Asymptotic and Oscillatory Behavior of Solutions of First Order Nonlinear Neutral Delay Differential Equations

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The authors consider the first order nonlinear neutral delay differential equation

\[ y(t) + P(t) y(t-r) = 0, \]

where \( P, Q, \) and \( f \) are continuous, \( Q(t) \geq 0, r \geq 0, \) \( f(u) > 0 \) if \( u \neq 0. \) They give sufficient conditions for all nonoscillatory solutions of (E) to converge to zero as \( t \to \infty. \) Two oscillation theorems for equation (E) are also proved. © 1991 Academic Press, Inc.

1. Introduction

In recent years delay differential equations have been studied extensively and the oscillatory theory for these equations is well developed. For example, see [1, 8, 9, 12–14, 16, and references therein]. In contrast, neutral delay differential equations, i.e., equations in which the highest order derivative of the unknown function appears both with and without delays, has received very little attention. For recent work on neutral delay equations we refer the reader to [2–7, 10–11, 15, 17]. A discussion of some applications of these equations and some of the differences in the behavior

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of their solutions and the solutions of delay differential equations can be found in [3–7].

In this paper we study the asymptotic behavior of the solutions of the first order neutral delay differential equation

$$\frac{d}{dt} \left[ y(t) + P(t) y(t - \tau) \right] + Q(t) f(y(t - \sigma)) = 0, \quad (1)$$

where $P, Q: [t_0, \infty) \to \mathbb{R}$ are continuous with neither $P$ nor $Q$ identically zero on any half line $[t, \infty)$, $\tau$ and $\sigma$ are nonnegative constants, and $f: \mathbb{R} \to \mathbb{R}$ is continuous.

Every solution $y(t)$ of (1) considered here is continuable and nontrivial, i.e., $y(t)$ is defined on $[t_1, \infty)$ for some $t_1 \geq t_0$ and $\sup\{|y(t)|: t \geq t_1\} > 0$ for every $t_1 \geq t_0$. Such a solution is said to be oscillatory if its set of zeros is unbounded above and is said to be nonoscillatory otherwise. We will say that (1) is oscillatory if all its nontrivial continuable solutions are oscillatory.

Here we obtain conditions that ensure the convergence to zero of the nonoscillatory solutions of (1) and sufficient conditions for (1) to be oscillatory. All previous known results of this type for Eq. (1) are for the special case when $f(u) = u$ and frequently require one or both of $P$ and $Q$ to be constant. Consequently, the results obtained here extend and/or generalize a number of known results.

2. ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS

In this section we study the asymptotic behavior of the nonoscillatory solutions of (1). We will assume throughout the remainder of the paper that

$$Q(t) \geq 0 \quad \text{and} \quad uf(u) > 0 \quad \text{for} \quad u \neq 0.$$

The condition

$$f(u) \text{ is bounded away from zero if } u \text{ is bounded away from zero} \quad (2)$$

will be needed in many of the results that follow.

We begin with a lemma that will be used frequently. Note that the cases in the lemma are not independent of each other.

LEMMA 1. Let $y(t)$ be a nonoscillatory solution of (1). Then the following statements are valid for

$$z(t) = y(t) + P(t) y(t - \tau).$$
(a) If \( y(t) \) is eventually positive (negative), then \( z \) is an eventually decreasing (increasing) function of \( t \).

(b) If \( y(t) \) is eventually positive (negative) and there exists a constant \( P_1 \) such that

\[
-1 < P_1 \leq P(t),
\]

then eventually \( z(t) > 0 \) (\( z(t) < 0 \)).

(c) If in addition to (2),

\[
\int_{-\infty}^{\infty} Q(s) \, ds = \infty
\]

and there exists a constant \( P_2 \) such that

\[
-1 < P_2 \leq P(t) \leq 0,
\]

then \( z(t) \to 0 \) as \( t \to \infty \).

(d) Suppose (2) and (4) hold and that there exists a constant \( P_3 \) such that

\[
P(t) \leq P_3 < -1.
\]

If \( y(t) \) is eventually positive (negative), then eventually \( z(t) < 0 \) (\( z(t) > 0 \)).

