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Broué's abelian defect group conjecture holds for the Harada–Norton sporadic simple group HN

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ABSTRACT

In representation theory of finite groups, there is a well-known and important conjecture due to M. Broué. He conjectures that, for any prime p , if a p -block A of a finite group G has an abelian defect group P , then A and its Brauer corresponding block B of the normaliser $N_G(P)$ of P in G are derived equivalent (Rickard equivalent). This conjecture is called *Broué's abelian defect group conjecture*.

We prove in this paper that Broué's abelian defect group conjecture is true for a non-principal 3-block A with an elementary abelian defect group P of order 9 of the Harada–Norton simple group HN. It then turns out that Broué's abelian defect group conjecture holds for all primes p and for all p -blocks of the Harada–Norton simple group HN.

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1. Introduction and notation

In representation theory of finite groups, one of the most important and interesting problems is to give an affirmative answer to a conjecture, which was introduced by M. Broué around 1988 [7], and is nowadays called *Broué's abelian defect group conjecture*. He actually conjectures the following:

1.1. Conjecture (*Broué's abelian defect group conjecture*). (See [7, 6.2 Question] and [21, Conjecture in p. 132].) Let p be a prime, and let (K, \mathcal{O}, k) be a splitting p -modular system for all subgroups of a finite group G . Assume that A is a block algebra of $\mathcal{O}G$ with a defect group P and that B is a block algebra of $\mathcal{O}N_G(P)$ such that B is the Brauer correspondent of A , where $N_G(P)$ is the normaliser of P in G . Then, A and B should be derived equivalent (Rickard equivalent) provided P is abelian.

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In fact, a stronger conclusion than 1.1 is expected. If G and H are finite groups and if A and B are block algebras of $\mathcal{O}G$ and $\mathcal{O}H$ (or kG and kH) respectively, we say that A and B are *splendidly Rickard equivalent* in the sense of Linckelmann [36,37], where he calls it a *splendid derived equivalence*, see the end of 1.8. Note that this is the same as that given by Rickard in [53] when A and B are the principal block algebras, which he calls a *splendid equivalence*.

1.2. Conjecture. (See Rickard [53], [54, Conjecture 4, in p. 193].) *Keep the notation, and suppose that P is abelian as in 1.1. Then, there should be a splendid Rickard equivalence between the block algebras A of $\mathcal{O}G$ and B of $\mathcal{O}N_G(P)$.*

There are several cases where the conjectures of Broué 1.1 and Rickard 1.2 are checked. For example we prove that 1.1 and 1.2 are true for the principal block algebra A of an arbitrary finite group G when the defect group P of A is elementary abelian of order 9 (and hence $p = 3$), see [24, (0.2) Theorem]. Then, it may be natural to ask what about the case of non-principal block algebras with the same defect group $P = C_3 \times C_3$. Namely, this paper should be considered as a continuation of such a project, which has already been accomplished for several cases in our previous papers for the O’Nan simple group and the Higman–Sims simple group in [27, 0.2 Theorem], for the Held simple group and the sporadic simple Suzuki group in [28, Theorem], and for the Janko’s simple group J_4 [29, Theorem 1.3], see also [44] and [32]. That is to say, our main theorem of this paper is the following:

1.3. Theorem. *Let G be the Harada–Norton simple group HN, and let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting 3-modular system for all subgroups of G , see the definition 1.8 below. Suppose that A is a non-principal block algebra of $\mathcal{O}G$ with a defect group P which is an elementary abelian group $C_3 \times C_3$ of order 9, and that B is a block algebra of $\mathcal{O}N_G(P)$ such that B is the Brauer correspondent of A . Then, A and B are splendidly Rickard equivalent, and hence Conjectures 1.1 and 1.2 of Broué and Rickard hold.*

As a matter of fact, the main result 1.3 above is obtained by proving the following:

1.4. Theorem. *Keep the notation and the assumption as in 1.3. Then, the non-principal block algebra A of $\mathcal{O}G$ with a defect group $P = C_3 \times C_3$ and the principal block algebra A' of $\mathcal{O}HS$ of the Higman–Sims simple group are Puig equivalent, that is A and A' are Morita equivalent which is realised by a ΔP -projective p -permutation $\mathcal{O}[G \times HS]$ -module, in other words, A and A' have isomorphic source algebras as interior P -algebras.*

Then, it turns out that, as a corollary to the main result (1.3), we eventually can prove that:

