The minimum span of L(2,1)-labelings of certain generalized Petersen graphs

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Abstract

In the classical channel assignment problem, transmitters that are sufficiently close together are assigned transmission frequencies that differ by prescribed amounts, with the goal of minimizing the span of frequencies required. This problem can be modeled through the use of an L(2,1)-labeling, which is a function \(f\) from the vertex set of a graph \(G\) to the non-negative integers such that \(|f(x) - f(y)| \geq 2\) if \(x\) and \(y\) are adjacent vertices and \(|f(x) - f(y)| \geq 1\) if \(x\) and \(y\) are at distance two. The goal is to determine the \(\lambda\)-number of \(G\), which is defined as the minimum span over all L(2,1)-labelings of \(G\), or equivalently, the smallest number \(k\) such that \(G\) has an L(2,1)-labeling using integers from \(\{0, 1, \ldots, k\}\). Recent work has focused on determining the \(\lambda\)-number of generalized Petersen graphs (GPGs) of order \(n\). This paper provides exact values for the \(\lambda\)-numbers of GPGs of orders 5, 7, and 8, closing all remaining open cases for orders at most 8. It is also shown that there are no GPGs of order 4, 5, 8, or 11 with \(\lambda\)-number exactly equal to the known lower bound of 5, however, a construction is provided to obtain examples of GPGs with \(\lambda\)-number 5 for all other orders. This paper also provides an upper bound for the number of distinct isomorphism classes for GPGs of any given order. Finally, the exact values for the \(\lambda\)-number of \(n\)-stars, a subclass of the GPGs inspired by the classical Petersen graph, are also determined. These generalized stars have a useful representation on Möbius strips, which is fundamental in verifying our results.

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1. Introduction

Graph labelings are commonly used to model the channel assignment problem [10], wherein one must assign frequencies to transmitters (radio, TV, cell phones, etc.) in a network so that interfering transmitters (interference usually due to geographic proximity) are assigned different frequencies. Each transmitter can be represented by a vertex in a graph, and edges connect pairs of vertices corresponding to interfering transmitters. Frequencies, represented by non-negative integers, must then be assigned to each vertex. In one variation, transmitters modeled with adjacent
references on L(2,1)-labelings is provided in two recently published surveys [2, 20].

Fig. 1. A 9-labeling of the Petersen graph and an alternative drawing.

vertices must be assigned frequencies that are at least two apart and transmitters modeled with vertices separated by distance two must be assigned frequencies that are at least one apart. This type of channel assignment is known in the literature as an L(2,1)-labeling, first introduced by Griggs and Yeh in 1992 [9]. An L(2,1)-labeling of a graph G is a function f from the vertex set of G to the set of non-negative integers such that |f(x) − f(y)| ≥ 2 if d(x, y) = 1 and |f(x) − f(y)| ≥ 1 if d(x, y) = 2, where d(x, y) denotes the distance between the pair of vertices x, y. A vast array of references on L(2,1)-labelings is provided in two recently published surveys [2, 20].

An L(2,1)-labeling of a graph G that uses labels in the set {0, 1, . . . , k} is called a k-labeling. In the context of the channel assignment problem, the main goal is to minimize function f from the vertex set of G to the set of non-negative integers such that |f(x)−f(y)|≥2 if d(x, y) = 1 and |f(x)−f(y)|≥1 if d(x, y) = 2, where d(x, y) denotes the distance between the pair of vertices x, y. A vast array of references on L(2,1)-labelings is provided in two recently published surveys [2, 20].

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Definition 1. A GPG of order n ≥ 3 consists of two disjoint cycles Cn, called inner and outer cycles, so that each vertex on the outer (respectively, inner) cycle is adjacent to exactly one vertex on the inner (respectively, outer) cycle. Equivalently, if G is a GPG of order n then G has vertices {w0, w1, . . . , wn−1}∪{v0, v1, . . . , vn−1} with edges {wi, wi+1} and {vi, vi+1} for all i = 0, 1, . . . , n − 1, where subscript addition is taken modulo n, and each wi (respectively, vi), i = 0, 1, . . . , n − 1 is adjacent to exactly one vj (respectively, wj) for some 0 ≤ j ≤ n − 1. The cycle on vertices {w0, w1, . . . , wn−1} (respectively, {v0, v1, . . . , vn−1}) will be called outer (respectively, inner) cycle.