(e) Suppose that (2) and (4) hold and that there exists a constant \( P_4 < 0 \) such that

\[
P_4 \leq P(t) \leq 0.
\]

If \( y(t) \) is eventually positive (negative), then \( \lim_{t \to \infty} z(t) \) exists and its value is either 0 or \( -\infty \) (0 or \( \infty \)).

(f) Suppose that in addition to (2), (4), and (6) there exists a constant \( P_5 \) such that \( P_5 \leq P(t) \), i.e.,

\[
P_5 \leq P(t) \leq P_3 < -1.
\]

If \( y(t) \) is eventually positive (negative), then \( z(t) \to -\infty \) (\( z(t) \to \infty \)) as \( t \to \infty \).

Proof. All parts will be proved for solutions that are eventually positive. The arguments for the case of eventually negative solutions are similar.

(a) Let \( y(t) \) be eventually positive; then it follows from (1) that

\[
z'(t) = -Q(t) f(y(t - \sigma)) \leq 0
\]

for all large \( t \). Thus \( z(t) \) is eventually decreasing.
(b) If \( y(t) \) is eventually positive and the conclusion does not hold, then, since by (a) \( z(t) \) is decreasing, it follows that eventually either \( z(t) \equiv 0 \) or \( z(t) < 0 \). Now \( z(t) \equiv 0 \) implies that \( z'(t) = -Q(t) f(y(t - \sigma)) \equiv 0 \) contradicting the fact that \( Q(t) \neq 0 \) on any half line. Moreover, if \( z(t) < 0 \), then \( y(t) < -P(t) y(t - \tau) \) so \( P(t) < 0 \). From (3) it follows that \( -1 < P_1 < 0 \) and \( y(t) < -P_1 y(t - \tau) \). We then have \( y(t + \tau) \leq -P_1 y(t) \) and by induction it follows that \( y(t + n\tau) \leq (-P_1)^n y(t) \) for all positive integers \( n \). Hence \( y(t) \to 0 \) as \( t \to \infty \). But this, together with \( |P(t)| < 1 \), implies that \( z(t) \to 0 \) as \( t \to \infty \) contradicting the fact that \( z(t) < 0 \) and decreasing. Hence \( z(t) > 0 \).

(c) Note first that (5) implies (3) with \( P_1 \) replaced by \( P_2 \). If \( y(t) \) is eventually positive, then (a) and (b) imply that eventually \( z(t) \) is positive and decreasing. Therefore, \( z(t) \to L \geq 0 \) as \( t \to \infty \). Now suppose that \( L > 0 \). Since (5) requires \( P(t) \leq 0 \), we have \( z(t) \leq y(t) \). Thus there exists \( t_1 \geq t_0 \) such that \( L \leq z(t - \sigma) \leq y(t - \sigma) \) for \( t \geq t_1 \). Then from (1) and (2) it follows that \( z'(t) \leq -L_1 Q(t) \) for some positive constant \( L_1 \). Integrating the last inequality we obtain

\[
z(t) \leq z(t_1) - L_1 \int_{t_1}^{t} Q(s) \, ds
\]

which implies, in view of (4), that \( z(t) \to -\infty \) as \( t \to \infty \) which is impossible.

(d) Suppose \( y(t) \) is eventually positive. If the conclusion does not hold, then eventually \( z(t) \geq 0 \) and therefore

\[
y(t) + P(t) y(t - \tau) \geq 0.
\]

So from (6) we have

\[
y(t) \geq -P_3 y(t - \tau)
\]

for all sufficiently large \( t \). It follows by induction that for each positive integer \( n \),

\[
y(t + n\tau) \geq (-P_3)^n y(t)
\]

which implies that \( y(t) \to \infty \) as \( t \to \infty \). Thus (1) and (2) imply that there exist \( t_2 \geq t_0 \) and a positive constant \( L_2 \) such that

\[
z'(t) = -Q(t) f(y(t - \sigma)) \leq -L_2 Q(t)
\]

for \( t \geq t_2 \). An integration yields

\[
0 \leq z(t) \leq z(t_2) - L_2 \int_{t_2}^{t} Q(s) \, ds \to -\infty
\]