1.5. Corollary. *Broué’s abelian defect group conjecture 1.1 and even Rickard’s splendid equivalence conjecture 1.2 are true for all primes p and for all block algebras of $\mathcal{O}G$ when $G = \text{HN}$.*

1.6. Starting point and strategy. A story of the birth of this paper is actually very similar to that of the Janko’s simple group J_4 which is given in [29, 1.6]. Namely, relatively recently G. Hiss, J. Müller, F. Noeske and J.G. Thackray [16] have determined the 3-decomposition matrix of the group HN with defect group $C_3 \times C_3$, see 4.1. Our starting point for this work was actually to realise that the 3-decomposition matrix for the non-principal block of HN with an elementary abelian defect group of order 9 is exactly the same as that for the principal 3-block of the Higman–Sims simple group HS. Furthermore, the generalised 3-decomposition matrices of these two blocks are the same. Therefore, it is natural to suspect whether these two 3-block algebras would be Morita equivalent not only over an algebraically closed field k of characteristic 3 but also over a complete discrete valuation ring \mathcal{O} whose residue field is k , and we might expect even that they are *Puig equivalent* (we shall give a precise definition of Puig equivalence in 1.8 below). Anyhow, since the two conjectures of Broué and Rickard in 1.1 and 1.2 respectively have been solved for the principal 3-block of HS in a paper of Okuyama [48] it turns out that Broué’s abelian defect group conjecture 1.1 and Rickard’s splendid equivalence conjecture 1.2 shall be solved also for the non-principal 3-block of HN with the same defect group $C_3 \times C_3$.

1.7. Contents. In Section 2, we shall give several fundamental lemmas, which are useful and powerful to prove our main results. In Sections 3 and 4, we shall investigate 3-modular representations for HN and we shall get trivial source (p -permutation) modules which are in the non-principal 3-block A of HN with a defect group $P = C_3 \times C_3$. In Section 5, we shall list data on Green correspondents of simples in the principal 3-block A' of HS, which are known by a result of [58, Theorem], see [48, Example 4.8]. Finally, in Sections 6–8, we shall give complete proofs of our main results 1.3, 1.4 and 1.5.

To achieve our results, next to theoretical reasoning we have to rely on fairly heavy computations. As tools, we use the computer algebra system GAP [11], to calculate with permutation groups as well as with ordinary and Brauer characters. We also make use of the data library [6], in particular allowing for easy access to the data compiled in [9,18,61], and of the interface [60] to the data library [62]. Moreover, we use the computer algebra system MeatAxe [55] to handle matrix representations over finite fields, as well as its extensions to compute submodule lattices [39], radical and socle series [42], homomorphism spaces and endomorphism rings [41], and direct sum decompositions [40]. We give more detailed comments on the relevant computations in the spots where they enter the picture.

1.8. Notation. Throughout this paper, we use the following notation and terminology. Let A be a ring. We denote by 1_A , $Z(A)$ and A^\times for the unit element of A , the centre of A and the set of all units in A , respectively. We denote by $\text{rad}(A)$ the Jacobson radical of A and by $\text{rad}^i(A)$ the i -th power $(\text{rad}(A))^i$ for any positive integer i while we define $\text{rad}^0(A) = A$. We write $\text{Mat}_n(A)$ for the matrix ring of all $n \times n$ -matrices whose entries are in A . Let B be another ring. We denote by $\text{mod-}A$, $A\text{-mod}$ and $A\text{-mod-}B$ the categories of finitely generated right A -modules, left A -modules and (A, B) -bimodules, respectively. We write M_A , ${}_A M$ and ${}_A M_B$ when M is a right A -module, a left A -module and an (A, B) -bimodule. However, by a module we mean a finitely generated right module unless otherwise stated. Let M and N be A -modules. We write $N \mid M$ if N is (isomorphic to) a direct summand of M as an A -module.