The well-known Petersen graph is shown with a 9-labeling in Fig. 1 in its classical form, along with a second rendition suggested by Definition 1.

In applications involving network models, one seeks to find a balance between network connectivity, efficiency, and reliability. The double-cycle structure of the GPGs is appealing for such applications since it is superior to a tree or cycle structure as it ensures network connectivity in case of any two independent node/connection failures while keeping the number of connections at a minimum level.

This paper is organized as follows. In Section 2, we present a constructive proof showing that every family of GPGs of order n contains graphs whose λ-number achieves the minimum value of 5, with the exception of n = 4, 5, 8, and 11. In Section 3, we provide the exact λ-numbers for all GPGs of orders 5, 7, and 8, thereby closing all cases with orders up to 8. We also provide an upper bound for the number of non-isomorphic GPGs of a given order. In Section 4, we provide the exact λ-numbers of n-stars, a subclass of GPGs inspired by the symmetry of the GPG prisms and the classical Petersen graph. We close by summarizing our findings in Section 5.

2. Meeting the general lower bound on the λ-number of GPGs

Since a GPG is a 3-regular graph, the following result due to Griggs and Yeh [9] establishes the lower bound of 5 for the λ-number of any GPG:

\[ \lambda(G) \leq A^2 + A - 2 \]

where A denotes the maximum degree of a vertex of G. To this date, the best known general upper bound \( \lambda(G) \leq A^2 \) is due to Gonçalves [7]. Since determining \( \lambda(G) \) is NP-hard [5], a significant body of literature on L(2,1)-labelings focuses on verifying Griggs and Yeh’s conjecture and finding exact values for \( \lambda(G) \) for particular classes of graphs. Of these, grid-like structures have been the subject of several papers due to their natural fit with practical applications: square grids or Cartesian products of paths [12–14], triangular grids [8], and generalized Petersen graphs (GPGs) [1, 3].

\[ \lambda(G) = \lambda(G)_{\text{in}} \]

where \( \lambda(G)_{\text{in}} \) denotes the distance between the pair of vertices x, y. A vast array of references on L(2,1)-labelings is provided in two recently published surveys [2, 20].

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**Result 1.** (Griggs and Yeh [9]). If a graph $G$ contains three vertices with maximum degree $\Delta \geq 2$ and one of them is adjacent to the other two vertices, then $\lambda(G)$ is at least $\Delta + 2$.

In this section, we present a constructive proof showing that every family of GPGs of order $n$ contains graphs with $\lambda$-number 5, with the exception of $n = 4, 5, 8,$ and 11. Our construction utilizes a particular case of a result by Georges and Mauro [4] on regular graphs. In Result 2, we present this result restricted to GPGs with $\lambda$-number 5 and provide its proof not only for completeness but also since it leads to Corollary 3, not explicitly mentioned in [4], which will be instrumental in our construction.

We first introduce some notation. Consider a $k$-labeling of a graph $G$. The *multiplicity* $m(i)$ of a label $i$ in $\{0, 1, \ldots, k\}$ is defined as the number of vertices in $G$ with label $i$. For $i, j$ in $\{0, 1, \ldots, k\}$, let $m(i, j)$ denote the number of vertices with label $i$ which are adjacent to a vertex with label $j$ and let $m(i, j^*)$ be the number of vertices with label $i$ which are not adjacent to a vertex with label $j$.

**Result 2.** (Georges and Mauro [4]). Let $G$ be a 3-regular graph with $\lambda(G) = 5$. For any 5-labeling of $G$ we have that $m(1) = m(2) = m(3) = m(4) \leq m(0) = m(5)$.

**Proof.** Let $G$ be a 3-regular graph with a 5-labeling. Every vertex labeled $i$ with $1 \leq i \leq 4$ is adjacent to three vertices with the three labels in $\{0, 1, \ldots, 5\} - \{i - 1, i, i + 1\}$, respectively, and therefore $m(i) \leq m(j)$ for each $j$ in $\{0, 1, \ldots, 5\} - \{i - 1, i, i + 1\}$. So, $m(1) \leq m(3) \leq m(1) \leq m(4) \leq m(2) \leq m(4) = m(1)$ and consequently $m(1) = m(2) = m(3) = m(4)$.

