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On parameters related to strong and weak domination in graphs

Gayla S. Domke[a], Johannes H. Hattingh[a,*], Lisa R. Markus[b], Elna Ungerer[c]

[a] Department of Mathematics and Statistics, University Plaza, Georgia State University, Atlanta, GA 30303-3083, USA

[b] Department of Mathematics, De Anza College, Cupertino, CA 95014, USA

[c] Department of Mathematics, Rand Afrikaans University, Auckland Park 2006, South Africa

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Abstract

Let G be a graph. Then $\mu(G) \leqslant |V(G)| - \delta(G)$ where $\mu(G)$ denotes the weak or independent weak domination number of G and $\mu(G) \leqslant |V(G)| - \Delta(G)$ where $\mu(G)$ denotes the strong or independent strong domination number of G. We give necessary and sufficient conditions for equality to hold in each case and also describe specific classes of graphs for which equality holds. Finally, we show that the problems of computing i_{w} and i_{st} are NP-hard, even for bipartite graphs.

Keywords: Domination; Strong; Weak

1. Introduction

For undefined terminology, the reader is referred to [1] or [8]. The symbols n and V will be reserved for the order of G and the vertex set of G, respectively.

A set $D \subseteq V$ is a *dominating set*, denoted DS, if every vertex not in D is adjacent to at least one vertex in D. A set $D \subseteq V$ is a *weak dominating set*, denoted WDS, if every vertex u not in D is adjacent to a vertex v in D where $\deg(v) \leqslant \deg(u)$. A set $D \subseteq V$ is a *strong dominating set*, denoted SDS, if every vertex u not in D is adjacent to a vertex v in D where $\deg(v) \geqslant \deg(u)$. The *domination number of G* (*weak domination number*,

* Corresponding author.
E-mail address: jhattingh@gsu.edu (J.H. Hattingh).

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strong domination number, respectively), denoted $\gamma(G)$ ($\gamma_w(G), \gamma_{st}(G)$, respectively), is the minimum size of a DS (WDS, SDS, respectively) of G.

A set $D \subseteq V$ is *an independent set*, denoted IS, if no two vertices of D are adjacent. A DS (WDS, SDS, respectively) when independent will be denoted IDS (IWDS, ISDS, respectively). The *independent domination number of G* (*independent weak domination number*, *independent strong domination number*, respectively), denoted $i(G)$ ($i_w(G), i_{st}(G)$, respectively), is the minimum size of an IDS (IWDS, ISDS, respectively) of G. Every graph admits an IWDS and an ISDS. For example, to find an IWDS, say D, in a graph G, apply the following algorithm:

$S := \emptyset$;
$D := \emptyset$;
`while` $S \neq V$
`begin`
  `Let` $v \in \{v \in V - S \mid \deg(v)$ `is as small as possible` $\}$;
  $S := S \cup N[v]$;
  $D := D \cup \{v\}$
`end`;

The concepts of weak and strong domination were introduced by Sampathkumar and Pushpa Latha in [13] in which the following motivation for strong and weak domination is offered. Consider a network of roads connecting a number of locations. In such a network, the degree of a vertex v is the number of roads meeting at v. Suppose $\deg(u) \geqslant \deg(v)$. Naturally, the traffic at u is heavier than that at v. If we consider the traffic between u and v, preference should be given to the vehicles going from u to v. Thus, in some sense, u strongly dominates v and v weakly dominates u. These concepts were further studied in [5–7,10–12].

Define V_δ (V_Δ, respectively) as $\{v \in V \mid \deg(v) = \delta(G)\}$ ($\{v \in V \mid \deg(v) = \Delta(G)\}$, respectively). Since every IDS of G is also a DS of G, $\gamma(G) \leqslant i(G)$. Moreover, since any maximal IS, say S, with $S \cap V_\Delta \neq \emptyset$ contains at most $n - \Delta(G)$ vertices and every maximal IS is also dominating, $i(G) \leqslant n - \Delta(G)$. Thus, $\gamma(G) \leqslant i(G) \leqslant n - \Delta(G)$. Graphs G for which equality holds in the bounds $\mu(G) \leqslant n - \Delta(G)$ where $\mu \in \{\gamma, i\}$ were studied in [2,3].