as \( t \to \infty \) which is a contradiction.
(e) For \( y(t) \) eventually positive we have from (1) that \( z'(t) \leq -Q(t) f(y(t - \sigma)) \leq 0 \) for all sufficiently large \( t \). Therefore \( z(t) \to L_3 \), where \( L_3 < \infty \). If \( L_3 > -\infty \), then integrating (1) over \([t, A]\) and then letting \( A \to \infty \) yields

\[
z(t) = L_3 + \int_t^\infty Q(s) f(y(s - \sigma)) \, ds
\]

which implies that \( \int_t^\infty Q(s) f(y(s - \sigma)) \, ds < \infty \). Then (2) and (4) together imply \( \lim \inf_{t \to \infty} y(t) = 0 \). Let \( t_3 > t_0 \) be such that \( y(t - \tau) > 0 \) for \( t \geq t_3 \).

If \( L_3 < 0 \), there exists \( t_4 \geq t_3 \) so that \( y(t_4 - \tau) < L_3/4|P_4| < L_3/4 \). We then have from (7) that

\[
y(t_4) \leq L_3 - L_3/4 - |P_4| L_3/4 < L_3/2
\]

contradicting \( y(t) > 0 \). If \( L_3 > 0 \), then (7) implies that \( y(t) > L_3 \) for \( t \geq t_3 \) and it then follows from (2) that there exists a constant \( L_4 > 0 \) so that

\[
z(t) > L_3 + L_4 \int_t^\infty Q(s) \, ds.
\]

But the last inequality implies that \( z(t) \to \infty \) as \( t \to \infty \) contradicting \( z(t) \to L_3 < \infty \). Thus either \( L_3 = 0 \) or \( L_3 = -\infty \).

(f) Note that (8) implies that (6) and (7) hold. Hence from (a), (d), and (e) we have \( z'(t) \leq 0 \), \( z(t) < 0 \), and \( z(t) \to 0 \) or \( -\infty \) as \( t \to \infty \). Clearly \( z(t) \to -\infty \) as \( t \to \infty \).

Next we obtain two theorems on the asymptotic behavior of the non-oscillatory solutions of (1) using the results obtained in Lemma 1 on the asymptotic behavior of \( z(t) \).

**Theorem 2.** Suppose there exists positive constants \( q, B, \) and \( P_6 \) such that

\[
Q(t) \geq q, \tag{9}
\]

\[
|f(u)| \geq B|u| \quad \text{for all } u, \tag{10}
\]

and

\[
0 \leq P(t) \leq P_6. \tag{11}
\]

If \( y(t) \) is a nonoscillatory solution of (1), then \( y(t) \to 0 \) as \( t \to \infty \).

**Proof.** Let \( y(t) \) be an eventually positive solution of (1), say \( y(t) > 0 \),
$y(t - \tau) > 0$ and $y(t - \sigma) > 0$ for $t \geq T \geq t_0$. By parts (a) and (b) of Lemma 1, $z(t)$ is eventually positive and decreasing so $z(t) \to L_5 \geq 0$ as $t \to \infty$. Integrating (1) over $[t, A]$ for $t \geq T$ and letting $A \to \infty$ yields

$$z(t) = L_5 + \int_t^\infty Q(s) f(y(s - \sigma)) \, ds$$

for $i \geq T$. Next observe that (9) and (10) imply that

$$qB \int_t^\infty y(s - \sigma) \, ds \leq \int_t^\infty Q(s) f(y(s - \sigma)) \, ds = z(t) - L_5$$

which implies that $y \in L^1[T, \infty)$. Since $z(t) = y(t) + P(t) y(t - \tau)$, (11) implies that $z \in L^1[T, \infty)$. But this, together with the fact that $z(t)$ is decreasing, implies that $L_5 = 0$. Therefore $y(t) \to 0$ as $t \to \infty$ since $P(t) \geq 0$. The proof when $y(t)$ is eventually negative is similar.

**Remark.** Theorem 2 reduces to one case of Theorem 2 in [7] when $f(u) \equiv u$. Also, Theorem 2 includes [3, Theorems 3 and 9; 4, Theorem 5] and extends special cases of [6, Theorem 1(b); 11, Theorem 1(b)].

