The proof of the following commutative lemma may be found in the appendix.

Lemma 8.5 (Lemma A.12). *Let I and J be homogeneous ideals of U. There is some $d \geqslant 0$ such that for all $n \in \mathbb{Z}$ for which $U/(I + \phi^n(J))$ is bounded, $\underline{\mathrm{Ext}}^i_U(U/I, U/\phi^n(J))$ has right bound $\leqslant d$.* $\square$

Proposition 8.6. *Assume the critical density condition, and let $N \in S$-*gr.

(1) $\dim_k \underline{\mathrm{Ext}}^p_S(P(c_i), N) < \infty$ *for $0 \leqslant p \leqslant t-1$ and any $i \in \mathbb{Z}$.*
(2) *Let $M \in S$-*dist. *Then* $\dim_k \underline{\mathrm{Ext}}^p_S(M, N) < \infty$ *for $0 \leqslant p \leqslant t-1$.*

Proof. (1) Since N is finitely generated, it is easy to see that each graded piece of $E = \underline{\mathrm{Ext}}^p_S(P(c_i), N)$ is finite dimensional over k. So it is enough to show that E is bounded. Note that E is automatically left bounded since N is [5, Proposition 3.1.1(c)]. It remains to show that E is right bounded. Using a finite filtration of N by cyclic modules, one reduces quickly to the case where N is cyclic, say $N = S/I$.

In case $I = 0$, $E = \underline{\mathrm{Ext}}^p_S(P(c_i), S) = 0$ for $0 \leqslant p \leqslant t-1$ by the Cohen–Macaulay property of S (Lemma 8.3(2)).

Now assume that $I \neq 0$. By Lemma 8.4(3), we have for each $n \geqslant 0$ the k-space isomorphism

$$\underline{\mathrm{Ext}}^p_S(S/\mathfrak{m}_{c_i}, S/I)_n \cong \underline{\mathrm{Ext}}^p_U(U/\mathfrak{m}_{c_i}, U/\phi^{-n}(I))_n.$$

Now $\phi^{-n}(I) \subseteq \mathfrak{m}_{c_i}$, or equivalently $I \subseteq \mathfrak{m}_{c_{i+n}}$, can hold for at most finitely many n, since the points $\{c_i\}_{i\in\mathbb{Z}}$ are critically dense. Thus for $n \gg 0$ we have $\phi^{-n}(I) \nsubseteq \mathfrak{m}_{c_i}$, and the module $U/(\phi^{-n}(I) + \mathfrak{m}_{c_i})$ is bounded. By Lemma 8.5, there is some fixed $d \geqslant 0$ such that $\underline{\mathrm{Ext}}^p_U(U/\mathfrak{m}_{c_i}, U/\phi^{-n}(I))_n = 0$ as long as $n \geqslant d$. We conclude that $\underline{\mathrm{Ext}}^p_S(S/\mathfrak{m}_{c_i}, S/I)_n = 0$ for $n \gg 0$, as we wish.

(2) Since $M \in S$-dist, we may choose a finite filtration of M with factors which are shifts of the point modules $P(c_i)$ or ${}_Sk$. Since S satisfies χ by Lemma 8.3(4), $\dim_k \underline{\mathrm{Ext}}^p_S(k, N) < \infty$ for all $p \geqslant 0$, and now the statement follows by part (1). $\square$

For an $\mathbb{N}$-graded algebra A, if L and N are $\mathbb{Z}$-graded right and left A-modules, respectively, then the k-space $\mathrm{Tor}^A_i(L, N)$ has a natural $\mathbb{Z}$-grading which we emphasize by using the notation $\underline{\mathrm{Tor}}^A_i(L, N)$. To study homological algebra over R, we will generally try to reduce to calculations over the ring S. In particular, we will often use the following convergent spectral sequence, which is valid for any graded modules ${}_RM$ and ${}_SN$ [25, Eq. (2.2)]:

$$\underline{\mathrm{Ext}}^p_S(\underline{\mathrm{Tor}}^R_q(S,M),N) \underset{p}{\Rightarrow} \underline{\mathrm{Ext}}^{p+q}_R(M,N). \tag{8.7}$$

We also note for reference the 5-term exact sequence arising from this spectral sequence [21, 11.2]:

$$\begin{aligned} 0 &\to \underline{\mathrm{Ext}}^1_S(S\otimes_R M,N) \to \underline{\mathrm{Ext}}^1_R(M,N) \to \underline{\mathrm{Hom}}_S(\underline{\mathrm{Tor}}^R_1(S,M),N) \\ &\to \underline{\mathrm{Ext}}^2_S(S\otimes_R M,N) \to \underline{\mathrm{Ext}}^2_R(M,N). \end{aligned} \tag{8.8}$$

In order to make effective use of the spectral sequence, we need some information about $\underline{\mathrm{Tor}}$.

Lemma 8.9. *Fix some* $M\in R$-gr. *Also, let* Q *be the right point module of* R *such that* $(S/R)_R \cong \bigoplus_{i=1}^\infty Q[-i]$ (Corollary 5.6(2), *applied to the right side, which is valid by Lemma* 4.2(1)). *Then*

(1) $\underline{\mathrm{Tor}}^R_q(S,M)\in S$-dist *for any* $q\geqslant 1$. *If* M *is torsion, then* $\underline{\mathrm{Tor}}^R_q(S,M)\in S$-dist *for* $q\geqslant 0$.
(2) $\dim_k \underline{\mathrm{Tor}}^R_q(Q,M)<\infty$ *for* $q\geqslant 1$.

Proof. (1) From the long exact sequence in $\underline{\mathrm{Tor}}^R_q(-,M)$ associated to the short exact sequence of R-bimodules $0\to R\to S\to S/R\to 0$, we see that

$$\underline{\mathrm{Tor}}^R_q(S,M)\cong \underline{\mathrm{Tor}}^R_q(S/R,M) \tag{8.10}$$

as left R-modules, for all $q\geqslant 2$. Also, at the bottom of the long exact sequence we have

$$0\to \underline{\mathrm{Tor}}^R_1(S,M)\to \underline{\mathrm{Tor}}^R_1(S/R,M)\to M\to \dots \tag{8.11}$$

Thus there is at least an injection of left R-modules $\underline{\mathrm{Tor}}^R_q(S,M)\to \underline{\mathrm{Tor}}^R_q(S/R,M)$ for all $q\geqslant 1$. Now computing $N=\underline{\mathrm{Tor}}^R_q(S/R,M)$ using a free resolution of M, it is a subfactor of some direct sum of copies of $(S/R)\in R$-Dist, so $N\in R$-Dist. Then $\underline{\mathrm{Tor}}^R_q(S,M)\in R$-Dist and thus in S-Dist for $q\geqslant 1$, using Lemma 8.2. But since we may calculate $\underline{\mathrm{Tor}}^R_q(S,M)$ using a resolution of M by free modules of finite rank, we have $\underline{\mathrm{Tor}}^R_q(S,M)\in S$-dist for $q\geqslant 1$.

If M is torsion, we need to show in addition that $\underline{\mathrm{Tor}}^R_0(S,M)$ is in S-dist. It is enough to show this for $M=k$, in which case one calculates immediately that $\underline{\mathrm{Tor}}^R_0(S,M)\cong S/SR_{\geqslant 1}\cong S/\mathfrak{m}_{c_0}=P(c_0)$ which is obviously in S-dist.

(2) As in part (1), $N=\underline{\mathrm{Tor}}^R_q(S/R,M)\in R$-Dist and $\underline{\mathrm{Tor}}^R_q(S,M)$ is in S-dist and thus in R-dist, by Lemma 8.2, for all $q\geqslant 1$. Since $M\in R$-gr, we get using (8.10) and (8.11) that $N\in R$-gr and so $N\in R$-dist for all $q\geqslant 1$.

Note that by the definition of R-dist, the Hilbert function of N is forced to satisfy $\dim_k N_m < C$ for some constant C, all $m \geqslant 0$. Then since $\underline{\mathrm{Tor}}$ commutes with direct sums [5, Proposition 2.4(1)], $N \cong \oplus_{i=1}^{\infty} \underline{\mathrm{Tor}}_q(Q, M)[-i]$ as graded vector spaces, and so we must have $\dim_k \underline{\mathrm{Tor}}_q^R(Q, M) < \infty$ for $q \geqslant 1$. $\square$

As an easy consequence of the spectral sequence, we may show that R and S have no nontrivial extensions by torsion modules in the category of R-modules.

Lemma 8.12. $\underline{\mathrm{Ext}}^1_R({}_Rk, R) = 0 = \underline{\mathrm{Ext}}^1_R({}_Rk, S)$.

Proof. Consider the long exact sequence in $\underline{\mathrm{Ext}}_R(k, -)$ associated to the short exact sequence $0 \to R \to S \to S/R \to 0$:

$$\ldots \to \underline{\mathrm{Hom}}_R(k, S/R) \to \underline{\mathrm{Ext}}^1_R(k, R) \to \underline{\mathrm{Ext}}^1_R(k, S) \to \ldots \ . \tag{8.13}$$

Now ${}_R(S/R)$ is torsionfree, since it is isomorphic to a direct sum of point modules by Corollary 5.6(2). Thus $\underline{\mathrm{Hom}}_R(k, S/R) = 0$.

To analyze the group $\underline{\mathrm{Ext}}^1_R(k, S)$, we use the beginning of the 5-term exact sequence (8.8) for $M = {}_Rk$ and $N = S$:

$$0 \to \underline{\mathrm{Ext}}^1_S(S \otimes {}_Rk, S) \to \underline{\mathrm{Ext}}^1_R(k, S) \to \underline{\mathrm{Hom}}_S(\underline{\mathrm{Tor}}^R_1(S, k), S) \to \ldots \ . \tag{8.14}$$

Now by Lemma 8.9(i), $\underline{\mathrm{Tor}}^R_i(S, k)$ is in S-dist for all $i \geqslant 0$; in particular, $\mathrm{GK}_S(S \otimes {}_Rk) \leqslant 1$ and $\mathrm{GK}_S(\underline{\mathrm{Tor}}^R_1(S, k)) \leqslant 1$. Then $\underline{\mathrm{Ext}}^1_S(S \otimes {}_Rk, S) = 0$ by the Cohen–Macaulay property of S (Lemma 8.3(2)) and $\underline{\mathrm{Hom}}_S(\underline{\mathrm{Tor}}^R_1(S, k), S) = 0$ since S is a domain with $\mathrm{GK}(S) = t + 1 > 1$. Thus by (8.14) $\underline{\mathrm{Ext}}^1_R(k, S) = 0$, and by (8.13) $\underline{\mathrm{Ext}}^1_R(k, R) = 0$ as well. $\square$

9. The maximal order property

Let A be a noetherian domain with Goldie quotient ring Q. We say A is a *maximal order* in Q if given any ring T with $A \subseteq T \subseteq Q$ and nonzero elements a, b of A with $aTb \subseteq A$, we have $T = A$. If A is commutative, then A is a maximal order if and only if A is integrally closed in its fraction field [17, Proposition 5.1.3].

We are interested in an equivalent formulation of the maximal order property. For any left ideal I of A, we define $\mathcal{O}_r(I) = \{q \in Q \mid Iq \subseteq I\}$ and $\mathcal{O}_l(I) = \{q \in Q \mid qI \subseteq I\}$. Then A is a maximal order if and only if $\mathcal{O}_r(I) = A = \mathcal{O}_l(I)$ for all nonzero ideals I of A [17, Proposition 5.1.4]. If A is an $\mathbb{N}$-graded algebra with a graded ring of fractions D, then for any homogeneous ideal I of A we may also define $\mathcal{O}^g_r(I) = \{q \in D \mid Iq \subseteq I\}$ and $\mathcal{O}^g_l(I) = \{q \in D \mid qI \subseteq I\}$. In the graded case we have the following criterion for the maximal order property.