Since any IWDS (ISDS, respectively) is a WDS (SDS, respectively) dominating set, we obtain $\gamma_w(G) \leqslant i_w(G)$ and $\gamma_{st}(G) \leqslant i_{st}(G)$. We show that $i_w(G) \leqslant n - \delta(G)$ and $i_{st}(G) \leqslant n - \Delta(G)$ from which the results $\gamma_w(G) \leqslant n - \delta(G)$ and $\gamma_{st}(G) \leqslant n - \Delta(G)$ of [13] follow immediately. Furthermore, we give necessary and sufficient conditions for equality to hold in each case and also describe specific classes of graphs for which equality holds.

In [5,6] it is shown that the problems of computing γ_{st} and γ_w are NP-hard. We close, by showing that the problems of computing i_w and i_{st} are also NP-hard, even for bipartite graphs.

2. Graphs G which satisfy $i_w(G) = |V(G)| - \delta(G)$ or $i_{st}(G) = |V(G)| - \Delta(G)$

In this section, we show that if G is a graph, then $i_w(G) \leqslant n - \delta(G)$ and $i_{st}(G) \leqslant n - \Delta(G)$ and give necessary and sufficient conditions for equality to hold in these inequalities. We also describe specific classes of graphs for which equality holds in the above inequalities.

We begin with independent weak domination.

Lemma 1. *Let G be a graph. If D is an IWDS of G, then $D \cap V_\delta \neq \emptyset$.*

Proof. Let $v \in V_\delta$. In order for v to be weakly dominated, $v \in D$ or v is adjacent to a vertex u in D which has degree less than or equal to that of v. In the latter case, $u \in V_\delta$ and the result follows. □

Proposition 2. *If G is a graph, then $i_w(G) \leqslant n - \delta(G)$.*

Proof. Let D be any IWDS and let $v \in D \cap V_\delta$ (by Lemma 1). Since D is independent, $D \cap N(v) = \emptyset$. Thus, $D \subseteq V - N(v)$ and the result follows. □

In view of Proposition 2, it is natural to ask for which graphs equality holds and we begin by giving a necessary condition.

Proposition 3. *Let G be a graph with $i_w(G) = n - \delta(G)$ and let $v \in V_\delta$. Then $V - N(v)$ is independent.*

Proof. Suppose, to the contrary, that $V - N(v)$ is dependent. Then, by applying the following algorithm, we obtain an IWDS of size at most $n - \delta(G) - 1$, which is a contradiction. □

```
S := N[v];
D := {v};
while S ≠ V
begin
      Let u ∈ {u ∈ V − S | deg(u) is as small as possible };
      S := S ∪ N[u];
      D := D ∪ {u}
end;
```

We now characterize those graphs for which $i_w(G) = n - \delta(G)$.

Theorem 4. *Let G be a graph. Then $i_w(G) = n - \delta(G)$ if and only if $V - N(v)$ is independent for every vertex $v \in V_\delta$.*

Proof. If $i_w(G) = n - \delta(G)$ and $v \in V_\delta$, then $V - N(v)$ is independent by Proposition 3.

For the converse, suppose $V - N(v)$ is independent for every vertex $v \in V_\delta$ and let D be a minimum IWDS of G. Then, by Lemma 1, $D \cap V_\delta$ contains a vertex, say v. Since $v \in D$, $N(v) \cap D = \emptyset$ and $D \subseteq V - N(v)$. Moreover, since $V - N(v)$ is independent, no vertex in $V - N(v)$ can weakly dominate any other vertex in $V - N(v)$. Thus, $D = V - N(v)$ and the result follows. □

We now characterize the connected triangle-free graphs G which satisfy $i_w(G)=n-\delta(G)$. As immediate consequences we will obtain characterizations of connected bipartite graphs G and trees G which satisfy $i_w(G)=n-\delta(G)$.

Proposition 5. *Let G be a connected triangle-free graph. Then $i_w(G)=n-\delta(G)$ if and only if $G\in\{K_1, K_{n-\delta(G),\delta(G)}\}$.*

Proof. Suppose $i_w(G)=n-\delta(G)$ and let $\delta=\delta(G)$. If $\delta=0$, then $G=K_1$. Suppose, therefore, $\delta\geqslant 1$. Let $v\in V_\delta$. By Theorem 4, $V-N(v)$ is independent. Furthermore, since G is triangle-free, $N(v)$ is independent. Since each vertex in $V-N(v)$ has degree at least $\delta=|N(v)|$, each vertex in $V-N(v)$ is adjacent to every vertex in $N(v)$. Thus, $G=K_{n-\delta,\delta}$.

