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# Generalized geometrical structures of odd dimensional manifolds ☆

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**Abstract**

We define an almost-cosymplectic-contact structure which generalizes cosymplectic and contact structures of an odd dimensional manifold. Analogously, we define an almost-coPoisson–Jacobi structure which generalizes a Jacobi structure. Moreover, we study relations between these structures and analyse the associated algebras of functions.

As examples of the above structures, we present geometrical dynamical structures of the phase space of a general relativistic particle, regarded as the 1st jet space of motions in a spacetime. We describe geometric conditions by which a metric and a connection of the phase space yield cosymplectic and dual coPoisson structures, in case of a spacetime with absolute time (a Galilei spacetime), or almost-cosymplectic-contact and dual almost-coPoisson–Jacobi structures, in case of a spacetime without absolute time (an Einstein spacetime).


**Résumé**

Nous définissons une structure presque cosymplectique-contact, qui généralise les structures cosymplectiques et de contact d'une variété de dimension impaire. D'une manière analogue, nous définissons une structure presque coPoisson–Jacobi, qui généralise la structure de Jacobi. Nous étudions aussi les relations entre ces structures et nous analysons les algèbres des fonctions associées.

Comme exemples de ces structures nous présentons les structures géométriques dynamiques de l'espace des phases d'une particule relativiste générale regardé comme jet du premier ordre des mouvements dans l'espace-temps. Nous décrivons les conditions géométriques pour lesquelles une métrique et une connexion de l'espace des phases produisent des structures cosymplectiques et coPoisson duales dans le cas d'un espace-temps (espace-temps de Galilée) ou presque cosymplectique-contact et presque coPoisson duales dans l'espace-temps sans temps absolu (espace-temps d'Einstein).


*Keywords:* Spacetime; Phase space; Phase connection; Schouten bracket; Frölicher–Nijenhuis bracket; Cosymplectic structure; coPoisson structure; Contact structure; Jacobi structure; Almost-cosymplectic-contact structure; Almost-coPoisson–Jacobi structure

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## Introduction

In [2,3,5–7] we studied geometrical structures on the phase space of a spacetime naturally induced (in the sense of [10]) by a metric and a phase connection. Some of these structures are well known and some are less standard. In the present paper, we generalize these structures on odd dimensional manifolds and study general properties of such structures.

First, in Section 1, we recall some standard structures and introduce new structures, namely almost-cosymplectic-contact, coPoisson and almost-coPoisson–Jacobi structures. In Section 2 we study algebras of functions which are associated with the new geometrical structures.

As examples of the above new structures, we study the geometrical structures on the phase space of a spacetime. Actually, the geometric objects arising in Section 3.1, in the framework of the Galilei's phase space [2,5,6], involve mainly the concepts of cosymplectic and (regular) coPoisson structures. On the other hand, the analogous geometric objects arising in Section 3.2, in the framework of the Einstein's phase space [3,7], involve mainly the concepts of almost-cosymplectic-contact and almost-coPoisson–Jacobi structures (eventually contact and Jacobi structures).

In the standard non-relativistic analytical mechanics, the usual phase space is defined by the vertical tangent space, or by the vertical cotangent space of spacetime. These spaces are even dimensional and equipped with a symplectic structure induced, respectively, by the metric, or by the canonical Liouville form. Passing to relativistic analytical mechanics, the above spaces are usually replaced by the tangent space, or by the cotangent space of spacetime. However, for physical reasons, the velocity of motions needs to be normalized through the time component, in the Galilei case, or through the metric, in the Einstein case. These constraints yield an odd dimensional phase space, where the symplectic structure is no longer the appropriate geometric framework. Moreover, we can get rid of normalization constraints, with all related complications, and also of the choice of units of measurement of time, by describing the phase space in terms of jets. In the Galilei case we deal with jets of sections (related to absolute time) and in the Einstein case we deal with jets of submanifolds (related to the Lorentz metric). Indeed, this will be the framework for the examples of the geometric structures discussed in the present paper.

## 1. Geometrical structures

We use the inner product $i$ of $k$-vectors with $r$-forms defined by $i_{X_1\wedge\cdots\wedge X_k}\beta = i_{X_k}\dots i_{X_1}\beta$, for each $r$-form $\beta$ and $k$ vector fields $X_1,\dots,X_k$, with $k\leqslant r$. We use the same symbol for the dual inner product of $k$-forms with $r$-vectors.

For the Schouten bracket we use the identity, [11,12,16],

$$i_{[P,Q]}\beta = (-1)^{q(p+1)} i_P d i_Q \beta + (-1)^p i_Q d i_P \beta - i_{P\wedge Q} d\beta,$$

for each $p$-vector $P$, $q$-vector $Q$ and $(p+q-1)$-form $\beta$. In particular, for each vector field $E$ and 2-vector $\Lambda$, we have $i_{[E,\Lambda]}\beta = i_E d i_\Lambda \beta - i_\Lambda d i_E \beta$, for each closed 2-form $\beta$, and $i_{[\Lambda,\Lambda]}\beta = 2 i_\Lambda d i_\Lambda \beta$, for each closed 3-form $\beta$.

In what follows, $\boldsymbol{M}$ is a $(2n+1)$-dimensional smooth manifold.

### *1.1. Covariant and contravariant pairs*

**Definition 1.1.** We define a *covariant pair* to be a pair $(\omega,\Omega)$ consisting of a 1-form $\omega$ and a 2-form $\Omega$ of constant rank $2r$, with $0\leqslant r\leqslant n$, such that $\omega\wedge\Omega^r \not\equiv 0$, and a *contravariant pair* to be a pair $(E,\Lambda)$ consisting of a vector field $E$ and a 2-vector $\Lambda$ of constant rank $2s$, with $0\leqslant s\leqslant n$, such that $E\wedge\Lambda^s\not\equiv 0$. Thus, by definition, we have $\Omega^r\not\equiv 0$, $\Omega^{r+1}\equiv 0$ and $\Lambda^s\not\equiv 0$, $\Lambda^{s+1}\equiv 0$.

We say that the pairs $(\omega,\Omega)$ and $(E,\Lambda)$ are *regular* if, respectively,

$$\omega\wedge\Omega^n\not\equiv 0 \quad\text{and}\quad E\wedge\Lambda^n\not\equiv 0.$$

Let us consider a covariant pair $(\omega,\Omega)$ and a contravariant pair $(E,\Lambda)$.

We define the following linear maps and subspaces:

$$\begin{aligned}
&\Omega^\flat : T\boldsymbol{M}\to T^*\boldsymbol{M} : X\mapsto X^\flat =: i_X\Omega, && \Lambda^\sharp : T^*\boldsymbol{M}\to T\boldsymbol{M} : \alpha\mapsto \alpha^\sharp =: i_\alpha\Lambda,\\
&\langle\omega\rangle =: \{\lambda\omega \mid \lambda\in\mathbb{R}\}\subset T^*\boldsymbol{M}, && \langle E\rangle =: \{\lambda E\mid \lambda\in\mathbb{R}\}\subset T\boldsymbol{M},\\
&\ker E =: \{\alpha\in T^*\boldsymbol{M}\mid \alpha(E)=0\}, && \ker\omega =: \{X\in T\boldsymbol{M}\mid \omega(X)=0\}.
\end{aligned}$$

We have $\dim(\operatorname{im}\Omega^\flat) = 2r$ and $\dim(\operatorname{im}\Lambda^\sharp) = 2s$.
If $(\omega, \Omega)$ is regular, then $r = n$, $\dim(\operatorname{im}\Omega^\flat) = 2n$, $\dim(\ker\Omega^\flat) = 1$, $\dim(\ker\omega) = 2n$.
If $(E, \Lambda)$ is regular, then $s = n$, $\dim(\operatorname{im}\Lambda^\sharp) = 2n$, $\dim(\ker\Lambda^\sharp) = 1$, $\dim(\ker E) = 2n$.

### 1.2. Structures given by covariant pairs

According to [12], a *pre cosymplectic structure* on $\boldsymbol{M}$ is defined by a regular covariant pair $(\omega, \Omega)$.

Two distinguished types of pre cosymplectic structures appear in the literature. Namely, we recall that a *cosymplectic structure* [1] and a *contact structure* [11] are defined by a covariant pair $(\omega, \Omega)$ such that, respectively,

$$d\omega = 0, \qquad d\Omega = 0, \qquad \omega \wedge \Omega^n \not\equiv 0, \tag{1.1}$$

$$\Omega = d\omega, \qquad \omega \wedge \Omega^n \not\equiv 0. \tag{1.2}$$

Thus, a contact structure is characterized just by a 1-form $\omega$ such that

$$\omega \wedge (d\omega)^n \not\equiv 0.$$

We can easily generalize the above structures in the following way.

**Definition 1.2.** We define an *almost-cosymplectic-contact structure* to be a covariant pair $(\omega, \Omega)$ such that

$$d\Omega = 0, \qquad \omega \wedge \Omega^n \not\equiv 0.$$

Clearly, for $d\omega = 0$ we obtain a cosymplectic structure and for $\Omega = d\omega$ a contact structure. So, almost-cosymplectic-contact structures are regular structures which generalize both cosymplectic and contact structures.

### 1.3. Structures given by contravariant pairs

Two distinguished types of contravariant pairs appear in the literature.

Namely, we recall that a *Jacobi structure* is defined by a contravariant pair $(E, \Lambda)$ such that

$$[E, \Lambda] = 0, \qquad [\Lambda, \Lambda] = -2E \wedge \Lambda,$$

where [, ] denotes the Schouten bracket.

In the particular case when $E = 0$, we obtain:

$$[\Lambda, \Lambda] = 0$$

and the pair $(E, \Lambda) =: (0, \Lambda)$ is called *Poisson structure*.

On the other hand, in the particular case when $\Lambda = 0$, we obtain $[E, \Lambda] = 0$ and $[\Lambda, \Lambda] = 0$ and the pair $(E, \Lambda) =: (E, 0)$ is called *trivial structure*.

In the following we assume $E \not\equiv 0$ and $\Lambda \not\equiv 0$.

**Remark 1.3.** In the literature (see for instance [12]) the condition $E \wedge \Lambda^s \not\equiv 0$ is considered just as a possible non-necessary property of the Jacobi pair $(E, \Lambda)$. So, our definition is a little more restrictive; however, the assumption $E \wedge \Lambda^s \not\equiv 0$ is quite reasonable and it is needed for our subsequent developments.

In the literature (see for instance [11,12,16]) the Jacobi structure is usually defined by the identities $[E, \Lambda] = 0$, $[\Lambda, \Lambda] = 2E \wedge \Lambda$. The difference in the sign in the second identity, with respect to our definition, is caused by the different convention on the inner product, hence by the different sign in definition of $\Lambda^\sharp$.

In order to exhibit a certain symmetry between geometric structures given by covariant and contravariant pairs, we introduce the following notions:

**Definition 1.4.** We define a *pre coPoisson structure* to be a contravariant pair $(E, \Lambda)$.

In particular, a *coPoisson structure* is defined by a contravariant pair $(E, \Lambda)$ such that

$$[E, \Lambda] = 0, \qquad [\Lambda, \Lambda] = 0.$$

**Definition 1.5.** We define an *almost-coPoisson–Jacobi structure* to be a 3-plet $(E, \Lambda, \omega)$, where $(E, \Lambda)$ is a contravariant pair and $\omega$ a 1-form, such that

$$[E, \Lambda] = -E \wedge \Lambda^\sharp(L_E \omega), \qquad [\Lambda, \Lambda] = 2E \wedge (\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega), \qquad i_E \omega = 1, \qquad i_\omega \Lambda = 0.$$

The 1-form $\omega$ is said to be the *fundamental 1-form* of the almost-coPoisson–Jacobi structure.

**Remark 1.6.** Almost-coPoisson–Jacobi structures generalize both coPoisson and Jacobi structures.

Indeed, if $d\omega = 0$, then we have $L_E \omega = i_E d\omega = 0$, hence from Definition 1.5 we obtain $[E, \Lambda] = 0$ and $[\Lambda, \Lambda] = 0$, i.e. $(E, \Lambda)$ turns out to be a coPoisson structure.

Moreover, if $L_E \omega = 0$ and $(\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega) = -\Lambda$, then we obtain $[E, \Lambda] = 0$ and $[\Lambda, \Lambda] = -2E \wedge \Lambda$, i.e. $(E, \Lambda)$ turns out to be a Jacobi structure.

**Proposition 1.7.** *Let $(E, \Lambda)$ be a regular contravariant pair. Then, there exists a unique 1-form $\omega$, such that $i_\omega(E \wedge \Lambda^n) = \Lambda^n$. Indeed, such an $\omega$ satisfies the equalities $i_E \omega = 1$ and $i_\omega \Lambda = 0$.*

*Thus, the 3-plet $(E, \Lambda, \omega)$ turns out to be an almost-coPoisson–Jacobi structure if and only if $[E, \Lambda] = -E \wedge \Lambda^\sharp(L_E \omega)$ and $[\Lambda, \Lambda] = 2E \wedge (\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega)$.*

Thus, a regular almost-coPoisson–Jacobi structure can be defined just as a suitable contravariant pair $(E, \Lambda)$, as the additional 1-form $\omega$ is naturally determined by the above pair itself.

### *1.4. Dual structures*

Let us consider a covariant pair $(\omega, \Omega)$ and a contravariant pair $(E, \Lambda)$.

**Definition 1.8.** The pairs $(\omega, \Omega)$ and $(E, \Lambda)$ are said to be *mutually dual* if they are regular, the maps

$$\Omega^\flat_{|\operatorname{im}(\Lambda^\sharp)} : \operatorname{im}(\Lambda^\sharp) \to \operatorname{im}(\Omega^\flat) \subset T^*\boldsymbol{M} \quad \text{and} \quad \Lambda^\sharp_{|\operatorname{im}(\Omega^\flat)} : \operatorname{im}(\Omega^\flat) \to \operatorname{im}(\Lambda^\sharp) \subset T\boldsymbol{M}$$

are isomorphisms, and

$$\big(\Omega^\flat_{|\operatorname{im}(\Lambda^\sharp)}\big)^{-1} = \Lambda^\sharp_{|\operatorname{im}(\Omega^\flat)}, \qquad \big(\Lambda^\sharp_{|\operatorname{im}(\Omega^\flat)}\big)^{-1} = \Omega^\flat_{|\operatorname{im}(\Lambda^\sharp)}, \qquad i_E \Omega = 0, \qquad i_\omega \Lambda = 0, \qquad i_E \omega = 1.$$

**Theorem 1.9.** *(See [12].) The relation of duality yields a bijection between regular covariant pairs $(\omega, \Omega)$ and regular contravariant pairs $(E, \Lambda)$.*

Thus, the geometric structures given by dual covariant and contravariant pairs are essentially the same.