Lemma 9.1. *Let A be an $\mathbb{N}$-graded noetherian domain which has a graded quotient ring D and Goldie quotient ring Q. Then A is a maximal order if and only if $\mathcal{O}_r^g(I) = A = \mathcal{O}_l^g(I)$ holds for all homogeneous nonzero ideals I of A.*

This result is stated in [26, Lemma 2], but the reference given there is faulty and so we will supply a brief proof here.

Proof. We may write $D \cong T[z, z^{-1}; \sigma]$ for some division ring T and automorphism σ of T. Then since T is a maximal order, it follows by Maury and Raynaud [16, Propositions IV.2.1,V.2.3] that D is a maximal order in Q.

Assume that $\mathcal{O}_r^g(J) = A = \mathcal{O}_l^g(J)$ for all homogeneous ideals J of A. Let I be any ideal of A, and let $q \in \mathcal{O}_r(I)$. Then DI is a 2-sided ideal of D [12, Theorem 9.20], and also $q \in \mathcal{O}_r(DI)$. Since D is a maximal order in Q, this forces $q \in D$.

Given any $d = \sum d_i \in D$ where $d_i \in D_i$, let n be maximal such that $d_n \neq 0$ and set $\widetilde{d} = d_n$. Let $\widetilde{I}$ be the 2-sided homogeneous ideal generated by $\widetilde{a}$ for all $a \in I$. Write $q = \sum_{i=m}^{n} d_i$; then since $Iq \subseteq I$, we have $\widetilde{I} d_n \subseteq \widetilde{I}$ and so $d_n \in \mathcal{O}_r^g(\widetilde{I}) = A$. Then $q - d_n \in \mathcal{O}_r(I)$. By induction on $n - m$ we get that $q - d_n \in A$ and so $q \in A$. Thus $\mathcal{O}_r(I) = A$, and an analogous argument gives $\mathcal{O}_l(I) = A$, so A is a maximal order. The opposite implication is trivial. $\square$

Let $S = S(\varphi)$ and $R = R(\varphi, c)$ and assume the critical density condition. Our next goal is to show that $R = R(\varphi, c)$ is a maximal order. First, we note that the ring S has this property.

Lemma 9.2. *$S = S(\varphi)$ is a maximal order.*

Proof. By Zhang [31, Theorem 5.11], S is ungraded Cohen–Macaulay and Auslander–regular, since U has both properties; also, since S is graded it is trivially stably free. By Stafford [22, Theorem 2.10], any ring satisfying these three properties is a maximal order. $\square$

We will also require the following lemma concerning the annihilators of modules in R-Dist.

Lemma 9.3. (1) *If $M \in R$-*Dist, *then either ${}_R M$ is torsion or else* $\operatorname{ann}_R M = 0$.

(2) *In particular, if I is a nonzero ideal of R then ${}_R(SIS/IS)$ is torsion.*

Proof. (1) Consider the S-point module $P(c_i)$ for some $i \in \mathbb{Z}$. By Lemma 3.2(1), $P(c_i)$ has point sequence $(c_i, c_{i-1}, c_{i-2}, \ldots)$. Then $\operatorname{ann}_R P(c_i) = \bigcap_{j=0}^{\infty} \mathfrak{m}_{c_{i-j}} \cap R$, and by the critical density of the points $\{c_i\}$ we conclude that $\operatorname{ann}_R P(c_i) = 0$. Now the statement follows easily from the definition of R-Dist.

(2) Since $M = {}_R(SIS/IS)$ is a homomorphic image of a direct sum of copies of (S/R), we have $M \in R$-Dist. Since also $IM = 0$, by part (1) ${}_R M$ is torsion. $\square$

Recall that R and S have the same graded quotient ring D (Lemma 4.7). For any graded left R-submodules M, N of D, we identify $\underline{\mathrm{Hom}}_R(M,N)$ with $\{d \in D \mid Md \subseteq N\}$. Similarly, if M, N are graded left S-submodules of D we identify $\underline{\mathrm{Hom}}_S(M,N)$ and $\{d \in D \mid Md \subseteq N\}$.

Proposition 9.4. *Let I be a nonzero homogeneous ideal of R. Then $\mathcal{O}_l^g(I) \subseteq S$ and $\mathcal{O}_r^g(I) \subseteq S$.*

Proof. Consider $\mathcal{O}_r^g(I)$ for some nonzero homogeneous ideal I of R. We have that

$$\mathcal{O}_r^g(I) = \{q \in D | Iq \subseteq I\} \subseteq \{q \in D | SIq \subseteq SI\} = \underline{\mathrm{Hom}}_S(SI, SI).$$

We will show that $\underline{\mathrm{Hom}}_S(SI, SI) \subseteq S$. Since S is a maximal order by Lemma 9.2, we know that $\mathcal{O}_r^g(SIS) = \underline{\mathrm{Hom}}_S(SIS, SIS) = S$. Set $M = SIS/SI \in S\text{-gr}$.

Now from the exact sequence of R-bimodules $0 \to SI \to SIS \to M \to 0$, we get the following long exact sequence in $\underline{\mathrm{Ext}}$:

$$\begin{aligned} 0 \to \underline{\mathrm{Hom}}_S(M, SIS) &\to \underline{\mathrm{Hom}}_S(SIS, SIS) \to \underline{\mathrm{Hom}}_S(SI, SIS) \\ &\to \underline{\mathrm{Ext}}_S^1(M, SIS) \to \dots \end{aligned}$$

in which the terms are again R-bimodules and the maps are all R-bimodule maps. By a right sided version of Lemma 9.3(2), which is valid by Lemma 4.2(1), M_R must be torsion. Then since the left R-structure of $\underline{\mathrm{Ext}}_S^1(M, SIS)$ comes from the right side of M, it follows from the fact that ${}_SM$ is finitely generated that the left R-module structure of $\underline{\mathrm{Ext}}_S^1(M, SIS)$ is also torsion.

Now $\underline{\mathrm{Hom}}_S(M, SIS) = 0$, since S is a domain and $\mathrm{GK}(M) < \mathrm{GK}(S)$. Thus $\underline{\mathrm{Hom}}_S(SI, SIS)$ is an R-subbimodule of D which is an essential left R-module extension of $\underline{\mathrm{Hom}}_S(SIS, SIS) = S$ by a torsion module. But by Lemma 8.12, $\underline{\mathrm{Ext}}_R^1(k, S) = 0$ and so S has no nontrivial torsion extensions. This forces $\underline{\mathrm{Hom}}_S(SI, SIS) = S$, and thus $\mathcal{O}_r^g(I) = \underline{\mathrm{Hom}}_S(SI, SI) \subseteq \underline{\mathrm{Hom}}_S(SI, SIS) = S$.

The proof that $\mathcal{O}_l^g(I) \subseteq S$ follows by applying the same argument in the extension of rings $R^{op} \subseteq S^{op}$, and again invoking Lemma 4.2(1). $\square$

Now we may complete the proof that R is a maximal order.

Theorem 9.5. *Assume the critical density condition, so that R is noetherian. Then R is a maximal order.*

Proof. Let I be any nonzero homogeneous ideal of R. Then $\underline{\mathrm{Hom}}_R({}_RI, {}_RI) = \mathcal{O}_r^g(I) \subseteq S$, by Proposition 9.4. Set $M = (\underline{\mathrm{Hom}}_R(I, I))/R$; then ${}_RM$ is a submodule of ${}_R(S/R)$, so $M \in R\text{-Dist}$. Since $IM = 0$, Proposition 9.3(1) implies that M is a torsion module. But $\underline{\mathrm{Ext}}_R^1(k, R) = 0$ by Lemma 8.12, and so R may not have any nontrivial torsion extensions. Since $R \subseteq \underline{\mathrm{Hom}}_R(I, I)$ is an essential extension, this forces $\mathcal{O}_r^g(I) =$

$\underline{\mathrm{Hom}}(I, I) = R$. Applying the same argument in R^{op}, we get $\mathcal{O}_l^g(I) = R$ as well. Thus R is a maximal order by Lemma 9.1. $\square$

10. The χ condition and R-proj

We begin this section by reviewing some definitions from the theory of noncommutative projective schemes which we will use in the next two sections. See [5] for more details.

Let A be a noetherian $\mathbb{N}$-graded ring which is finitely generated in degree one. Let A-Tors be the full subcategory of torsion objects in A-Gr. Then A-Tors is a *localizing subcategory* of A-Gr, which means that the quotient category $A\text{-Qgr} = A\text{-Gr}/A\text{-Tors}$ is defined, and the exact *quotient functor* π: $A\text{-Gr} \to A\text{-Qgr}$ has a right adjoint ω, which is called the *section functor*. For torsionfree $M \in A\text{-Gr}$ we may describe $\omega\pi(M)$ explicitly as the unique largest essential extension M' of M such that M'/M is torsion. For all $\mathcal{M} \in A\text{-qgr}$, $\omega(\mathcal{M})$ is torsionfree and $\pi\omega(\mathcal{M}) \cong \mathcal{M}$.

The *noncommutative projective scheme* A-Proj is defined to be the ordered pair $(A\text{-Qgr}, \pi(A))$, where $\pi(A)$ is called the *distinguished object*. We write $A\text{-Proj} \cong B\text{-Proj}$ if there is an equivalence of categories $A\text{-Qgr} \sim B\text{-Qgr}$ under which the distinguished objects correspond. We also work with the subcategories of noetherian objects $A\text{-gr}, A\text{-tors}, A\text{-qgr} = A\text{-gr}/A\text{-tors}$, and we set $A\text{-proj} = (A\text{-qgr}, \pi(A))$.

The category A-Qgr has enough injectives and so Ext groups are defined in this category. The shift functor $M \mapsto M[1]$, which is an autoequivalence of the category A-Gr, descends naturally to an autoequivalence of A-Qgr. For $\mathcal{M}, \mathcal{N} \in A\text{-Qgr}$ we define

$$\underline{\mathrm{Ext}}^p(\mathcal{M}, \mathcal{N}) = \bigoplus_{i=-\infty}^{\infty} \mathrm{Ext}^p_{A\text{-Qgr}}(\mathcal{M}, \mathcal{N}[i]).$$

Now we define cohomology and graded cohomology for A-Proj by setting $\mathrm{H}^i(\mathcal{N}) = \mathrm{Ext}^i(\pi(A), \mathcal{N})$ and $\underline{\mathrm{H}}^i(\mathcal{N}) = \underline{\mathrm{Ext}}^i(\pi(A), \mathcal{N})$ for $\mathcal{N} \in A\text{-Qgr}$. The section functor ω may be described using cohomology as $\omega(\mathcal{M}) = \underline{\mathrm{H}}^0(\mathcal{M})$ for all $\mathcal{M} \in A\text{-Qgr}$.

For making explicit computations, it is useful to note that for all $M, N \in A\text{-gr}$ and all $p \geqslant 0$,

$$\underline{\mathrm{Ext}}^p_{A\text{-Qgr}}(\pi(M), \pi(N)) \cong \lim_{n \to \infty} \underline{\mathrm{Ext}}^p_A(M_{\geqslant n}, N). \tag{10.1}$$

In case $M = A$, we have the following additional formula for all $p \geqslant 1$ (see [5, Proposition 7.2(2)]):

$$\underline{\mathrm{H}}^p(\pi(N)) \cong \lim_{n \to \infty} \underline{\mathrm{Ext}}^{p+1}_A(A/A_{\geqslant n}, N). \tag{10.2}$$

If A is commutative this last formula amounts to the usual correspondence between sheaf cohomology in A-proj and local cohomology for the ring A.

Recall the χ conditions which were defined in Section 8. Artin and Zhang proved the following noncommutative analog of Serre's theorem:

Theorem 10.3 (Artin and Zhang [5, Theorem 4.5]). *Let A be a right noetherian $\mathbb{N}$-graded algebra, and let $\mathscr{A} = \pi(A)$ be the distinguished object of A-proj. Then $B = \bigoplus_{i \geqslant 0} \mathrm{H}^0(\mathscr{A}[i])$ is naturally a graded ring and there is a canonical homomorphism $\psi : A \to B$. If A satisfies χ_1 then ψ is an isomorphism in large degree, and B-proj $\cong A$-proj.*

In other words, if A satisfies χ_1 then the noncommutative projective scheme A-proj, together with the shift functor $M \mapsto M[1]$, determines the ring A up to a finite-dimensional vector space.