To complete the proof, we must show that $m(0) = m(5)$. First notice that for $i$ in $\{2, 3, 4\}$, a vertex labeled $i$ must be adjacent to a vertex labeled 0 and thus $m(i) = m(0) - m(0, i^*)$. But this implies that $m(0, 2^*) = m(0, 3^*) = m(0, 4^*) = m(0) - m(2)$ since $m(2) = m(3) = m(4)$. Observe that if a vertex labeled 0 is not adjacent to a vertex labeled $i$ in $\{2, 3, 4\}$ then it must be adjacent to a vertex labeled 5. Therefore

$$m(0, 5) = m(0, 2^*) + m(0, 3^*) + m(0, 4^*) = 3(m(0) - m(2)).$$

(1)

For $j$ in $\{1, 2, 3\}$, $j$ must be adjacent to a vertex labeled 5 and thus $m(j) = m(5) - m(5, j^*)$. But this implies that $m(5, 1^*) = m(5, 2^*) = m(5, 3^*) = m(5) - m(2)$ since $m(1) = m(2) = m(3)$. Also notice that if a vertex labeled 5 is not adjacent to a vertex labeled $j$ in $\{1, 2, 3\}$ then it must be adjacent to a vertex labeled 0. Therefore

$$m(5, 0) = m(5, 1^*) + m(5, 2^*) + m(5, 3^*) = 3(m(5) - m(2)).$$

(2)

Since $m(0, 5) = m(5, 0)$, Eqs. (1) and (2) imply that $m(0) = m(5)$. \[\square\]

The following corollary will motivate the construction in Theorem 4, the main result of this section.

**Corollary 3.** Let $G$ be a GPG of order $n$ with a 5-labeling. Then there are non-negative integers $p$ and $q$ such that

(a) $n = 2p + q$ where $m(1) = m(2) = m(3) = m(4) = p$ and $m(0) = m(5) = q$;

(b) $m(0, i^*) = m(5, j^*) = q - p$ for $i = 2, 3, 4$ and $j = 1, 2, 3$;

(c) $m(0, 5) = m(5, 0) = 3(q - p)$;

(d) $p \leq q \leq 1.5p$.

**Proof.** Clearly $G$ is a 3-regular graph. Therefore item (a) follows from Result 2 and by noting that $G$ has $2n = 4p + 2q$ vertices. Items (b) and (c) are verified in the proof of Result 2. The first inequality in (d) also follows from Result 2. The second inequality in (d) follows from (c) since $3(q - p) = m(0, 5) \leq m(0) = q$. \[\square\]

**Theorem 4.** The lower bound 5 for the $\lambda$-number of GPGs of order $n$ is tight if and only if $n$ is not in $\{4, 5, 8, 11\}$.

**Proof.** Assume that $G$ is a GPG of order $n$ with $\lambda$-number 5. By inspection, no value in $\{4, 5, 8, 11\}$ can be written as $n = 2p + q$ where $p$ and $q$ are non-negative integers satisfying $p \leq q \leq 1.5p$, as required by items (a)
Fig. 2. Block matrices $A$ and $B$ for the construction in the proof of Theorem 4.

Fig. 3. Examples for $n=6$ and 10 of the construction in the proof of Theorem 4.

and (d) in Corollary 3. Thus $n$ is not in $\{4, 5, 8, 11\}$. Therefore, there are no GPGs of order 4, 5, 8, or 11 that have $\lambda$-number 5.

Henceforth, we must assume $n$ is not in $\{4, 5, 8, 11\}$. To prove that the lower bound 5 for the $\lambda$-number of GPGs of order $n$ is tight it is sufficient to construct a GPG of order $n$ and provide a 5-labeling for it. We will use the two matrices of labels in Fig. 2 to construct a GPG of order $n$ with a 5-labeling. In each matrix $A$ and $B$ in Fig. 2, the first row will be used to label consecutive vertices on the outer cycle, and the second row will be used to label consecutive vertices on the inner cycle. The line segments in these matrices determine the adjacencies between a vertex on the outer and a vertex on the inner cycle as follows. For matrix $A$, the vertices labeled 3, 1, 5 on the outer cycle are adjacent to the vertices labeled 0, 4, 2 on the inner cycle, respectively. For matrix $B$, the first three vertices labeled 3, 1, 5 on the outer cycle are adjacent to the first three vertices labeled 0, 4, 2 on the inner cycle, respectively, and the last four vertices labeled 0, 3, 1, 5 on the outer cycle are adjacent to the last four vertices labeled 2, 5, 4, 0 on the inner cycle, respectively.