In the literature $E$ is called the *fundamental vector field* [12], or the *Reeb vector field* [13], and $\Lambda$ the *fundamental 2-tensor* of $(\omega, \Omega)$.

**Note 1.10.** Summing up, for the convenience of the reader, we provide a schematic table with the main structures discussed above:

1a) *cosymplectic structure* = covariant pair $(\omega, \Omega)$, such that

$$d\omega = 0, \qquad d\Omega = 0, \qquad \omega \wedge \Omega^n \not\equiv 0;$$

1b) *contact structure* = covariant pair $(\omega, \Omega)$, such that

$$\Omega = d\omega, \qquad \omega \wedge \Omega^n \not\equiv 0;$$

1c) *almost-cosymplectic-contact structure* = covariant pair $(\omega, \Omega)$, such that

$$d\Omega = 0, \qquad \omega \wedge \Omega^n \not\equiv 0;$$

2a) *Jacobi structure* = contravariant pair $(E, \Lambda)$, such that

$$[E, \Lambda] = 0, \qquad [\Lambda, \Lambda] = -2E \wedge \Lambda;$$

2b) *Poisson structure* = contravariant pair $(E, \Lambda)$, such that

$$E = 0, \qquad [\Lambda, \Lambda] = 0;$$

2c) *coPoisson structure* = contravariant pair $(E, \Lambda)$, such that

$$[E, \Lambda] = 0, \qquad [\Lambda, \Lambda] = 0;$$

2d) *almost-coPoisson–Jacobi structure* = 3-plet $(E, \Lambda, \omega)$, such that $(E, \Lambda)$ is a contravariant pair and $\omega$ a 1-form such that

$$[E, \Lambda] = -E \wedge \Lambda^\sharp(L_E\omega), \quad [\Lambda, \Lambda] = 2E \wedge \big(\Lambda^\sharp \otimes \Lambda^\sharp\big)(d\omega), \qquad i_E\omega = 1, \qquad i_\omega \Lambda = 0;$$

3) *dual* pairs = regular pairs $(\omega, \Omega)$ and $(E, \Lambda)$, such that

$$\big(\Omega^\flat_{|\operatorname{im}(\Lambda^\sharp)}\big)^{-1} = \Lambda^\sharp_{|\operatorname{im}(\Omega^\flat)}, \qquad i_E\Omega = 0, \qquad i_\omega\Lambda = 0, \qquad i_E\omega = 1.$$

### *1.5. Relations between structures*

Now, let us consider dual pairs $(\omega, \Omega)$ and $(E, \Lambda)$ and state some results.

**Lemma 1.11.** *We have*:

$$\langle E\rangle = \ker \Omega^\flat, \qquad \operatorname{im}\big(\Lambda^\sharp\big) = \ker\omega \quad \textit{and} \quad \langle\omega\rangle = \ker\Lambda^\sharp, \qquad \operatorname{im}\big(\Omega^\flat\big) = \ker E.$$

**Proof.** 1) We have $\langle E\rangle \subset \ker\Omega^\flat$; hence, $\dim(\ker\Omega^\flat) = 1 = \dim\langle E\rangle$ implies $\langle E\rangle = \ker\Omega^\flat$.

If $X \in \sec(\boldsymbol{M}, \operatorname{im}(\Lambda^\sharp))$, then there exists $\alpha \in \sec(\boldsymbol{M}, T^*\boldsymbol{M})$, such that $i_\alpha\Lambda = X$; hence,

$$\omega(X) = \omega(i_\alpha\Lambda) = \Lambda(\alpha, \omega) = -i_\alpha\Lambda^\sharp(\omega) = 0, \quad \text{hence } X \in \sec(\boldsymbol{M}, \ker\omega).$$

Then, $\dim(\operatorname{im}\Lambda^\sharp) = 2n = \dim(\ker\omega)$ implies $\operatorname{im}(\Lambda^\sharp) = \ker\omega$.

2) In the same way we prove the other two identities. □

**Proposition 1.12.** *We have the splittings*:

$$T\boldsymbol{M} = \langle E\rangle \oplus \operatorname{im}\big(\Lambda^\sharp\big) \quad \textit{and} \quad T^*\boldsymbol{M} = \langle\omega\rangle \oplus \operatorname{im}\big(\Omega^\flat\big).$$

*Accordingly, for each $X \in \sec(\boldsymbol{M}, T\boldsymbol{M})$ and $\alpha \in \sec(\boldsymbol{M}, T^*\boldsymbol{M})$, we have the splittings*:

$$X = \omega(X)E + \big(X - \omega(X)E\big) \quad \textit{and} \quad \alpha = \alpha(E)\omega + \big(\alpha - \alpha(E)\omega\big).$$

*Thus, the maps*

$$\Lambda^\sharp \circ \Omega^\flat : T\boldsymbol{M} \to \operatorname{im}\big(\Lambda^\sharp\big) \quad \textit{and} \quad \Omega^\flat \circ \Lambda^\sharp : T^*\boldsymbol{M} \to \operatorname{im}\big(\Omega^\flat\big)$$

*are the "orthogonal" projections of the splittings of $T\boldsymbol{M}$ and $T^*\boldsymbol{M}$.*

**Proof.** The equalities $\dim\langle E\rangle + \dim\operatorname{im}(\Lambda^\sharp) = 1 + 2n$ and $\langle E\rangle \cap \operatorname{im}(\Lambda^\sharp) = \langle E\rangle \cap \ker\omega = 0$ yield $T\boldsymbol{M} = \langle E\rangle \oplus \operatorname{im}(\Lambda^\sharp)$.

Clearly, we have:

$$\omega(X)E \in \sec\big(\boldsymbol{M}, \langle E\rangle\big), \qquad X - \omega(X)E \in \sec\big(\boldsymbol{M}, \operatorname{im}(\Lambda^\sharp)\big) = \sec(\boldsymbol{M}, \ker\omega).$$

Then, we obtain:

$$X - \omega(X)E = \big(\Lambda^\sharp \circ \Omega^\flat\big)\big(X - \omega(X)E\big) = \big(\Lambda^\sharp \circ \Omega^\flat\big)(X).$$

The dual result can be obtained in the same way. □

**Proposition 1.13.** *For each $X, Y \in \sec(\boldsymbol{M}, T\boldsymbol{M})$ and $\alpha, \beta \in \sec(\boldsymbol{M}, T^*\boldsymbol{M})$, we have*:

$$\Omega\big(\alpha^\sharp, \beta^\sharp\big) = -\Lambda(\alpha, \beta) \quad \textit{and} \quad \Lambda\big(X^\flat, Y^\flat\big) = -\Omega(X, Y), \tag{1.3}$$

*i.e.*

$$\big(\Lambda^\sharp \otimes \Lambda^\sharp\big)(\Omega) = -\Lambda \quad \textit{and} \quad \big(\Omega^\flat \otimes \Omega^\flat\big)(\Lambda) = -\Omega. \tag{1.4}$$

**Proof.** We have:

$$\Omega\big(\Lambda^\sharp(\alpha), \Lambda^\sharp(\beta)\big) = i_{\Lambda^\sharp(\beta)}\Omega^\flat\big(\Lambda^\sharp(\alpha)\big) = i_{\Lambda^\sharp(\beta)}\big(\alpha - \alpha(E)\omega\big) = \Lambda\big(\beta, \alpha - \alpha(E)\omega\big) = -\Lambda(\alpha, \beta).$$

The second identity can be proved in the same way. □

**Lemma 1.14.** *Let us consider the functions $f, g, h \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$, the* closed *forms $\alpha, \beta, \gamma \in \sec(\boldsymbol{M}, T^*\boldsymbol{M})$, and the induced vector fields $X, Y, Z \in \sec(\boldsymbol{M}, T\boldsymbol{M})$, given by,*

$$X =: \alpha^\sharp + fE, \qquad Y =: \beta^\sharp + gE, \qquad Z =: \gamma^\sharp + hE, \tag{1.5}$$

*where $f = \omega(X)$, $g = \omega(Y)$, $h = \omega(Z)$.*

*Then, the following equality holds*:

$$\begin{aligned} d\Omega(X, Y, Z) = {}& \big(i_{E\wedge(\Lambda^\sharp\otimes\Lambda^\sharp)(d\omega)} - \tfrac{1}{2} i_{[\Lambda,\Lambda]}\big)(\alpha\wedge\beta\wedge\gamma) + f(i_{[E,\Lambda]} + i_{E\wedge(L_E\omega)^\sharp})(\beta\wedge\gamma) \\ & + g(i_{[E,\Lambda]} + i_{E\wedge(L_E\omega)^\sharp})(\gamma\wedge\alpha) + h(i_{[E,\Lambda]} + i_{E\wedge(L_E\omega)^\sharp})(\alpha\wedge\beta). \end{aligned} \tag{1.6}$$

**Proof.** Let $\tilde\alpha, \tilde\beta, \tilde\gamma$ be the projections of $\alpha, \beta, \gamma$ on $\sec(\boldsymbol{M}, \mathrm{im}(\Omega^\flat)) \subset \sec(\boldsymbol{M}, T^*\boldsymbol{M})$.

We have:

$$\begin{aligned} d\Omega(X, Y, Z) &= d\Omega\big(\alpha^\sharp + \omega(X)E, \beta^\sharp + \omega(Y)E, \gamma^\sharp + \omega(Z)E\big) \\ &= d\Omega\big(\alpha^\sharp, \beta^\sharp, \gamma^\sharp\big) + \omega(X)d\Omega\big(E, \beta^\sharp, \gamma^\sharp\big) + \omega(Y)d\Omega\big(\alpha^\sharp, E, \gamma^\sharp\big) + \omega(Z)d\Omega\big(\alpha^\sharp, \beta^\sharp, E\big). \end{aligned}$$

Then, we obtain:

$$\begin{aligned} d\Omega\big(\alpha^\sharp, \beta^\sharp, \gamma^\sharp\big) &= \alpha^\sharp.\Omega\big(\beta^\sharp, \gamma^\sharp\big) + \beta^\sharp.\Omega\big(\gamma^\sharp, \alpha^\sharp\big) + \gamma^\sharp.\Omega\big(\alpha^\sharp, \beta^\sharp\big) \\ &\quad - \Omega\big(\big[\alpha^\sharp, \beta^\sharp\big], \gamma^\sharp\big) - \Omega\big(\big[\beta^\sharp, \gamma^\sharp\big], \alpha^\sharp\big) - \Omega\big(\big[\gamma^\sharp, \alpha^\sharp\big], \beta^\sharp\big) \\ &= -\alpha^\sharp.\Lambda(\beta,\gamma) - \beta^\sharp.\Lambda(\gamma,\alpha) - \gamma^\sharp.\Lambda(\alpha,\beta) + i_{[\alpha^\sharp,\beta^\sharp]}i_{\gamma^\sharp}\Omega + i_{[\beta^\sharp,\gamma^\sharp]}i_{\alpha^\sharp}\Omega + i_{[\gamma^\sharp,\alpha^\sharp]}i_{\beta^\sharp}\Omega \\ &= -i_{\alpha^\sharp}d\big(\Lambda(\beta,\gamma)\big) - i_{\beta^\sharp}d\big(\Lambda(\gamma,\alpha)\big) - i_{\gamma^\sharp}d\big(\Lambda(\alpha,\beta)\big) \\ &\quad + (i_{\alpha^\sharp}di_{\beta^\sharp} - i_{\beta^\sharp}di_{\alpha^\sharp})i_{\gamma^\sharp}\Omega + (i_{\beta^\sharp}di_{\gamma^\sharp} - i_{\gamma^\sharp}di_{\beta^\sharp})i_{\alpha^\sharp}\Omega + (i_{\gamma^\sharp}di_{\alpha^\sharp} - i_{\alpha^\sharp}di_{\gamma^\sharp})i_{\beta^\sharp}\Omega \\ &\quad - i_{\alpha^\sharp\wedge\beta^\sharp}di_{\gamma^\sharp}\Omega - i_{\beta^\sharp\wedge\gamma^\sharp}di_{\alpha^\sharp}\Omega - i_{\gamma^\sharp\wedge\alpha^\sharp}di_{\beta^\sharp}\Omega \\ &= \Lambda\big(\alpha, d\big(\Lambda(\beta,\gamma)\big)\big) + \Lambda\big(\beta, d\big(\Lambda(\gamma,\alpha)\big)\big) + \Lambda\big(\gamma, d\big(\Lambda(\alpha,\beta)\big)\big) \\ &\quad - d\tilde\alpha\big(\beta^\sharp, \gamma^\sharp\big) - d\tilde\beta\big(\gamma^\sharp, \alpha^\sharp\big) - d\tilde\gamma\big(\alpha^\sharp, \beta^\sharp\big) \\ &= -i_\Lambda di_\Lambda(\alpha\wedge\beta\wedge\gamma) + \alpha(E)d\omega\big(\beta^\sharp, \gamma^\sharp\big) + \beta(E)d\omega\big(\gamma^\sharp, \alpha^\sharp\big) + \gamma(E)d\omega\big(\alpha^\sharp, \beta^\sharp\big) \\ &= -i_\Lambda di_\Lambda(\alpha\wedge\beta\wedge\gamma) + \alpha(E)\big(\Lambda^\sharp\otimes\Lambda^\sharp\big)(d\omega)(\beta,\gamma) \\ &\quad + \beta(E)\big(\Lambda^\sharp\otimes\Lambda^\sharp\big)(d\omega)(\gamma,\alpha) + \gamma(E)\big(\Lambda^\sharp\otimes\Lambda^\sharp\big)(d\omega)(\alpha,\beta) \\ &= (i_{E\wedge(\Lambda^\sharp\otimes\Lambda^\sharp)(d\omega)} - i_\Lambda di_\Lambda)(\alpha\wedge\beta\wedge\gamma) \\ &= \big(i_{E\wedge(\Lambda^\sharp\otimes\Lambda^\sharp)(d\omega)} - \tfrac{1}{2}i_{[\Lambda,\Lambda]}\big)(\alpha\wedge\beta\wedge\gamma). \end{aligned}$$

Similarly, we obtain:

$$\begin{aligned} d\Omega\big(\alpha^\sharp, \beta^\sharp, E\big) &= E.\Omega\big(\alpha^\sharp, \beta^\sharp\big) - \Omega\big(\big[\beta^\sharp, E\big], \alpha^\sharp\big) - \Omega\big([E, \alpha^\sharp, \beta^\sharp\big) \\ &= -E.\Lambda(\alpha,\beta) + i_{[\beta^\sharp,E]}i_{\alpha^\sharp}\Omega + i_{[E,\alpha^\sharp]}i_{\beta^\sharp}\Omega \\ &= -E.\Lambda(\alpha,\beta) + (i_{\beta^\sharp}di_E - i_Edi_{\beta^\sharp} - i_{\beta^\sharp\wedge E}d)\tilde\alpha + (i_Edi_{\alpha^\sharp} - i_{\alpha^\sharp}di_E - i_{E\wedge\alpha^\sharp}d)\tilde\beta \\ &= E.\Lambda(\alpha,\beta) + i_{\beta^\sharp\wedge E}d\big(\alpha(E)\omega\big) + i_{E\wedge\alpha^\sharp}d\big(\beta(E)\omega\big) \\ &= i_Edi_\Lambda(\alpha\wedge\beta) - \Lambda\big(d\big(\alpha(E)\big), \beta\big) + \Lambda\big(d\big(\beta(E)\big), \alpha\big) - \alpha(E)d\omega\big(E, \beta^\sharp\big) + \beta(E)d\omega\big(E, \alpha^\sharp\big) \\ &= (i_{[E,\Lambda]} + i_{E\wedge(L_E\omega)^\sharp})(\alpha\wedge\beta). \end{aligned}$$

Then, the above equalities imply (1.6). □

It is well known [9,12] that if $(\omega, \Omega)$ is contact, then $(E, \Lambda)$ is Jacobi. Thus, the geometric structures given by dual contact and regular Jacobi pairs are essentially the same. But we obtain the equivalence of structures also for other types of dual covariant and contravariant pairs.