In this section, we will analyze the χ conditions for $R = R(\varphi, c)$, assuming the critical density condition throughout. We shall show that R satisfies χ_1, but that χ_i fails for R for $i \geqslant 2$. In particular, R satisfies the noncommutative Serre's theorem, and is the first example of a noetherian algebra which satisfies χ_1 but not all of the χ conditions.

We will say that $\chi_i(M)$ *holds* for a particular module $M \in A$-Gr if $\underline{\mathrm{Ext}}^j_A(k, M) < \infty$ for $0 \leqslant j \leqslant i$. The reader may easily prove the following simple facts.

Lemma 10.4. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in R-gr, and let $N \in R$-gr.*

(1) *If $\chi_1(M')$ and $\chi_1(M'')$ hold then $\chi_1(M)$ holds.*
(2) *If $\chi_1(M)$ holds then $\chi_1(M')$ holds.*
(3) *If $\dim_k N < \infty$ then $\chi_1(N)$ holds.* $\square$

To prove χ_1 for R we will reduce to the case of S-modules.

Proposition 10.5. *Suppose that $N \in S$-gr. Then $\chi_1({}_R N)$ holds.*

Proof. Consider the first 3 terms of the 5-term exact sequence (8.8) for $M = {}_R k$:

$$0 \to \underline{\mathrm{Ext}}^1_S(S \otimes_R k, N) \to \underline{\mathrm{Ext}}^1_R(k, N) \to \underline{\mathrm{Hom}}_S(\underline{\mathrm{Tor}}^R_1(S, k), N) \to \dots .$$

Now $\underline{\mathrm{Tor}}^R_i(S, k)$ is in S-dist for any $i \geqslant 0$, by Lemma 8.9(1). Then by Proposition 8.6, we conclude that $\dim_k \underline{\mathrm{Ext}}^j_S(\underline{\mathrm{Tor}}^R_i(S, k)), N) < \infty$ for $j = 0, 1$ and $i \geqslant 0$. Thus $\dim_k \underline{\mathrm{Ext}}^1_R(k, N) < \infty$. $\square$

The following result completes the proofs of Theorems 1.3 and 1.5 from the introduction.

Theorem 10.6. *Assume the critical density condition and let* $R = R(\varphi, c)$.

(1) *R satisfies* χ_1.
(2) $\underline{\mathrm{Ext}}^2_R(k, R)$ *is not bounded, and* χ_i *fails for all* $i \geqslant 2$.

Proof. (1) By Lemma 10.4(1) and induction it is enough to show that $\chi_1(M)$ holds for all graded cyclic R-modules M.

Let R/I be an arbitrary graded cyclic left R-module. If $I = 0$, then $\chi_1({}_R R)$ holds by Lemma 8.12. Assume then that $I \neq 0$. Consider the exact sequence (5.15). Now $\chi_1({}_R(S/SI))$ holds by Proposition 10.5. By Lemma 5.16(1), both $(SI \cap R)/I$ and $S/(R + SI)$ have finite filtrations with factors which are either torsion or shifted R-point modules with a compatible S-module structure. Then $\chi_1((SI \cap R)/I)$ and $\chi_1(S/(R + SI))$ hold, by Proposition 10.5 and Lemma 10.4(1),(3). Finally, applying Lemma 10.4(1),(2) to (5.15) we get that $\chi_1(R/I)$ holds.

(2) Consider the long exact sequence in $\underline{\mathrm{Ext}}_R(k, -)$ that arises from the short exact sequence of R-modules $0 \to R \to S \to S/R \to 0$:

$$\ldots \to \underline{\mathrm{Ext}}^1_R(k, S) \to \underline{\mathrm{Ext}}^1_R(k, S/R) \to \underline{\mathrm{Ext}}^2_R(k, R) \to \ldots \quad . \tag{10.7}$$

Now $\underline{\mathrm{Ext}}^1_R(k, S) = 0$, by Lemma 8.12. On the other hand,

$$\underline{\mathrm{Ext}}^1_R(k, S/R) \cong \bigoplus_{i=1}^{\infty} \underline{\mathrm{Ext}}^1_R(k, P(c_{-1}))[-i]$$

by Corollary 5.6(2), since $\underline{\mathrm{Ext}}$ commutes with direct sums in the second coordinate [5, Proposition 3.1(1)(b)]. By Theorem 6.6(2), it is clear that the point module $P(c_{-1})$ has a nontrivial extension by $k[1]$, since any point module $P(c_0, e)$ defined there satisfies $(P(c_0, e)[1])_{\geqslant 0} \cong {}_R P(c_{-1})$. Thus $\underline{\mathrm{Ext}}^1_R(k, P(c_{-1})) \neq 0$, and so $\oplus_{i=1}^{\infty} \underline{\mathrm{Ext}}^1_R(k, P(c_{-1}))[-i] \cong \underline{\mathrm{Ext}}^1_R(k, S/R)$ is not right bounded. Then by the exact sequence (10.7), $\underline{\mathrm{Ext}}^2_R(k, R)$ is also not right bounded. In particular, $\dim_k \underline{\mathrm{Ext}}^2_R(k, R) = \infty$ and χ_i fails for R for all $i \geqslant 2$ by definition. $\square$

We see next that the failure of χ_i for R for $i \geqslant 2$ is reflected in the cohomology of R-proj. We recall the noncommutative version of Serre's finiteness theorem which was proved by Artin and Zhang, which we have restated slightly.

Theorem 10.8 (Artin and Zhang [5, Theorem 7.4]). *Let A be a left noetherian finitely $\mathbb{N}$-graded algebra which satisfies χ_1. Then A satisifes χ_i for some $i \geqslant 2$ if and only if the following two conditions hold*:

(1) $\dim_k \mathrm{H}^j(\mathcal{N}) < \infty$ *for all* $0 \leqslant j < i$ *and all* $\mathcal{N} \in A$-qgr.
(2) $\underline{\mathrm{H}}^j(\mathcal{N})$ *is right bounded for all* $1 \leqslant j < i$ *and all* $\mathcal{N} \in A$-qgr.

Proof. This follows immediately from the proof of [5, Theorem 7.4]. $\square$

Lemma 10.9. *Let A be a left noetherian finitely $\mathbb{N}$-graded algebra satisfying χ_i. Then* $\dim_k \operatorname{Ext}^j(\mathcal{M}, \mathcal{N}) < \infty$ *for $0 \leqslant j < i$ and for all $\mathcal{M}, \mathcal{N} \in A$*-qgr.

Proof. Let $\mathcal{A} = \pi(A)$. Since any $M \in A$-gr is an image of some finite sum of shifts of A, in A-qgr there is an exact sequence

$$0 \to \mathcal{M}' \to \mathcal{F} \to \mathcal{M} \to 0,$$

where we have $\mathcal{F} = \bigoplus_{i=1}^n \mathcal{A}[d_i]$ for some integers $d_i \in \mathbb{Z}$. Then $\operatorname{Ext}^j(\mathcal{F}, \mathcal{N}) \cong \bigoplus_{i=1}^n \operatorname{Ext}^j(\mathcal{A}, \mathcal{N}[-d_i]) = \bigoplus_{i=1}^n \mathrm{H}^j(\mathcal{N}[-d_i])$ and so $\dim_k \operatorname{Ext}^j(\mathcal{F}, \mathcal{N}) < \infty$ for all $0 \leqslant j < i$ by Theorem 10.8.

We induct on j. If $j = 0$ then there is an exact sequence $0 \to \operatorname{Hom}(\mathcal{M}, \mathcal{N}) \to \operatorname{Hom}(\mathcal{F}, \mathcal{N})$ from which it follows that $\dim_k \operatorname{Hom}(\mathcal{M}, \mathcal{N}) < \infty$. For $0 < j < i$, there is the long exact sequence

$$\ldots \to \operatorname{Ext}^{j-1}(\mathcal{M}', \mathcal{N}) \to \operatorname{Ext}^j(\mathcal{M}, \mathcal{N}) \to \operatorname{Ext}^j(\mathcal{F}, \mathcal{N}) \to \ldots$$

and since $\dim_k \operatorname{Ext}^{j-1}(\mathcal{M}', \mathcal{N}) < \infty$ by the induction hypothesis, we have $\dim_k \operatorname{Ext}^j(\mathcal{M}, \mathcal{N}) < \infty$ as well. This completes the induction step and the proof. $\square$

We can now make the failure of the Serre's finiteness theorem for R-proj explicit.

Lemma 10.10. *Let $\mathcal{R} = \pi(R) \in R$*-qgr *be the distinguished object of* R-proj. *Then* $\dim_k \mathrm{H}^1(\mathcal{R}) = \infty$.

Proof. Set $\mathcal{S} = \pi(S) \in R$-Qgr. The exact sequence $0 \to R \to S \to S/R \to 0$ descends to an exact sequence $0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{S}/\mathcal{R} \to 0$ in R-qgr. For $\mathcal{M} \in R$-Qgr, the cohomology $\mathrm{H}^0(\mathcal{M})$ may be identified with the zeroeth graded piece of the module $\omega(\mathcal{M})$, where ω is the section functor. Recall also that for torsionfree $M \in A$-Gr, $\omega\pi(M)$ is the largest essential extension of M by a torsion module. Since $\underline{\operatorname{Ext}}^1_R(k, S) = 0$ by Lemma 8.12, ${}_RS$ has no nontrivial torsion extensions and so $\omega(\mathcal{S}) = S$. In particular, $\dim_k \mathrm{H}^0(\mathcal{S}) = \dim_k S_0 = 1$. On the other hand, $S/R = \bigoplus_{i=1}^{\infty} P(c_{-1})[-i]$ is an infinite direct sum of shifted R-point modules by Corollary 5.6(2). For each $i \geqslant 0$, by Lemma 6.6(2) there is some R-point module $P(c_i, e_i)$ which satisfies $P(c_i, e_i)_{\geqslant i+1} \cong P(c_{-1})[-i-1]$. Then $M = \bigoplus_{i=0}^{\infty} P(c_i, e_i)$ is an essential extension of S/R by a torsion module, so $M \subseteq \omega(\mathcal{S}/\mathcal{R})$ and it follows that $\dim_k \mathrm{H}^0(\mathcal{S}/\mathcal{R}) \geqslant \dim_k M_0 = \infty$. Now the long exact sequence in cohomology forces $\dim_k \mathrm{H}^1(\mathcal{R}) = \infty$ as well. $\square$

The following result, which proves Theorem 1.4 from the introduction, shows that the category R-qgr is necessarily quite different from any of the standard examples.

Theorem 10.11. *Assume the critical density condition.*

(1) *Suppose that A is a left noetherian finitely $\mathbb{N}$-graded k-algebra which satisfies χ_2. Then the categories A*-qgr *and R*-qgr *are not equivalent.*

(2) R-qgr *is not equivalent to* coh X, *the category of coherent sheaves on* X, *for any commutative projective scheme* X.

Proof. (1) The proof is immediate from Lemmas 10.9 and 10.10.

(2) This follows from part (1) and the usual commutative Serre's theorem. $\square$

11. Global and cohomological dimension of R-proj

In this section, our goal is to show that R-proj has finite global dimension, and thus finite cohomological dimension. We will also give upper bounds for these numbers.