Let $n = 3p + r$ with $0 \leq r < 3$. In order to build a GPG of order $n$, we arrange $p - 2r$ copies of matrix $A$ followed by $r$ copies of matrix $B$. As $n$ is not in $\{4, 5, 8, 11\}$, it is then the case that $p \geq 2r$ as required (since $r \geq 1$ implies $p \geq 2$, and $r=2$ implies $p \geq 4$). This new matrix has $3(p-2r)+7r=n$ columns and can be used to 5-label a GPG of order $n$ as follows: the labels in the first row are assigned to the $3p + r$ consecutive vertices on the outer cycle, while the labels in the second row, along with the line segments representing adjacencies, determine the remainder of the 5-labeling of the GPG. In Fig. 3, we illustrate the results of this construction for $n = 6$ and 10. For $n = 6$, we have $p = 2$ and $r = 0$, which leads us to arrange two copies of $A$. For $n = 10$, we have $p = 3$ and $r = 1$, which leads us to arrange one copy of $A$ followed by one copy of $B$.

We note that one can verify that the prescribed values of $p$ and $q = p + r$ satisfy all conditions in Corollary 3.

We remark that the GPG presented in the proof of Theorem 4 is not unique. For example, a block matrix obtained from any permutation of $p - 2r$ copies of matrix $A$ and $r$ copies of matrix $B$ also provides a GPG of order $n = 3p + r$ with a 5-labeling. The choices for $p$ and $q$ in Corollary 3 are not unique either. For example, for $n = 21$, the construction in Theorem 4 prescribes that $p = 7$, $r = 0$, and consequently $q = 7$. However, one could also choose $p = 6$ and $q = 9$, and a construction similar to the one presented in Theorem 4, with $r = q - p = 3$, provides a GPG of order $n = 3p + r$ with a 5-labeling.

3. Exact $\lambda$-numbers for GPGs of order 5, 7, 8

In this section, we determine the exact $\lambda$-numbers for GPGs of order 5, 7, and 8. We also provide an upper bound for the number of non-isomorphic GPGs of any given order $n$. Previously, Georges and Mauro [3] determined the exact $\lambda$-numbers for GPGs of order 3 and 4, while Adams, et al. determined the exact $\lambda$-numbers for GPGs of order 6 [1].
Result 5. Let $G$ be a GPG of order $n$.

(a) (Georges and Mauro [3]) If $n = 3$, then $\lambda(G) = 5$ and $G$ is isomorphic to $H_1$ in Fig. 4. If $n = 4$, then $\lambda(G) = 6$ if $G$ is isomorphic to $H_2$ in Fig. 4, otherwise $\lambda(G) = 7$ and $G$ is isomorphic to $H_3$ in Fig. 4.

(b) (Adams et al. [1]) If $n = 6$, then $\lambda(G) = 5$ if $G$ is isomorphic to either $G_1$ or $G_2$ in Fig. 5, otherwise $\lambda(G) = 6$.

Recall that Georges and Mauro [3] showed that the $\lambda$-number for all GPGs is bounded above by 8, excluding the case of the Petersen graph with $\lambda$-number 9. They also showed that this upper bound can be lowered to 7 for GPGs of order at most 6, and conjectured that the upper bound of 7 also holds for GPGs of orders greater than 6. Adams et al. recently proved the conjecture to be true for orders 7 and 8, and also lowered the upper bound to 6 for GPGs of order 6 [1].

In determining the $\lambda$-number for GPGs of orders 5, 7, and 8, as well as in the case of order 6 as completed in earlier work [1], it is imperative to partition the GPGs of a given order into isomorphism classes. By examining only one representative from each isomorphism class, we can greatly reduce the difficulty of determining the $\lambda$-numbers of GPGs of a given order. Table 1 displays the number of non-isomorphic GPGs of orders 3, 4, 5, and 6; the results for orders 3, 4, and 5 were found by Georges and Mauro [3], while we obtained the result for order 6 with the aid of a computer program.
Fig. 6. GPGs of order 5 with \( \lambda \)-number 6.

We demonstrate the benefit of using isomorphism classes in the concise proof of our following theorem:

**Theorem 6.** Let \( G \) be a GPG of order 5. Then \( \lambda(G) = 9 \) if \( G \) is isomorphic to the Petersen graph, otherwise \( \lambda(G) = 6 \).

