JOURNAL OF APPROXIMATION THEORY 81, 23–37 (1995)

On Convergence of Hermite–Fejér Interpolation Polynomials

H.-B. KNOOP AND XIN-LONG ZHOU

Department of Mathematics, University of Duisburg, 47048 Duisburg, Germany

Communicated by Doron S. Lubinsky

Received April 30, 1993; accepted in revised form March 14, 1994

Hermite–Fejér interpolation operators based on the zeros of Jacobi polynomials, in general, are not uniformly convergent for all $f \in C[-1, 1]$. In this work we investigate weighted approximation by such operators with Jacobi weight functions. A necessary and sufficient condition for the weight function is given such that these interpolation polynomials converge in the weighted maximum norm for all $f \in C[-1, 1]$. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let $\{x_k\}_{k=1}^n$ be the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree n, where α, $\beta > -1$. For later purposes we normalize the Jacobi polynomial with $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. The so-called Hermite–Fejér interpolation polynomial $H_n^{(\alpha,\beta)}(f, \cdot)$ of degree at most $2n-1$ is defined by

$$H_n^{(\alpha,\beta)}(f, x_k) = f(x_k), \qquad H_n^{(\alpha,\beta)\prime}(f, x_k) = 0, \qquad k = 1, 2, ..., n.$$

for $f \in C[-1, 1]$. In the following we will use the following notation: $\|\cdot\|$ denotes the sup-norm on the whole interval $[-1, 1]$, and $\|\cdot\|_I$ or $\|\cdot\|_{[c,d]}$ (for $I = [c, d]$, $-1 \leqslant c < d \leqslant 1$) will be the sup-norm with respect to the subinterval I. Furthermore, $\|\cdot\|_{L_p}$ will denote the L_p-norm with respect to the interval $[-1, 1]$.

It is known (e.g., see [8]) that

$$\|H_n^{(\alpha,\beta)}(f) - f\| \to 0, \qquad \forall f \in C[-1, 1]$$

is equivalent to the condition α, $\beta < 0$. On the other hand, if we consider subsets of $[-1, 1]$, we have the following (see [8]): for all α, $\beta > -1$ and all closed subintervals $I \subset (-1, 1)$ fixed,

$$\|H_n^{(\alpha,\beta)}(f) - f\|_I \to 0, \qquad \forall f \in C[-1, 1].$$

0021-9045/95 $6.00

This means that divergence only happens near the points ± 1. The following statement characterizes the behaviour of Hermite–Fejér interpolation polynomials via L_p norms for $0 < p \leqslant \infty$ (see [5]):

$$\|H_n^{(\alpha,\beta)}(f) - f\|_{L_p} \to 0 \qquad \forall f \in C[-1, 1]$$

if and only if

$$\max\{\alpha, \beta\} < \frac{1}{p}. \tag{1.1}$$

A different approach is to consider the convergence in weighted spaces, e.g., with Jacobi weights, because the weights vanish at the end points ± 1. It may be expected that by a suitable choice of weight function these operators turn out to have properties similar to those in the case $-1 < \alpha$, $\beta < 0$.

Put

$$w_{a,b}(x) = (1-x)^a (1+x)^b, \qquad a, b \geqslant 0,$$

and

$$\|f\|_{a,b} = \max_{-1 \leqslant x \leqslant 1} |w_{a,b}(x) f(x)|.$$

Háy and Vértesi [3] showed that if $a = \alpha + 1/2$, $b = \beta + 1/2$, then for all $f \in C[-1, 1]$

$$\|H_n^{(\alpha,\beta)}(f) - f\|_{a,b} \to 0. \tag{1.2}$$

In the present paper we will give a necessary and sufficient condition on a and b such that (1.2) is true for all $f \in C[-1, 1]$. First we have the following

THEOREM 1.1. *For* α, $\beta > -1$, *and* a, $b \geqslant 0$, *in order that*

$$\lim_{n \to \infty} \|w_{a,b}(H_n^{(\alpha,\beta)}(f) - f)\|_{[0,1]} = 0$$

holds for all functions in $C[-1, 1]$ *it is sufficient that* $\alpha < a$.

Thus, it follows from this theorem that (1.2) holds for each $f \in C[-1, 1]$ if $\alpha < a$ and $\beta < b$. On the other hand, in the case $\alpha \geqslant a$ or $\beta \geqslant b$, (1.2) no longer holds for some $f \in C[-1, 1]$. Hence, it is natural to ask what is the situation for these α and β. The following theorem anwers this question in part.