**Theorem 1.15.** *The following facts hold*:

(1) $(\omega, \Omega)$ *is an almost-cosymplectic-contact pair if and only if* $(E, \Lambda, \omega)$ *is an almost-coPoisson–Jacobi* 3*-plet*;
(2) $(\omega, \Omega)$ *is a cosymplectic pair if and only if* $(E, \Lambda)$ *is a coPoisson pair*;
(3) $(\omega, \Omega)$ *is a contact pair if and only if* $(E, \Lambda)$ *is a Jacobi pair.*

**Proof.** Let us consider a point $x \in \boldsymbol{M}$. All 1-forms on $\boldsymbol{M}$ can be obtained, pointwisely, from closed 1-forms. Then, according to the splitting (1.5), all vectors $X, Y, Z \in T_x\boldsymbol{M}$ can be obtained, pointwisely, by means of closed forms; conversely, all closed forms $\alpha, \beta, \gamma \in \sec(\boldsymbol{M}, T^*\boldsymbol{M})$ can be obtained, pointwisely, from all vectors above.

Therefore, from Lemma 1.14 we deduce the following facts, by means of a pointwise reasoning, by taking into account the fact that the equality (1.6) involves the vectors $X, Y, Z$ and the forms $\alpha, \beta, \gamma$ only pointwisely and by considering their arbitrariness at $x \in \boldsymbol{M}$.

1) $d\Omega = 0$ if and only if $[\Lambda, \Lambda] = 2E \wedge (\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega)$ and $[E, \Lambda] = -E \wedge (L_E\omega)^\sharp$, i.e. the pair $(\omega, \Omega)$ is almost-cosymplectic-contact if and only if the 3-plet $(E, \Lambda, \omega)$ is almost-coPoisson–Jacobi.

2) Moreover, if $d\Omega = 0$ and $d\omega = 0$ then $[E, \Lambda] = 0$ and $[\Lambda, \Lambda] = 0$, i.e. $(E, \Lambda)$ is coPoisson.

On the other hand, if $d\Omega = 0$ and $(E, \Lambda)$ is coPoisson, then $(\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega) = 0$ and $(L_E\omega)^\sharp = 0$, i.e. $d\omega(\alpha^\sharp, \beta^\sharp) = 0$ and $d\omega(E, \alpha^\sharp) = 0$, for all 1-forms $\alpha, \beta \in \sec(\boldsymbol{M}, T^*\boldsymbol{M})$. Then, from the splitting $T\boldsymbol{M} = \langle E\rangle \oplus \operatorname{im}(\Lambda^\sharp)$, we have $d\omega = 0$ and the pair $(\omega, \Omega)$ is cosymplectic.

Hence the pair $(\omega, \Omega)$ is cosymplectic if and only if the pair $(E, \Lambda)$ is coPoisson.

3) Finally, if $d\omega = \Omega$, hence $d\Omega = 0$, we have $[E, \Lambda] = -E \wedge (L_E\omega)^\sharp = 0$ and $[\Lambda, \Lambda] = 2E \wedge (\Lambda^\sharp \otimes \Lambda^\sharp)(\Omega) = -2E \wedge \Lambda$, hence the pair $(E, \Lambda)$ is Jacobi.

On the other hand, if $d\Omega = 0$ and the pair $(E, \Lambda)$ is Jacobi, then $(\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega) = -\Lambda$ and $(L_E\omega)^\sharp = 0$, i.e. $d\omega(\alpha^\sharp, \beta^\sharp) = -\Lambda(\alpha, \beta)$ and $d\omega(E, \alpha^\sharp) = 0$, hence $d\omega = \Omega$, i.e. the pair $(\omega, \Omega)$ is contact.

Thus, the pair $(\omega, \Omega)$ is contact if and only if the pair $(E, \Lambda)$ is Jacobi. □

### *1.6. Darboux's charts*

First, let us consider an almost-cosymplectic-contact structure $(\omega, \Omega)$.

**Note 1.16.** [11] In a neighborhood of each $x \in \boldsymbol{M}$ there exists a local chart (a *Darboux's chart*) $(t, x^i, x^{i+n})$, with $i = 1, \ldots, n$, adapted to an almost-cosymplectic-contact structure $(\omega, \Omega)$, i.e. such that

$$\omega = dt + \sum_{1\leqslant i\leqslant n} \left(\omega^i dx^i + \omega^{i+n} dx^{i+n}\right), \qquad \Omega = \sum_{1\leqslant i\leqslant n} dx^i \wedge dx^{i+n}, \tag{1.7}$$

where $\omega^i, \omega^{i+n} \in \operatorname{map}(\boldsymbol{M}, \mathbb{R})$.

Indeed, the above almost-cosymplectic-contact pair is cosymplectic if, for instance, $\omega^i = \omega^{i+n} = 0$ [1] and contact if, for instance, $\omega^i = -x^{i+n}$ and $\omega^{i+n} = 0$.

Then, let us consider an almost-coPoisson–Jacobi structure $(E, \Lambda, \omega)$. We can find Darboux's charts adapted to this structure, analogously to the case of almost-cosymplectic-contact structures.

**Lemma 1.17.** *Let* $\alpha, \beta \in \sec(\boldsymbol{M}, T^*\boldsymbol{M})$*. Then, we have*:

$$\begin{aligned}
\left[E, \alpha^\sharp\right] &= \left(L_E\alpha - \alpha(E)(L_E\omega)\right)^\sharp + \Lambda(L_E\omega, \alpha)E, \\
\left[\alpha^\sharp, \beta^\sharp\right] &= \left(d\Lambda(\alpha, \beta) + 2i_{\beta^\sharp}d\alpha - \alpha(E)(i_{\beta^\sharp}d\omega) - 2i_{\alpha^\sharp}d\beta + \beta(E)(i_{\alpha^\sharp}d\omega)\right)^\sharp \\
&\quad - d\omega\left(\alpha^\sharp, \beta^\sharp\right)E.
\end{aligned}$$

**Proof.** For each $h \in \operatorname{map}(\boldsymbol{M}, \mathbb{R})$, we have:

$$\begin{aligned}
[E, \alpha^\sharp].h &= E.(\alpha^\sharp.h) - \alpha^\sharp.(E.h) = E.\Lambda(\alpha, dh) - \Lambda(\alpha, d(E.h)) \\
&= i_{[E,\Lambda]}(\alpha \wedge dh) + \Lambda(L_E\alpha, dh) \\
&= -i_{E\wedge(L_E\omega)^\sharp}(\alpha \wedge dh) + \Lambda(L_E\alpha, dh) \\
&= -i_E\alpha((L_E\omega)^\sharp.h) + \Lambda(L_E\omega, \alpha)(E.h) + \Lambda(L_E\alpha, dh) \\
&= (L_E\alpha - \alpha(E)(L_E\omega))^\sharp.h + \Lambda(L_E\omega, \alpha)E.h,
\end{aligned}$$

and

$$\begin{aligned}
[\alpha^\sharp, \beta^\sharp].h &= \alpha^\sharp.(\beta^\sharp.h) - \beta^\sharp.(\alpha^\sharp.h) \\
&= \Lambda(\alpha, d\Lambda(\beta, dh)) - \Lambda(\beta, d\Lambda(\alpha, dh)) \\
&= -\tfrac{1}{2} i_{[\Lambda,\Lambda]}(\alpha \wedge \beta \wedge dh) - \Lambda(dh, d\Lambda(\alpha, \beta)) + 2d\alpha(\beta^\sharp, dh^\sharp) - 2d\beta(\alpha^\sharp, dh^\sharp) \\
&= (d\Lambda(\alpha, \beta) + 2i_{\beta^\sharp}d\alpha - 2i_{\alpha^\sharp}d\beta)^\sharp.h - i_{E\wedge(\Lambda^\sharp\otimes\Lambda^\sharp)d\omega}(\alpha \wedge \beta \wedge dh) \\
&= (d\Lambda(\alpha, \beta) + 2i_{\beta^\sharp}d\alpha - \alpha(E)(i_{\beta^\sharp}d\omega) - 2i_{\alpha^\sharp}d\beta + \beta(E)(i_{\alpha^\sharp}d\omega))^\sharp.h \\
&\quad - d\omega(\alpha^\sharp, \beta^\sharp)E.h. \qquad \square
\end{aligned}$$

**Proposition 1.18.** *If $f, g \in \operatorname{map}(\boldsymbol{M}, \mathbb{R})$, then*

$$\begin{aligned}
[E, df^\sharp] &= (d(E.f) - (E.f)(L_E\omega))^\sharp + \Lambda(L_E\omega, df)E, \\
[df^\sharp, dg^\sharp] &= (d\Lambda(df, dg) - (E.f)(i_{dg^\sharp}d\omega) + (E.g)(i_{df^\sharp}d\omega))^\sharp - d\omega(df^\sharp, dg^\sharp)E.
\end{aligned}$$

**Proof.** There are consequences of the above Lemma 1.17, by putting $\alpha = df$ and $\beta = dg$. $\square$

**Theorem 1.19.** *The $(2s+1)$-dimensional distribution $\langle E\rangle \oplus \operatorname{im}\Lambda^\sharp$ is completely integrable and $(E, \Lambda, \omega)$ induces a regular almost-coPoisson–Jacobi structure on the integral submanifolds of $\langle E\rangle \oplus \operatorname{im}\Lambda^\sharp$.*

**Proof.** By the above Lemma 1.17, the distribution $\langle E\rangle \oplus \operatorname{im}\Lambda^\sharp$ is involutive and of constant rank, so it is completely integrable.

Let us consider a $(2s+1)$-dimensional integral submanifold $\iota : \boldsymbol{N} \hookrightarrow \boldsymbol{M}$ passing through $x \in \boldsymbol{M}$.

If $\tilde f, \tilde g \in \operatorname{map}(\boldsymbol{N}, \mathbb{R})$, then we can extend them (locally) to $f, g \in \operatorname{map}(\boldsymbol{M}, \mathbb{R})$, such that $\tilde f = f \circ \iota$, $\tilde g = g \circ \iota$. Then, we define $E_{\boldsymbol{N}} \in \sec(\boldsymbol{N}, T\boldsymbol{N})$ and $\Lambda_{\boldsymbol{N}} \in \sec(\boldsymbol{N}, \Lambda^2 T\boldsymbol{N})$ by:

$$E_{\boldsymbol{N}}.\tilde f = E.f, \qquad \Lambda_{\boldsymbol{N}}(d\tilde f, d\tilde g) = \Lambda(df, dg) = (df^\sharp).g = -(dg)^\sharp.f.$$

Indeed, the above $E_{\boldsymbol{N}}$ and $\Lambda_{\boldsymbol{N}}$ depend only on $\tilde f, \tilde g$, since they are computed along the integral curves of $E, (df)^\sharp, (dg)^\sharp$ through $x$ and these curves belong to $\boldsymbol{N}$.

Clearly, $(E_{\boldsymbol{N}}, \Lambda_{\boldsymbol{N}})$ satisfy the equalities:

$$E_{\boldsymbol{N}}(\iota^*\alpha) = E(\alpha) \circ \iota, \qquad \Lambda_{\boldsymbol{N}}(\iota^*\alpha, \iota^*\beta) = \Lambda(\alpha, \beta) \circ \iota, \quad \forall \alpha, \beta \in \sec(\boldsymbol{M}, T^*\boldsymbol{M}).$$

Then, from the naturality of the Schouten bracket [10], we have:

$$\begin{aligned}
[E_{\boldsymbol{N}}, \Lambda_{\boldsymbol{N}}](\iota^*\alpha, \iota^*\beta) &= [E, \Lambda](\alpha, \beta) \circ \iota, \\
[\Lambda_{\boldsymbol{N}}, \Lambda_{\boldsymbol{N}}](\iota^*\alpha, \iota^*\beta, \iota^*\gamma) &= [\Lambda, \Lambda](\alpha, \beta, \gamma) \circ \iota.
\end{aligned}$$

Let us set $\omega_{\boldsymbol{N}} = \iota^*\omega$ and $\Lambda^{\sharp_N} : T^*\boldsymbol{N} \to T\boldsymbol{N} : \iota^*\alpha \mapsto (\iota^*\alpha)^{\sharp_N} =: i_{\iota^*\alpha}\Lambda_{\boldsymbol{N}}$.

Then, $i_E\omega = 1$ implies $i_{E_N}\omega_{\boldsymbol{N}} = 1$ and $i_\omega\Lambda = 0$ implies $i_{\omega_N}\Lambda_{\boldsymbol{N}} = 0$.