**Proof.** It is well-known that any graph isomorphic to the Petersen graph has \( \lambda \)-number 9. Excluding graphs isomorphic to the Petersen graph, Georges and Mauro showed that there are three non-isomorphic GPGs of order 5, and they provided representatives from each of these classes [3]. In Fig. 6, we provide 6-labelings for each of these representatives.

From Result 1, the \( \lambda \)-number of GPGs of any order is at least 5, however, it follows from Theorem 4 that the \( \lambda \)-number of GPGs of order 5 is strictly greater than 5. Hence, we conclude that the three graphs in Fig. 6 have \( \lambda \)-number exactly 6. □

To determine the \( \lambda \)-numbers of GPGs of orders 7 and 8, we needed an upper bound for the number of distinct isomorphism classes of these GPGs. In order to find an appropriate upper bound, we referred to work completed nearly half a century ago by Golomb and Welch [6], who studied polygonal paths formed on \( n \) vertices equally spaced on a circle:

Given \( n \) equally spaced points on a circle, one may pick a first vertex in \( n \) ways, a second vertex in \( (n-1) \) ways, \( \ldots \), an \( n \)th vertex in 1 way, and return to the starting point in 1 way, for a total of \( n! \) polygonal paths. Two polygonal paths that differ only in starting point or orientation will be called identical polygons. If, besides possible difference in starting point and orientation, two polygons differ only by a plane rotation, they will be termed equivalent. If, in addition to possible differences of these three types, two polygons differ only by a reflection through some axis, they will be called similar. Using a combinatorial formula of Polya, it has been possible to obtain explicit expressions... for the number of classes \( S(n) \) of similar \( n \)-gons... It is convenient to separate the even from the odd values of \( n \). In all cases, summation is extended over the divisors \( d \) of \( n \), and \( \phi(a) \) is Euler’s totient function...

\[
S_{\text{odd}}(n) = \frac{1}{4n^2} \left( \sum_{d|n} \phi^2 \left( \frac{n}{d} \right) \cdot d! \cdot \left( \frac{n}{d} \right)^d + 2^{\frac{n-1}{2}} \cdot n^2 \cdot \frac{(n-1)}{2}! \right),
\]

\[
S_{\text{even}}(n) = \frac{1}{4n^2} \left( \sum_{d|n} \phi^2 \left( \frac{n}{d} \right) \cdot d! \cdot \left( \frac{n}{d} \right)^d + 2^{n/2} \cdot \frac{n(n+6)}{4} \cdot \frac{n}{2}! \right).
\]

We have established a relationship between Golomb and Welch’s \( n \)-gons inscribed within circles and GPGs of order \( n \). An inscribed \( n \)-gon can be obtained from a GPG of order \( n \) as follows. Recall that each GPG of order \( n \) contains an inner cycle with \( n \) vertices and an outer cycle with \( n \) vertices, and each inner cycle vertex is connected to exactly one outer cycle vertex and vice versa. Contracting the edges connecting vertices from the inner to the outer cycles leaves one cycle comprised of edges from the original outer cycle and \( n \) vertices, along with the additional edges representing the geometry of the original inner cycle. In the contracted graph, the edges from the original outer cycle correspond to the circle surrounding the \( n \)-gon, while the edges representing the geometry of the original inner cycle correspond to the closed polygonal path defined as the \( n \)-gon. Conversely, a GPG of order \( n \) can be obtained from an \( n \)-gon by...
Table 2
Number of similar \( n \)-gons
\[
\begin{array}{|c|c|}
\hline
n & S(n) \\
\hline
3 & 1 \\
4 & 2 \\
5 & 4 \\
6 & 12 \\
7 & 39 \\
8 & 202 \\
\hline
\end{array}
\]

Fig. 7. Two isomorphic GPGs that generate two non-similar polygons.

Table 3
The 28 GPGs of order 7
\[
\begin{array}{cccc}
\end{array}
\]

replacing each of the \( n \) points on the circle with two new points (vertices) connected by a line segment (edges), with one point on the outside circle (the outer cycle) and the other on the polygonal path (the inner cycle), so that the circle and polygonal path do not have points in common. If two \( n \)-gons are similar, that is, they differ only by either (i) the starting point, (ii) orientation, (iii) plane rotation, or (iv) reflection through some axis, then the corresponding GPGs obtained as described above are isomorphic with the natural graph isomorphism counterparts of (i) through (iv). It follows that non-isomorphic GPGs of order \( n \) generate non-similar \( n \)-gons. Consequently, \( GPG(n) \leq S(n) \) (cf. Tables 1 and 2).