Let Π_n be the set of algebraic polynomials of degree at most n, and let

$$E_n(f)_{a,b} = \inf_{P \in \Pi_n} \|f - P\|_{a,b}$$

be the best approximation constant with respect to Π_n and the norm $\|\cdot\|_{a,b}$. If $a=b=0$ we shall write $E_n(f)$ instead of $E_n(f)_{0,0}$. Using this notation we have

THEOREM 1.2. *Let* $\alpha > a$ *and* $f \in C[-1, 1]$ *satisfy*

$$E_n(f)_{a,0} = o(n^{-2\alpha+2a}).$$

Then

$$\lim_{n\to\infty} \|w_{a,0}(H_n^{(\alpha,\beta)}(f)-f)\|_{[0,1]} = 0 \tag{1.3}$$

if and only if

$$\int_{-1}^{1} (f(t)-f(1))\, w'_{\alpha-j,\beta+1}(t)\, dt = 0, \qquad j=0, 1, ..., [\alpha-a]. \tag{1.4}$$

Remark 1.3. (1) If $[\alpha-a] \neq \alpha-a$ or $a>0$, then condition (1.4) is equivalent to

$$\int_{-1}^{1} f(t)\, w'_{\alpha-j,\beta+1}(t)\, dt = 0, \qquad j=0, 1, 2, ..., [\alpha-a].$$

(2) Denote by $\omega_r(f, \cdot)$ the rth modulus of smoothness of f with respect to the norm $\|\cdot\|$. In [7] Szabados proved that $\omega(f^{[2\alpha]}, h) = o(h^{2\alpha-[2\alpha]})$; $f^{(j)}(1)=0$, $j=1, 2, ..., [\alpha]$; and (1.4) imply (1.3) with $w_{a,0}=1$ (that means $a=0$). As $E_n(f) = O(\omega_r(f, 1/n))$, we see that the condition $f^{(j)}(1)=0$, $j=1, 2, ..., [\alpha]$, is not necessary.

(3) Let $\beta \geqslant b$ and $E_n(f)_{0,b} = o(n^{2\beta+2b})$. Then

$$\lim_{n\to\infty} \|w_{0,b}(H_n^{(\alpha,\beta)}(f)-f\|_{[-1,0]} = 0$$

is equivalent to

$$\int_{-1}^{1} (f(t)-f(-1))\, w'_{\alpha+1,\beta-j}(t)\, dt = 0, \qquad j=0, 1, ..., [\beta-b].$$

From Theorems 1.1 and 1.2 we get

THEOREM 1.4. *For* $\alpha, \beta > -1$, *in order that* (1.2) *holds for all* f *in* $C[-1, 1]$ *it is necessary and sufficient that* $\alpha < a$ *and* $\beta < b$.

It is interesting to see that, by Theorem 1.4, (1.1) actually implies

$$\|H_n^{(\alpha,\beta)}(f)-f\|_{1/p,1/p} \to 0, \qquad \forall f \in C[-1, 1].$$

Hence, for those polynomials L_p convergence is equivalent to convergence in the weighed maximum norm with $w(x) = (1-x^2)^{1/p}$.

Furthermore, from Theorem 1.2 we have now the following result, some special cases of which can be found in [2, 4, 6, 7, 10, and 11].

THEOREM 1.5. *For* α, $\beta > -1$ *we have the following equivalent statements*:

(1) *if* $\alpha = a$, $\beta < b$, *then*

$$\|H_n^{(\alpha,\beta)}(f) - f\|_{a,b} \to 0 \Leftrightarrow \int_{-1}^{1} (f(t) - f(1))\, w'_{\alpha,\beta+1}(t)\, dt = 0,$$

(2) *if* $\alpha < a$, $\beta = b$, *then*

$$\|H_n^{(\alpha,\beta)}(f) - f\|_{a,b} \to 0 \Leftrightarrow \int_{-1}^{1} (f(t) - f(-1))\, w'_{\alpha+1,\beta}(t)\, dt = 0,$$

(3) *if* $\alpha = a$ *and* $\beta = b$, *then*

$$\|H_n^{(\alpha,\beta)}(f) - f\|_{a,b} \to 0$$

if and only if

$$\int_{-1}^{1} (f(t) - f(1))\, w'_{\alpha,\beta+1}(t)\, dt = 0$$

and

$$\int_{-1}^{1} (f(t) - f(-1))\, w'_{\alpha+1,\beta}(t)\, dt = 0.$$

2. LEMMAS

We write throughout this paper

$$\omega^{(\alpha,\beta)}(\theta) := \left|\sin\frac{\theta}{2}\right|^{2\alpha+1} \left|\cos\frac{\theta}{2}\right|^{2\beta+1}.$$

Furthermore, the conjugate function of $g \in C_{2\pi}$ is given by

$$\overline{g(\theta)} := \frac{1}{2\pi}\int_0^{\pi} \frac{g(\theta+u) - g(\theta-u)}{\tan(u/2)}\, du.$$

From this definition one can easily deduce that, if $g \in C_{2\pi}$ is an odd function, then

$$\overline{g(\theta)} = \frac{1}{\pi} \int_0^\pi \frac{g(u) \sin u}{\cos \theta - \cos u} \, du,$$

and if $g \in C_{2\pi}$ is even then

$$\overline{g(\theta)} = \frac{\sin \theta}{\pi} \int_0^\pi \frac{g(u)}{\cos \theta - \cos u} \, du.$$

The following estimates can be found in [8] and [11].