Conversely, if $G=K_{n-\delta,\delta}$, then the partite set of cardinality $n-\delta$ is a minimum IWDS of G, so that $i_w(G)=n-\delta$, as required. □

As immediate consequences we obtain

Corollary 6. *Let G be a connected bipartite graph. Then $i_w(G)=n-\delta(G)$ if and only if $G\in\{K_1, K_{\delta(G),n-\delta(G)}\}$.*

Corollary 7. *Let T be a tree. Then $i_w(T)=n-\delta(T)$ if and only if $T\in\{K_1, K_{1,n-1}\}$.*

We now present similar results involving the independent strong domination number. The proofs of these results are along similar lines to those above and are, therefore, omitted.

Lemma 8. *Let G be a graph. If D is an ISDS of G, then $D\cap V_\Delta\neq\emptyset$.*

Proposition 9. *If G is a graph, then $i_{st}(G)\leqslant n-\Delta(G)$.*

Proposition 10. *Let G be a graph such that $i_{st}(G)=n-\Delta(G)$ and let $v\in V_\Delta$. Then $V-N(v)$ is independent.*

Theorem 11. *Let G be a graph. Then $i_{st}(G)=n-\Delta(G)$ if and only if $V-N(v)$ is independent for every vertex $v\in V_\Delta$.*

We conclude this section by characterizing the connected triangle-free graphs G which satisfy $i_{st}(G)=n-\Delta(G)$. As immediate consequences we will obtain characterizations of connected bipartite graphs G and trees G which satisfy $i_{st}(G)=n-\Delta(G)$.

Proposition 12. *Let G be a connected triangle-free graph. Then $i_{st}(G)=n-\Delta(G)$ if and only if $G=K_1$ or G is a spanning subgraph of the graph $K_{\Delta(G),n-\Delta(G)}$ and if $n\neq 2\Delta(G)$, then V_Δ is contained in the partite set of $K_{\Delta(G),n-\Delta(G)}$ of size $n-\Delta(G)$.*

Proof. Suppose $\gamma_{st}(G)=n-\Delta(G)$ and let $\Delta=\Delta(G)$. If $\Delta=0$, then $G=K_1$. Suppose, therefore, that $\Delta\geqslant 1$. Let D be a minimum ISDS of G. Then, by Lemma 8, there is a vertex v (say) in $D\cap V_\Delta$. But then $D\cap N(v)=\emptyset$, $N(v)$ is independent (since G is triangle-free) and $V-N(v)$ is independent (by Proposition 10). It follows that G is a bipartite graph with bipartition $(V-N(v),N(v))$ and hence a spanning subgraph of $K_{n-\Delta,\Delta}$. It remains to show that if $n\neq 2\Delta$, then the V_Δ is contained in the partite set $V-N(v)$ of size $n-\Delta$. We do this by showing that no vertex in $N(v)$ has maximum degree. For suppose, to the contrary, that $u\in N(v)\cap V_\Delta$. Then, since $n\neq 2\Delta$ and G is connected, there is a vertex in $V-N[v]-N[u]$ which is adjacent to a vertex in $N(v)$. But then $V-N(u)$ is not independent, which is contrary to our assumption.

Conversely, suppose $G=K_1$ or G is a spanning subgraph of $K_{\Delta,n-\Delta}$ and suppose if $n\neq 2\Delta(G)$, then the V_Δ is contained in the partite set of $K_{\Delta(G),n-\Delta(G)}$ of size $n-\Delta(G)$. If $v\in V_\Delta$, then, since G is bipartite, $V-N(v)$ is independent. Hence, by Theorem 11, $i_{st}(G)=n-\Delta$. □

As immediate consequences we obtain

Corollary 13. *Let G be a connected bipartite graph. Then $i_{st}(G)=n-\Delta(G)$ if and only if $G=K_1$ or G is a spanning subgraph of $K_{\Delta(G),n-\Delta(G)}$ and if $n\neq 2\Delta(G)$, then V_Δ is contained in the partite set of $K_{\Delta(G),n-\Delta(G)}$ of size $n-\Delta(G)$.*

Corollary 14. *Let T be a tree. Then $i_{st}(T)=n-\Delta(T)$ if and only if whenever T is rooted at a vertex of degree $\Delta(T)$, then the height of the tree is at most two.*

3. Graphs G which satisfy $\gamma_w(G)=|V(G)|-\delta(G)$

Since $\gamma_w(G)\leqslant i_w(G)$ for any graph G, the next result, originally given by Sampathkumar and Pushpa Latha, follows immediately.