Let us recall the definitions of these concepts:

Definition 11.1. Let A be a connected finitely generated $\mathbb{N}$-graded algebra. The *global dimension* of A-qgr (or A-proj) is

$$\mathrm{gldim}(A\text{-qgr}) = \sup\{i \mid \mathrm{Ext}^i(\mathscr{M}, \mathscr{N}) \neq 0 \text{ for some } \mathscr{M}, \mathscr{N} \in A\text{-qgr}\}.$$

The *cohomological dimension* of A-proj is

$$\mathrm{cd}(A\text{-proj}) = \sup\{i \mid \mathrm{H}^i(\mathscr{N}) \neq 0 \text{ for some } \mathscr{N} \in A\text{-qgr}\}.$$

If A-qgr has finite global dimension, then it is immediate that A-proj has finite cohomological dimension. We remark that it is not known if there exists any graded algebra A such that $\mathrm{cd}(A\text{-proj}) = \infty$.

Now let $S = S(\varphi)$ and $R = R(\varphi, c)$, and assume the critical density condition as usual. It is easy to compute the global and cohomological dimensions of S-proj:

Lemma 11.2. $\mathrm{cd}(S\text{-proj}) = \mathrm{gldim}(S\text{-qgr}) = \mathrm{GK}(S) - 1 = t$.

Proof. Since S is a Zhang twist of the polynomial ring U, we have an isomorphism $S\text{-proj} \cong (\mathrm{coh}\, \mathbb{P}^t, \mathcal{O}_{\mathbb{P}^t})$, and the values of both dimensions for the commutative scheme $\mathbb{P}^t$ are well known. $\square$

The main machinery we will use to study Ext groups in R-qgr is Proposition 11.6 below, which needs the spectral sequence given in the following lemma.

Lemma 11.3. *For any system in* R-Gr *of the form* $\ldots \to M_n \to \ldots \to M_1 \to M_0$ *and any* $N \in S$-Gr *there is a convergent spectral sequence of the form*

$$E_2^{pq} = \lim_{n \to \infty} \underline{\mathrm{Ext}}_S^p(\underline{\mathrm{Tor}}_q^R(S, M_n), N) \underset{p}{\Rightarrow} \lim_{n \to \infty} \underline{\mathrm{Ext}}_R^{p+q}(M_n, N).$$

Proof. Consider the spectral sequence (8.7) for arbitrary ${}_R M \in R$-Gr:

$$\underline{\mathrm{Ext}}^p_S(\underline{\mathrm{Tor}}^R_q(S, M), N) \underset{p}{\Rightarrow} \underline{\mathrm{Ext}}^{p+q}_R(M, N).$$

Let $\mathscr{C}$ be the category of all $\mathbb{N}$-indexed directed systems of modules in R-Gr of the form

$$\ldots \to M_n \to \ldots \to M_1 \to M_0.$$

Let $\mathscr{D}$ be the analogous category of directed systems of modules in S-Gr. Both of these categories have enough projectives and injectives. For example, if P is a projective object of R-Gr, then any object in $\mathscr{C}$ of the form

$$\ldots \to 0 \to 0 \to 0 \to P \xrightarrow{\cong} P \xrightarrow{\cong} \ldots \xrightarrow{\cong} P \tag{11.4}$$

is projective, and clearly every object in $\mathscr{C}$ is an image of a direct sum of objects of this form. See [30, Exercises 2.3.7, 2.3.8] for more details. The functor $S \otimes_R - :$ R-Gr $\to S$-Gr extends to a functor $G : \mathscr{C} \to \mathscr{D}$. We also have a functor $F : \mathscr{D} \to \mathrm{Ab}$ defined by $\{L_n\}_{n \in \mathbb{N}} \mapsto \lim_{n \to \infty} \underline{\mathrm{Hom}}_S(L_n, N)$, where Ab is the category of abelian groups. It is easy to see that G is right exact; since Ab has exact direct limits [30, Theorem 2.6.15], F is left exact. Finally, G sends any direct sum of objects in $\mathscr{C}$ of the form in (11.4) to a projective object in $\mathscr{D}$. Then corresponding to the composition of functors $F \circ G$ is a Grothendieck spectral sequence (see [21, Theorem 11.40])

$$E_2^{p,q} = R^p F(L_q G(M_\cdot)) \underset{p}{\Rightarrow} R^{p+q}(FG)(M_\cdot)$$

which we leave to the reader to show unravels to the spectral sequence required by the lemma. $\square$

To get the most out of the spectral sequence, we also note the following simple lemma.

Lemma 11.5. *Let A be a connected graded noetherian ring, and let $\ldots \to M_n \to \ldots \to M_1 \to M_0$ a directed system of modules in A-gr. For each n, let $\tau(M_n)$ be the torsion submodule of M_n. Then for any $N \in A$-gr and $p \geqslant 0$,*

$$\lim_{n \to \infty} \underline{\mathrm{Ext}}^p_A(M_n, N) \cong \lim_{n \to \infty} \underline{\mathrm{Ext}}^p_A(M_n/\tau(M_n), N)$$

as k-spaces.

Proof. Since direct limits are exact in the category of abelian groups [30, Theorem 2.6.15], there is a long exact sequence in $\lim_{n \to \infty} \underline{\mathrm{Ext}}(-, N)$ arising from short exact sequence of complexes $0 \to \tau(M.) \to M. \to M./\tau(M.) \to 0$. Given fixed n, the module $\tau(M_n)$ is bounded and so for some $n' \gg n$ the natural map $\tau(M_{n'}) \to \tau(M_n)$ is zero, and then the natural map $\underline{\mathrm{Ext}}^p_A(\tau(M_n), N) \to \underline{\mathrm{Ext}}^p_A(\tau(M_{n'}), N)$ is zero. Thus

$\lim_{n\to\infty} \underline{\mathrm{Ext}}^p_A(\tau(M_n), N) = 0$ for all $p \geqslant 0$, and the desired result follows from the long exact sequence. $\square$

Proposition 11.6. *Let $N \in S$-Gr, and let $M \in R$-gr.*

(1) *As graded vector spaces, for all $m \geqslant 0$ we have*

$$\underline{\mathrm{Ext}}^m_{R\text{-Qgr}}(\pi(M), \pi(N)) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^m_S(S \otimes_R M_{\geqslant n}, N).$$

(2) *In case $M = R$, for $m \geqslant 1$ we have*

$$\underline{\mathrm{H}}^m(\pi(N)) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^{m+1}_S(S/SR_{\geqslant n}, N).$$

Proof. (1) We use the spectral sequence of Lemma 11.3:

$$E_2^{pq} = \lim_{n\to\infty} \underline{\mathrm{Ext}}^p_S(\underline{\mathrm{Tor}}^R_q(S, M_{\geqslant n}), N) \underset{p}{\Rightarrow} \lim_{n\to\infty} \underline{\mathrm{Ext}}^{p+q}_R(M_{\geqslant n}, N).$$

Our goal is to show that $E_2^{pq} = 0$ for any pair of indices p, q with $q \geqslant 1$.

Fix $q \geqslant 1$. For fixed $n \geqslant 0$, we claim first that there is some $n' \geqslant n$ such that the natural map $\psi_1 : \underline{\mathrm{Tor}}^R_q(S, M_{\geqslant n'}) \to \underline{\mathrm{Tor}}^R_q(S, M_{\geqslant n})$ is 0. As in Lemma 8.9, there is a right point module Q of R such that $S/R \cong \bigoplus_{i=1}^{\infty} Q[-i]$ as right R-modules. Now by the fact that $\underline{\mathrm{Tor}}$ commutes with direct sums [5, Proposition 2.4(1)] we get a commutative diagram

$$\begin{array}{ccc}
\underline{\mathrm{Tor}}^R_q(S, M_{\geq n'}) & \xrightarrow{\psi_1} & \underline{\mathrm{Tor}}^R_q(S, M_{\geq n}) \\
\downarrow & & \downarrow \\
\underline{\mathrm{Tor}}^R_q(S/R, M_{\geq n'}) & \xrightarrow{\psi_2} & \underline{\mathrm{Tor}}^R_q(S/R, M_{\geq n}) \\
\downarrow \cong & & \downarrow \cong \\
\bigoplus_{i=1}^{\infty} \underline{\mathrm{Tor}}^R_q(Q, M_{\geq n'})[-i] & \xrightarrow{\psi_3} & \bigoplus_{i=1}^{\infty} \underline{\mathrm{Tor}}^R_q(Q, M_{\geq n})[-i]
\end{array}$$

where the top two vertical maps are at least injections (see Lemma 8.9) and the ψ_i are the natural maps.

Now $T^n = \underline{\mathrm{Tor}}^R_q(Q, M_{\geqslant n})$ is bounded, by Lemma 8.9(2). Also, clearly the left bound $l(n)$ of T^n satisfies $\lim_{n\to\infty} l(n) = \infty$. It follows that for $n' \gg n$ the natural map $\theta : T^{n'} \to T^n$ is 0. The restriction of the map ψ_3 to any summand is just a shift of the map θ, so $\psi_3 = 0$ for $n' \gg n$. Finally, the commutative diagram gives $\psi_1 = 0$ for $n' \gg n$. This proves the claim.

Write ${}_nE_2^{pq} = \underline{\mathrm{Ext}}_S^p(\underline{\mathrm{Tor}}_q^R(S, M_{\geqslant n}), N)$. Since $\psi_1 = 0$ for $n' \gg n$, the natural map ${}_nE_2^{pq} \to {}_{n'}E_2^{pq}$ is also zero for $n' \gg n$. Since n was arbitrary, we have $E_2^{pq} = \lim_{n\to\infty} {}_nE_2^{pq} = 0$.

Therefore only the E_2^{pq} with $q = 0$ are possibly nonzero, and the spectral sequence collapses, giving an isomorphism of vector spaces for all $m \geqslant 1$ as follows (using also (10.1)):

$$\begin{aligned}\underline{\mathrm{Ext}}^m_{R\text{-Qgr}}(\pi(M), \pi(N)) &\cong \lim_{n\to\infty} \underline{\mathrm{Ext}}_R^m(M_{\geqslant n}, N) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}_S^m(\underline{\mathrm{Tor}}_0^R(S, M_{\geqslant n}), N) \\ &= \lim_{n\to\infty} \underline{\mathrm{Ext}}_S^m(S \otimes_R M_{\geqslant n}, N).\end{aligned}$$

(2) Assume that $M = R$. Setting $T^n = \underline{\mathrm{Tor}}_q^R(Q, R/R_{\geqslant n})$ in this case (where Q is as in part (1)) it is obvious that T^n is bounded, since $R/R_{\geqslant n}$ is, and it is still true for $q \geqslant 1$ that the left bound $l(n)$ of T^n satisfies $\lim_{n\to\infty} l(n) = \infty$ [5, Proposition 2.4(6)]. Then the same argument as in part (1) shows that the spectral sequence

$$E_2^{pq} = \lim_{n\to\infty} \underline{\mathrm{Ext}}_S^p(\underline{\mathrm{Tor}}_q^R(S, R/R_{\geqslant n}), N) \underset{p}{\Rightarrow} \lim_{n\to\infty} \underline{\mathrm{Ext}}_R^{p+q}(R/R_{\geqslant n}, N)$$

also collapses, so we have (using (10.2)) that

$$\begin{aligned}\underline{\mathrm{H}}_R^m(\pi(N)) &\cong \lim_{n\to\infty} \underline{\mathrm{Ext}}_R^{m+1}(R/R_{\geqslant n}, N) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}_S^{m+1}(\underline{\mathrm{Tor}}_0^R(S, R/R_{\geqslant n}), N) \\ &= \lim_{n\to\infty} \underline{\mathrm{Ext}}_S^{m+1}(S/SR_{\geqslant n}, N)\end{aligned}$$

for all $m \geqslant 1$. $\square$

Armed with the preceding results, we can now show that R-qgr has finite global dimension, and calculate upper bounds on the values of the global dimension and cohomological dimension for R-proj. This proves Theorem 1.6 from the introduction.

Theorem 11.7. *Assume the critical density condition for $R = R(\varphi, c)$, and recall that $t = \mathrm{GK}(R) - 1$. Then*

(1) $\mathrm{gldim}(R\text{-qgr}) \leqslant t + 1$.
(2) $\mathrm{cd}(R\text{-proj}) \leqslant t$.