Lemma 2.1. *We have*

$$\omega^{(\alpha,\beta)}(\theta)(P_n^{(\alpha,\beta)}(\cos \theta))^2 \leqslant \frac{C}{n}, \qquad \forall \theta \in \left[\frac{c}{n}, \pi - \frac{c}{n}\right], \tag{2.1}$$

and

$$|(\omega^{(\alpha,\beta)}(\theta))^{-1/2} G_n^{(\alpha,\beta)}(\theta)| \leqslant \frac{C}{\sqrt{n}}, \qquad \forall \theta \in \left[\frac{c}{n}, \pi - \frac{c}{n}\right], \tag{2.2}$$

where

$$G_n^{(\alpha,\beta)}(\theta) = \overline{P_n^{(\alpha,\beta)}(\cos \theta)\, \omega^{(\alpha,\beta)}(\theta)}.$$

Furthermore, let $M_n := \{\theta \mid \theta = \arccos x,\ P_n^{(\alpha,\beta)\prime}(x) = 0\}$, *then there exists* $C > 0$, *which does not depend on* n *and* θ, *such that*

$$\frac{C^{-1}}{n} \leqslant \omega^{(\alpha,\beta)}(\theta)(P_n^{(\alpha,\beta)}(\cos \theta))^2 \leqslant \frac{C}{n}, \qquad \forall \theta \in M_n. \tag{2.3}$$

The following lemma was essentially given in [12].

Lemma 2.2. *For* $T \in \Pi_{2n-1}$ *the following identity holds*:

$$\begin{aligned} &H_n^{(\alpha,\beta)}(T, x) - T(x) \\ &\quad = \frac{\pi n!\, \Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\, \Gamma(n+\beta+1)} \\ &\quad\quad \times \{(P_n^{(\alpha,\beta)}(x))^2\, \overline{t'(\theta)\, \omega^{(\alpha,\beta)}(\theta)} - t'(\theta)\, P_n^{(\alpha,\beta)}(x)\, G_n^{(\alpha,\beta)}(\theta)\}. \end{aligned}$$

Here $\theta = \arccos x$, $t(\theta) = T(\cos \theta)$, *and* Γ *is the gamma function.*

Proof. For brevity we write $H = H_n^{(\alpha,\beta)}$, $P = P_n^{(\alpha,\beta)}$, and

$$C_n = n!\,\Gamma(n+\alpha+\beta+1)\{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)\}^{-1}$$

The Hermite interpolation formula implies

$$H(T,x) - T(x) = -P(x)\sum_{k=1}^{n} T'(x_k)(1-x_k^2)\frac{P(x)}{x-x_k}\frac{1}{(1-x_k^2)(P'(x_k))^2}.$$

Let $\lambda_{n,k} := \lambda_{n,k}(\alpha,\beta)$ be the coeffients of the Gauss–Jacobi quadrature formula. Then

$$2^{1+\alpha+\beta}\frac{1}{(1-x_k^2)(P'(x_k))^2} = C_n\lambda_{n,k}.$$

As for x fixed, the function

$$P(x)\frac{T'(u)(1-u^2) - T'(x)(1-x^2)}{x-u} + T'(x)(1-x^2)\frac{P(x)-P(u)}{x-u}$$

is an algebraic polynomial in u of degree at most $2n-1$, and $P(x_k) = 0$, $k = 1, 2, ..., n$, we obtain

$$\begin{aligned} &H(T,x) - T(x) \\ &= \frac{-C_n}{2^{1+\alpha+\beta}}\left\{P^2(x)\int_{-1}^{1}\frac{T'(u)(1-u^2)-T'(x)(1-x^2)}{x-u}(1-u)^\alpha(1+u)^\beta\,du\right. \\ &\quad \left. + P(x)\,T'(x)(1-x^2)\int_{-1}^{1}\frac{P(x)-P(t)}{x-u}(1-u)^\alpha(1+u)^\beta\,du\right\}. \end{aligned}$$

Now letting $u = \cos\eta$ and $x = \cos\theta$ and making use of the representation of the conjugate function mentioned at the beginning of this section, we deduce from above the assertion of the lemma. ∎

LEMMA 2.3. *Suppose* $0 \leqslant r_1 < \alpha + 3/2$, $0 \leqslant r_2 < \beta + 3/2$, *and* $r \geqslant 0$. *Let* $T \in \Pi_n$, $t(\theta) = T(\cos\theta)$. *Then, for* $0 \leqslant \theta \leqslant \pi/2$,

$$\begin{aligned} &\frac{1}{\omega^{(\alpha,\beta)}(\theta)}\overline{t'(\theta)\,\omega^{(\alpha,\beta)}(\theta)} \\ &= \frac{1}{4\pi}\int_0^{\theta/2}\frac{\Delta_u^2 t(\theta)}{\sin^2(u/2)}\,du \\ &\quad + \frac{1}{\pi\omega^{(\alpha,\beta)}(\theta)}\sum_{j=0}^{m}(1-\cos\theta)^j\int_{3\theta/2}^{\pi}\frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{j+1}}\,du \\ &\quad + O(\|\varphi T'\|_{r_1,r_2}\theta^{-2r_1} + \|\varphi T'\|_{r,r_2}\theta^{-2r}), \end{aligned}$$

where $m > \alpha - r - 1/2$, $\varphi(x) = \sqrt{1-x^2}$, *and* Δ_h^r *is the* r*th symmetric difference operator with stepsize* h.