Moreover, for each $\alpha, \beta \in \sec(\boldsymbol{M}, T^*\boldsymbol{M})$, we have:

$$\iota^*(L_E\omega) = L_{E_N}\omega_{\boldsymbol{N}} \quad \text{and} \quad d\omega_{\boldsymbol{N}}((\iota^*\alpha)^{\sharp_N}, (\iota^*\beta)^{\sharp_N}) = d\omega(\alpha^\sharp, \beta^\sharp) \circ \iota.$$

Then, we have:

$$\begin{aligned}
[E_N, \Lambda_N](\iota^*\alpha, \iota^*\beta) &= [E, \Lambda](\alpha, \beta) \circ \iota \\
&= -\big(E \wedge (L_E\omega)^\sharp\big)(\alpha, \beta) \circ \iota = -E(\alpha)\Lambda(L_E\omega, \beta) \circ \iota + E(\beta)\Lambda(L_E\omega, \alpha) \circ \iota \\
&= -E_N(\iota^*\alpha)\Lambda_N\big(\iota^*(L_E\omega), \iota^*\beta\big) + E_N(\iota^*\beta)\Lambda_N\big(\iota^*(L_E\omega), \iota^*\alpha\big) \\
&= -\big(E_N \wedge (L_{E_N}\omega_N)^{\sharp_N}\big)(\iota^*\alpha, \iota^*\beta).
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
[\Lambda_N, \Lambda_N](\iota^*\alpha, \iota^*\beta, \iota^*\gamma) &= [\Lambda, \Lambda](\alpha, \beta, \gamma) \circ \iota \\
&= 2\big(E \wedge \big(\Lambda^\sharp \otimes \Lambda^\sharp\big)\, d\omega\big)(\alpha, \beta, \gamma) \circ \iota \\
&= 2\big(E(\alpha)\, d\omega\big(\beta^\sharp, \gamma^\sharp\big) - E(\beta)\, d\omega\big(\alpha^\sharp, \gamma^\sharp\big) + E(\gamma)\, d\omega\big(\alpha^\sharp, \beta^\sharp\big)\big) \circ \iota \\
&= 2\big(E_N(\iota^*\alpha) d\omega_N\big((\iota^*\beta)^{\sharp_N}, (\iota^*\gamma)^{\sharp_N} - E_N(\iota^*\beta) d\omega_N\big((\iota^*\alpha)^{\sharp_N}, (\iota^*\gamma)^{\sharp_N}\big) \\
&\quad + E_N(\iota^*\gamma) d\omega_N\big((\iota^*\alpha)^{\sharp_N}, (\iota^*\beta)^{\sharp_N}\big)\big) \\
&= 2\big(E_N \wedge \big(\Lambda_N^\sharp \otimes \Lambda_N^\sharp\big) d\omega_N\big)(\iota^*\alpha, \iota^*\beta, \iota^*\gamma).
\end{aligned}$$

Hence, $(E_N, \Lambda_N, \omega_N)$ is a regular almost-coPoisson–Jacobi 3-plet on $N$. □

**Proposition 1.20.** *In a neighborhood of each $x \in M$ there exists a local chart (a* Darboux's chart*) $(W; t, x^i, x^{i+n})$, with $i = 1, \dots, n$, adapted to the almost-coPoisson–Jacobi 3-plet $(E, \Lambda, \omega)$ i.e. such that*

$$\begin{aligned}
E &= \frac{\partial}{\partial t}, \\
\Lambda &= \sum_{1 \leqslant i \leqslant s} \frac{\partial}{\partial x^{i+n}} \wedge \frac{\partial}{\partial x^i} - \sum_{1 \leqslant i \leqslant s} \left(\omega^{i+n} \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x^i} - \omega^i \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x^{i+n}}\right), \\
\omega &= dt + \sum_{1 \leqslant i \leqslant n} \big(\omega^i dx^i + \omega^{i+n} dx^{i+n}\big),
\end{aligned} \tag{1.8}$$

*where $\omega^i$, $\omega^{i+n} \in \mathrm{map}(M, \mathbb{R})$.*

**Proof.** First, let us suppose that $\Lambda$ be of rank $2s = 2n$ and let us consider a Darboux's chart adapted to the dual almost-cosymplectic-contact pair $(\omega, \Omega)$. Then, from (1.7) we can easily see that (1.8) is satisfied.

Next, let us suppose that $2s < 2n$.

Let $s = 0$. Then, $i_E\omega = 1$ implies that there exists a chart $(t, x^i, x^{i+n})$ such that

$$E = \frac{\partial}{\partial t}, \qquad \Lambda = 0, \qquad \omega = dt + \sum_{1 \leqslant i \leqslant n} \big(\omega^i dx^i + \omega^{i+n} dx^{i+n}\big).$$

Let $s > 0$. Then, let us consider an integral submanifold $N$ of the distribution $\langle E \rangle \oplus \operatorname{im} \Lambda^\sharp$. There exists a coordinate neighborhood $(W; t, x^i, x^{i+n})$ of each $x \in N$, with $i = 1, \dots, n$, such that $N$ is given by $x^j = 0, x^{j+n} = 0$, with $j = s+1, \dots, n$, and such that the coordinate neighborhood $(W \cap N; t, x^i, x^{n+i})$, with $i = 1, \dots, s$, is the Darboux's chart on $N$ adapted to $(E_N, \Lambda_N, \omega_N)$. □

**Remark 1.21.** It is easy to see that $(E, \Lambda, \omega)$ given by (1.8) satisfies the conditions for almost-coPoisson–Jacobi 3-plets.

**Remark 1.22.** Let $(E, \Lambda)$ be a contravariant pair with $s < n$. Then, there exists a 1-form $\omega$ which satisfies $i_E\omega = 1$ and $i_\omega\Lambda = 0$ (hence also $i_\omega(E \wedge \Lambda^s) = \Lambda^s$). But such a form is not unique.

Moreover, the 3-plet $(E, \Lambda, \omega)$ turns out to be almost-coPoisson–Jacobi if and only if the equalities $[E, \Lambda] = -E \wedge (L_E\omega)^\sharp$ and $[\Lambda, \Lambda] = E \wedge (\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega)$ are satisfied. We can see it in adapted Darboux's charts; in fact, if the coordinate expressions of $E$ and $\Lambda$ are given by (1.8), then the functions $\omega^i, \omega^{i+n}$, with $i = 1, \dots, s$, are given uniquely by $\Lambda$, but $\omega^i, \omega^{i+n}$, with $i = s+1, \dots, n$, are arbitrary, so $\omega$ is not unique.

**Note 1.23.** The almost-coPoisson–Jacobi 3-plet given in Darboux's charts by (1.8) is coPoisson if, for instance, $\omega^i = \omega^{i+n} = 0$, with $i = 1, \dots, s$, and is Jacobi if, for instance, $\omega^i = -x^{i+n}$, $\omega^{i+n} = 0$, with $i = 1, \dots, s$.

## 2. Lie algebra of functions

Next, we study the algebras of functions associated with the geometrical structure given by a pre coPoisson pair.

### 2.1. Poisson algebra of functions

First, let us start by considering just a 2-vector $\Lambda \in \mathrm{sec}(\boldsymbol{M}, \Lambda^2 T\boldsymbol{M})$.

**Definition 2.1.** The *Poisson bracket* of functions $f, g \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$ is defined as

$$\{f, g\} =: i_{df \wedge dg} \Lambda = i_\Lambda (df \wedge dg) = \Lambda(df, dg). \tag{2.1}$$

We have very well known properties (see, for instance, [16]).

**Proposition 2.2.** *For each $f, g, h \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$, we have*:

$$\big\{\{f, g\}, h\big\} + \big\{\{g, h\}, f\big\} + \big\{\{h, f\}, g\big\} = \frac{1}{2} i_{[\Lambda,\Lambda]}(df \wedge dg \wedge dh), \tag{2.2}$$

*i.e. the following facts are equivalent*:

(1) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$, $\forall f, g, h \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$.
(2) *The bracket* $\{,\}$ *is a Lie bracket.*
(3) $(\Lambda)$ *is a Poisson structure, i.e.* $[\Lambda, \Lambda] = 0$.
(4) $[df^\sharp, dg^\sharp].h = d\{f, g\}^\sharp.h$, $\forall f, g, h \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$.

Thus, a Poisson structure yields a Lie algebra of functions (the *Poisson algebra* of functions) and the map:

$$\Lambda^\sharp \circ d : \mathrm{map}(\boldsymbol{M}, \mathbb{R}) \to \mathrm{sec}(\boldsymbol{M}, T\boldsymbol{M})$$

is a Lie algebra homomorphism with respect to the Poisson bracket of functions and the Lie bracket of vector fields.

**Corollary 2.3.** *If $(E, \Lambda)$ is a coPoisson pair, then $\Lambda$ defines a Poisson algebra of functions.*

### 2.2. Jacobi algebra of functions

Then, let us consider a contravariant pair $(E, \Lambda)$.

**Remark 2.4.** If $(E, \Lambda)$ is a Jacobi pair with $s > 0$, then the Poisson bracket does not satisfy the Jacobi identity. In fact, the Jacobi identity turns out to be equivalent to $E \wedge \Lambda = 0$. But this condition conflicts with our hypothesis $E \wedge \Lambda^s \not\equiv 0$.

**Definition 2.5.** The *Hamiltonian lift* of a function $f \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$ is defined to be the vector field:

$$X_f =: i_{df} \Lambda - f E = df^\sharp - f E. \tag{2.3}$$

**Definition 2.6.** The *Jacobi bracket* of two functions $f, g \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$ is defined as

$$[f, g] =: \{f, g\} - f E.g + g E.f = \Lambda(df, dg) - f E.g + g E.f. \tag{2.4}$$

**Lemma 2.7.** *For each $f, g \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$, we have*:

$$E.\{f, g\} = \{E.f, g\} + \{f, E.g\} + i_{[E,\Lambda]}(df \wedge dg). \tag{2.5}$$

**Proof.** We have:

$$\begin{aligned} i_{[E,\Lambda]}(df \wedge dg) &= i_E d i_\Lambda (df \wedge dg) - i_\Lambda d i_E (df \wedge dg) \\ &= i_E d i_\Lambda (df \wedge dg) - i_\Lambda d(E.f dg - E.g df) \\ &= i_E d i_\Lambda (df \wedge dg) - i_\Lambda \big(d(E.f) \wedge dg\big) - i_\Lambda \big(df \wedge d(E.g)\big) \\ &= E.\{f,g\} - \{E.f, g\} - \{f, E.g\}. \qquad \square \end{aligned}$$

**Proposition 2.8.** *(See [11,12,16].) For each* $f, g, h \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$*, we have*

$$\begin{aligned} &\big[[f,g],h\big] + \big[[g,h],f\big] + \big[[h,f],g\big] \\ &\quad = \left(\frac{1}{2} i_{[\Lambda,\Lambda]} + i_{E\wedge\Lambda}\right)(df \wedge dg \wedge dh) + i_{[E,\Lambda]}\big(f dg \wedge dh + g\, dh \wedge df + h\, df \wedge dg\big). \end{aligned} \tag{2.6}$$

**Proposition 2.9.** *(See [9].) The Jacobi bracket defines a Lie algebra of functions if and only if* $[E, \Lambda] = 0$ *and* $[\Lambda, \Lambda] = -2E \wedge \Lambda$.

*So, a Jacobi pair* $(E, \Lambda)$ *defines a Lie algebra of functions with respect to the Jacobi bracket* (*the* Jacobi algebra *of functions*).

**Remark 2.10.** A coPoisson pair does not define a Lie algebra of functions with respect to the Jacobi bracket. Indeed, for a coPoisson pair, we have

$$\big[[f,g],h\big] + \big[[g,h],f\big] + \big[[h,f],g\big] = i_{E\wedge\Lambda}(df \wedge dg \wedge dh),$$

so, in general, the Jacobi identity is not satisfied. Indeed, it is satisfied in the particular case when $E \wedge \Lambda = 0$, but this condition conflicts with our hypothesis $E \wedge \Lambda^s \not\equiv 0$.

**Proposition 2.11.** *(See [12].) For each* $f, g, h \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$*, we have*:

$$\begin{aligned} &\big([X_f, X_g] - X_{[f,g]}\big).h \\ &\quad = -\left(\frac{1}{2} i_{[\Lambda,\Lambda]} + i_{E\wedge\Lambda}\right)(df \wedge dg \wedge dh) - f i_{[E,\Lambda]}(dg \wedge dh) - g i_{[E,\Lambda]}(df \wedge dh). \end{aligned}$$

*Hence, the following facts are equivalent*:

(1) $[X_f, X_g] = X_{[f,g]}$, $\forall f, g \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$*;*
(2) *the Hamiltonian lift of functions with respect to a pair* $(E, \Lambda)$ *is a Lie algebra homomorphism with respect to the Jacobi bracket and the Lie bracket of vector fields*;
(3) *the pair* $(E, \Lambda)$ *is a Jacobi structure, i.e.* $[E, \Lambda] = 0$ *and* $[\Lambda, \Lambda] = -2E \wedge \Lambda$.

**Note 2.12.** Summing up, for the convenience of the reader, we provide a schematic table with the main Lie brackets discussed above:

1) for a Poisson structure $(\Lambda)$, we have the *Poisson bracket*, the *Hamiltonian lift* and a Lie algebra homomorphism, according to the equalities,

$$\{f,g\} =: \Lambda(df, dg), \qquad X_f =: i_{df}\Lambda, \qquad [X_f, X_g] = X_{\{f,g\}};$$

2) for a Jacobi structure $(E, \Lambda)$, we have the *Jacobi bracket*, the *Hamiltonian lift* and a Lie algebra homomorphism, according to the equalities,

$$[f,g] =: \{f,g\} - f E.g + g E.f, \qquad X_f =: i_{df}\Lambda - f E, \qquad [X_f, X_g] = X_{[f,g]}.$$

*2.3. Uniqueness of the Jacobi structure*

Now, we revisit the well known Proposition 2.9 [9] in the context of our almost-coPoisson–Jacobi structures. Actually, we prove that an almost-coPoisson–Jacobi 3-plet $(E, \Lambda, \omega)$ defines a Lie algebra of functions with respect to the Jacobi bracket if and only if the pair $(E, \Lambda)$ is Jacobi.

Let us consider an almost-coPoisson–Jacobi 3-plet $(E, \Lambda, \omega)$.