From now on, let k be a field and assume that A is a finite-dimensional k -algebra. Suppose that M is an A -module. Then, we denote by $\text{soc}(M)$ the socle of M . We define $\text{soc}_0(M) = 0$ and $\text{soc}_1(M) = \text{soc}(M)$. Then, we define $\text{soc}_i(M)$ by $\text{soc}_i(M)/\text{soc}_{i-1}(M) = \text{soc}(M/\text{soc}_{i-1}(M))$ for any integer $i \geq 2$. Similarly, we write $\text{rad}^i(M)$ for $M \cdot \text{rad}^i(A)$ for any integer $i \geq 0$. By using this, we define $L_i(M)$ by $\text{rad}^{i-1}(M)/\text{rad}^i(M)$ for $i = 1, 2, \dots$. We call $L_i(M)$ the i -th Loewy layer of M . We denote by $j(M)$ the Loewy length of M , namely $j(M)$ is the least positive integer j satisfying $\text{rad}^j(M) = 0$. We write $P(M)$ and $I(M)$ for a projective cover and an injective hull (envelope) of M , respectively, and we write Ω for the Heller operator (functor), namely, ΩM is the kernel of the projective cover $P(M) \twoheadrightarrow M$. Dually, $\Omega^{-1}M$ is the cokernel of the injective hull $M \hookrightarrow I(M)$. For simple A -modules S_1, \dots, S_n (some of which are possibly isomorphic) we write that $M = a_1 \times S_1 + \dots + a_n \times S_n$, as composition factors for positive integers a_1, \dots, a_n when the set of all composition factors are a_1 times S_1, \dots, a_n times S_n . For an A -module M and a simple A -module S , we denote by $c_M(S)$ the multiplicity of all composition factors of M which are isomorphic to S . We write $c(S, T)$ for $c_{P(S)}(T)$ for simple A -modules S and T , namely, this is so-called the Cartan invariant with respect to S and T .

To describe the structure of an A -module, we either indicate the radical and socle series, in cases where these series coincide and are sufficient for our analysis, or we draw an Alperin diagram [1]. An A -module need not have an Alperin diagram, but if it does then it is a compact way to give a more detailed structural description of the module under consideration; note that the Alperin diagram is closely related to the Hasse diagram of the incidence relation amongst the local submodules in the sense of [43], hence for explicit examples is easily determined using the techniques described in [39]. Note, however, that by giving any kind of diagram an A -module in general is not uniquely determined up to isomorphism.

Let N be another A -module. Then, $\text{Hom}_A(M, N)$ is the set of all right A -module-homomorphisms from M to N , which canonically is a k -vector space, and we denote by $\text{PHom}_A(M, N)$ the set of all (relatively) projective homomorphisms in $\text{Hom}_A(M, N)$, which is a k -subspace of $\text{Hom}_A(M, N)$. Hence, we can define the factor space, that is, we write $\underline{\text{Hom}}_A(M, N)$ for the factor space $\text{Hom}_A(M, N)/\text{PHom}_A(M, N)$. By making use of this, as is well known, we can construct the stable

module category $\underline{\text{mod}}\text{-}A$, which is a quotient category of $\text{mod}\text{-}A$ such that the set of all morphisms is given by $\underline{\text{Hom}}_A(M, N)$.

In this paper, G is always a finite group and we fix a prime number p . Assume that $(\mathcal{K}, \mathcal{O}, k)$ is a splitting p -modular system for all subgroups of G , that is to say, \mathcal{O} is a complete discrete valuation ring of rank one such that its quotient field is \mathcal{K} which is of characteristic zero and its residue field $\mathcal{O}/\text{rad}(\mathcal{O})$ is k which is of characteristic p , and that \mathcal{K} and k are splitting fields for all subgroups of G . We mean by an $\mathcal{O}G$ -lattice a finitely generated right $\mathcal{O}G$ -module which is a free \mathcal{O} -module. We sometimes call it just an $\mathcal{O}G$ -module. Let X be a kG -module. Then, we write X^\vee for the k -dual of X , namely, $X^\vee = \text{Hom}_k(X, k)$ which is again a right kG -module via $(x)(\varphi g) = (xg^{-1})\varphi$ for $x \in X, \varphi \in X^\vee$ and $g \in G$. Similarly, we write χ^\vee for the dual (complex conjugate) of χ for an ordinary character χ of G . Let H be a subgroup of G , and let M and N be an $\mathcal{O}G$ -lattice and an $\mathcal{O}H$ -lattice, respectively. Then, let $M \downarrow_H^G = M \downarrow_H$ be the restriction of M to H , and let $N \uparrow_H^G = N \uparrow^G$ be the induction (induced module) of N to G , that is, $N \uparrow^G = (N \otimes_{\mathcal{O}H} \mathcal{O}G)_{\mathcal{O}G}$. Similar for kG - and kH -modules.