Proof. As $t'(\theta)\,\omega^{(\alpha,\beta)}(\theta)$ is an odd function of θ, we have

$$\overline{t'(\theta)\,\omega^{(\alpha,\beta)}(\theta)} = \frac{1}{\pi}\int_0^\pi \frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{\cos\theta - \cos u}\,du$$
$$=: \frac{1}{\pi}\left\{\int_0^{\theta/2} + \int_{\theta/2}^{3\theta/2} + \int_{3\theta/2}^{\pi}\right\}.$$

We note that there exists $C>0$ such that $C^{-1}u^{2\alpha+1} \leqslant \omega^{(\alpha,\beta)}(u) \leqslant Cu^{2\alpha+1}$ for all u satisfying $0 \leqslant u \leqslant \pi/2$ (in short, $\omega^{(\alpha,\beta)}(u) \sim u^{2\alpha+1}$). Thus, since $|t'(\theta)| = |T'(x)|\,\varphi(x)$, we have

$$\left|\int_0^{\theta/2} \frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{\cos\theta - \cos u}\,du\right| \leqslant C\,\|\varphi T'\|_{r_1,r_2}\int_0^{\theta/2}\frac{u^{2\alpha+2-2r_1}}{\theta^2}\,du$$
$$\leqslant C\,\|\varphi T'\|_{r_1,r_2}\cdot\theta^{2\alpha+1-2r_1}.$$

Since for $0<\theta\leqslant\pi/2$,

$$|\omega^{(\alpha,\beta)}(u) - \omega^{(\alpha,\beta)}(\theta)| \leqslant C\,|u-\theta|\,\theta^{2\alpha}, \qquad \forall u \in \left[\frac{\theta}{2}, \frac{3\theta}{2}\right],$$

for the second integral we have the relation

$$\int_{\theta/2}^{3\theta/2} \frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{\cos\theta - \cos u}\,du$$
$$= \omega^{(\alpha,\beta)}(\theta)\int_{\theta/2}^{3\theta/2}\frac{t'(u)\sin u}{\cos\theta-\cos u}\,du + O\left(\theta^{2\alpha}\int_{\theta/2}^{3\theta/2}\frac{|t'(u)|\sin u}{\theta+u}\,du\right)$$
$$= \omega^{(\alpha,\beta)}(\theta)\int_{\theta/2}^{3\theta/2}\frac{t'(u)\sin u}{\cos\theta-\cos u}\,du + O(\theta^{2\alpha+1-2r_1}\,\|\varphi T'\|_{r_1,r_2}).$$

On the other hand, using the formula

$$\frac{\sin u}{\cos\theta-\cos u} = -\frac{1}{2}\left\{\frac{1}{\tan(\theta-u)/2} - \frac{1}{\tan(\theta+u)/2}\right\},$$

we deduce

$$\int_{\theta/2}^{3\theta/2}\frac{t'(u)\sin u}{\cos\theta-\cos u}\,du = -\frac{1}{2}\int_{\theta/2}^{3\theta/2}\frac{t'(u)}{\tan(\theta-u)/2}\,du$$
$$+\frac{1}{2}\int_{\theta/2}^{3\theta/2}\frac{t'(u)}{\tan(\theta+u)/2}\,du. \qquad (2.4)$$

Taking $v=\theta-u$ in the first integral of the right-hand side of (2.4) and integrating by parts, we get

$$\int_{\theta/2}^{3\theta/2} \frac{t'(u)}{\tan(\theta-u)/2}\,du = -\int_0^{\theta/2} \frac{\Delta_{2v} t'(\theta)}{\tan(v/2)}\,dv$$
$$= \frac{1}{2}\int_0^{\theta/2} \frac{\Delta_v^2 t(\theta)}{\sin^2(v/2)}\,dv + O(\|\varphi T'\|_{r_1, r_2}\,\theta^{-2r_1}).$$

To estimate the second integral of the right-hand side of (2.4), we use the estimate $|\tan(\theta+u)/2| \sim \theta$ if $u \in [\theta/2, 3\theta/2]$ to obtain

$$\left|\int_{\theta/2}^{3\theta/2} \frac{t'(u)}{\tan(\theta+u)/2}\,du\right| \leqslant C\,\|\varphi T'\|_{r_1, r_2}\,\theta^{-2r_1}.$$

Finally, since

$$\frac{1}{\cos\theta-\cos u} = \frac{1}{1-\cos u}\sum_{j=0}^{m}\left(\frac{1-\cos\theta}{1-\cos u}\right)^j + \frac{1}{\cos\theta-\cos u}\left(\frac{1-\cos\theta}{1-\cos u}\right)^{m+1},$$

and

$$(1-\cos\theta)^{m+1}\int_{3\theta/2}^{\pi} \frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{(\cos\theta-\cos u)(1-\cos u)^{m+1}}\,du$$
$$= O\left(\theta^{2m+2}\int_{\theta}^{\pi} \frac{\|\varphi T'\|_{r, r_2}\,u^{2\alpha-2r}}{u^{2m+2}}\,du\right)$$
$$= O(\|\varphi T'\|_{r, r_2}\,\theta^{2\alpha+1-2r}),$$

we arrive at

$$\int_{3\theta/2}^{\pi} \frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{\cos\theta-\cos u}\,du = \sum_{j=0}^{m}(1-\cos\theta)^j\int_{3\theta/2}^{\pi}\frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{j+1}}\,du$$
$$+ O(\|\varphi T'\|_{r, r_2}\,\theta^{2\alpha+1-2r}).$$