**Lemma 2.13.** *The following facts are equivalent*:

(1) *for each* $f, g, h \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$,

$$\big(\tfrac{1}{2} i_{[\Lambda,\Lambda]} + i_{E\wedge\Lambda}\big)(df \wedge dg \wedge dh) + i_{[E,\Lambda]}(f\,dg \wedge dh + g\,dh \wedge df + h\,df \wedge dg) = 0,$$

(2) *for each* $f, g \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$,

$$\{f, g\} = -d\omega(X_f, X_g). \tag{2.7}$$

**Proof.** We have:

$$i_{\Lambda^\sharp(L_E\omega)} df = -d\omega\big(E, df^\sharp\big).$$

Then,

$$\begin{aligned}
&\big(\tfrac{1}{2} i_{[\Lambda,\Lambda]} + i_{E\wedge\Lambda}\big)(df \wedge dg \wedge dh) + i_{[E,\Lambda]}(f\,dg \wedge dh + g\,dh \wedge df + h\,df \wedge dg)\\
&\quad = i_{E\wedge(\Lambda + (\Lambda^\sharp\otimes\Lambda^\sharp)(d\omega))}(df \wedge dg \wedge dh) + i_{E\wedge\Lambda^\sharp(L_E\omega)}(f\,dg \wedge dh + g\,dh \wedge df + h\,df \wedge dg)\\
&\quad = (E.f)\big(\Lambda(dg, dh) + (\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega)(dg, dh)\big)\\
&\qquad + (E.g)\big(\Lambda(dh, df) + \big(\Lambda^\sharp \otimes \Lambda^\sharp\big)(d\omega)(dh, df)\big)\\
&\qquad + (E.h)\big(\Lambda(df, dg) + \big(\Lambda^\sharp \otimes \Lambda^\sharp\big)(d\omega)(df, dg)\big)\\
&\qquad + f(E.g)d\omega\big(E, dh^\sharp\big) - f(E.h)d\omega\big(E, dg^\sharp\big)\\
&\qquad + g(E.h)d\omega\big(E, df^\sharp\big) - g(E.f)d\omega\big(E, dh^\sharp\big)\\
&\qquad + h(E.f)d\omega\big(E, dg^\sharp\big) - h(E.g)d\omega\big(E, df^\sharp\big)\\
&\quad = (E.f)\big(\{g, h\} + d\omega\big(dg^\sharp, dh^\sharp\big) - g d\omega\big(E, dh^\sharp\big) + h d\omega\big(E, dg^\sharp\big)\big)\\
&\qquad + (E.g)\big(\{h, f\} + d\omega\big(dh^\sharp, df^\sharp\big) - h d\omega\big(E, df^\sharp\big) + f d\omega\big(E, dh^\sharp\big)\big)\\
&\qquad + (E.h)\big(\{f, g\} + d\omega\big(df^\sharp, dg^\sharp\big) - f d\omega\big(E, dg^\sharp\big) + g d\omega\big(E, df^\sharp\big)\big)\\
&\quad = (E.f)\big(\{g, h\} + d\omega\big(dg^\sharp - gE, dh^\sharp - hE\big)\big)\\
&\qquad + (E.g)\big(\{h, f\} + d\omega\big(dh^\sharp - hE, df^\sharp - fE\big)\big)\\
&\qquad + (E.h)\big(\{f, g\} + d\omega\big(df^\sharp - fE, dg^\sharp - gE\big)\big). \qquad \square
\end{aligned}$$

**Proposition 2.14.** *The almost-coPoisson–Jacobi structure $(E, \Lambda, \omega)$ yields a Lie algebra of functions with respect to the Jacobi bracket if and only if the Poisson bracket satisfies* (2.7).

**Proof.** It follows from the above Lemma 2.13 end from Lemma 2.8. $\square$

**Corollary 2.15.** *A Jacobi pair $(E, \Lambda)$ yields a Lie algebra with respect to the Jacobi bracket.*

*A coPoisson pair $(E, \Lambda)$ yields a Lie algebra with respect to the Jacobi bracket if and only if $\Lambda = 0$.*

**Proof.** Let $(E, \Lambda)$ be a Jacobi pair. Then, for each $\alpha, \beta \in \sec(\boldsymbol{M}, T^*\boldsymbol{M})$, we have:

$$d\omega\big(\alpha^\sharp, \beta^\sharp\big) = -\Lambda(\alpha, \beta) \quad \text{and} \quad d\omega\big(E, \alpha^\sharp\big) = 0,$$

hence, for each $f, g \in \mathrm{map}(\boldsymbol{M}, \mathbb{R})$, we obtain

$$\{f, g\} =: \Lambda(df, dg) = -d\omega\big(df^\sharp, dg^\sharp\big) = -d\omega(X_f, X_g),$$

hence condition (2.7) is satisfied.

Let $(E, \Lambda)$ be a coPoisson pair. Then, we have $d\omega = 0$, hence condition (2.7) is satisfied if and only if $\{f, g\} = 0$, i.e. if and only if $\Lambda = 0$. $\square$

**Theorem 2.16.** *An almost-coPoisson–Jacobi* 3*-plet* $(E, \Lambda, \omega)$ *yields a Lie algebra of functions with respect to the Jacobi bracket if and only if the pair* $(E, \Lambda)$ *is Jacobi.*

**Proof.** It is sufficient to prove that (2.7) implies that the pair $(E, \Lambda)$ is Jacobi.

We can prove it in a local chart.

In a Darboux's chart adapted to an almost-coPoisson–Jacobi 3-plet $(E, \Lambda, \omega)$ according to (1.8) we have:

$$\begin{aligned} X_f = {} & \Big(-f + \sum_{1\leqslant i\leqslant s} \Big(\omega^{i+n}\frac{\partial f}{\partial x^i} - \omega^i \frac{\partial f}{\partial x^{i+n}}\Big)\Big)\frac{\partial}{\partial t} \\ & + \sum_{1\leqslant i\leqslant s} \Big(\frac{\partial f}{\partial x^{i+n}} - \omega^{i+n}\frac{\partial f}{\partial t}\Big)\frac{\partial}{\partial x^i} + \sum_{1\leqslant i\leqslant s} \Big(\omega^i\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x^i}\Big)\frac{\partial}{\partial x^{i+n}}. \end{aligned} \tag{2.8}$$

Then,

$$\begin{aligned} d\omega(X_f, X_g) = {} & \Big(f\frac{\partial g}{\partial t} - g\frac{\partial f}{\partial t}\Big).\sum_{i=1}^{s}\Big(\frac{\partial \omega^i}{\partial t}\omega^{i+n} - \frac{\partial \omega^{i+n}}{\partial t}\omega^i\Big) \\ & + \sum_{1\leqslant i\leqslant s}\Big(f\frac{\partial g}{\partial x^i} - g\frac{\partial f}{\partial x^i}\Big)\frac{\partial \omega^{i+n}}{\partial t} - \sum_{i=1}^{s}\Big(f\frac{\partial g}{\partial x^{i+n}} - g\frac{\partial f}{\partial x^{i+n}}\Big)\frac{\partial \omega^i}{\partial t} \\ & + \sum_{1\leqslant i,j\leqslant s}\Big(\frac{\partial f}{\partial t}\frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial t}\frac{\partial f}{\partial x^i}\Big).\Big(\omega^{j+n}\omega^{i+n}\frac{\partial \omega^j}{\partial t} - \omega^j\omega^{i+n}\frac{\partial \omega^{j+n}}{\partial t} \\ & + \omega^{j+n}\frac{\partial \omega^{i+n}}{\partial x^j} - \omega^{j+n}\frac{\partial \omega^j}{\partial x^{i+n}} + \omega^j\frac{\partial \omega^{j+n}}{\partial x^{i+n}} - \omega^j\frac{\partial \omega^{i+n}}{\partial x^{j+n}}\Big) \\ & + \sum_{1\leqslant i,j\leqslant s}\Big(\frac{\partial f}{\partial t}\frac{\partial g}{\partial x^{i+n}} - \frac{\partial g}{\partial t}\frac{\partial f}{\partial x^{i+n}}\Big).\Big(\omega^j\omega^i\frac{\partial \omega^{j+n}}{\partial t} - \omega^{j+n}\omega^i\frac{\partial \omega^j}{\partial t} \\ & + \omega^{j+n}\frac{\partial \omega^j}{\partial x^i} - \omega^{j+n}\frac{\partial \omega^i}{\partial x^j} + \omega^j\frac{\partial \omega^{j+n}}{\partial x^{i+n}} - \omega^j\frac{\partial \omega^{j+n}}{\partial x^i} + \omega^j\frac{\partial \omega^i}{\partial x^{j+n}} - \omega^{j+n}\frac{\partial \omega^j}{\partial x^{i+n}}\Big) \\ & + \sum_{1\leqslant i,j\leqslant s}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}\Big(\omega^{j+n}\frac{\partial \omega^{i+n}}{\partial t} - \omega^{i+n}\frac{\partial \omega^{j+n}}{\partial t} + \frac{\partial \omega^{i+n}}{\partial x^{j+n}} - \frac{\partial \omega^{j+n}}{\partial x^{i+n}}\Big) \\ & + \sum_{1\leqslant i,j\leqslant s}\Big(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^{j+n}} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial x^{j+n}}\Big).\Big(\omega^{i+n}\frac{\partial \omega^j}{\partial t} - \omega^j\frac{\partial \omega^{i+n}}{\partial t} + \frac{\partial \omega^{i+n}}{\partial x^j} - \frac{\partial \omega^j}{\partial x^{i+n}}\Big) \\ & + \sum_{1\leqslant i,j\leqslant s}^{s}\frac{\partial f}{\partial x^{i+n}}\frac{\partial g}{\partial x^{j+n}}\Big(\omega^j\frac{\partial \omega^i}{\partial t} - \omega^i\frac{\partial \omega^j}{\partial t} + \frac{\partial \omega^j}{\partial x^i} - \frac{\partial \omega^i}{\partial x^j}\Big). \end{aligned}$$

On the other hand,

$$\begin{aligned} \{f, g\} = \sum_{1\leqslant i\leqslant s}\Big( & \frac{\partial f}{\partial x^{i+n}}\frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial x^{i+n}}\frac{\partial f}{\partial x^i} \\ & - \omega^{i+n}\Big(\frac{\partial f}{\partial t}\frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial t}\frac{\partial f}{\partial x^i}\Big) + \omega^i\Big(\frac{\partial f}{\partial t}\frac{\partial g}{\partial x^{i+n}} - \frac{\partial g}{\partial t}\frac{\partial f}{\partial x^{i+n}}\Big)\Big). \end{aligned}$$

Now, if we assume $\{f, g\} = -d\omega(X_f, X_g)$, then we obtain the following system of partial differential equations, by comparing the above expressions, for all $i, j = 1, \dots, s$,

$$0 = \frac{\partial \omega^{i+n}}{\partial t}, \qquad 0 = \frac{\partial \omega^i}{\partial t}, \qquad 0 = \sum_{1 \leqslant j \leqslant s} \left( \omega^j \frac{\partial \omega^{j+n}}{\partial x^{i+n}} - \omega^{j+n} \frac{\partial \omega^j}{\partial x^{i+n}} \right),$$

$$0 = \left( \frac{\partial \omega^j}{\partial x^i} - \frac{\partial \omega^i}{\partial x^j} \right), \qquad 0 = \left( \frac{\partial \omega^{i+n}}{\partial x^{j+n}} - \frac{\partial \omega^{j+n}}{\partial x^{i+n}} \right), \qquad \delta^i_j = \left( \frac{\partial \omega^{i+n}}{\partial x^j} - \frac{\partial \omega^j}{\partial x^{i+n}} \right).$$

Now, if we use the above identities, then we obtain $[E, \Lambda] = 0$ and $[\Lambda, \Lambda] = -2E \wedge \Lambda$, so, $(E, \Lambda)$ is a Jacobi pair. □

## 3. Examples: dynamical structures

As examples of the geometric structures analyzed above, now we discuss the dynamical structures arising on the phase space of a spacetime in classical relativistic theories. We consider the relativistic Galilei and the Einstein spacetimes, emphasizing the analogies and the differences between the two cases.

In order to make our theory explicitly independent from units of measurement, we introduce the "spaces of scales" [8]. Roughly speaking, a space of scales $\mathbb{S}$ has the algebraic structure of $\mathbb{R}^+$ but has no distinguished "basis". We can naturally define the tensor product of spaces of scales and the tensor product of spaces of scales and vector spaces. We can also naturally define rational tensor powers $\mathbb{S}^{m/n}$ of a space of scales $\mathbb{S}$. Moreover, we can make a natural identification $\mathbb{S}^* \simeq \mathbb{S}^{-1}$.

The basic objects of our theory (the metric field, the phase 2-form, the phase 2-vector, etc.) will be valued into *scaled* vector bundles, that is into vector bundles multiplied tensorially with spaces of scales. In this way, each tensor field carries explicit information on its "scale dimension". Actually, we assume the following basic spaces of scales: the space of *time intervals* $\mathbb{T}$, the space of *lengths* $\mathbb{L}$ and the space of *masses* $\mathbb{M}$. Moreover, we consider the following "universal scales": the *speed of light* $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$ and the *Planck constant* $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

A *time unit* is defined to be an element $u_0 \in \mathbb{T}$, or, equivalently, its dual $u^0 \in \mathbb{T}^*$.

### *3.1. Galilei spacetime*

First, we study the geometrical structures arising on the phase space of a Galilei spacetime [2,5,6,14].

#### *3.1.1. Spacetime*

We assume *absolute time* to be an affine 1-dimensional space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} =: \mathbb{T} \otimes \mathbb{R}$.

We assume spacetime to be an oriented $(3 + 1)$-dimensional fibred manifold $\boldsymbol{E}$ equipped with a *time fibring* $t : \boldsymbol{E} \to \boldsymbol{T}$.

A *spacetime chart* is defined to be a chart $(x^\lambda) \equiv (x^0, x^i)$ of $\boldsymbol{E}$, adapted to the orientation, to the fibring, to the affine structure of $\boldsymbol{T}$ and to a time unit $u_0$. Greek indices will span all spacetime coordinates and Latin indices will span the fibre coordinates. In the following, we shall always refer to spacetime charts. The induced local bases of $T\boldsymbol{E}$ and $T^*\boldsymbol{E}$ are denoted, respectively, by $(\partial_\lambda)$ and $(d^\lambda)$.

The vertical restriction of forms will be denoted by the "check" symbol $^\vee$.

The differential of the time fibring is the scaled 1-form $dt : \boldsymbol{E} \to \bar{\mathbb{T}} \otimes T^*\boldsymbol{E}$, with coordinate expression $dt = u_0 \otimes d^0$.

We assume spacetime to be equipped with a *scaled spacelike Riemannian metric* $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$. The contravariant metric is denoted by $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E})$.