It is interesting to observe that the conclusion of Theorem 2 was obtained in [2, Theorem 4] with conditions (9)–(11) replaced by condition (4), the requirement that $f$ be increasing, and either

(i) $P(t) \geq 0$ and $P(t) \to 0$ as $t \to \infty$, or

(ii) $P(t) \to \infty$ as $t \to \infty$.

Theorem 2 implies that all nonoscillatory solutions of

$$\frac{d}{dt} [y(t) + y(t - \ln 2)] + 3[3/2 + \sin y(t)] y(t) / [3/2 + \sin e^{-t}] = 0, \quad t \geq 1$$

converge to zero as $t \to \infty$, while [2, Theorem 4] does not apply since $P(t) \equiv 1$. Here $y(t) = e^{-t}$ is such a solution.

**Theorem 3.** If (2), (4), and (5) are satisfied, then every nonoscillatory solution of (1) tends to zero as $t \to \infty$.

**Proof.** Let $y(t)$ be an eventually positive solution of (1). By [2, Theorem 2(i)] $y(t)$ is bounded so there exists $A_1 > 0$ such that

$$\limsup_{t \to \infty} y(t) = A_1.$$  

This ensures the existence of an increasing sequence $\{t_n\}$ with $t_1$ large enough so that $y(t) > 0$ for $t > t_1 - \tau$, and with $t_n \to \infty$ and $y(t_n) \to A_1$ as $n \to \infty$. Then from (5) we have

$$z(t_n) = y(t_n) + P(t_n) y(t_n - \tau) \geq y(t_n) + P_2 y(t_n - \tau)$$

for $i \geq T$. Next observe that (9) and (10) imply that
If \( A_1 > 0 \), then there exists a positive number \( \varepsilon \) such that \((1 - P_2)\varepsilon < (1 + P_2)A_1\) and hence
\[-P_2(A_1 + \varepsilon) < A_1 - \varepsilon.\]
Furthermore, for all sufficiently large \( n \), \( y(t_n - \tau) < A_1 + \varepsilon \) so we have
\[A_1 - \varepsilon > -P_2(A_1 + \varepsilon) > -P_2 y(t_n - \tau) \geq y(t_n) - z(t_n)\]
for all such \( n \). Note next that we have from Lemma 1(c) that \( z(t_n) \to 0 \) as \( n \to \infty \). Thus letting \( n \to \infty \) in the last inequality leads to a contradiction and we conclude that \( y(t) \to 0 \) as \( t \to \infty \). The argument when \( y(t) \) is eventually negative is similar.

**Remark.** Theorem 3 extends [4, Theorem 3], one part of [7, Theorem 2] and special cases of [6, Theorem 1(b); 10, Theorems 1 and 5; 11, Theorem 1(b)]. Also, Theorem 5 reduces to [7, Theorem 3, part (iv)] when \( f(u) \equiv u \).

Theorem 3 implies that all nonoscillatory solutions of
\[
\frac{d}{dt} \left[ y(t) - y(t - 2)/2 \right] + (t - 1)^3(t^2 - 8t + 8) y(t - 1)/2[t(t - 2)]^2 = 0, \quad t \geq 3,
\]
converge to zero \( t \to \infty \). None of the results in [4, 6, 7, 10, 11] apply to this example.

We now have obtained results guaranteeing that, under other appropriate conditions, all nonoscillatory solutions of (1) converge to zero if either
\[
0 \leq P(t) \leq P_6 \quad (11)
\]
or
\[
-1 < P_2 \leq P(t) \leq 0 \quad (5)
\]
holds. It is natural to ask if the same conclusion holds if \( P(t) \) is not bounded above
\[
P(t) \text{ is not bounded above} \quad (12)
\]
or if
\[
P(t) \leq -1. \quad (13)
\]
In the case of (12) a partial answer is given by [2, Theorem 4] as was discussed following Theorem 2 above. As to (13), if (4) and (8) hold, then [2, Theorem 6] shows that every nonoscillatory solution \( y(t) \) of (1) satisfies \( |y(t)| \to \infty \) as \( t \to \infty \).

3. Oscillation Theorems

In this section we give two sets of conditions which are sufficient to ensure that (1) is oscillatory. Both results are for the case where \( P(t) \leq 0 \) and \( f \) is increasing.