The reversed inequality is not true in general, since there are non-similar \( n \)-gons that can be generated from isomorphic GPGs. For example, in Fig. 7, the two GPGs \( G_1 \) and \( G_2 \) are isomorphic since we can switch the placement of the outer and inner cycles in \( G_1 \) to obtain \( G_2 \). However, the 7-gons \( H_1 \) and \( H_2 \) obtained from \( G_1 \) and \( G_2 \), respectively, were shown not to be similar by Golomb and Welch [6].

From the list of GPGs generated from the 39 non-similar 7-gons [6, p. 350], we eliminated 11 isomorphic GPGs that can be obtained by switching outer and inner cycles as in Fig. 7. Table 3 contains the remaining 28 GPGs using the following notation. Given a GPG of order 7, number the consecutive vertices in the outer cycle in order, clockwise, with integers 1, 2, ..., 7 and number each vertex on the inner cycle with the same integer as the vertex on the outer cycle adjacent to it. Now, each bracketed permutation of integers 1, 2, ..., 7 in Table 3 completely describes a GPG by providing the order in which the numbers assigned to the vertices of the inner cycle appear consecutively if you go around the cycle clockwise. For example, Fig. 8 shows the drawings of [1236754] and [1246375]. (Note that Fig. 8 shows L(2,1)-labelings for the drawings of these graphs, not the numberings used to describe the graphs.)
With the assistance of a computer program written to produce $L(2,1)$-labelings, we verified that 26 GPGs of order 7 have $\lambda$-number 6, while the two GPGs in bold in the fourth column of Table 3 have $\lambda$-number 5. Fig. 8 shows these two GPGs and corresponding 5-labelings. We note that using Corollary 3 and an exhaustive case discussion, it is possible to verify directly that the only GPGs of order 7 with $\lambda$-number 5 are, up to isomorphisms, the ones in Fig. 8. Since the proof is tedious and the techniques used are straightforward, we omitted it for the sake of brevity.

**Theorem 7.** Let $G$ be a GPG of order 7. Then $\lambda(G) = 5$ if $G$ is isomorphic to $G_1$ or $G_2$ in Fig. 8, otherwise $\lambda(G) = 6$.

Using a similar approach, we generated the GPGs of order 8 from every possible 8-permutation, and then systematically eliminated GPGs that are isomorphic via rotations, reflections, or the switching of outer and inner cycles. We were left with 127 GPGs and verified that only three of them do not have 6-labelings. Out of these three, two are isomorphic through a more complicated isomorphism. In Fig. 9 we show these three graphs: $H_1$ and $H_2$ are not isomorphic since $H_1$ contains 4-cycles and $H_2$ does not; $H_2$ can be obtained from $H_3$ by drawing the edges in bold as the outer cycle and the dashed edges as the inner cycle. Since the $\lambda$-number of GPGs of order 8 is at most 7 [1], these three GPGs without 6-labelings must have $\lambda$-number exactly equal to 7. These results are summarized in Theorem 8:

**Theorem 8.** Let $G$ be a GPG of order 8. Then $\lambda(G) = 7$ if $G$ is isomorphic to $H_1$ or $H_2$ in Fig. 9, otherwise $\lambda(G) = 6$.

4. A subclass of GPGs: the $n$-stars

A *prism* is a GPG wherein the edges between vertices on the outer and inner cycles are precisely $\{w_i, v_i\}$ for $i = 0, 1, \ldots, n - 1$, according to the notation given in Definition 1. Equivalently, a prism is the Cartesian product of a path of length 2 and a cycle, denoted $P_2 \square C_n$. As these prisms and other similar grid-like structures arising from the Cartesian products of paths and/or cycles have potential for applications, the $\lambda$-numbers of these Cartesian products have been well studied, as reviewed in Section 1. In particular, the joint effort of several authors focusing specifically on the prisms gives the following result:

**Result 9.** (Jha et al. [15], Kuo and Yan [17], Georges and Mauro [4], Klavžar and Vesel [16]). Let $G$ be the prism, $P_2 \square C_n$. Then $\lambda(G) = 5$ if $n$ is a multiple of 3, otherwise $\lambda(G) = 6$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figures}
\caption{GPGs of order 7 with $\lambda$-number 5.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figures2}
\caption{GPGs of order 8 with $\lambda$-number 7.}
\end{figure}
Motivated by this success in completely characterizing the $\lambda$-number of prisms of any order $n$, we defined another subclass of GPGs that was inspired both by the symmetry of the prisms and the original Petersen graph itself.