The estimates above imply the assertion of this lemma. ▮

3. Proof of the Theorems

Proof of Theorem 1.1. We observe that, as the factor $(1+x)^b$ has no effect on the convergence, one can assume $b>\beta$. To show the assertion of this theorem we use the following result obtained by Vértesi [9]: there exists $T \in \Pi_{2n-1}$ such that $\|T-f\| \leqslant CE_n(f)$, $T(x_k)=f(x_k)$, $k=1, 2, ..., n$. Thus

$$\|H_n^{(\alpha,\beta)}(f)-f\|_{a,b} \leqslant \|H_n^{(\alpha,\beta)}(T)-T\|_{a,b}+CE_n(f).$$

As $H_n^{(\alpha,\beta)}T-T \in \Pi_{2n-1}$, we get for $c>0$ and sufficiently large n (see [1, Theorem 8.4.8])

$$\|H_n^{(\alpha,\beta)}(T)-T\|_{a,b} \leqslant C\, \|w_{a,b}(H_n^{(\alpha,\beta)}(T)-T)\|_{[-1+c/n^2,\, 1-c/n^2]},$$

where C does not depend on n and T. Thus, to complete the proof, it is enough to verify

$$\|w_{a,b}(H_n^{(\alpha,\beta)}(T)-T)\|_{[-1+c/n^2,\, 1-c/n^2]}=o(1).$$

Moreover, as $\|f-T\| \leqslant CE_n(f)$, we have $t'(\theta)=o(n)$ for $t(\theta)=T(\cos\theta)$. Hence, making use of Lemmas 2.1 and 2.2, we only need to prove

$$\max_{1/n \leqslant \theta \leqslant \pi/2} \left| \frac{w_{a,b}(\cos\theta)}{\omega^{(\alpha,\beta)}(\theta)}\, \overline{t'(\theta)\,\omega^{(\alpha,\beta)}(\theta)} \right| = o(n). \tag{3.1}$$

To this end, we choose $r_1=0$, $r_2=0$, $r=a$, and $m=0$ in Lemma 2.3. Then, for $\theta \in [1/n, \pi/2]$, one has

$$\begin{aligned}
&\frac{w_{a,b}(\cos\theta)}{\omega^{(\alpha,\beta)}(\theta)}\, \overline{t'(\theta)\,\omega^{(\alpha,\beta)}(\theta)} \\
&\quad = \frac{w_{a,b}(\cos\theta)}{4\pi} \int_0^{\theta/2} \frac{\Delta_u^2 t(\theta)}{\sin^2 u/2}\, du \\
&\qquad + \frac{w_{a,b}(\cos\theta)}{\pi\omega^{(\alpha,\beta)}(\theta)} \int_{3\theta/2}^{\pi} \frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{1-\cos u}\, du + O(\|t'\|).
\end{aligned} \tag{3.2}$$

Now the first term on the right-hand side of (3.2) is $o(n)$ as

$$\int_0^{\theta/2} \frac{\Delta_u^2 t(\theta)}{\sin^2(u/2)}\, du = o(n), \qquad \forall \theta \in \left[\frac{1}{n}, \frac{\pi}{2}\right].$$

Furthermore, without loss of generality, we may assume $\alpha \geqslant 0$. Integration by parts yields

$$\begin{aligned}
&\int_{3\theta/2}^{\pi} \frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{1-\cos u}\, du \\
&\quad = -\int_{3\theta/2}^{\pi} (t(u)-t(0)) \left(\frac{\omega^{(\alpha,\beta)}(u)\sin u}{1-\cos u}\right)' du + O(\|t'\|\, \theta^{2\alpha+1}) \\
&\quad = O(\|t\| + \|t'\|\, \theta^{2\alpha+1}).
\end{aligned} \tag{3.3}$$

Now, as $a > \alpha$ and $w_{a,b}(\cos\theta)/\omega^{(\alpha,\beta)}(\theta) = O(\theta^{2a-2\alpha-1})$ for $\theta \in [1/n, \pi/2]$, we deduce from the above that the second term of the right-hand side of (3.2) is $o(n)$. Therefore (3.1) holds. The proof is complete. ∎

Proof of Theorem 1.2. First we note that $f \in C[-1, 1]$ and $E_n(f)_{a,0} = o(n^{-2\alpha+2a})$ imply $E_n(f) = o(n^{-2\alpha+4a})$. In case $-2\alpha+4a \geqslant 0$, this is trivial because $f \in C[-1, 1]$. Otherwise we take $P_n^* \in \Pi_n$, $n = 1, 2, \ldots$, such that

$$\|f - P_n^*\|_{a,0} = o(n^{-2\alpha+2a}).$$

Thus

$$\|f - P_n^*\|_{[-1+c/n^2,\, 1-c/n^2]} = o(n^{-2\alpha+4a})$$

and (see [1, Theorem 8.4.8])

$$\begin{aligned}\|P^*_{2^{k+1}n} - P^*_{2^k n}\| &\leqslant C\, \|P^*_{2^{k+1}n} - P^*_{2^k n}\|_{[-1+c2^{-2k}n^{-2},\, 1-c2^{-2k}n^{-2}]}\\ &= o((2^k n)^{-2\alpha+4a}),\end{aligned}$$

which then imply

$$\|f - P_n^*\| \leqslant \sum_{k=0}^{\infty} \|P^*_{2^{k+1}n} - P^*_{2^k n}\| = o(n^{-2\alpha+4a}).$$