One theorem requires that \( f \) satisfies the sublinearity condition
\[
\int_{0}^{\infty} \left[ \frac{1}{f(u)} \right] \, du < \infty \quad \text{for all} \quad \alpha > 0, \tag{15}
\]
while the other requires the superlinearity condition
\[
\int_{-\infty}^{0} \left[ \frac{1}{f(u)} \right] \, du < \infty \quad \text{and} \quad \int_{c}^{\infty} \left[ \frac{1}{f(u)} \right] \, du < \infty \quad \text{for all} \quad c > 0. \tag{16}
\]

**Theorem 4.** If (4), (5), (14), and (15) hold, then (1) is oscillatory.

**Proof.** Suppose that the conclusion of the theorem is false; then (1) has a nonoscillatory solution \( y(t) \). Suppose that \( y(t) > 0 \) for \( t \geq t_1 > t_0 \). From parts (a) and (b) of Lemma 1, there exists \( T \geq t_1 \) such that \( y(t - \tau) > 0 \), \( y(t - \sigma) > 0 \), \( z(t) > 0 \) and decreasing on \([T, \infty)\). Note next that (5) implies \( z(t) \leq y(t) \) and therefore we have from (1) and (14) that

\[
z'(t) + Q(t) f(z(t - \sigma)) \leq 0 \tag{17}
\]
for \( t \geq T \). Since \( z(t) \) is decreasing we have

\[
z'(t) + Q(t) f(z(t)) \leq 0,
\]
or

\[
z'(t)/f(z(t)) + Q(t) \leq 0\tag{18}
\]
for \( t \geq T \). Integrating the last inequality we obtain
\[
-\int_{z(t)}^{z(T)} \left[ \frac{1}{f(u)} \right] \, du + \int_{T}^{t} Q(s) \, ds \leq 0
\]
which is impossible in view of (4) and (15).
If we assume that \( y(t) \) is eventually negative, the proof is similar and will be omitted.

**Theorem 5.** Suppose that (4), (8), (14), and (16) are satisfied. If, in addition,

\[
\sigma \leq \tau, \tag{17}
\]

then (1) is oscillatory.

**Proof.** Suppose (1) has a nonoscillatory solution \( y(t) \). If \( y(t) > 0 \) for \( t \geq t_1 > t_0 \), then parts (a) and (f) of Lemma 1 ensure the existence of \( T_1 \geq t_1 \) such that \( y(t - \tau) > 0 \) and \( z(t) < 0 \) on \( [T_1, \infty) \) and such that \( z(t) \) is decreasing on \( [T_1, \infty) \). Also observe that (17) implies that \( y(t - \sigma) \) is positive for \( t \geq T_1 \). Then from (8) we have

\[
P_5 y(t - \tau) \leq P(t) y(t - \tau) < z(t) < 0,
\]

so

\[
0 < z(t + \tau)/P_5 < y(t)
\]

for \( t \geq T_1 \). This, together with (14), implies that \( f(z(t + \tau - \sigma)/P_5) \leq f(y(t - \sigma)) \) for \( t \geq T \geq T_1 + \sigma \). Note next that (17) implies that \( t + \tau - \sigma \geq t \) from which it follows that

\[
f(z(t)/P_5) \leq f(z(t + \tau - \sigma)/P_5) \leq f(y(t - \sigma)).
\]

Thus we have

\[
Q(t) f(z(t)/P_5) \leq Q(t) f(y(t - \sigma)) = -z'(t).
\]

Therefore

\[
z'(t)/f(z(t)/P_5) + Q(t) \leq 0.
\]

Integrating the last inequality leads to

\[
P_5 \int_{z(T)/P_5}^{z(t)/P_5} \left[1/f(u)\right] du + \int_{T}^{t} Q(s) ds \leq 0
\]

which, in view of (4), contradicts (16). This completes the proof when \( y(t) \) is eventually positive; the proof when \( y(t) \) is eventually negative is similar.

**Remark.** The oscillation results in [3–7, 10–11] are all for equations with \( f(u) = u \) so that neither (15) nor (16) is satisfied. The results in [17]
are all for second order equations. Consequently, Theorems 4 and 5 are not included in any of those results.

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