**Definition 10.** For each odd $n$, $n \geq 5$, an $n$-star is a GPG wherein the edges between vertices on the outer and inner cycles are precisely $\{w(n-1)i/2, v_i\}$ for $i = 0, 1, \ldots, n - 1$, where subscripts are taken modulo $n$ and the notation is as introduced in Definition 1.

It follows that the $n$-stars are well defined since $\{(n - 1)i/2 \mod n, i = 0, 1, \ldots, n - 1\} = \{0, 1, 2, \ldots, n - 1\}$ for all odd $n$. We note that the 5-star is precisely the Petersen graph, and we demonstrate the 7-star and 9-star in Fig. 10 in a way that illustrates the connection with the usual drawing of the Petersen graph.

Every $n$-star has a nice representation on a Möebius strip. For instance, the 7-star and the 9-star of Fig. 10 can be drawn on Möebius strips as shown in Fig. 11. The vertices in the inner cycle of the $n$-stars of Fig. 10 appear as solid dots in Fig. 11 and the vertices on the outer cycle as white dots. The Möebius representation of an $n$-star allows us to see its structure in a new light as an interconnection of 5-cycles. The importance of the Möebius representation of the $n$-stars becomes apparent in the proof of the following Lemma as well as in the proof of Theorem 12, the main result of this section.

**Lemma 11.** Let $G$ be an $n$-star with $n \geq 7$ and $n \neq 11$. Then $G$ has a 6-labeling. If $n$ is a multiple of 3, then $G$ has a 5-labeling.

**Proof.** Let $G$ be an $n$-star with $n \geq 7$ and $n \neq 11$. Since $n$ is odd, exactly one of the three integers $n - 1, n - 3, n - 5$ is a multiple of 6. Moreover, since $n \geq 7$ and $n \neq 11$, it follows that exactly one of the three integers $n - 7, n - 9, n - 17$ is a non-negative multiple of 6. We can then write $n = 6k + m$, where $k$ is a non-negative integer and $m$ is in $\{7, 9, 17\}$. Consider the graphs $A_m$, $m = 7, 9, 17$ and $B$ shown in Fig. 12 with given 5- or 6-labelings. Construct a Möebius strip representation of the $n$-star by using a copy of $A_m$ followed by $k$ consecutive copies of $B$ in a row and connecting these copies through the addition of the following edges: $A_m$ is linked to the first copy of $B$, if one exists, by connecting the right-most diamond-shaped vertex $\blacklozenge$, triangular-shaped vertex $\blacktriangle$, and square-shaped vertex $\blacksquare$ of $A_m$ to the respective left-most $\blacklozenge$, $\blacktriangle$, $\blacksquare$, vertices of $B$; two consecutive copies of $B$ are connected in similar form; the right-most $\blacklozenge$, $\blacktriangle$, $\blacksquare$, vertices of the sequence of $A_m$ and the $k$ consecutive copies of $B$ are then connected to the respective left-most $\blacklozenge$, $\blacktriangle$, $\blacksquare$, vertices of $A_m$. By inspection, one can verify that this construction provides a 6-labeling of $G$. When $n$ is a multiple
Fig. 12. 5- and 6-labelings of $A_m, m = 7, 9, 17$ and $B$.

Fig. 13. Impossible label assignments described in the proof of Theorem 12.

of 3, then $m = 9$ and the constructed 6-labeling is actually a 5-labeling since the label 6 is never used in $A_9$ or in $B$, concluding our proof. □

**Theorem 12.** Let $G$ be an $n$-star. Then

(a) $\lambda(G) = 9$ if $n = 5$,

(b) $\lambda(G) = 7$ if $n = 11$,

(c) $\lambda(G) = 5$ if $n$ is a multiple of 3,

(d) $\lambda(G) = 6$ otherwise.