Hence (see [11, Theorem 1.5]) there exists $T \in \Pi_{2n-1}$ such that

$$T(\pm 1) = f(\pm 1), \qquad T(x_k) = f(x_k), \qquad k = 1, 2, \ldots, n,$$

and

$$\|T - f\|_{a,0} = o(n^{-2\alpha+2a}).$$

Now as before

$$(H_n^{(\alpha,\beta)}(f) - f)\, w_{a,0} = (H_n^{(\alpha,\beta)}(T) - T)\, w_{a,0} + o(1). \tag{3.4}$$

On the other hand, we note that if $b > \beta$, then, by Theorem 1.1,

$$\|(H_n^{(\alpha,\beta)}(f) - f)\, w_{a,b}\|_{[-1,0]} = o(1).$$

Thus, we deduce the following relationship: for $b > \beta$,

$$\|(H_n^{(\alpha,\beta)}(f) - f)\, w_{a,0}\|_{[0,1]} = o(1) \Leftrightarrow \|H_n^{(\alpha,\beta)}(f) - f\|_{a,b} = o(1).$$

Therefore, by (3.4), we obtain that in the case $b > \beta$, formula (1.3) is equivalent to

$$\|H_n^{(\alpha,\beta)}(T) - T\|_{a,b} = o(1),$$

or

$$\|w_{a,b}(H_n^{(\alpha,\beta)}(T)-T)\|_{[-1+c/n^2,\,1-c/n^2]}=o(1).$$

As

$$\|w_{a,b}(H_n^{(\alpha,\beta)}(T)-T)\|_{[-1+c/n^2,\,0]}=o(1),$$

we see that (1.3) is equivalent to

$$\|w_{a,0}(H_n^{(\alpha,\beta)}(T)-T)\|_{[0,\,1-c/n^2]}=o(1). \tag{3.5}$$

Thus we only need to prove that (3.5) holds if and only if (1.4) holds. Now Lemmas 2.1–2.3 show for $r_1=r=a$, $r_2=0$, and $m=[\alpha-a]+1$ in Lemma 2.3 and $\theta=\arccos x$, $\theta\in[c/n,\pi/2]$,

$$\begin{aligned}
H_n^{(\alpha,\beta)}(T,x)-T(x)
&=\frac{n!\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}\,(P_n^{(\alpha,\beta)}(x))^2\,\omega^{(\alpha,\beta)}(\theta)\\
&\quad\times\left\{\frac{1}{4\pi}\int_0^{\theta/2}\frac{\Delta_u^2 t(\theta)}{\sin^2 u/2}\,du\right.\\
&\quad\left.+\frac{1}{\pi\omega^{(\alpha,\beta)}(\theta)}\sum_{j=0}^{m}(1-\cos\theta)^j\int_{3\theta/2}^{\pi}\frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{j+1}}\,du\right\}\\
&\quad+o(\theta^{-2a}).
\end{aligned}$$

Next, as

$$w_{a,0}(\cos\theta)\int_0^{\theta/2}\frac{\Delta_u^2 t(\theta)}{\sin^2 u/2}=o(n),$$

and for $j=0,1,2,\ldots,m-1$ one has

$$\begin{aligned}
(1-\cos\theta)^j\int_0^{3\theta/2}\frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{j+1}}\,du&=o\left(n\theta^{2j}\int_0^{\theta}\frac{u^{2\alpha+2-2a}}{u^{2j+2}}\,du\right)\\
&=o(n\theta^{2\alpha+1-2a}),
\end{aligned}$$

we see that if we can prove

$$(1-\cos\theta)^{m+a}\,(P_n^{(\alpha,\beta)}(x))^2\int_{3\theta/2}^{\pi}\frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{m+1}}\,du=o(1), \tag{3.6}$$

then it follows from the above that for some constant $C_n \sim 1$,

$$H_n^{(\alpha,\beta)}(T,x)-T(x)$$
$$=C_n(P_n^{(\alpha,\beta)}(x))^2\sum_{j=0}^{m-1}(1-\cos\theta)^j\int_0^\pi\frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{j+1}}\,du+o(\theta^{-2a}).$$

Integrating by parts and by the conditions $E_n(f)_{a,0}=o(n^{-2\alpha+2a})$ and $t(0)=f(1)$ we obtain for some constants $d_j \sim 1$, $j=0, 1, ..., m-1$,

$$H_n^{(\alpha,\beta)}(T,x)-T(x)$$
$$=C_n(P_n^{(\alpha,\beta)}(x))^2\sum_{j=0}^{m-1}d_j(1-x)^j\int_{-1}^{1}(f(t)-f(1))\,w'_{\alpha-j,\beta+1}(t)\,dt$$
$$+o(w_{a,0}^{-1}(x)).$$

In view of this formula and (2.3) it is not difficult to obtain that (3.5) holds if and only if (1.4) holds.