Proof. (1) Given any $M \in R$-gr and $N \in S$-Gr, Proposition 11.6(1) and Lemma 11.5 show that

$$\underline{\mathrm{Ext}}^p_{R\text{-qgr}}(\pi(M), \pi(N)) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}_S^p(S \otimes M_{\geqslant n}, N) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}_S^p(N'_n, N),$$

where $N'_n = (S \otimes M_{\geqslant n})/\tau(S \otimes M_{\geqslant n})$. For all $n \geqslant 0$, the S-module N'_n is torsionfree. By the graded Auslander–Buchsbaum formula [11, Exercise 19.8], a torsionfree module in U-gr has projective dimension at most $(\dim U) - 1 = t$, since it has depth

$\geqslant 1$; by the equivalence of categories $U\text{-Gr} \sim S\text{-Gr}$ we infer that each N'_n has projective dimension at most t over S. Then for $p > t$ each term of the direct limit is 0 and so $\underline{\mathrm{Ext}}^p_{R\text{-qgr}}(\pi(M), \pi(N)) = 0$.

Now if L is an arbitrary graded cyclic R-module, then either $L = R$, in which case we have the exact sequence $0 \to R \to S \to S/R \to 0$, or $L = R/I$ for $I \neq 0$, in which there is the exact sequence (5.15): $0 \to (SI \cap R)/I \to R/I \to S/SI \to S/(R + SI) \to 0$. Since S/R, $(SI \cap R)/I$, and $S/(R + SI)$ all have finite filtrations with factors which have an S-structure compatible with their left R-structure (Lemmas 5.6 and 5.16), we conclude that $\underline{\mathrm{Ext}}^p_{R\text{-qgr}}(\pi(M), \pi(L)) = 0$ for $p > (t+1)$. Since any $L' \in R\text{-gr}$ has a finite filtration by cyclic modules, we see that $\underline{\mathrm{Ext}}^p_{R\text{-qgr}}(\pi(M), \pi(L')) = 0$ for all $M, L' \in R\text{-gr}$ and $p > (t+1)$, and so $\mathrm{gldim}(R\text{-qgr}) \leqslant t+1$.

(2) We will show that for any $N \in S\text{-Gr}$, $\underline{\mathrm{Ext}}^p_{R\text{-qgr}}(\pi(R), \pi(N)) = 0$ for $p > (t-1)$. Then the bound $\mathrm{cd}(R\text{-proj}) \leqslant t$ will follow by a similar argument as in part (1). Let $J^{(n)}$ be the left S-ideal $\mathfrak{m}_{c_0} \cap \mathfrak{m}_{c_1} \cap \ldots \cap \mathfrak{m}_{c_{n-1}}$. By Theorem 4.4, $R_n = (J^{(n)})_n$, and furthermore $J^{(n)}$ is generated in degrees $\leq n$, by Lemma 4.3. It is easy to see then that $J^{(n)}$ is the largest extension of $SR_{\geqslant n}$ inside S by a torsion module.

Now by Proposition 11.6(2) and Lemma 11.5, we have

$$\underline{\mathrm{Ext}}^p_{R\text{-qgr}}(\pi(R), \pi(N)) \cong \lim_{n \to \infty} \underline{\mathrm{Ext}}^{p+1}_S(S/SR_{\geqslant n}, N) \cong \lim_{n \to \infty} \underline{\mathrm{Ext}}^{p+1}_S(S/J^{(n)}, N).$$

But each $S/J^{(n)}$ is torsionfree and hence has projective dimension at most t, by the same argument as in part (1), so every term in the direct limit is zero when $p > (t-1)$. $\square$

Before leaving the subject of cohomological dimension, we wish to mention another approach to cohomology for noncommutative graded algebras which is provided by the work of Van Oystaeyen and Willaert on schematic algebras [27–29]. An algebra graded A is called *schematic* if it has enough Ore sets to give an open cover of A-proj; we shall not concern ourselves here with the formal definition. For such algebras one can define a noncommutative version of Čech cohomology which gives the same cohomology groups as the cohomology theory we studied above.

It turns out that the theory of schematic algebras is of no help in computing the cohomology of R-proj. Indeed, if A is a connected $\mathbb{N}$-graded noetherian schematic algebra then $\underline{\mathrm{Ext}}^n_A({}_Ak, A)$ is torsion as a right A-module for all $n \in \mathbb{N}$ [29, Proposition 3], hence finite dimensional over k. But we saw in Theorem 10.6 that $\dim_k \underline{\mathrm{Ext}}^2_R({}_Rk, R) = \infty$. Thus we have incidentally proven the following proposition.

Proposition 11.8. *Assume the critical density condition. Then $R = R(\varphi, c)$ is a connected $\mathbb{N}$-graded noetherian domain, generated in degree one, which is not schematic.*

The previously known nonschematic algebras have not been generated in degree one [29, p. 12].

12. The critical density property

We saw in Theorem 5.12 that the noetherian property for $R(\varphi, c)$ depends on the critical density of the set of points $\mathscr{C} = \{\varphi^i(c)\}_{i \in \mathbb{Z}}$, and we have been assuming that $\mathscr{C}$ is critically dense ever since. In this section, we justify this assumption by showing that the critical density of $\mathscr{C}$ holds for generic choices of φ and c. We will only concern ourselves with pairs (φ, c) such that Hypothesis 4.1 holds, that is such that c has infinite order under φ. Next, we will show that in case $\operatorname{char} k = 0$, the set $\mathscr{C}$ is critically dense if and only if it is dense, which is a much simpler condition to check. Finally, we discuss rings generated by Eulerian derivatives, which was the context in which rings of the form $R(\varphi, c)$ first appeared in the literature [14]. We translate our earlier results into this language, and show that they solve several open questions in [14].

Throughout this section we write $c_i = \varphi^{-i}(c)$. As we did earlier in Section 7, we think of automorphisms of $\mathbb{P}^t$ as elements of $\mathrm{PGL}_{t+1}(k) = \mathrm{GL}_{t+1}(k)/k^{\times}$, which act on the left on column vectors of homogeneous coordinates for $\mathbb{P}^t$. We write $\operatorname{diag}(p_0, p_1, \ldots, p_t)$ for the automorphism which is represented by a diagonal matrix with diagonal entries $p_0, p_1, \ldots, p_t$.

Let us define precisely our (somewhat nonstandard) intended meaning of the word "generic".

Definition 12.1. A subset U of a variety X is *generic* if its complement is contained in a countable union of proper closed subvarieties of X.

If the base field k is uncountable, a generic subset is intuitively very large. For example, if $k = \mathbb{C}$ then a property which holds generically holds "almost everywhere" in the sense of Lebesgue measure. For any results below which involve genericity we will assume that k is uncountable.

In the next theorem we will prove that $\mathscr{C}$ is critically dense for φ a suitably general diagonal matrix, and c chosen from an open set of $\mathbb{P}^t$. The proof will depend on the following combinatorial lemma.

Lemma 12.2. *Fix $d \geq 1$, and set $N = \binom{t+d}{d}$. Let $U = k[x_0, x_1, \ldots, x_t]$ be the polynomial ring, and give monomials in U the lexicographic order with respect to some fixed ordering of the variables. Let $f_1, f_2, \ldots, f_N$ be the monomials of degree d in $U = k[x_0, x_1, \ldots, x_t]$, enumerated so that $f_1 < f_2 < \ldots f_N$ in the lex order. Fix some sequence of distinct nonnegative integers $a_1 < a_2 < \cdots < a_N$. Then the polynomial* $\det(f_i^{a_j}) \in U$ *is nonzero.*

Proof. Set $F = \det(f_i^{a_j}) \in U$. Let S_N be the symmetric group on N elements, with identity element 1; then F is a sum of terms of the form $h_\sigma = \pm \prod_{i=1}^{N} f_i^{a_{\sigma(i)}}$ for $\sigma \in S_N$. It is straightforward to check that the monomial $f_1^{a_1} f_2^{a_2} \cdots f_N^{a_N}$ is the unique largest in the lex order occurring among the h_σ, and that it occurs only in h_1 and thus may not be cancelled by any other term. $\square$

The next theorem includes, in particular, the result of Proposition 5.13 which we stated earlier without proof.

Theorem 12.3. *Let* $\varphi = \operatorname{diag}(1, p_1, p_2, \dots, p_t)$ *with* $\{p_1, p_2, \dots p_t\}$ *algebraically independent over the prime subfield of* k. *Let* $c = (b_0 : b_1 : \cdots : b_t) \in \mathbb{P}^t$ *with* $b_i \neq 0$ *for all* $0 \leqslant i \leqslant t$. *Then* $\mathscr{C} = \{\varphi^i(c)\}_{i\in\mathbb{Z}}$ *is critically dense and* $R(\varphi, c)$ *is noetherian.*

Proof. We have the explicit formula $c_{-n} = (b_0 : b_1 p_1^n : b_2 p_2^n : \cdots : b_t p_t^n)$. We will actually prove that the set of points $\mathscr{C}$ is in *general position*: that is, that at most $\binom{n+d}{d}$ of the points lie on any degree d hypersurface of $\mathbb{P}^t$. This will obviously imply that $\mathscr{C}$ is critically dense.

Suppose that $\mathscr{C}$ fails to be in general position. Then there is some $d \geqslant 1$ and a sequence of $N = \binom{t+d}{d}$ integers $a_1 < a_2 < \cdots < a_N$ such that the points $c_{a_1}, c_{a_2}, \dots, c_{a_N}$ lie on a degree d hypersurface in $\mathbb{P}^t$. We may assume that the a_i are nonnegative, since if the $\{c_{a_i}\}$ lie on a degree d hypersurface then the same is true of the points $\{\varphi^{-m}(c_{a_i})\} = \{c_{a_i+m}\}$ for any $m \in \mathbb{Z}$. Let $f_1, f_2, \dots, f_N$ be the distinct degree d monomials in the variables x_i of U. It follows that $\det(f_i(c_{a_j})) = 0$.

Given the explicit formula for c_n, one calculates that

$$\det(f_i(c_{a_j})) = B[\det(f_i^{a_j})](1 : p_1^{-1} : p_2^{-1} : \cdots : p_t^{-1}) = 0,$$

where B is a monomial in the b_i and hence is nonzero by hypothesis. Now by Lemma 12.2 the polynomial $\det(f_i^{a_j})$ is a nonzero homogeneous element of U, which clearly has coefficients in the prime subfield of k. Thus $p_1^{-1}, p_2^{-1}, \dots, p_t^{-1}$ satisfy some nonzero nonhomogeneous relation with coefficients in the prime subfield of k, contradicting the hypothesis on the $\{p_i\}$.

Thus the set $\mathscr{C}$ must be in general position, and so is critically dense. Then certainly c must also have infinite order under φ, so that Hypothesis 4.1 holds. Now $R(\varphi, c)$ is noetherian by Theorem 5.12. $\square$

Next, let us show that for generic choices of φ and c (in the sense of Definition 12.1), the ring $R(\varphi, c)$ is noetherian. Because of Lemma 4.2(2), for every fixed $c \in \mathbb{P}^t$ we get the same class of rings $\{R(\varphi, c) \mid \varphi \in \operatorname{Aut} \mathbb{P}^t\}$. Thus we might as well fix some arbitrary c and vary φ only.

Theorem 12.4. *Assume that the base field* k *is uncountable. Fix* $c \in \mathbb{P}^t$. *There is a generic subset* Y *of* $X = \operatorname{Aut} \mathbb{P}^t$ *such that* $R(\varphi, c)$ *is noetherian for all* $\varphi \in Y$.

Proof. By Lemma 4.2(2) there is no harm in assuming that $c = (1 : 1 : \cdots : 1)$. Choose some homogeneous coordinates $(z_{ij})_{0 \leqslant i,j \leqslant t}$ for $X \subseteq \mathbb{P}(M_{t+1}(k))$. Just as in the proof of Theorem 12.3, we see that $\mathscr{C} = \{c_i\}_{i\in\mathbb{Z}}$ fails to be in general position if and only if there exists some $d \geqslant 1$ and some choice of $N = \binom{t+d}{d}$ nonnegative integers $a_1 < a_2 < \cdots < a_N$ such that $\det f_i(c_{a_j}) = 0$, where the f_i are the degree d monomials in U.