Corollary 15 (Sampathkumar and Pushpa [13]). *If G is a graph, then $\gamma_w(G)\leqslant n-\delta(G)$.*

Note that if $\gamma_w(G)=n-\delta(G)$, then it is also true that $i_w(G)=n-\delta(G)$.

In order to completely characterize all graphs G for which $\gamma_w(G)=n-\delta(G)$, we need the following lemma.

Lemma 16. *Let G be a graph. If $v\in V_\delta$ and $V-N(v)$ is independent, then $V-N(v)\subseteq V_\delta$.*

Proof. Let $u\in V-N(v)$. Since $V-N(v)$ is independent, $N(u)\subseteq N(v)$ and $|N(u)|=\delta(G)$, and the result follows. □

Theorem 17. *Let G be a graph. Then $\gamma_w(G)=n-\delta(G)$ if and only if one of the following conditions is satisfied*

(1) $\delta(G)=n-1$, *i.e.* $G=K_n$,
(2) $\delta(G)=n-2$,
(3) $\delta(G)\leqslant n-3$ *and if* $v\in V_\delta$, *then* $V-N(v)$ *is independent and every vertex in* $N(v)$ *has degree at least* $\delta(G)+1$.

Proof. Suppose $\gamma_w(G)=n-\delta(G)$ and $\delta(G)\leqslant n-3$. Let $v\in V_\delta$. By Proposition 3, $V-N(v)$ is independent. Moreover, since $V-N(v)$ is independent, $V-N(v)\subseteq V_\delta$ (cf. Lemma 16). Note that each vertex in $N(v)$ is adjacent to every vertex in $V-N(v)$. Suppose $u\in N(v)\cap V_\delta$. But then $\{u,v\}$ is a WDS of G, so that $\gamma_w(G)\leqslant 2<3\leqslant n-\delta(G)$, which is a contradiction. Thus, every vertex of $N(v)$ has degree at least $\delta(G)+1$.

Conversely, suppose that one of conditions (1), (2) or (3) is satisfied. If $\delta(G)=n-1$, then $G=K_n$ and $\gamma_w(G)=1=n-(n-1)=n-\delta(G)$. Suppose $\delta(G)=n-2$. Let $\{u,v\}$ be two non-adjacent vertices of G. Then $\{u,v\}$ is a WDS of G, so that $\gamma_w(G)\leqslant 2=n-(n-2)=n-\delta(G)$. Clearly, no single vertex can weakly dominate all the other vertices of G. Thus, $\gamma_w(G)=2=n-\delta(G)$.

Let $v\in V_\delta$ and assume condition (3) holds. As before, $V-N(v)\subseteq V_\delta$, while each vertex of $N(v)$ has degree at least $\delta(G)+1$. It follows that each vertex of $V-N(v)$ is in every WDS of G. Thus, $\gamma_w(G)\geqslant|V-N(v)|=n-\delta(G)$. Clearly, $V-N(v)$ is a WDS of G, so that $\gamma_w(G)=n-\delta(G)$, as required. □

We now characterize the connected triangle-free graphs G which satisfy $\gamma_w(G)=n-\delta(G)$. As immediate consequences we will obtain characterizations of connected bipartite graphs G and trees G which satisfy $\gamma_w(G)=n-\delta(G)$.

Proposition 18. *Let G be a connected triangle-free graph. Then $\gamma_w(G)=n-\delta(G)$ if and only if $G\in\{K_1,K_{1,1},K_{2,2}\}\cup\{K_{\delta(G),n-\delta(G)}$ where $\delta(G)\neq n/2\}$.*

Proof. Suppose $\gamma_w(G)=n-\delta(G)$. Let $\delta=\delta(G)$. If $\delta=0$, then $G=K_1$. Suppose, therefore, that $\delta\geqslant 1$. First consider the case when $\delta\leqslant n-3$. Let $v\in V_\delta$. By Theorem 17, $V-N(v)$ is independent and hence each vertex of $V-N(v)$ is adjacent to each vertex of $N(v)$. Furthermore, since G is triangle-free, $N(v)$ is independent. Thus, G is a complete bipartite graph with bipartition $(V-N(v),N(v))$, so that $G=K_{n-\delta,\delta}$. Since every vertex of $N(v)$ has degree at least $\delta+1$ (cf. Theorem 17), $|V-N(v)|\geqslant\delta+1$, so that $n=|V-N(v)|+|N(v)|\geqslant(\delta+1)+\delta=2\delta+1$, whence $\delta\neq n/2$.