It thus remains to show (3.6). We note $m=[\alpha-a]+1$. Thus if $\alpha-a<[\alpha-a]+1/2$

$$\int_{3\theta/2}^{\pi}\frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{m+1}}\,du=O\left(\|\varphi T'\|_{a,0}\int_\theta^\pi\frac{u^{2\alpha-2a}}{u^{2m}}\,du\right)$$
$$=O(\|\varphi T'\|_{a,0}\,\theta^{2\alpha-2a-2m+1}).$$

This estimate implies (3.6) due to Lemma 2.1.

If $\alpha-a\geqslant[\alpha-a]+1/2$, then $\alpha-a=[\alpha-a]+\xi$, $1/2\leqslant\xi<1$. Integration by parts gives

$$\int_{3\theta/2}^{\pi}\frac{t'(u)\,\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{m+1}}\,du$$
$$=-\left(t\left(\frac{3\theta}{2}\right)-t(0)\right)\frac{\omega^{(\alpha,\beta)}(3\theta/2)\sin(3\theta/2)}{(1-\cos(3\theta/2))^{m+1}}$$
$$-\int_{3\theta/2}^{\pi}(t(u)-t(0))\left(\frac{\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{m+1}}\right)'du=:-I_1-I_2.$$

We first consider I_1. Put $g(u)=f(\cos u)$. Then $t(0)=g(0)$ and

$$\left(g\left(\frac{3\theta}{2}\right)-t\left(\frac{3\theta}{2}\right)\right)\left(1-\cos\frac{3\theta}{2}\right)^a=o(n^{-2\alpha+2a}).$$

Since $\theta\in[c/n, \pi/2]$,

$$\frac{\omega^{(\alpha,\beta)}(3\theta/2)\sin(3\theta/2)}{(1-\cos(3\theta/2))^{m+1}}\sim\theta^{2\alpha-2m}.$$

By Lemma 2.1,

$$\begin{aligned}(1-\cos\theta)^{m+a}&\,(P_n^{(\alpha,\beta)}(x))^2 I_1\\ &= O(\theta^{2m+2a}(P_n^{(\alpha,\beta)}(x))^2\,\omega(g,\theta)\,\theta^{2\alpha-2m})\\ &\quad+o(n^{-2\alpha+2a}\theta^{2m+2a}(P_n^{(\alpha,\beta)}(x))^2\,\theta^{2\alpha-2m-2a})\\ &=o(1),\end{aligned}$$

where $\omega(g,\theta)$ is the modulus of continuity of g with the step θ.

To estimate I_2, we replace $t(u)$ by $g(u)$ and calculate it in case $\alpha=m$ and $\alpha\neq m$ separately. First note that

$$\begin{aligned}(1-\cos\theta)^{m+a}&\,(P_n^{(\alpha,\beta)}(x))^2 I_2\\ &=(1-\cos\theta)^{m+a}\,(P_n^{(\alpha,\beta)}(x))^2\\ &\quad\times\int_{3\theta/2}^{\pi}(g(u)-g(0))\left(\frac{\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{m+1}}\right)'du+o(1).\end{aligned}$$

Furthermore, if $\alpha=m$, the integral is bounded and as $\frac{1}{2}\leqslant\xi<1$, one gets $0<a\leqslant 1/2$. Hence,

$$(1-\cos\theta)^{m+a}\,(P_n^{(\alpha,\beta)}(x))^2=O\left(\frac{\theta^{2a-1}}{n}\right)=o(1).$$

Thus (3.6) still holds. If $\alpha\neq m$, we have, by Lemma 2.1,

$$\begin{aligned}(1-\cos\theta)^{m+a}\,(P_n^{(\alpha,\beta)}(x))^2&\int_{3\theta/2}^{\pi}(g(u)-g(0))\left(\frac{\omega^{(\alpha,\beta)}(u)\sin u}{(1-\cos u)^{m+1}}\right)'du\\ &=O\left(\frac{\theta^{2m+2a-2\alpha-1}}{n}\int_{\theta}^{\pi}\omega(g,u)\,u^{2\alpha-2m-1}\,du\right).\qquad(3.7)\end{aligned}$$

Thus if $\alpha-m>0$, since $m>\alpha-a$, the right-hand side of (3.6) can be estimated by $o(1)$. If $\alpha-m<0$, observe that

$$\lim_{\delta\to 0}\delta^{2m-2\alpha}\int_{\delta}^{\pi}\omega(g,u)\,u^{2\alpha-2m-1}\,du=0.$$

Hence for any $\varepsilon>0$, there exists $\delta_0>0$ so that

$$\delta^{2m-2\alpha}\int_{\delta}^{\pi}\omega(g,u)\,u^{2\alpha-2m-1}\,du<\varepsilon,\qquad\forall\delta\leqslant\delta_0.$$