We have the coordinate expressions:

$$g = g_{ij} \check{d}^i \otimes \check{d}^j, \quad \text{with } g_{ij} \in \operatorname{map}\big(\boldsymbol{E}, \mathbb{L}^2 \otimes \mathbb{R}\big),$$
$$\bar{g} = g^{ij} \partial_i \otimes \partial_j, \quad \text{with } g^{ij} \in \operatorname{map}\big(\boldsymbol{E}, \mathbb{L}^{-2} \otimes \mathbb{R}\big).$$

*3.1.2. Phase space*

A *motion* is defined to be a section $s : \boldsymbol{T} \to \boldsymbol{E}$. The *1st differential* of a motion $s$ is defined to be the map $ds : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E}$. We have $dt(ds) = 1$.

We assume as *phase space* the 1st jet space $J_1\boldsymbol{E}$ of motions.

A space time chart $(x^\lambda)$ induces naturally a chart $(x^\lambda, x^i_0)$ on $J_1\boldsymbol{E}$.

The *velocity* of a motion $s$ is defined to be its 1st jet $j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E}$.

We define the *contact map* to be the unique fibred morphism $д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ over $\boldsymbol{E}$ such that $д \circ j_1 s = ds$, for each motion $s$. We have $д \lrcorner dt = 1$. The coordinate expression of $д$ is:

$$д = u^0 \otimes д_0 \equiv u^0 \otimes \left(\partial_0 + x^i_0 \partial_i\right).$$

The map $д$ is injective. Accordingly, the 1st jet space can be naturally identified with the subbundle $J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$, of scaled vectors which project on $\boldsymbol{1} : \boldsymbol{T} \to \bar{\mathbb{T}}^* \otimes \bar{\mathbb{T}}$. Thus, the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$ turns out to be affine and associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E}$. Indeed, $J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$ is the fibred submanifold over $\boldsymbol{E}$ characterized by the constraint $\dot{x}^0_0 = 1$.

We define also the *complementary contact map* $\theta =: 1 - д \circ dt : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$. The coordinate expression of $\theta$ is:

$$\theta = \theta^i \otimes \partial_i \equiv \left(d^i - x^i_0 d^0\right) \otimes \partial_i .$$

*3.1.3. Vertical bundle of the phase space*

Let $V_0 J_1\boldsymbol{E} \subset V J_1\boldsymbol{E} \subset T J_1\boldsymbol{E}$ be the vertical tangent subbundle over $\boldsymbol{E}$ and the vertical tangent subbundle over $\boldsymbol{T}$, respectively. The affine structure of the phase space yields the equality $V_0 J_1\boldsymbol{E} = J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{T}^* \otimes V\boldsymbol{E})$, hence the natural map $\nu : J_1\boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V_0 J_1\boldsymbol{E})$, with coordinate expression $\nu = u_0 \otimes \check{d}^i \otimes \partial^0_i$.

*3.1.4. Spacetime connections*

We define a *spacetime connection* to be a torsion free linear connection $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ of the bundle $T\boldsymbol{E} \to \boldsymbol{E}$. Its coordinate expression is of the type

$$K = d^\lambda \otimes \left(\partial_\lambda + K_\lambda{}^\mu{}_\nu \dot{x}^\nu \dot{\partial}_\mu\right), \quad \text{with } K_\lambda{}^\mu{}_\nu = K_\nu{}^\mu{}_\lambda \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}).$$

A spacetime connection $K$ is said to be *time preserving* if it preserves the time fibring, i.e. if $\nabla dt = 0$. In coordinates, this reads $K_\lambda{}^0{}_\mu = 0$.

A time preserving spacetime connection $K$ is said to be *metric* if it preserves the metric $g$, i.e. if $\nabla g = 0$. In coordinates, it reads:

$$\begin{aligned} K_0{}^i{}_0 &= -g^{ij} 2\phi_{0,0j}, \\ K_0{}^i{}_h = K_h{}^i{}_0 &= -\frac{1}{2} g^{ij} (2\phi_{0,hj} + \partial_0 g_{hj}), \\ K_k{}^i{}_h = K_h{}^i{}_k &= -\frac{1}{2} g^{ij} (\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk}), \end{aligned}$$

where $\phi \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \Lambda^2 T^*\boldsymbol{E})$ is a scaled spacetime 2-form (which depends on $K$ and on the chosen chart).

The vertical restriction of a metric spacetime connection $K$ is just the Levi Civita connection of the spacetime fibres.

A spacetime connection $K$ is said to be a *Galilei connection* if it is time preserving, metric and such that its curvature tensor $R$ fulfills a symmetry condition which in coordinates reads $R_\lambda{}^i{}_\mu{}^j = R_\mu{}^j{}_\lambda{}^i$, where $R_\lambda{}^i{}_\mu{}^j =: g^{jp} R_\lambda{}^i{}_{\mu p}$.

*3.1.5. Phase connections*

We define a *phase connection* to be a connection of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$.

A phase connection can be represented, equivalently, by a tangent valued form $\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T J_1\boldsymbol{E}$, which is projectable over $\boldsymbol{1} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E}$, or by the complementary vertical valued form $\nu[\Gamma] : J_1\boldsymbol{E} \to T^* J_1\boldsymbol{E} \otimes V J_1\boldsymbol{E}$, respectively, with coordinate expressions:

$$\Gamma = d^\lambda \otimes \left(\partial_\lambda + \Gamma_\lambda{}^i{}_0 \partial^0_i\right), \qquad \nu[\Gamma] = \left(d^i_0 - \Gamma_\lambda{}^i{}_0 d^\lambda\right) \otimes \partial^0_i, \quad \text{with } \Gamma_\lambda{}^i{}_0 \in \operatorname{map}(J_1\boldsymbol{E}, \mathbb{R}).$$

The coordinate expression of an affine phase connection $\Gamma$ is $\Gamma_\lambda{}^i{}_0 = \Gamma_\lambda{}^i{}_0{}^0{}_p x_0^p + \Gamma_\lambda{}^i{}_0{}^0{}_0$.

We can prove [4] that there is a natural bijective map $\chi : K \mapsto \Gamma$ between time preserving linear spacetime connections $K$ and affine phase connections $\Gamma$, with coordinate expression $\Gamma_\lambda{}^i{}_0{}^0{}_\mu = K_\lambda{}^i{}_\mu$.

*3.1.6. Dynamical phase connection*

The space of 2–jets of motions $J_2\boldsymbol{E}$ can be naturally regarded as the affine subbundle $J_2\boldsymbol{E} \subset \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$, which projects on $д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$.

A *dynamical phase connection* is defined to be a 2nd-order connection, i.e. a section $\gamma : J_1\boldsymbol{E} \to J_2\boldsymbol{E}$, or, equivalently, a section $\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$, which projects on $д$.

The coordinate expression of a dynamical phase connection is of the type:

$$\gamma = u^0 \otimes \left(\partial_0 + x_0^i \partial_i + \gamma_0{}^i{}_0 \partial_i^0\right), \quad \text{with } \gamma_0{}^i{}_0 \in \operatorname{map}(J_1\boldsymbol{E}, \mathbb{R}).$$

If $\gamma$ is a dynamical phase connection, then we have $\gamma \lrcorner dt = 1$.

The contact map $д$ and a phase connection $\Gamma$ yield the section $\gamma \equiv \gamma[д, \Gamma] =: д \lrcorner \Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$, which turns out to be a dynamical phase connection, with coordinate expression,

$$\gamma_0{}^i{}_0 = \Gamma_0{}^i{}_0 + \Gamma_j{}^i{}_0 x_0^j.$$

In particular, a time preserving spacetime connection $K$ yields the dynamical phase connection $\gamma =: \gamma[д, K] =: д \lrcorner \chi(K)$, with coordinate expression:

$$\gamma_{00}^i = K_h{}^i{}_k x_0^h x_0^k + 2K_h{}^i{}_0 x_0^h + K_0{}^i{}_0.$$

*3.1.7. Phase 2-form and 2-vector*

The metric $g$ and a phase connection $\Gamma$ yield the scaled 2-form $\Omega$, called (*scaled*) *phase 2-form*, and the scaled vertical 2-vector $\Lambda$, called (*scaled*) *phase 2-vector*,

$$\Omega = \Omega[g, \Gamma] =: g \lrcorner \big(\nu[\Gamma] \wedge \theta\big) : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \Lambda^2 T^* J_1\boldsymbol{E},$$
$$\Lambda = \Lambda[g, \Gamma] =: \bar{g} \lrcorner (\Gamma \wedge \nu) : J_1\boldsymbol{E} \to \mathbb{T} \otimes \mathbb{L}^{-2} \otimes \Lambda^2 V J_1\boldsymbol{E},$$

with coordinate expressions:

$$\Omega[g, \Gamma] = g_{ij} u^0 \otimes \left(d_0^i - \Gamma_\lambda{}^i{}_0 d^\lambda\right) \wedge \left(d^j - x_0^j d^0\right),$$
$$\Lambda[g, \Gamma] = g^{ij} u_0 \otimes \left(\partial_i + \Gamma_i{}^h{}_0 \partial_h^0\right) \wedge \partial_j^0.$$

We can easily see that $dt \wedge \Omega^3 \not\equiv 0$ and $\gamma \wedge \Lambda^3 \not\equiv 0$.

There is a unique dynamical phase connection $\gamma$, such that $\gamma \lrcorner \Omega[g, \Gamma] = 0$. Namely, $\gamma = \gamma[д, \Gamma]$.

In particular, a metric spacetime connection $K$ yields the (scaled) phase 2-form $\Omega \equiv \Omega[g, K] =: \Omega[g, \chi(K)]$ and the (scaled) phase 2-vector $\Lambda \equiv \Lambda[g, K] =: \Lambda[g, \chi(K)]$ with coordinate expressions

$$\begin{aligned}\Omega = {} & -g_{ij} u^0 \otimes \left(d^i - x_0^i d^0\right) \wedge d_0^j + \left(\tfrac{1}{2}\partial_j g_{hk} x_0^h x_0^k + \partial_0 g_{hj} x_0^h + \phi_{0,0j}\right) u^0 \otimes d^0 \wedge d^j \\ & + \left(\tfrac{1}{2}(\partial_i g_{hj} - \partial_j g_{hi}) x_0^h + \tfrac{1}{2}\phi_{0,ij}\right) u^0 \otimes d^i \wedge d^j,\end{aligned}$$
$$\Lambda = g^{ij} u_0 \otimes \partial_i \wedge \partial_j^0 - \frac{1}{2} g^{ih} g^{jk} \left((\partial_k g_{lr} - \partial_h g_{lk}) x_0^l + \phi_{0,kh}\right) u_0 \otimes \partial_i^0 \wedge \partial_j^0.$$

*3.1.8. Dynamical structures of the phase space*

We have the following result [2,5].

**Theorem 3.1.** *Let us consider a spacetime connection $K$ and the induced objects $\Gamma =: \chi(K)$, $\gamma =: \gamma[д, \Gamma]$, $\Omega =: \Omega[g, \Gamma]$ and $\Lambda =: \Lambda[g, \Gamma]$. Then, the following assertions are equivalent.*

(1) *$K$ is a Galilei connection.*
(2) *$\Omega$ is closed, i.e. $(-dt, \Omega)$ is a scaled cosymplectic pair.*
(3) *$[\gamma, \Lambda] = 0$ and $[\Lambda, \Lambda] = 0$, i.e. $(-\gamma, \Lambda)$ is a scaled (regular) coPoisson pair.*

*Moreover, the cosymplectic pair $(-dt, \Omega)$ and the coPoisson pair $(-\gamma, \Lambda)$ are mutually dual.*

**Remark 3.2.** If $K$ is a time preserving spacetime connection, then the induced pairs $(-dt, \Omega[g, K])$ and $(-\gamma[д, K], \Lambda[g, K])$ are scaled.

On the other hand, some results of the general theory of geometrical structures developed in the first two sections requires unscaled pairs.

Indeed, if we refer to a particle of mass $m \in \mathbb{M}$ and consider the universal scales $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$ and $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$, then we obtain unscaled pairs in the following natural way.

We have the unscaled spacetime 1-form:

$$\frac{mc^2}{\hbar} dt : \boldsymbol{E} \to T^*\boldsymbol{E}.$$

Moreover, the rescaled contact map $Д =: \frac{\hbar}{mc^2} д : J_1\boldsymbol{E} \to T\boldsymbol{E}$ yields the unscaled phase vector field:

$$\gamma \equiv \gamma[Д, K] = \frac{\hbar}{mc^2}\gamma[д, K] : \boldsymbol{E} \to TJ_1\boldsymbol{E}.$$

Furthermore, the rescaled metric $G =: \frac{m}{\hbar} g : \boldsymbol{E} \to \mathbb{T} \otimes V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}$ yields the unscaled phase 2-form and phase 2-vector:

$$\Omega \equiv \Omega[G, K] = \frac{m}{\hbar}\Omega[g, K] : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E},$$
$$\Lambda \equiv \Lambda[G, K] = \frac{\hbar}{m}\Lambda[g, K] : J_1\boldsymbol{E} \to \Lambda^2 T J_1\boldsymbol{E}.$$

Thus, if $K$ is a Galilei spacetime connection, then $(-\frac{mc^2}{\hbar} dt, \Omega)$ and $(-\frac{\hbar}{mc^2}\gamma, \Lambda)$ turn out to be mutually dual unscaled cosymplectic and coPoisson pairs of the phase space.

Indeed, the Plank constant does not play any direct role in classical mechanics; nevertheless, such a scale is necessary for getting unscaled objects as above.

### 3.2. *Einstein spacetime*

Then, we study the geometrical structures arising on the phase space of an Einstein spacetime [3,7].

#### 3.2.1. *Spacetime*

We assume *spacetime* to be an oriented 4-dimensional manifold $\boldsymbol{E}$ equipped with a scaled Lorentzian metric $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$, with signature $(-+++)$; we suppose spacetime to be time oriented. The contravariant metric is denoted by $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes (T\boldsymbol{E} \otimes T\boldsymbol{E})$.

A *spacetime chart* is defined to be a chart $(x^\lambda) \equiv (x^0, x^i) \in \operatorname{map}(\boldsymbol{E}, \mathbb{R} \times \mathbb{R}^3)$ of $\boldsymbol{E}$, which fits the orientation of spacetime and such that the vector field $\partial_0$ is timelike and time oriented and the vector fields $\partial_1, \partial_2, \partial_3$ are spacelike. Greek indices $\lambda, \mu, \ldots$ will span spacetime coordinates, while Latin indices $i, j, \ldots$ will span spacelike coordinates. In the following, we shall always refer to spacetime charts. The induced local bases of $T\boldsymbol{E}$ and $T^*\boldsymbol{E}$ are denoted, respectively, by $(\partial_\lambda)$ and $(d^\lambda)$. We have the coordinate expressions:

$$g = g_{\lambda\mu} d^\lambda \otimes d^\mu, \quad \text{with } g_{\lambda\mu} \in \operatorname{map}\big(\boldsymbol{E}, \mathbb{L}^2 \otimes \mathbb{R}\big),$$
$$\bar{g} = g^{\lambda\mu} \partial_\lambda \otimes \partial_\mu, \quad \text{with } g^{\lambda\mu} \in \operatorname{map}\big(\boldsymbol{E}, \mathbb{L}^{-2} \otimes \mathbb{R}\big).$$

#### 3.2.2. *Jets of submanifolds*

In view of the definition of the phase space, let us consider a manifold $\boldsymbol{M}$ of dimension $n$ and recall a few basic facts concerning jets of submanifolds [15].