Suppose $\delta=n-2$ and let $\{u,v\}$ be non-adjacent vertices of G. It follows that u and v are adjacent to every vertex of $V-\{u,v\}$. Let $x\in V-\{u,v\}$. If $\deg(x)=n-1$, then, since G is triangle-free, $G=K_{1,2}$. Suppose, therefore, that $\deg(x)=n-2$ and let y be the vertex that is not adjacent to x. Since G is triangle-free, $V=\{u,v,x,y\}$ and $G=K_{2,2}$.

Finally, consider the case when $\delta=n-1$. Then, since G is triangle-free, $G\in\{K_1,K_{1,1}\}$.

The converse is clear. □

As immediate consequences we obtain

Corollary 19. *Let G be a connected bipartite graph. Then* $\gamma_w(G)=n-\delta(G)$ *if and only if* $G\in\{K_1,K_{1,1},K_{2,2}\}\cup\{K_{\delta(G),n-\delta(G)}$ *where* $\delta(G)\neq n/2\}$.

Corollary 20. *Let T be a tree. Then* $\gamma_w(G)=n-\delta(T)$ *if and only if* $T\in\{K_1,K_{1,n-1}\}$.

4. Graphs G which satisfy $\gamma_{st}(G)=|V(G)|-\Delta(G)$

In this section, we give necessary and sufficient conditions for equality to hold in the inequality $\gamma_{st}(G)\leqslant n-\Delta(G)$. We also describe specific classes of graphs for which equality holds in the above inequality.

Since $\gamma_{st}(G)\leqslant i_{st}(G)$ for any graph G, the next result, originally given by Sampathkumar and Pushpa Latha follows immediately.

Corollary 21 (Sampathkumar and Pushpa [13]). *If G is a graph, then* $\gamma_{st}(G)\leqslant n-\Delta(G)$.

Note that if $\gamma_{st}(G)=n-\Delta(G)$, then it is also true that $i_{st}(G)=n-\Delta(G)$.

We are now in a position to characterize all graphs for which $\gamma_{st}(G)=n-\Delta(G)$.

Theorem 22. *Let G be a graph. Then* $\gamma_{st}(G)=n-\Delta(G)$ *if and only if for every* $v\in V_\Delta$ *the following two conditions hold*:

(1) $V-N(v)$ *is independent*, *and*
(2) *If* $u\in N(v)$ *is adjacent to vertices x and y in* $V-N[v]$, *then* $\deg(u)<\max\{\deg(x),\deg(y)\}$.

Proof. Suppose $\gamma_{st}(G)=n-\Delta(G)$ and let $v\in V_\Delta(G)$. Then, by Proposition 10, condition (1) holds. To show that condition (2) holds, let $u\in N(v)$ and let $x,y\notin N[v]$ be adjacent to u. If $\deg(u)\geqslant\max\{\deg(x),\deg(y)\}$, then $V-(N(v)\cup\{x,y\})\cup\{u\}$ is a SDS of G. Hence, $\gamma_{st}(G)\leqslant n-|N(v)|-2+1=n-\Delta(G)-1$, which is a contradiction. Thus, $\deg(u)<\max\{\deg(x),\deg(y)\}$, as required.

For the converse, suppose that for every vertex in V_Δ conditions (1) and (2) hold. Let D be a minimum SDS. Every SDS contains a vertex in V_Δ (cf. Lemma 1). Let v be such a vertex in D. Then no vertex of $N(v)$ can strongly dominate two or more vertices of $V-N[v]$. However, each of these vertices is strongly dominated, so that $\gamma_{st}(G)\geqslant 1+|V-N[v]|=n-\Delta(G)$, and the proof is complete. □

We conclude this section by characterizing those trees T which satisfy $\gamma_{st}(T)=n-\Delta(T)$.