Each condition $\det f_i(c_{a_j}) = 0$ is a closed condition in the coordinates of X; moreover it does not hold identically, otherwise for no choice of φ would $\mathscr{C}$ be in general position, in contradiction to the proof of Theorem 12.3. There are countably many such conditions, and so the complement Y of the union of all of these closed subsets is generic by definition. Thus for $\varphi \in Y$ we have that $\mathscr{C}$ is in general position and so $R(\varphi, c)$ is noetherian, by Theorem 5.12. $\square$

It is not hard, in contrast to the preceding theorems, to come by examples of (φ, c) for which the c_i are distinct but not even dense, much less critically dense. One such example should suffice to illustrate this situation.

Example 12.5. Suppose that $t = 2$ and $\operatorname{char} k = 0$. Let

$$\varphi = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

with $c = (0 : 0 : 1)$. Then $\mathscr{C}$ is not dense in $\mathbb{P}^2$.

Proof. One easily calculates the formula $c_{-n} = (n(n-1)/2 : n : 1)$ for $n \in \mathbb{Z}$. Since $\operatorname{char} k = 0$, the c_i are obviously distinct. But the polynomial $f = x_0x_2 + \frac{1}{2}x_2x_1 - \frac{1}{2}x_1^2$ vanishes at $(n(n-1)/2 : n : 1)$ for every $n \in \mathbb{Z}$, and so $\{c_i\}_{i \in \mathbb{Z}}$ is not dense. $\square$

A similar argument will show more generally that if the Jordan canonical form of a matrix representing φ has a Jordan block of size $\geqslant 3$ or more than one Jordan block of size 2, then given any $c \in \mathbb{P}^t$, the set $\mathscr{C}$ is not dense in $\mathbb{P}^t$. See [20] for further details.

12.1. Improvements in characteristic zero

Theorems 12.3 and 12.4 hold for an algebraically closed field of arbitrary characteristic. In case where $\operatorname{char} k = 0$, we will show that one can get a better result by invoking the following theorem of Cutkosky and Srinivas.

Theorem 12.6 (Cutkosky and Srinivas [10, Theorem 7]). *Let G be a connected commutative algebraic group defined over an algebraically closed field k of characteristic 0. Suppose that $g \in G$ is such that the cyclic subgroup $H = \langle g \rangle$ is dense in G. Then any infinite subset of H is dense in G.*

The theorem has the following consequence.

Proposition 12.7. *Let $\operatorname{char} k = 0$. Then $\mathscr{C} = \{\varphi^i(c)\}_{i \in \mathbb{Z}}$ is critically dense in $\mathbb{P}^t$ if and only if $\mathscr{C}$ is Zariski dense in $\mathbb{P}^t$.*

Proof. If $\mathscr{C}$ is critically dense in $\mathbb{P}^t$, then $\mathscr{C}$ is of course dense in $\mathbb{P}^t$ by definition.

Now assume that $\mathscr{C}$ is dense. Choose a matrix $L \in \mathrm{GL}_{t+1}(k)$ to represent φ (so L is unique up to scalar multiple). Now set $V = \sum_{i \in \mathbb{Z}} kL^i \subseteq M_{t+1}(k)$. Let $\tilde{c} \in \mathbb{A}^{t+1}$ be a particular choice of coordinates for c, and think of elements of $\mathbb{A}^{t+1}$ as column vectors. Then the linear evaluation map $\widetilde{\psi} : V \to \mathbb{A}^{t+1}$ defined by $N \mapsto N\tilde{c}$ descends to a map $\psi : \mathbb{P}V \to \mathbb{P}^t$ which sends φ^i to $\varphi^i(c)$ for all $i \in \mathbb{Z}$. The map $\widetilde{\psi}$ must be surjective, else $\mathscr{C}$ would lie on a proper linear subspace of $\mathbb{P}^t$. Note also that since L satisfies its characteristic polynomial, $\dim_k V \leqslant t+1$. This forces $\widetilde{\psi}$ to be an isomorphism, and so ψ is an isomorphism of projective spaces. In particular, writing $H = \{\varphi^i\}_{i \in \mathbb{Z}}$, we have via the automorphism ψ that H is dense in $\mathbb{P}V$.

Now let $G = \mathbb{P}V \cap \mathrm{PGL}_{t+1}(k)$. Then G is an algebraic group, since it is the closure of the subgroup H of $\mathrm{PGL}_{t+1}(k)$ [8, Proposition 1.3]. Since any two elements of V commute, G is commutative. Note also that G is an open subset of the projective space $\mathbb{P}V$, so G is irreducible and in particular connected. Finally, we always assume that the field k is algebraically closed, so the hypotheses of Theorem 12.6 are all satisfied.

Now H is dense in G, so H is critically dense in G by Theorem 12.6. Then as a subset of $\mathbb{P}V$, H is critically dense in $\mathbb{P}V$. Finally, applying ψ again we get that $\mathscr{C}$ is critically dense in $\mathbb{P}^t$. $\square$

Thus in case $\operatorname{char} k = 0$, the question of the noetherian property for $R(\varphi, c)$ reduces to the question of the density of $\mathscr{C} = \{\varphi^i(c)\}_{i \in \mathbb{Z}}$, which is easy to analyze for particular choices of φ and c. in particular, $\mathscr{C}$ will be dense if and only if c is not contained in a proper closed set $X \subsetneq \mathbb{P}^t$ with $\varphi(X) = X$. Let us note some specific examples. Note that part (1) of the following example is a significant improvement over Theorem 12.3 if the field has zero characteristic.

Example 12.8. *Let* $\operatorname{char} k = 0$.

(1) *Suppose that* $\varphi = \operatorname{diag}(1, p_1, \ldots, p_t)$, *and that the multiplicative subgroup of* $k^\times$ *generated by* $p_1, p_2, \ldots p_t$ *is* $\cong \mathbb{Z}^t$. *Let* c *be the point* $(a_0 : a_1 : \cdots : a_t)$. *Then* $R(\varphi, c)$ *is noetherian if and only if* $a_i \neq 0$ *for all* $0 \leqslant i \leqslant t$.
(2) *Let*

$$\varphi = \begin{bmatrix} 1 & 1 & & & \\ 0 & 1 & & & \\ & & p_2 & & \\ & & & \cdots & \\ & & & & p_t \end{bmatrix}$$

such that the multiplicative subgroup of $k^\times$ generated by the $p_2, \dots, p_t$ is $\cong \mathbb{Z}^{t-1}$. Let $c = (a_0 : a_1 : \cdots : a_t) \in \mathbb{P}^t$. Then $R(\varphi, c)$ is noetherian if and only if $a_i \neq 0$ for all $1 \leqslant i \leqslant t$.

Proof. (1) Let ϕ be the automorphism of U corresponding to φ; explicitly (up to scalar multiple), $\phi(x_i) = p_i x_i$, if we set $p_0 = 1$. Suppose that J is a graded ideal of U with $\phi(J) = J$. Then if we choose $m \gg 0$ such that $J_m \neq 0$, then there is some $0 \neq f \in J_m$ with $\phi(f) \in kf$, since the action of ϕ on the finite-dimensional vector space J_m has an eigenvector. If $f = \sum b_I x_I$ (where I is a multi-index), then $\phi(f) = \sum b_I p_I x_I$. The hypothesis on the p_i forces p_I to be distinct for distinct multi-indices I of degree m, so f must be a scalar multiple of a single monomial in the x_i. Thus any closed set $X \subsetneq \mathbb{P}^t$ with $\varphi(X) = X$ is contained in the union of hyperplanes $\bigcup_{i=0}^t \{x_i = 0\}$. It follows that if all $a_i \neq 0$ then $\mathscr{C}$ is dense. Conversely, if some $a_i = 0$ then $\mathscr{C}$ is contained in the hyperplane $x_i = 0$ and $\mathscr{C}$ is not dense. Now the result follows from Lemma 12.7 and Theorem 5.12.

(2) The automorphism ϕ of U corresponding to φ is given by $\phi(x_0) = x_0 + x_1$, $\phi(x_1) = x_1$, and $\phi(x_i) = p_i x_i$ for $2 \leqslant i \leqslant t$. If J is a graded ideal of U with $\phi(J) = J$, then as above there is some $0 \neq f \in J_m$ with $\phi(f) \in kf$. We leave it to the reader to show that this forces f to be scalar multiple of a monomial in $x_1, x_2, \dots, x_t$ only; the rest of the proof is as in part (1). $\square$

Lemma 12.7 and Example 12.8(1) fail in positive characteristic. The next example, which we thank Mel Hochster for suggesting, shows this explicitly.

Example 12.9. *Let k have characteristic $p > 0$ and let $y \in k$ be transcendental over the prime subfield $\mathbb{F}_p$. Suppose that $t = 2$, and let $\varphi = \operatorname{diag}(1, y, y+1)$ and $c = (1:1:1)$. The multiplicative subgroup of $k^\times$ generated by y and $y+1$ is isomorphic to $\mathbb{Z}^2$, but both sets of points $\{c_i\}_{i \geqslant 0}$ and $\{c_i\}_{i \leqslant 0}$ are dense but not critically dense in $\mathbb{P}^t$. The ring $R(\varphi, c)$ is neither left nor right noetherian.*

Proof. It is easy to see since y is transcendental over $\mathbb{F}_p$ that the multiplicative subgroup of $k^\times$ generated by y and $y+1$ is isomorphic to $\mathbb{Z}^2$.

We have $c_{-n} = \varphi^n(c) = (1, y^n, (y+1)^n)$, so c has infinite order under φ. If $n = p^j$ for some $j \geqslant 0$, then $(y+1)^n = y^n + 1$. Therefore c_{-n} is on the line $X = \{x_0 + x_1 - x_2 = 0\}$ for all $n = p^j$. On the other hand, suppose that $n \geqslant 0$ is not a power of p. Then some binomial coefficient $\binom{n}{i}$ with $0 < i < n$ is not divisible by p, and the binomial expansion of $(y+1)^n$ contains the nonzero term $\binom{n}{i} y^i$. Since y is transcendental over $\mathbb{F}_p$, this implies $(y+1)^n \neq y^n + 1$ and so c_{-n} is not on the line X. Thus for $n \geqslant 0$, c_{-n} is on X if and only if n is a power of p. It follows that the set of points $\{c_i\}_{i \leqslant 0}$ is not critically dense. By Theorem 5.12, R is not right noetherian.

Put $\mathscr{D} = \{c_i\}_{i \leqslant 0}$, and consider the Zariski closure $\bar{\mathscr{D}}$ of this set of points. Since the line X contains infinitely many points of $\mathscr{D}$, $X \subseteq \bar{\mathscr{D}}$. For all $n \in \mathbb{Z}$, $\varphi^n(X)$ also contains infinitely many points of $\mathscr{D}$, and so $\bigcup_{n \in \mathbb{Z}} \varphi^n(X) \subseteq \bar{\mathscr{D}}$. Finally, one checks that the $\varphi^n(X)$ are distinct lines for all $n \in \mathbb{Z}$. It follows that $\bar{\mathscr{D}} = \mathbb{P}^2$, and $\mathscr{D}$ is a Zariski dense set.