Let $k \geqslant 0$ be an integer. A *k-jet* of 1-dimensional submanifolds of $\boldsymbol{M}$ at $x \in \boldsymbol{M}$ is defined to be an equivalence class of 1-dimensional submanifolds touching each other at $x$ with a contact of order $k$. The $k$-jet of a 1-dimensional submanifold $s : \boldsymbol{N} \hookrightarrow \boldsymbol{M}$ at $x \in \boldsymbol{N}$ is denoted by $j_k s(x)$. The set of all $k$-jets of all 1-dimensional submanifolds at $x \in \boldsymbol{M}$ is denoted by $J_{kx}(\boldsymbol{M}, 1)$. The set $J_k(\boldsymbol{M}, 1) =: \bigsqcup_{x \in \boldsymbol{M}} J_{kx}(\boldsymbol{M}, 1)$ is said to be the *k-jet space* of 1-dimensional submanifolds of $\boldsymbol{M}$. In particular, for $k = 0$, we have the natural identification $J_0(\boldsymbol{M}, 1) = \boldsymbol{M}$, given by $j_0 s(x) = x$, for each 1-dimensional submanifold $s : \boldsymbol{N} \hookrightarrow \boldsymbol{M}$. For each integers $k \geqslant h \geqslant 0$, we have the natural projection $\pi^k_h : J_k(\boldsymbol{M}, 1) \to J_h(\boldsymbol{M}, 1) : j_k s(x) \mapsto j_h s(x)$.

A chart of $\boldsymbol{M}$ is said to be *divided* if the set of its coordinate functions is divided into two subsets of 1 and $n-1$ elements. Our typical notation for a divided chart will be $(x^0, x^i)$, with $1 \leqslant i \leqslant n-1$. A divided chart and a 1-dimensional submanifold $s : \boldsymbol{N} \hookrightarrow \boldsymbol{M}$ are said to be *related* if the map $\breve{x}^0 =: x^0|_{\boldsymbol{N}} \in \mathrm{map}(\boldsymbol{N}, \mathbb{R})$ is a chart of $\boldsymbol{N}$. In such a case, the submanifold $\boldsymbol{N}$ is locally characterized by $s^i \circ (\breve{x}^0)^{-1} =: (x^i \circ s) \circ (\breve{x}^0)^{-1} \in \mathrm{map}(\mathbb{R}, \mathbb{R})$. In particular, if the divided chart is adapted to the submanifold, then the chart and the submanifold are related.

Let us consider a divided chart $(x^0, x^i)$ of $\boldsymbol{M}$.

Then, for each submanifold $s : \boldsymbol{N} \hookrightarrow \boldsymbol{M}$ which is related to this chart, the chart yields naturally the local fibred chart $(x^0, x^i; x^i_{\underline{\alpha}})_{1 \leqslant |\underline{\alpha}| \leqslant k} \in \mathrm{map}(J_k(\boldsymbol{M}, 1),\ \mathbb{R}^n \times \mathbb{R}^{k(n-1)})$ of $J_k(\boldsymbol{M}, 1)$, where $\underline{\alpha} =: (h)$ is a multi-index of "range" 1 and "length" $|\underline{\alpha}| = h$ and the functions $x^i_{\underline{\alpha}}$ are defined by $x^i_{\underline{\alpha}} \circ j_1 \boldsymbol{N} =: \partial_{0\ldots 0} s^i$, with $1 \leqslant |\underline{\alpha}| \leqslant k$.

We can prove the following facts:

1) the above charts $(x^0, x^i; x^i_{\underline{\alpha}})$ yield a smooth structure of $J_k(\boldsymbol{M}, 1)$;

2) for each 1-dimensional submanifold $s : \boldsymbol{N} \subset \boldsymbol{M}$ and for each integer $k \geqslant 0$, the subset $j_k s(\boldsymbol{N}) \subset J_k(\boldsymbol{M}, 1)$ turns out to be a smooth 1-dimensional submanifold;

3) for each integers $k \geqslant h \geqslant 1$, the maps $\pi^k_h : J_k(\boldsymbol{M}, 1) \to J_h(\boldsymbol{M}, 1)$ turn out to be smooth bundles.

We shall always refer to such divided charts $(x^0, x^i)$ of $\boldsymbol{M}$ and to the induced fibred charts $(x^0, x^i; x^i_{\underline{\alpha}})$ of $J_k(\boldsymbol{M}, 1)$.

Let $m_1 \in J_1(\boldsymbol{M}, 1)$, with $m_0 = \pi^1_0(m_1) \in \boldsymbol{M}$. Then, the tangent spaces at $m_0$ of all 1-dimensional submanifolds $s : \boldsymbol{N} \hookrightarrow \boldsymbol{M}$, such that $j_1 s(m_0) = m_1$, coincide. Accordingly, we denote by $T[m_1] \subset T_{m_0}\boldsymbol{M}$ the tangent space at $m_0$ of the above 1-dimensional submanifolds $\boldsymbol{N}$ generating $m_1$. We have the natural fibred isomorphism $J_1(\boldsymbol{M}, 1) \to \mathrm{Grass}(\boldsymbol{M}, 1) : m_1 \mapsto T[m_1] \subset T_{m_0}\boldsymbol{M}$ over $\boldsymbol{M}$ of the 1st jet bundle with the Grassmannian bundle of dimension 1. If $s : \boldsymbol{N} \hookrightarrow \boldsymbol{M}$ is a 1-dimensional submanifold, then we obtain $T[j_1 s] = \mathrm{span}\langle \partial_0 + \partial_0 s^i \, \partial_i \rangle$, with reference to a related chart.

### *3.2.3. Phase space*

A *motion* is defined to be a 1-dimensional timelike submanifold $s : \boldsymbol{T} \hookrightarrow \boldsymbol{E}$.

For every arbitrary choice of a "*proper time origin*" $t_0 \in \boldsymbol{T}$, we obtain the "*proper time scaled function*" given by the equality $\sigma : \boldsymbol{T} \to \bar{\mathbb{T}} : t \mapsto \frac{1}{c} \int_{[t_0, t]} \| \frac{ds}{d\breve{x}^0} \| \, d\breve{x}^0$.

This map yields, at least locally, a bijection $\boldsymbol{T} \to \bar{\mathbb{T}}$, hence a (local) affine structure of $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}}$. Indeed, this (local) affine structure does not depend on the choice of the proper time origin and of the spacetime chart.

Let us choose a time origin $t_0 \in \boldsymbol{T}$ and consider the associated proper time scaled function $\sigma : \boldsymbol{T} \to \bar{\mathbb{T}}$ and the induced linear isomorphism $T\boldsymbol{T} \to \boldsymbol{T} \times \bar{\mathbb{T}}$. Moreover, let us consider a spacetime chart $(x^\lambda)$ and the induced chart $(\breve{x}^0) \in \mathrm{map}(\boldsymbol{T}, \mathbb{R})$. Let us set $\partial_0 s^\lambda =: \frac{ds^\lambda}{d\breve{x}^0}$.

The *1st differential* of the motion $s$ is defined to be the map $ds =: \frac{ds}{d\sigma} : \boldsymbol{T} \to \bar{\mathbb{T}}^* \otimes T\boldsymbol{E}$.

We have $g(ds, ds) = -c^2$.

We assume as *phase space* the subspace $\mathcal{J}_1\boldsymbol{E} \subset J_1(\boldsymbol{E}, 1)$ consisting of all 1-jets of motions.

For each 1-dimensional submanifold $s : \boldsymbol{T} \subset \boldsymbol{E}$ and for each $x \in \boldsymbol{T}$, we have $j_1 s(x) \in \mathcal{J}_1\boldsymbol{E}$ if and only if $T[j_1 s(x)] = T_x\boldsymbol{T}$ is timelike.

Any spacetime chart $(x^0, x^i)$ is related to each motion $s$. Hence, the fibred chart $(x^0, x^i, x^i_0)$ is defined on tubelike open subsets of $\mathcal{J}_1\boldsymbol{E}$.

We shall always refer to the above fibred charts.

The *velocity* of a motion $s$ is defined to be its 1-jet $j_1 s : \boldsymbol{T} \to \mathcal{J}_1(\boldsymbol{E}, 1)$.

We define the *contact map* to be the unique fibred morphism $\text{д} : \mathcal{J}_1\boldsymbol{E} \to \bar{\mathbb{T}}^* \otimes T\boldsymbol{E}$ over $\boldsymbol{E}$, such that $\text{д} \circ j_1 s = ds$, for each motion $s$. We have $g(\text{д}, \text{д}) = -c^2$. The coordinate expression of д is:

$$\text{д} = c\alpha^0 \big(\partial_0 + x^i_0 \partial_i\big), \quad \text{where } \alpha^0 =: 1 / \sqrt{\left| g_{00} + 2 g_{0j} x^j_0 + g_{ij} x^i_0 x^j_0 \right|}.$$

The map $\text{д} : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ is injective. Indeed, it makes $\mathcal{J}_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$ the fibred submanifold over $\boldsymbol{E}$ characterized by the constraint $g_{\lambda\mu} \dot{x}^\lambda_0 \dot{x}^\mu_0 = -(c_0)^2$.

We define the *time form* to be the map $\tau =: -\frac{1}{c^2} g^\flat(д) : \mathcal{J}_1\boldsymbol{E} \to \bar{\mathbb{T}} \otimes T^*\boldsymbol{E}$. We have $\tau(д) = 1$, and $\bar{g}(\tau, \tau) = -\frac{1}{c^2}$. The coordinate expression of $\tau$ is:

$$\tau = \tau_\lambda d^\lambda, \quad \text{where } \tau_\lambda = -\frac{\alpha^0}{c}\big(g_{0\lambda} + g_{i\lambda} x_0^i\big).$$

We define also the *complementary contact map* $\theta =: 1 - д \otimes \tau : \mathcal{J}_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E}$. The coordinate expression of $\theta$ is:

$$\theta = d^\lambda \otimes \partial_\lambda + (\alpha^0)^2\big(g_{0\lambda} + g_{i\lambda} x_0^i\big) d^\lambda \otimes \big(\partial_0 + x_0^j \partial_j\big).$$

*3.2.4. Vertical bundle of the phase space*

Let $V\mathcal{J}_1\boldsymbol{E} \subset T\mathcal{J}_1\boldsymbol{E}$ be the vertical tangent subbundle over $\boldsymbol{E}$. The vertical prolongation of the contact map yields the mutually inverse linear fibred isomorphisms:

$$\nu_\tau : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T} \otimes V_\tau^*\boldsymbol{E} \otimes V\mathcal{J}_1\boldsymbol{E} \quad \text{and} \quad \nu_\tau^{-1} : \mathcal{J}_1\boldsymbol{E} \to V^*\mathcal{J}_1\boldsymbol{E} \otimes \mathbb{T}^* \otimes V_\tau\boldsymbol{E},$$

with coordinate expressions,

$$\nu_\tau = \frac{1}{c\alpha^0}\big(d^i - x_0^i d^0\big) \otimes \partial_i^0, \qquad \nu_\tau^{-1} = c\alpha^0 d_0^i \otimes \big(\partial_i - c\alpha^0 \tau_i \big(\partial_0 + x_0^p \partial_p\big)\big).$$

Thus, for each $Y \in \sec(\mathcal{J}_1\boldsymbol{E}, V\mathcal{J}_1\boldsymbol{E})$ and $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, we obtain:

$$\nu_\tau^{-1}(Y) \in \mathrm{fib}(\mathcal{J}_1\boldsymbol{E}, \mathbb{T}^* \otimes V_\tau\boldsymbol{E}) \quad \text{and} \quad \nu_\tau(X) \in \sec(\mathcal{J}_1\boldsymbol{E}, \mathbb{T} \otimes V\mathcal{J}_1\boldsymbol{E}),$$

with coordinate expressions,

$$\nu_\tau^{-1}(Y) = c\alpha^0 Y_0^i\big(\partial_i - c\alpha^0 \tau_i \big(\partial_0 + x_0^p \partial_p\big)\big) \quad \text{and} \quad \nu_\tau(X) = \frac{1}{c\alpha^0} \tilde{X}^i \partial_i^0,$$

where $\tilde{X}^i = X^i - x_0^i X^0$.

*3.2.5. Spacetime connections*

We define a *spacetime connection* to be a torsion free linear connection $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ of the bundle $T\boldsymbol{E} \to \boldsymbol{E}$. Its coordinate expression is of the type:

$$K = d^\lambda \otimes \big(\partial_\lambda + K_\lambda{}^\nu{}_\mu \dot{x}^\mu \dot{\partial}_\nu\big), \quad \text{with } K_\mu{}^\nu{}_\lambda = K_\lambda{}^\nu{}_\mu \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}).$$

We denote by $K[g]$ the *Levi Civita connection*, i.e. the torsion free linear spacetime connection such that $\nabla g = 0$.

*3.2.6. Phase connections*

We define a *phase connection* to be a connection of the bundle $\mathcal{J}_1\boldsymbol{E} \to \boldsymbol{E}$.