A *subdivision* of an edge uv is obtained by introducing a new vertex w and replacing the edge uv with the edges uw and wv. A tree is a *spider* if it is the tree formed from $K_{1,m}$ by subdividing every edge. We will call these subdivided edges *legs* of the spider.

A tree is a *wounded spider* if it is formed from $K_{1,m}$ by subdividing at most $m-1$ of the edges. Thus, a star is a wounded spider.

Corollary 23. *Let T be a tree. Then $\gamma_{\rm st}(T)=n-\Delta(T)$ if and only if T is a wounded spider or a spider which has at least three legs.*

Proof. Suppose $\gamma_{\rm st}(T)=n-\Delta(T)$ and let $v\in V_\Delta$. Since $V-N(v)$ is independent and T does not contain any cycles, each vertex of $V-N[v]$ has degree 1. Suppose $u\in(v)$ is adjacent to vertices x and y in $V-N[v]$. By Theorem 22, $\deg(u)<\max\{\deg(x),\deg(y)\}=1$, a contradiction. Hence, every vertex $u\in N(v)$ is adjacent to at most one vertex in $V-N[v]$, so that $\deg(u)\leqslant 2$. It is easy to see that for a spider T with two legs, $\gamma_{\rm st}(T)=2<5-3=n-\Delta(T)$ and a spider with one leg is just $K_{1,2}$ (a wounded spider). Thus, T is a wounded spider or a spider which has at least three legs.

It is easy to check that if T is a wounded spider or a spider which has at least three legs, then $\gamma_{\rm st}(T)=n-\Delta(T)$. $\square$

5. Complexity results

In [5,6] it was shown that the problems of computing $\gamma_{\rm st}$ and $\gamma_{\rm w}$ are NP-hard. In this section, we show that the problems of computing $i_{\rm w}$ and $i_{\rm st}$ are also NP-hard. In each case we will state the corresponding decision problem in the standard Instance–Question form [4] and indicate the polynomial-time reduction used to prove that it is NP-complete.

Independent weak dominating set (IWDS)

Instance: A graph G and a positive integer ℓ.

Question: Does G have an IWDS of cardinality at most ℓ?

Theorem 24. *IWDS is NP-complete, even for bipartite graphs.*

Proof. The reduction is from EXACT COVER BY 3-SETS (X3C) with the additional requirement that each variable appears in at least two subsets [9]. Given an instance $X=\{x_1,\dots,x_{3q}\}$ and $\mathscr{C}=\{C_1,\dots,C_m\}$ of X3C, where $C_j\subseteq X$ and $|C_j|=3$ for $1\leqslant j\leqslant m$, construct a bipartite graph G as follows. Corresponding to each variable x_i, we associate the graph $F_i\cong P_5$ with z_i,y_i,x_i,y_i',z_i' being consecutive vertices in F_i. Corresponding to each set C_j we associate the graph $H_j\cong P_5$ with c_j,d_j,e_j,f_j,g_j being consecutive vertices on H_j. The construction of G is completed by joining x_i and c_j if and only if the variable x_i occurs in the set C_j. Finally, set $\ell=2m+7q$.

Suppose $\mathscr{C}$ has an exact 3-cover, say $\mathscr{C}'$. Then it is easily verified that $D=\bigcup_{i=1}^{3q}\{z_i,z_i'\}\cup\bigcup_{C_j\in\mathscr{C}'}\{e_j,c_j\}\cup\bigcup_{C_j\notin\mathscr{C}'}\{d_j\}\cup\bigcup_{j=1}^{m}\{g_j\}$ is an IWDS of cardinality ℓ.

Conversely, suppose D is a minimum IWDS of cardinality at most ℓ. Note that all end vertices are in D. Let $j\in\{1,\dots,m\}$. Then, since f_j is adjacent to g_j, which is an end vertex, and D is independent, $f_j\notin D$. Since e_j is dominated and D is independent, $|D\cap\{e_j,d_j\}|=1$. Note that $D\cap\bigcup_{i=1}^{3q}\{y_i,y_i'\}=\emptyset$. Thus, $|D\cap(\bigcup_{i=1}^{3q}\{x_i\}\cup$