Analogously, one may check that if Y is the curve $\{x_1x_2 + x_2x_0 - x_0x_1\} = 0$, then $c_n \in Y$ for $n \geqslant 0$ if and only if $n = p^j$ for some $j \geqslant 0$, so that $\{c_i\}_{i \geqslant 0}$ is not critically dense. Yet a similar proof as above shows that $\{c_i\}_{i \geqslant 0}$ is Zariski dense. According to Theorem 5.12, R is also not left noetherian. $\square$

13. Algebras generated by Eulerian derivatives

The original motivation for our study of the algebras $R(\varphi, c)$ came from the results of Jordan [14] on algebras generated by two Eulerian derivatives. In this final section we show that Jordan's examples are special cases of the algebras $R(\varphi, c)$, and so we may use our previous results to answer the main open question of [14], namely whether algebras generated by two Eulerian derivatives are ever noetherian. In fact, we will prove that an algebra generated by a generic finite set of Eulerian derivatives is noetherian.

Fix a Laurent polynomial algebra $k[y^{\pm 1}] = k[y, y^{-1}]$ over the base field k.

Definition 13.1. For $p \in k \backslash \{0, 1\}$, we define the operator $D_p \in \operatorname{End}_k k[y^{\pm 1}]$ by the formula $f(y) \mapsto \frac{f(py) - f(y)}{py - y}$. For $p = 1$, we define $D_1 \in \operatorname{End}_k k[y^{\pm 1}]$ by the formula $f \mapsto df/dy$. For any $p \neq 0$, we call D_p an *Eulerian Derivative*.

It is also useful to let y^{-1} be notation for the operator $y^{-1} : y^i \mapsto y^{i-1}$ for $i \in \mathbb{Z}$.

We now consider algebras generated by a finite set of Eulerian derivatives. There are naturally two cases, depending on whether D_1 is one of the generators.

Theorem 13.2. *Suppose that* $\{p_1, \dots p_t\} \in k \backslash \{0, 1\}$ *are distinct, and assume that the multiplicative subgroup of* $k^\times$ *these scalars generate is* $\cong \mathbb{Z}^t$. *Let* $R = k\langle D_{p_1}, D_{p_2}, \dots, D_{p_t}\rangle$. *Then* $R \cong R(\varphi, c)$ *for*

$$\varphi = \operatorname{diag}(1, p_1^{-1}, p_2^{-1}, \dots, p_t^{-1}) \quad \textit{and} \quad c = (1 : 1 : \cdots : 1).$$

R is noetherian if either $\operatorname{char} k = 0$ *or if the* $\{p_i\}$ *are algebraically independent over the prime subfield of* k.

Proof. Set $p_0 = 1$ and let $w_i = y^{-1} + (p_i - 1)D_{p_i}$ for $0 \leqslant i \leqslant t$. The automorphism ϕ of U corresponding to φ is given (up to scalar multiple) by $\phi : x_i \mapsto p_i^{-1}x_i$ for $0 \leqslant i \leqslant t$. An easy calculation shows that $S(\varphi)$ has relations $\{x_jx_i - p_j^{-1}p_ix_ix_j\}$ for $0 \leqslant i < j \leqslant t$; clearly these relations generate the ideal of relations for $S(\varphi)$, since $S(\varphi)$ has the Hilbert function of a polynomial ring in $t + 1$ variables.

As in [14, Section 2], it is straightforward to prove the identities $w_j w_i - p_j^{-1} p_i w_i w_j$ for all $0 \leqslant i, j \leqslant t$, so there is a surjection of algebras given by

$$\psi : S(\varphi) \to k\langle y^{-1}, D_{p_1}, \dots, D_{p_t} \rangle \subseteq \operatorname{End} k[y^{\pm 1}]$$
$$x_i \mapsto w_i, \quad 0 \leqslant i \leqslant t.$$

Now the hypothesis that the $\{p_i\}$ generate a rank t subgroup of $k^\times$ ensures that ψ is injective: this is proved for the case $t = 2$ in [14, Proposition 1]; the proof in general is analogous. Thus ψ is an isomorphism of algebras. Then one checks that the image under ψ of the subalgebra $R(\varphi, c)$ of $S(\varphi)$ is $k\langle D_{p_1}, D_{p_2}, \dots, D_{p_t} \rangle = R$.

The noetherian property for R follows from Example 12.8(1) in case char $k = 0$, or from Theorem 12.3 if the $\{p_i\}$ are algebraically independent over the prime subfield of k. □

The case where D_1 is one of the generators is very similar, and we only sketch the proof.

Theorem 13.3. *Assume that* char $k = 0$. *Let* $\{p_2, p_3, \dots p_t\} \in k \backslash \{0, 1\}$ *be distinct, and assume that the multiplicative subgroup of* $k^\times$ *that the* $\{p_i\}$ *generate is* $\cong \mathbb{Z}^{t-1}$. *Let* $R = k\langle D_1, D_{p_2}, D_{p_3}, \dots, D_{p_t} \rangle$. *Then* $R \cong R(\varphi, c)$ *for*

$$\varphi = \begin{bmatrix} 1 & 1 & & & \\ 0 & 1 & & & \\ & & p_2^{-1} & & \\ & & & \cdots & \\ & & & & p_t^{-1} \end{bmatrix} \quad \textit{and} \quad c = (0 : 1 : 1 : \cdots : 1).$$

The ring R *is noetherian.*

Proof. Let $p_1 = 1$, and let $w_i = y^{-1} + (p_i - 1) D_{p_i}$ for $1 \leqslant i \leqslant t$. As in the preceding proposition, one calculates the relations for the algebra $S(\varphi)$, and using these and the identities in [14, Section 2], one gets an algebra surjection

$$\psi : S(\varphi) \to k\langle y^{-1}, D_1, D_{p_2}, \dots, D_{p_t} \rangle,$$
$$x_0 \mapsto -D_1,$$
$$x_i \mapsto w_i, \ 1 \leqslant i \leqslant t.$$

The hypothesis on the $\{p_i\}$ implies that ψ is an isomorphism, by an analogous proof to that of [14, Proposition 1]. Then $R(\varphi, c)$ is mapped isomorphically onto R. The noetherian property for R follows from Example 12.8(2). □

The results above easily imply that a ring generated by a generic set of Eulerian derivatives is noetherian.

Theorem 13.4. *Assume that k is uncountable. Let V_i be the closed set $\{y_i = 0\}$ in $\mathbb{A}^t$. There is a generic subset $Y \subseteq \mathbb{A}^t \setminus \bigcup_{i=1}^t V_i$ such that if $(p_1, p_2, \ldots, p_t) \in Y$ then $R = k\langle D_{p_1}, D_{p_2}, \ldots D_{p_t}\rangle$ is noetherian.*

Proof. Let $k[y_1, y_2, \ldots y_t]$ be the coordinate ring of $\mathbb{A}^t$, and write $V(f)$ for the vanishing set in $\mathbb{A}^t$ of $f \in k[y_1, y_2, \ldots y_t]$. Let $\mathbb{F}$ be the prime subfield of k, and set $A = \mathbb{F}[y_1, y_2, \ldots y_t]$. The set Y of points $(p_1, p_2, \ldots, p_t) \subseteq \mathbb{A}^t$ where the $\{p_i\}$ are algebraically independent over $\mathbb{F}$ is the complement in $\mathbb{A}^t$ of $\bigcup_{f \in A} V(f)$. But since $\mathbb{F}$ is countable, A is also countable and so Y is a generic subset of $\mathbb{A}^t$ (Definition 12.1). Now apply Theorem 13.2. $\square$

We can also produce an example of a ring generated by Eulerian derivatives that is *not* noetherian, the existence of which was also an open question in [14].

Proposition 13.5. *Assume that* $\operatorname{char} k = p > 0$ *and that k has transcendence degree at least* 1 *over its prime subfield $\mathbb{F}_p$. Then there exist scalars $p_1, p_2 \in k$ such that the ring $k\langle D_{p_1}, D_{p_2}\rangle$ is not noetherian.*

Proof. Let $y \in k$ be transcendental over $\mathbb{F}_p$. Consider the ring $R(\varphi, c)$ of Example 12.9, where $\varphi = \operatorname{diag}(1, y, y+1)$ and $c = (1:1:1)$. As in Example 12.9, the scalars $y, y+1$ generate a rank 2 multiplicative subgroup of the field k, so setting $p_1 = y$ and $p_2 = y+1$ we have $R(\varphi, c) \cong k\langle D_{p_1}, D_{p_2}\rangle$ by Theorem 13.2. But as we saw in Example 12.9, $R(\varphi, c)$ is not noetherian. $\square$

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Appendix A. Castelnuovo–Mumford regularity

In this appendix, we discuss the notion of Castelnuovo–Mumford regularity and use some recent results in this subject to prove the technical lemmas in the main body of the paper. We thank Jessica Sidman for alerting us to these methods.

Let $U = k[x_0, x_1, \ldots, x_t]$ be a polynomial ring over an algebraically closed field k, graded as usual with $\deg(x_i) = 1$ for all i. All multiplication in this appendix is commutative, and so we omit the $\circ$ notation which we introduced in Section 3. Generally speaking, the notion of regularity for a U-module M is a convenient way of encapsulating information about the degrees of the generators of all of the syzygies of M.

Definition A.1 (Eisenbud [11, p. 509]). Let $M \in U\text{-gr}$. Take a minimal graded free resolution of M:

$$0 \to \bigoplus_{i=1}^{r_{(t+1)}} U[-e_{i,t+1}] \to \cdots \to \bigoplus_{i=1}^{r_0} U[-e_{i,0}] \to M \to 0.$$

If $e_{i,j} \leqslant m + j$ for all i, j then we say that M is *m-regular*. The *regularity* of M, $\operatorname{reg} M$, is the smallest integer m for which M is m-regular (if $M = 0$ then we set $\operatorname{reg} M = -\infty$).

There are other equivalent characterizations of regularity, with different advantages; see for example [7, Definition 3.2].

Regularity behaves well with respect to exact sequences.

Lemma A.2 (Eisenbud [11, Corollary 20.19]). *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence in* U-gr. *Then*

(1) $\operatorname{reg} M' \leqslant \max(\operatorname{reg} M, \operatorname{reg} M'' + 1)$.
(2) $\operatorname{reg} M \leqslant \max(\operatorname{reg} M', \operatorname{reg} M'')$.
(3) $\operatorname{reg} M'' \leqslant \max(\operatorname{reg} M' - 1, \operatorname{reg} M)$.

Let us define some related notions. For I a graded ideal of U, we define the *saturation* of I to be

$$I^{\text{sat}} = \{x \in U \mid (U_{\geqslant n})x \subseteq I \text{ for some } n\}.$$

The ideal I^{sat} is the unique largest extension of I inside U by a torsion module. If $I^{\text{sat}} = I$ then we say that I is *saturated*.

A module or ideal that is m-regular stabilizes in degree m in the following important ways.

Lemma A.3. (1) *If* $M \in U$-gr *is m-regular, then M is generated in degrees less than or equal to m.*

(2) *If I is a graded ideal of U then* $(\operatorname{sat} I)_{\geqslant m} = I_{\geqslant m}$ *for* $m = \operatorname{reg} I$.

(3) *If I is a graded ideal of U, then I is m-regular if and only if* $I_{\geqslant m}$ *is m-regular.*

(4) *If* $M \in U$-gr *then the Hilbert function of* $F(n) = \dim_k M_n$ *of M agrees with the Hilbert polynomial of M in degrees* $\geqslant (\operatorname{reg} M + 1) - (t + 1 - \operatorname{pd}(M))$, *where* $\operatorname{pd}(M)$ *is the projective dimension of M.*

Proof. (1) is immediate from Definition A.1, and (2) and (3) follow from [7, Definition 3.2]. For (4), note that the Hilbert function of U, $f(n) = \dim_k U_n$, agrees with its Hilbert polynomial $(n+t)(n+t-1)\cdots(n+1)/t!$ for $n \geqslant -t$. Then calculating the Hilbert function of M from its minimal free resolution, the result follows from the definition of regularity. $\square$

The regularity of an ideal $I \subseteq U$ might be much greater than the minimal generating degree of I, but at least there is the following bound.

Lemma A.4 (Bayer and Mumford [7, Proposition 3.8]). *Let I be a homogeneous ideal of U, and let d be the minimal generating degree of I. Then* $\operatorname{reg} I \leqslant (2d)^{t!}$.