A phase connection can be represented, equivalently, by a tangent valued form $\Gamma : \mathcal{J}_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\mathcal{J}_1\boldsymbol{E}$, which is projectable over $\mathbf{1} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E}$, or by the complementary vertical valued form $\nu[\Gamma] : \mathcal{J}_1\boldsymbol{E} \to T^*\mathcal{J}_1\boldsymbol{E} \otimes V\mathcal{J}_1\boldsymbol{E}$, or by the vector valued form $\nu_\tau[\Gamma] =: \nu_\tau^{-1} \circ \nu[\Gamma] : \mathcal{J}_1\boldsymbol{E} \to T^*\mathcal{J}_1\boldsymbol{E} \otimes (\mathbb{T}^* \otimes V_\tau\boldsymbol{E})$. Their coordinate expressions are:

$$\Gamma = d^\lambda \otimes \big(\partial_\lambda + \Gamma_\lambda{}^i{}_0 \partial_i^0\big), \qquad \nu[\Gamma] = \big(d_0^i - \Gamma_\lambda{}^i{}_0 d^\lambda\big) \otimes \partial_i^0,$$
$$\nu_\tau[\Gamma] = c\alpha^0\big(d_0^i - \Gamma_\lambda{}^i{}_0 d^\lambda\big) \otimes \big(\partial_i - c\alpha^0 \tau_i \big(\partial_0 + x_0^p \partial_p\big)\big), \quad \text{with } \Gamma_\lambda{}^i{}_0 \in \mathrm{map}(\mathcal{J}_1\boldsymbol{E}, \mathbb{R}).$$

We define the *curvature* of a phase connection $\Gamma$ to be the vertical valued 2-form:

$$R = R[\Gamma] =: -[\Gamma, \Gamma] : \mathcal{J}_1\boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\mathcal{J}_1\boldsymbol{E},$$

where [, ] is the Frölicher–Nijenhuis bracket.

We can prove that there is a natural map $\chi : K \mapsto \Gamma$ between linear spacetime connections $K$ and phase connections $\Gamma$, with coordinate expression:

$$\Gamma_\lambda{}^i{}_0 = K_\lambda{}^i{}_0 + K_\lambda{}^i{}_p x_0^p - x_0^i\big(K_\lambda{}^0{}_0 + K_\lambda{}^0{}_p x_0^p\big).$$

*3.2.7. Dynamical phase connection*

The space of 2-jets of motions $\mathcal{J}_2\boldsymbol{E}$ can be naturally regarded as the affine subbundle $\mathcal{J}_2\boldsymbol{E} \subset \mathbb{T}^* \otimes T\mathcal{J}_1\boldsymbol{E}$, which projects on д$: \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$.

A *dynamical phase connection* is defined to be a 2nd-order connection, i.e. a section $\gamma : \mathcal{J}_1\boldsymbol{E} \to \mathcal{J}_2\boldsymbol{E}$, or, equivalently, a section $\gamma : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\mathcal{J}_1\boldsymbol{E}$, which projects on д.

The coordinate expression of a dynamical phase connection is of the type:

$$\gamma = c\alpha^0\big(\partial_0 + x_0^i\partial_i + \gamma_0{}^i{}_0\partial_i^0\big), \quad \text{with } \gamma_0{}^i{}_0 \in \operatorname{map}(\mathcal{J}_1\boldsymbol{E}, \mathbb{R}).$$

If $\gamma$ is a dynamical phase connection, then we have $\gamma \lrcorner \tau = 1$.

The contact map д and a phase connection $\Gamma$ yield the section $\gamma \equiv \gamma[\text{д}, \Gamma] =: \text{д} \lrcorner \Gamma : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\mathcal{J}_1\boldsymbol{E}$, which turns out to be a dynamical phase connection, with coordinate expression,

$$\gamma_0{}^i{}_0 = \Gamma_0{}^i{}_0 + \Gamma_j{}^i{}_0 x_0^j.$$

In particular, a linear spacetime connection $K$ yields the dynamical phase connection $\gamma =: \gamma[\text{д}, K] =: \text{д} \lrcorner \chi(K)$, with coordinate expression:

$$\begin{aligned}\gamma_0{}^i{}_0 = {} & K_0{}^i{}_0 + K_0{}^i{}_h x_0^h + K_h{}^i{}_0 x_0^h + K_h{}^i{}_k x_0^h x_0^k \\ & - x_0^i\big(K_0{}^0{}_0 + K_0{}^0{}_h x_0^h + K_h{}^0{}_0 x_0^h + K_h{}^0{}_k x_0^h x_0^k\big).\end{aligned}$$

*3.2.8. Phase 2-form and 2-vector*

The metric $g$ and a phase connection $\Gamma$ yield the scaled 2-form $\Omega$, called (*scaled*) *phase* 2-*form*, and the scaled vertical 2-vector $\Lambda$, called (*scaled*) *phase* 2-*vector*,

$$\begin{aligned}\Omega &=: \Omega[g, \Gamma] =: g \lrcorner \big(\nu_\tau[\Gamma] \wedge \theta\big) : \mathcal{J}_1\boldsymbol{E} \to \big(\mathbb{T}^* \otimes \mathbb{L}^2\big) \otimes \Lambda^2 T^*\mathcal{J}_1\boldsymbol{E}, \\ \Lambda &=: \Lambda[g, \Gamma] =: \bar{g} \lrcorner (\Gamma \wedge \nu_\tau) : \mathcal{J}_1\boldsymbol{E} \to \big(\mathbb{T} \otimes \mathbb{L}^{-2}\big) \otimes \Lambda^2 T\mathcal{J}_1\boldsymbol{E},\end{aligned}$$

with coordinate expressions

$$\Omega = c\alpha^0\big(g_{i\mu} + c^2\tau_i\tau_\mu\big)\big(d_0^i - \Gamma_\lambda{}^i{}_0 d^\lambda\big) \wedge d^\mu, \qquad \Lambda = \frac{1}{c\alpha^0}\big(g^{j\lambda} - x_0^j g^{0\lambda}\big)\big(\partial_\lambda + \Gamma_\lambda{}^i{}_0\partial_i^0\big) \wedge \partial_j^0.$$

We can easily see that $-c^2\tau \wedge \Omega^3 \not\equiv 0$ and $-\frac{1}{c^2}\gamma \wedge \Lambda^3 \not\equiv 0$.

There is a unique dynamical phase connection $\gamma$, such that $\gamma \lrcorner \Omega[g, \Gamma] = 0$. Namely, $\gamma = \gamma[\text{д}, \Gamma]$.

In particular, a metric and time preserving spacetime connection $K$ yields the (scaled) phase 2-form $\Omega[g, K] =: \Omega[g, \chi(K)]$ and the (scaled) phase 2-vector $\Lambda[g, K] =: \Lambda[g, \chi(K)]$ with coordinate expressions

$$\begin{aligned}\Omega &= -c\alpha^0\big(g_{i\mu} + c^2\tau_i\tau_\mu\big)\big(d_0^i - \big(K_\lambda{}^i{}_0 + K_\lambda{}^i{}_j x_0^j - K_\lambda{}^0{}_0 x_0^i - K_\lambda{}^0{}_j x_0^i x_0^j\big)d^\lambda\big) \wedge d^\mu, \\ \Lambda &= \frac{1}{c\alpha^0}\big(g^{h\lambda} - g^{0\lambda}x_0^h\big)\big(\partial_\lambda + \big(K_\lambda{}^i{}_0 + K_\lambda{}^i{}_j x_0^j - K_\lambda{}^0{}_0 x_0^i - K_\lambda{}^0{}_j x_0^i x_0^j\big)\partial_i^0\big) \wedge \partial_h^0.\end{aligned}$$

*3.2.9. Dynamical structures of the phase space*

Let us consider a phase connection $\Gamma$ and the induced phase objects $\gamma =: \gamma[\text{д}, \Gamma]$, $\Omega =: \Omega[g, \Gamma]$, and $\Lambda =: \Lambda[g, \Gamma]$.

We define the Lie derivatives:

$$L_\Gamma\tau = (i_\Gamma d - d i_\Gamma)\tau \quad \text{and} \quad L_R\tau = (i_R d + d i_R)\tau.$$

Then, the following results holds [7]:

**Theorem 3.3.** *The following assertions are equivalent.*

(1) $L_{\nu_\tau(X)} L_\Gamma \tau = 0$, $\forall X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, *and* $L_R\tau = 0$.
(2) $d\Omega = 0$, *i.e.* $(-c^2\tau, \Omega)$ *is a* (*scaled*) *almost-cosymplectic-contact pair.*
(3) $[-\frac{1}{c^2}\gamma, \Lambda] = \frac{1}{c^2}\gamma \wedge \Lambda^\sharp(L_\gamma\tau)$ *and* $[\Lambda, \Lambda] = 2\gamma \wedge (\Lambda^\sharp \otimes \Lambda^\sharp)(d\tau)$, *i.e.* $(-\frac{1}{c^2}\gamma, \Lambda, -c^2\tau)$ *is a* (*scaled regular*) *almost-coPoisson–Jacobi* 3-*plet.*

*Moreover, the almost-cosymplectic-contact pair* $(-c^2\tau, \Omega)$ *and the* (*regular*) *almost-coPoisson–Jacobi* 3*-plet* $(-\frac{1}{c^2}\gamma, \Lambda, -c^2\tau)$ *are mutually dual.*

**Lemma 3.4.** *We have*:

$$\Omega - c^2 L_\Gamma \tau = -c^2 d\tau.$$

**Theorem 3.5.** *The following assertions are equivalent*:

(1) $L_\Gamma \tau = 0$.
(2) $\Omega = -c^2\, d\tau$, *i.e.* $(-c^2\tau, \Omega)$ *is a* (*scaled*) *contact pair.*
(3) $[-\frac{1}{c^2}\gamma, \Lambda] = 0$ *and* $[\Lambda, \Lambda] = \frac{2}{c^2}\gamma \wedge \Lambda$, *i.e.* $(-\frac{1}{c^2}\gamma, \Lambda)$ *is a* (*scaled regular*) *Jacobi pair.*

*Moreover, the contact pair* $(-c^2\tau, \Omega)$ *and the* (*regular*) *Jacobi pair* $(-\frac{1}{c^2}\gamma, \Lambda)$ *are mutually dual.*

Next, let us consider a linear spacetime connection $K$ and the induced phase objects $\Gamma =: \chi(K)$, $\gamma =: \gamma[д, \Gamma]$, $\Omega =: \Omega[g, \Gamma]$, and $\Lambda =: \Lambda[g, \Gamma]$.

**Theorem 3.6.** *The following assertions are equivalent*:

(1) $L_{\chi(K)}\tau = 0$.
(2) $g(Z,Z)((\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + g(T(X,Y),Z)) + \frac{1}{2}g(Z,X)(\nabla_Y g)(Z,Z) - \frac{1}{2}g(Z,Y)(\nabla_X g)(Z,Z) = 0$, *for each* $X, Y, Z \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, *where* $T$ *is the torsion of* $K$.
(3) $\Omega = -c^2\, d\tau$, *i.e.* $(-c^2\tau, \Omega)$ *is a* (*scaled*) *contact pair.*
(4) $[-\frac{1}{c^2}\gamma, \Lambda] = 0$ *and* $[\Lambda, \Lambda] = \frac{2}{c^2}\gamma \wedge \Lambda$, *i.e.* $(-\frac{1}{c^2}\gamma, \Lambda)$ *is a* (*scaled regular*) *Jacobi pair.*

*Moreover, if the above conditions are fulfilled, then the contact pair* $(-c^2\tau, \Omega)$ *and the* (*regular*) *Jacobi pair* $(-\frac{1}{c^2}\gamma, \Lambda)$ *are mutually dual.*

**Corollary 3.7.** *Let* $K$ *be a torsion free spacetime connection. If* $\nabla g$ *and* $g \otimes \nabla g$ *are symmetric* $(0,3)$ *and* $(0,5)$ *tensor fields, respectively, then* $(-c^2\tau, \Omega)$ *and* $(-\frac{1}{c^2}\gamma, \Lambda)$ *are mutually dual contact and Jacobi pairs, respectively.*

Eventually, let us consider the Levi Civita spacetime connection $K[g]$ and the induced phase objects $\Gamma \equiv \Gamma[g] =: \chi(K)$, $\gamma \equiv \gamma[д, g] =: \gamma[д, \Gamma]$, $\Omega \equiv \Omega[g] =: \Omega[g, \Gamma]$, and $\Lambda[g] =: \Lambda[g, \Gamma]$.
Then, the equality $\nabla g = 0$ and Theorem 3.6 yield the following result.

**Theorem 3.8.** *We have*:

(1) $\Omega = -c^2\, d\tau$, *i.e.* $(-c^2\tau, \Omega)$ *is a* (*scaled*) *contact pair.*
(2) $[-\frac{1}{c^2}\gamma, \Lambda] = 0$ *and* $[\Lambda, \Lambda] = \frac{2}{c^2}\gamma \wedge \Lambda$, *i.e.* $(-\frac{1}{c^2}\gamma, \Lambda)$ *is a* (*scaled regular*) *Jacobi pair.*

*Moreover, the contact pair* $(-c^2\tau, \Omega)$ *and the* (*regular*) *Jacobi pair* $(-\frac{1}{c^2}\gamma, \Lambda)$ *are mutually dual.*

**Remark 3.9.** If $K$ is a spacetime connection, then the induced pairs $(-c^2\tau, \Omega)$ and $(-\frac{1}{c^2}\gamma, \Lambda)$ are scaled.
On the other hand, some results of the general theory of geometrical structures developed in the first two sections requires unscaled pairs.
Indeed, if we refer to a particle of mass $m \in \mathbb{M}$ and consider the universal scales $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$ and $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$, then we obtain unscaled pairs in the following natural way.
We have the unscaled spacetime 1-form:

$$-\frac{mc^2}{\hbar}\tau : \boldsymbol{E} \to T^*\boldsymbol{E}.$$

Moreover, the rescaled contact map $Д =: \frac{\hbar}{mc^2} д : \mathcal{J}_1 \boldsymbol{E} \to T\boldsymbol{E}$ yields the unscaled phase vector field:

$$-\gamma[Д, K] = -\frac{\hbar}{mc^2}\gamma[д, K] : \boldsymbol{E} \to T\mathcal{J}_1\boldsymbol{E}.$$

Furthermore, the rescaled metric $G =: \frac{m}{\hbar} g : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}$ yields the unscaled phase 2-form and phase 2-vector:

$$\Omega \equiv \Omega[G, K] = \frac{m}{\hbar}\Omega[g, K] : \mathcal{J}_1\boldsymbol{E} \to \Lambda^2 T^*\mathcal{J}_1\boldsymbol{E},$$

$$\Lambda \equiv \Lambda[G, K] = \frac{\hbar}{m}\Lambda[g, K] : \mathcal{J}_1\boldsymbol{E} \to \Lambda^2 T\mathcal{J}_1\boldsymbol{E}.$$

Thus, if $K$ is the Levi Civita spacetime connection, then $(-\frac{mc^2}{\hbar}\tau, \frac{m}{\hbar}\Omega)$ and $(-\frac{\hbar}{mc^2}\gamma, \frac{\hbar}{m}\Lambda)$ turn out to be mutually dual unscaled contact and Jacobi pairs of the phase space.

Indeed, the Plank constant does not play any direct role in classical mechanics; nevertheless, such a scale is necessary for getting unscaled objects as above.

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