$\bigcup_{j=1}^{m}\{c_j\})| \leqslant (7q+2m)-6q-2m=q$. We show that $D\cap\bigcup_{i=1}^{3q}\{x_i\}=\emptyset$. Suppose $|D\cap\bigcup_{i=1}^{3q}\{x_i\}|=\ell'$. Then $|D\cap\bigcup_{i=1}^{m}\{c_i\}|\leqslant q-\ell'$, so that $|N[D\cap\bigcup_{j=1}^{m}\{c_j\}]\cap\bigcup_{i=1}^{3q}\{x_i\}|\leqslant 3q-3\ell'$. Thus, $|\bigcup_{i=1}^{3q}\{x_i\}-(N[D\cap\bigcup_{j=1}^{m}\{c_j\}]\cap\bigcup_{i=1}^{3q}\{x_i\})-(D\cap\bigcup_{i=1}^{3q}\{x_i\})|\geqslant 3q-(3q-3\ell')-\ell'=2\ell'$. If $\ell'>0$, then some x_i is not dominated by D, which is a contradiction. Thus, $\ell'=0$.

Let $\mathscr{C}'=\{C_j\,|\,c_j\in D\}$. Then, since D is an IWDS of G, $\mathscr{C}'$ is an exact three cover for X. □

Given a positive integer m, we construct a graph H as follows. Let d,e,f,g,h be consecutive vertices on the path P_5. Take $m+1$ copies of the graph K_2 and denote the vertex set of the ith copy by $\{d_i,e_i\}$. The construction of H is completed by joining d_i to d and e_i to e for $i=1,\dots,m+1$. The graph H will prove to be useful in showing that the following problem is NP-complete.

Independent strong dominating set (ISDS)
Instance: A graph G and a positive integer ℓ.
Question: Does G have an ISDS of cardinality at most ℓ?

We first prove a useful property concerning the graph H.

Lemma 25. *Suppose F is a graph that contains H as an induced subgraph with $\deg_F(d)=m+3$ and $\deg_H(v)=\deg_F(v)$ for all $v\in V(H)-\{d\}$. If D is a minimum ISDS of F, then $|D\cap V(H)|=m+3$.*

Proof. Let D be a minimum ISDS of F.

Consider, firstly, the case when $h\in D$. We show that $f\notin D$. For suppose, to the contrary, that $f\in D$. Since $\deg(d)=\deg(e)=m+3$, $\deg(e_i)=2$ for all $i=1,\dots,m+1$, and e are strongly dominated by a vertex in D, it follows that $d\in D$. Since D is independent, $D\cap\{d_1,\dots,d_{m+1}\}=\emptyset$. In order for D to dominate e_i for $i=1,\dots,m+1$, we have that $\{e_1,\dots,e_{m+1}\}\subseteq D$. But then $D'=D-\{f,h\}\cup\{g\}$ is an ISDS set of F such that $|D'|<|D|$, which is a contradiction. Thus, $f\notin D$, $e\in D$ and $\{e_1,\dots,e_{m+1}\}\cap D=\emptyset$. Since D is independent, $d\notin D$ and, in order for d_i, $i=1,\dots,m+1$, to be dominated by D, $\{d_1,\dots,d_{m+1}\}\subseteq D$. Thus, $|D\cap V(H)|=m+3$, as required.

Consider, secondly, the case when $h\notin D$. Then $g\in D$, while $f\notin D$. Note that $|\{e,d\}\cap D|=1$. On the one hand, if $e\in D$, then $\{e_1,\dots,e_m\}\cap D=\emptyset$, $d\notin D$ and $\{d_1,\dots,d_{m+1}\}\subseteq D$, so that $|D\cap V(H)|=m+3$. On the other hand, if $e\notin D$, then $d\in D$, $\{d_1,\dots,d_{m+1}\}\cap D=\emptyset$ and $\{e_1,\dots,e_{m+1}\}\subseteq D$, so that $|D\cap V(H)|=m+3$. □

Theorem 26. *ISDS is NP-complete, even for bipartite graphs.*

Proof. The reduction is from EXACT COVER BY 3-SETS (X3C). Given an instance $X=\{x_1,\dots,x_{3q}\}$ and $\mathscr{C}=\{C_1,\dots,C_m\}$ of X3C, where $C_j\subseteq X$ and $|C_j|=3$ for $1\leqslant j\leqslant m$, construct a bipartite graph G as follows. Corresponding to each variable x_i we associate a copy of K_1 with vertex x_i. Corresponding to each set C_j we associate the graph F_j constructed as follows. Take a single vertex c_j and m copies of the graph H, constructed above, and join the vertex c_j to each copy's d vertex. Note that each copy of H occurs as an induced subgraph in F_j with its d vertex having degree $m+3$. The construction of G is completed by joining x_i and c_j if and only if the variable x_i occurs in the set C_j. Finally, set $\ell=(m+3)m^2+q$.