Let δ_0 be fixed. We see that if $\theta \in [c/n, \delta_0]$, then

$$\frac{\theta^{2m+2a-2\alpha-1}}{n} \int_\theta^\pi \omega(g, u)\, u^{2\alpha-2m-1}\, du$$

$$\leqslant C\delta_0^{2m-2\alpha} \int_{\delta_0}^\pi \omega(g, u)\, u^{2\alpha-2m-1}\, du < C\varepsilon.$$

If $\theta \in [\delta_0, \pi/2]$, the integral on the right-hand side of (3.7) is bounded. Hence we have the same result if n is sufficiently large. Therefore in all cases we obtain that the right-hand side of (3.7) is $o(1)$. Thus (3.6) holds, and the proof is complete. ∎

Proof of Theorem 1.4. It follows from Theorem 1.2 that (1.2) implies $\alpha < a$ and $\beta < b$. Theorem 1.1 shows that this condition is sufficient. ∎

Proof of Theorem 1.5. Put $f^*(x) = f(-x)$. Then $H_n^{(\alpha,\beta)}(f, -x) = H_n^{(\beta,\alpha)}(f^*, x)$. Thus (1) and (2) are equivalent. We only need to verify (1). Recalling the proof of Theorem 1.2, we know that for $b > \beta$

$$\|(H_n^{(\alpha,\beta)}(f) - f)\, w_{a,0}\|_{[0,1]} = o(1) \Leftrightarrow \|H_n^{(\alpha,\beta)}(f) - f\|_{a,b} = o(1).$$

Hence we have to show

$$\|(H_n^{(\alpha,\beta)}(f) - f)\, w_{\alpha,0}\|_{[0,1]} = o(1) \Leftrightarrow \int_{-1}^1 (f(t) - f(1))\, w'_{\alpha,\beta+1}(t)\, dt = 0.$$

But this can be obtained directly from Theorem 1.2 if we choose $a = \alpha$ there.

To prove (3) we note

$$\|(H_n^{(\alpha,\beta)}(f) - f)\, w_{\alpha,\beta}\|_{[0,1]} = o(1) \Leftrightarrow \|(H_n^{(\alpha,\beta)}(f) - f)\, w_{\alpha,0}\|_{[0,1]} = o(1)$$

and

$$\|(H_n^{(\alpha,\beta)}(f) - f)\, w_{\alpha,\beta}\|_{[-1,0]} = o(1) \Leftrightarrow \|(H_n^{(\alpha,\beta)}(f) - f)\, w_{0,\beta}\|_{[-1,0]} = o(1).$$

Hence (3) follows from (1) and (2), which completes the proof. ∎

REFERENCES

1. Z. DITZIAN AND V. TOTIK, "Moduli of Smoothness," Springer-Verlag, New York/Berlin, 1987.
2. E. EGERVÁRY AND P. TURÁN, Notes on interpolation. V. On the stability of interpolation, *Acta Math. Hungar.* **9** (1958), 259–267.
3. B. HÁY AND P. VÉRTESI, Interpolation in spaces of weighted maximum norm, *Studia Sci. Math. Hungar.* **14** (1979), 1–9.

4. H.-B. KNOOP, "Hermite–Fejér-Interpolation mit Randbedingungen," Habilitationsschrift, Universität Duisburg, 1981.
5. P. NEVAI AND P. VÉRTESI, Mean convergence of Hermite–Fejér interpolation, *J. Math. Anal. Appl.* **105** (1985), 26–58.
6. A. SCHÖNHAGE, Zur Konvergenz der Stufenpolynome über den Nullstellen der Legendre-Polynome, *in* "Linear Operators and Approximation, Proc. Conf. Math. Research Institute Oberwolfach, 1971" (P. L. Butzer, J. P. Kahane, and B. Sz.-Nagy, Ed.), pp. 448–451, Birkhäuser, Basel/Stuttgart, 1972.
7. J. SZABADOS, On Hermite–Fejér interpolation for the Jacobi abscissas, *Acta Math. Hungar.* **23** (1972), 449–464.
8. G. SZEGÖ, "Orthogonal Polynomials," 4th ed., Colloq. Publ., Vol. 23, Amer. Math. Soc. Providence, RI, 1975.
9. P. VÉRTESI, Convergent interpolation processes for arbitrary systems of nodes, *Acta Math. Hungar.* **33** (1979), 223–234.
10. P. VÉRTESI, Convergence criteria for Hermite–Fejér interpolation based on Jacobi abscissas, *in* "Functions, Series, and Operators, Proc. International Conference in Budapest 1980, Vol. II," Colloq. Soc. János Bolyai, Vol. 35 (B. Sz.-Nagy and J. Szabados, Eds.), pp. 1253–1258, North-Holland, Amsterdam/New York, 1983.
11. X. ZHOU, "On Hermite–Fejér-type operators," thesis, Universität Duisburg, Duisburg, 1991.
12. X. ZHOU, Weighted approximation by Hermite–Fejér interpolation, *in* "Approximation Theory, Proc. International Conference in Kecskemét, Hungary, 1990," Colloq. Math. Soc. János Bolyai, Vol. 58 (K. Tandori and J. Szabados, Eds.), pp. 785–798, North-Holland, Amsterdam/New York, 1990.