The key ingredients in the proofs of our needed lemmas will be the following recent theorems of Conca and Herzog concerning the regularity of products.

Theorem A.5 (Conca and Herzog [9, Theorem 2.5]). *If I is a graded ideal of U with* $\dim U/I \leqslant 1$, *then for any $M \in U$-gr we have* $\operatorname{reg} IM \leqslant \operatorname{reg} I + \operatorname{reg} M$.

Theorem A.6 (Conca and Herzog [9, Theorem 3.1]). *Let $I_1, I_2, \ldots, I_e$ be (not necessarily distinct) nonzero ideals of U generated by linear forms. Then* $\operatorname{reg}(I_1 I_2 \cdots I_e) = e$.

We may now make the following observations about the regularity of an ideal of a finite set of points with multiplicites.

Lemma A.7. *Let $d_1, d_2, \ldots, d_n$ be distinct points of $\mathbb{P}^t$ with ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$. Let $0 < e_i$ for $1 \leqslant i \leqslant n$ and set $e = \sum e_i$. Let $J = \mathfrak{m}_1^{e_1} \cap \mathfrak{m}_2^{e_2} \cap \cdots \cap \mathfrak{m}_n^{e_n}$.*

(1) $\operatorname{reg} J \leqslant e$.
(2) *If the points $d_1, d_2, \ldots, d_n$ do not all lie on a line, then* $\operatorname{reg} J \leqslant e-1$.

Proof. (1) Let $I = \mathfrak{m}_1^{e_1}\mathfrak{m}_2^{e_2}\cdots\mathfrak{m}_n^{e_n}$. It is easy to see that J is saturated, and that J/I is torsion, so that J is the saturation of I. By Theorem A.6, $\operatorname{reg} I \leqslant e$. Then $J_{\geqslant e} = (I^{\mathrm{sat}})_{\geqslant e} = I_{\geqslant e}$ by Lemma A.3(2). By Lemma A.3(3), $\operatorname{reg} J \leqslant e$.

(2) The hypothesis on the points forces some three of the points $\{d_i\}$ to be non-collinear (in particular $n \geqslant 3$); by relabeling we may assume that d_1, d_2, d_3 do not lie on a line. Then one may check that $U/(\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3)_{\geqslant 1}$ is isomorphic to $(U/\mathfrak{m}_1 \oplus U/\mathfrak{m}_2 \oplus U/\mathfrak{m}_3)_{\geqslant 1}$ and so this module is 1-regular. Then by Lemma A.3(2), $U/(\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3)$ is 1-regular, so using Lemma A.2 we get that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3$ is 2-regular.

Now $L = (\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3)(\mathfrak{m}_1^{e_1-1} \cap \mathfrak{m}_2^{e_2-1} \cap \mathfrak{m}_3^{e_3-1} \cap \mathfrak{m}_4^{e_4} \cap \cdots \cap \mathfrak{m}_n^{e_n})$ is $(e-1)$-regular, using part (1) and Theorem A.5. Since $L^{\mathrm{sat}} = J$, a similar argument as in part (1) shows that J is $(e-1)$-regular. $\square$

Now we prove the series of results which we used in the body of the paper. The first is an immediate corollary of the proof of part (1) of the preceding lemma.

Lemma A.8 (Lemma 4.3). *Let $m_1, m_2, \dots, m_n$ be the ideals of U corresponding to distinct points $d_1, \dots, d_n$ in $\mathbb{P}^t$. Then $(\prod_{i=1}^n \mathfrak{m}_i)_{\geqslant n} = (\bigcap_{i=1}^n \mathfrak{m}_i)_{\geqslant n}$.*

Next is a simple Hilbert function calculation, which is presumably well known. Because this lemma is so fundamental above, we include a brief proof which uses the methods of regularity.

Lemma A.9 (Lemma 4.5). *Let $e_i > 0$ for all $1 \leqslant i \leqslant n$, and let $e = \sum e_i$. Set $J = \bigcap_{i=1}^n \mathfrak{m}_i^{e_i}$ for some distinct point ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$. Then $\dim_k J_m = \binom{m+t}{t} - \sum_i \binom{e_i+t-1}{t}$ for all $m \geqslant e-1$.*

In particular, if $J = \bigcap_{i=1}^n \mathfrak{m}_i$ then $\dim_k J_m = \binom{m+t}{t} - n$ for $m \geqslant n-1$.

Proof. We leave it to the reader to show that the Hilbert polynomial of the module $U/\mathfrak{m}_i^{e_i}$ is the constant $\binom{e_i+t-1}{t}$, for example by induction on e_i. Then since the points d_i are distinct, the Hilbert polynomial of J is $H(m) = \binom{m+t}{t} - \sum_i \binom{e_i+t-1}{t}$.

By Lemma A.7(1) $\operatorname{reg} J \leqslant e$. By the Auslander–Buchsbaum formula we have since $\operatorname{depth}(U/J) = 1$ that $\operatorname{pd}(U/J) = t$, and so $\operatorname{pd}(J) = t-1$. Now it follows from Lemma A.3(4) that the Hilbert function of J agrees with its Hilbert polynomial in degrees $\geqslant e-1$. $\square$

Lemma A.10 (Lemma 6.3). *Let the points $d_1, d_2, \dots, d_n, d_{n+1} \in \mathbb{P}^t$ be distinct, and assume that the points $d_1, \dots, d_n$ do not all lie on a line. Let $\mathfrak{m}_i \subseteq U$ be the homogeneous ideal corresponding to d_i.*

(1) $(\bigcap_{i=1}^n \mathfrak{m}_i)_{n-1} (\mathfrak{m}_{n+1})_1 = (\bigcap_{i=1}^{n+1} \mathfrak{m}_i)_n$.
(2) $(\bigcap_{i=1}^n \mathfrak{m}_i)_{n-1} (\mathfrak{m}_1)_1 = (\bigcap_{i=2}^n \mathfrak{m}_i \cap \mathfrak{m}_1^2)_n$.
(3) $(\bigcap_{i=2}^n \mathfrak{m}_i \cap \mathfrak{m}_1^2)_n (\mathfrak{m}_{n+1})_1 = (\bigcap_{i=2}^{n+1} \mathfrak{m}_i \cap \mathfrak{m}_1^2)_{n+1}$.
(4) *Let $b_1, b_2 \in \mathbb{P}^t$, with corresponding ideals* $\mathfrak{n}_1, \mathfrak{n}_2$, be such that $b_j \neq d_i$ for $j = 1, 2$ and $1 \leqslant i \leqslant n$. Then $(\bigcap_{i=1}^n \mathfrak{m}_i \cap \mathfrak{n}_1)_n = (\bigcap_{i=1}^n \mathfrak{m}_i \cap \mathfrak{n}_2)_n$ implies $b_1 = b_2$.

Proof. (1) Set $K = \bigcap_{i=1}^n \mathfrak{m}_i$, $L = \mathfrak{m}_{n+1}$, and $M = \bigcap_{i=1}^{n+1} \mathfrak{m}_i$. By Lemma A.7, we have that $\operatorname{reg} K \leqslant n-1$ and $\operatorname{reg} L \leqslant 1$. Then by Theorem A.5, $\operatorname{reg}(KL) \leqslant n$. Since clearly $M = (KL)^{\mathrm{sat}}$, by Lemma A.3(2) it follows that $(KL)_n = M_n$. Finally, by Lemma A.3(1), K is generated in degrees $\leqslant n-1$ and L is generated in degree 1. Thus $K_{n-1} L_1 = (KL)_n = M_n$.

(2)–(3) The proofs of these parts are very similar to the proof of (1) and are omitted.

(4) The ideals $K = (\bigcap_{i=1}^n \mathfrak{m}_i \cap \mathfrak{n}_1)$ and $L = (\bigcap_{i=1}^n \mathfrak{m}_i \cap \mathfrak{n}_2)$ are each n-regular by Lemma A.7, so both are generated in degrees $\leqslant n$. Now since $b_1 \neq d_i$ for all i, if $b_1 \neq b_2$ then the ideals K and L must differ in large degree, so they must differ in degree n. $\square$

We close with a simple application of regularity to the analysis of bounds for $\underline{\mathrm{Ext}}$ groups.

Lemma A.11. *Given $M, N \in U$-gr, there is a constant $d \in \mathbb{Z}$, depending only on* $\operatorname{reg} M$ *and* $\operatorname{reg} N$, *such that* $\operatorname{reg}(\underline{\mathrm{Ext}}_U^i(M,N)) < d$ *for all $i \geqslant 0$.*

Proof. Take a minimal graded free resolution of M:

$$0 \to \bigoplus_{i=1}^{r_{(t+1)}} U[-e_{i,t+1}] \to \cdots \to \bigoplus_{i=1}^{r_0} U[-e_{i,0}] \to M \to 0.$$

By the definition of regularity, $e_{i,j} \leqslant \operatorname{reg} M + (t+1)$ for all $i,j \geqslant 0$. Applying $\underline{\mathrm{Hom}}(-,N)$ to the complex with the M term deleted produces a complex

$$0 \to L_0 \xrightarrow{\psi^0} L_1 \xrightarrow{\psi^1} \cdots \xrightarrow{\psi^t} L_{t+1} \to 0$$

where $L_j = \oplus_{i=1}^{r_j} N[-e_{i,j}]$. Then $\operatorname{reg} L_j \leqslant (\operatorname{reg} M + \operatorname{reg} N + t + 1)$ for all $j \geqslant 0$.

Now consider the map $\psi^i : L_i \to L_{i+1}$ for some $i \geqslant 0$. Certainly L_i is generated in degrees less than or equal to $\operatorname{reg} L_i$, by Lemma A.3(1). Then $\operatorname{Im} \psi^i$ is also generated in degrees less than or equal to $\operatorname{reg} L_i$. By Lemma A.4, $\operatorname{reg}(\operatorname{Im} \psi^i) \leqslant f(\operatorname{reg} L_i)$ where $f(x) = (2x)^{t!}$. By Lemma A.2(1),

$$\operatorname{reg}(\ker \psi^i) \leqslant \max(\operatorname{reg} L_i, \operatorname{reg}(\operatorname{Im} \psi^i) + 1) \leqslant f(\operatorname{reg} L_i) + 1.$$

Finally, $\underline{\mathrm{Ext}}^i(M,N) \cong \ker \psi^i / \operatorname{Im} \psi^{i-1}$ and so by A.2(3),

$$\begin{aligned} \operatorname{reg}(\underline{\mathrm{Ext}}^i(M,N)) &\leqslant \max(f(\operatorname{reg} L_i) + 1, f(\operatorname{reg} L_{i-1})) \\ &\leqslant f(\operatorname{reg} M + \operatorname{reg} N + t + 1) + 1 \end{aligned}$$

and thus we may take $d = f(\operatorname{reg} M + \operatorname{reg} N + t + 1) + 1$. $\square$

The final lemma is an easy corollary of the preceding one.

Lemma A.12 (Lemma 8.5). *Let I, J be any homogeneous ideals of U, and let ϕ be an automorphism of U. Then there is some fixed $d \geqslant 0$ such that for all $n \in \mathbb{Z}$ such that $U/(I + \phi^n(J))$ is bounded, $\underline{\mathrm{Ext}}_U^i(U/I, U/\phi^n(J))$ is also bounded for all i, with d as a right bound.*

Proof. If $U/(I + \phi^n(J))$ is bounded, then $E^n = \underline{\mathrm{Ext}}_U^i(U/I, U/\phi^n(J))$ is certainly also bounded, since it is killed by $I + \phi^n(J)$. It is clear from the definition of regularity that the modules $\{U/\phi^n(J)\}_{n \in \mathbb{Z}}$ all have the same regularity, and so by Lemma A.11 there is some bound $d \geqslant 0$ such that $\operatorname{reg} E^n \leqslant d$ for all $n \in \mathbb{Z}$. Then if E^n is bounded, d is a right bound for it, by Lemma A.3(4). $\square$

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