Suppose $\mathscr{C}$ has an exact 3-cover, say $\mathscr{C}'$. Construct an ISDS D of G as follows. Include in D all those vertices c_j for which $C_j\in\mathscr{C}'$. Let $j\in\{1,\ldots,m\}$ and let H' be any of the (induced) copies of H occurring in F_j. If $c_j\in D$, include the vertices $d_1,\ldots,d_{m+1}$, e and g in D. This accounts for $m+3$ vertices in H' and $m(m+3)+1$ vertices in F_j. If $c_j\notin D$, include the vertices $d,e_1,\ldots,e_{m+1},g$ in D. Note that c_j is strongly dominated by d. This accounts for $m+3$ vertices in H' and $m(m+3)$ vertices in F_j. Since $\mathscr{C}'$ is an exact 3-cover, each x_i is strongly dominated by some $c_j\in D$. It is easy to verify that D is an ISDS of G of cardinality $(m(m+3)+1)q+m(m+3)(m-q)=m^2(m+3)+q=\ell$, as required.

Conversely, suppose D is a minimum ISDS of cardinality at most ℓ. By Lemma 25, for each of the m^2 induced copies H' of H, we have $|D\cap V(H')|=m+3$. Thus, $|D\cap(\{c_1,\ldots,c_m\}\cup\{x_1,\ldots,x_{3q}\})|\leqslant m^2(m+3)+q-m^2(m+3)=q$. Suppose $|D\cap\{x_1,\ldots,x_{3q}\}|=\ell'$. Then $|D\cap\{c_1,\ldots,c_m\}|\leqslant q-\ell'$, so that $|N[D\cap\{c_1,\ldots,c_m\}]\cap\{x_1,\ldots,x_{3q}\}|\leqslant 3q-3\ell'$. Thus, $|\{x_1,\ldots,x_{3q}\}-(D\cap\{c_1,\ldots,c_m\})-(N[D\cap\{c_1,\ldots,c_m\}]\cap\{x_1,\ldots,x_{3q}\})|\geqslant 2\ell'$. If $\ell'>0$, then some vertex x_i is not dominated by D, which is a contradiction. Thus, $\ell'=0$.

Let $\mathscr{C}'=\{C_j\,|\,c_j\in D\}$. Then, since D is an *ISDS* of G, $\mathscr{C}'$ is an exact three cover for X. $\square$

Although the problems of computing γ_{st}, γ_{w}, i_{w} and i_{st} are NP-hard, the problems of determining for a graph of order n whether $\mu(G)=n-\delta(G)$ for $\mu\in\{\gamma_{\mathrm{w}},i_{\mathrm{w}}\}$ or $\mu(G)=n-\Delta(G)$ for $\mu\in\{\gamma_{\mathrm{st}},i_{\mathrm{st}}\}$ are all in P, as may be seen by Theorems 4, 11, 17 and 22. We may also see this as follows, and use $\mu=\gamma_{\mathrm{w}}$ as an illustration. Generate, in polynomial time, all the $\binom{n}{n-\delta(G)-1}$ subsets of cardinality $n-\delta(G)-1$ of $V(G)$ and verify for each of these subsets, in polynomial time, whether it is a WDS or not. If we do not find a WDS, then $\gamma_{\mathrm{w}}(G)=n-\delta(G)$.

Linear time algorithms for computing $i_{\mathrm{w}}(T)$ and $i_{\mathrm{st}}(T)$ for a tree T are readily obtained using the methodology of Wimer [14]. We omit the details since a similar algorithm (computing $\gamma_{\mathrm{w}}(T)$ for a tree T) is presented in [6] and can be easily adapted to compute $i_{\mathrm{w}}(T)$ and $i_{\mathrm{st}}(T)$ for any tree T. As remarked in [5], the same is true for computing $\gamma_{\mathrm{st}}(T)$ for any tree T.

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