**Proof.** In item (a), $G$ is the well-known Petersen graph. To establish item (b), there is a laborious but straightforward case proof that we omit for brevity; we also verified this result with the assistance of a computer program. If $n$ is a multiple of 3, then Lemma 11 together with Result 1 imply item (c).

To show that item (d) holds, let us assume that $G$ is an $n$-star, $n \neq 5, n \neq 11$ and $n$ is not a multiple of 3. From Lemma 11, we know that there exists a 6-labeling for $G$, and from Result 1, we know that $\lambda(G) \geq 5$; therefore, to show that $\lambda(G) = 6$ it suffices to show that $G$ does not have a 5-labeling. By contradiction, assume $G$ has a 5-labeling. By Corollary 3, item (a), we have that $n = 2p + q$ and $m(1) = m(2) = m(3) = m(4) = p \leq q = m(0) = m(5)$, where $m(i)$ is the number of vertices with label $i$. If $p = q$, then the number of vertices $2n$ in the $n$-star must be a multiple of 6, and consequently $n$ must be a multiple of 3, contradicting our assumption. So, $p < q$. By Corollary 3, item (b), we have that $m(0, 3^*) = q - p \geq 1$, where $m(i, j^*)$ is the number of vertices labeled $i$ that do not have a neighbor labeled $j$. Therefore there is a vertex $u$ labeled 0 and vertices $v, w, z$ labeled 4, 2, 5, respectively, with $u$ adjacent to $v, w, z$. Notice that in the Möbius representation of the $n$-star, each edge belongs to exactly one 5-cycle or exactly two 5-cycles. Suppose first that the edge $\{u, v\}$ is the common edge between the two different 5-cycles $v, u, z, A, B$ and $v, u, w, C, D$. Since
A, B are adjacent and \(v, u, z\) have labels 4, 0, 5, respectively, the labels of \(A, B\) must be 3, 1, respectively. On the other hand, the only possible label for \(D\) would be 1 since \(D\) is adjacent to \(v\) with label 4, and \(D\) is at distance two of both \(u\) and \(w\) with labels 0, 2, respectively. But \(D\) and \(B\) would be two vertices at distance two with the same label 1, a contradiction. Therefore, the edge \(\{u, v\}\) must belong to exactly one 5-cycle. We then have two cases to consider: either edge \(\{u, z\}\) belongs to exactly two 5-cycles or it belongs to a single 5-cycle. These two situations are described in Fig. 13, respectively, where the labels 0, 4, 2, 5 in bold are labels for \(u, v, w, z\), respectively, the remaining labels are labels forced by the labels of \(u, v, w, z\), and the question mark (?) indicates that there are no possible labels for the corresponding vertex. Therefore we reached the expected contradiction and item (d) holds.

5. Conclusions

Motivated by the channel assignment problem, we studied the \(\lambda\)-number of GPGs. We determined for which \(n\) there exists a GPG of order \(n\) whose \(\lambda\)-number meets the known lower bound of 5. In particular, we showed that such a GPG exists for every order \(n\) except 4, 5, 8, and 11. This result is useful because it shows that we can achieve the minimum required number of labels for a GPG configuration with almost any number of vertices. When combined with Result 9, which implies that there exist GPGs of orders 4, 5, 8, and 11 that achieve \(\lambda\)-number 6, the lower bound for GPGs of all orders is now tight: a tight lower bound of 5 holds for the \(\lambda\)-number of GPGs of all orders except 4, 5, 8, and 11, and a tight lower bound of 6 holds for these exceptional orders.

Next, we determined the exact \(\lambda\)-numbers for all GPGs of order 5, 7, and 8, thereby closing all remaining open cases up to \(n = 8\). The method utilized to find these \(\lambda\)-numbers involved determining an upper bound for the number of non-isomorphic GPGs by comparing them with similar \(n\)-gons as studied nearly half a century ago by Golomb and Welch [6].

Finally, we defined \(n\)-stars to be a subclass of GPGs whose symmetries were inspired by both the prisms and the Petersen graph. We determined the exact \(\lambda\)-numbers for all \(n\)-stars, and introduced the idea of using a Möbius strip representation of these special GPGs.

Future work may involve tightening the upper bound provided by the similar polygons and utilizing the notion of non-isomorphic classes of GPGs in order to determine the exact \(\lambda\)-numbers for GPGs of orders greater than 8.

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