We denote by $\text{Irr}(G)$ and $\text{IBr}(G)$ the sets of all irreducible ordinary and Brauer characters of G , respectively. Let A be a block algebra (p -block) of $\mathcal{O}G$. Then, we write $\text{Irr}(A)$ and $\text{IBr}(A)$ for the sets of all characters in $\text{Irr}(G)$ and $\text{IBr}(G)$ which belong to A , respectively. We often mean by $\text{IBr}(A)$ the set of all non-isomorphic simple kG -modules belonging to A . For ordinary characters χ and ψ of G , we denote by $(\chi, \psi)^G$ the inner product of χ and ψ in usual sense. Let X and Y be kG -modules. Then, we write $[X, Y]^G$ for $\dim_k[\text{Hom}_{kG}(X, Y)]$. We denote by k_G the trivial kG -module. Similar for \mathcal{O}_G . For A -modules M and N we write $[M, N]^A$ for $\dim_k[\text{Hom}_A(M, N)]$.

We say that M is a *trivial source* (p -permutation) kG -module if M is an indecomposable kG -module whose source is k_Q , where Q is a vertex of M . Let G' be another finite group, and let V be an $(\mathcal{O}G, \mathcal{O}G')$ -bimodule. Then we can regard V as a right $\mathcal{O}[G \times G']$ -module via $v(g, g') = g^{-1}vg'$ for $v \in V, g \in G$ and $g' \in G'$. Similar for (kG, kG') -bimodules. We denote by ΔG the diagonal copy of G in $G \times G$, namely, $\Delta G = \{(g, g) \in G \times G \mid g \in G\}$. Let A and A' be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$, respectively. Then, we say that A and A' are *Puig equivalent* if A and A' have a common defect group P (and hence $P \subseteq G \cap G'$) and if there is a Morita equivalence between A and A' which is induced by an (A, A') -bimodule \mathfrak{M} such that, as a right $\mathcal{O}[G \times G']$ -module, \mathfrak{M} is a p -permutation (trivial source) module and ΔP -projective. Similar for blocks of kG and kG' . Due to a result of Puig (and independently of Scott), see [52, Remark 7.5], this is equivalent to a condition that A and A' have source algebras which are isomorphic as interior P -algebras, see [37, Theorem 4.1]. For an $(\mathcal{O}G, \mathcal{O}G')$ -bimodule V and a common subgroup Q of G and G' , we set $V^Q = \{v \in V \mid qv = vq, \forall q \in Q\}$. If Q is a p -group, the Brauer construction is defined to be a quotient $V(Q) = V^Q / [\sum_{R \leq Q} \text{Tr}_R^Q(V^R) + \text{rad}(\mathcal{O} \cdot V^Q)]$ where Tr_R^Q is the usual trace map. The Brauer homomorphism $\text{Br}_Q : (\mathcal{O}G)^Q \rightarrow kC_G(Q)$ is obtained from composing the canonical epimorphism $(\mathcal{O}G)^Q \twoheadrightarrow (\mathcal{O}G)(Q)$ and a canonical isomorphism $(\mathcal{O}G)(Q) \xrightarrow{\cong} kC_G(Q)$.

We say that A and A' are *stably equivalent of Morita type* if there exists an (A, A') -bimodule \mathfrak{M} such that ${}_A\mathfrak{M}$ and $\mathfrak{M}_{A'}$ are both projective, ${}_A(\mathfrak{M} \otimes_{A'} \mathfrak{M}^\vee)_A \cong {}_A A_A \oplus (\text{projective } (A, A)\text{-bimodule})$ and ${}_{A'}(\mathfrak{M}^\vee \otimes_A \mathfrak{M})_{A'} \cong {}_{A'} A'_{A'} \oplus (\text{projective } (A', A')\text{-bimodule})$. We say that A and A' are *splendidly stably equivalent of Morita type* if A and A' have a common defect group P and the stable equivalence of Morita type is induced by an (A, A') -bimodule \mathfrak{M} which is a p -permutation (trivial source) $\mathcal{O}[G \times G']$ -module and is ΔP -projective, see [37, Theorem 3.1]. We say that A and A' are *Rickard equivalent* if A and A' are derived equivalent, namely, $D^b(\text{mod}\text{-}A)$ and $D^b(\text{mod}\text{-}A')$ are equivalent as triangulated categories. We say that A and A' are *splendidly Rickard equivalent* if A and A' are derived equivalent by a complex $M^\bullet \in C^b(A\text{-mod}\text{-}A')$ and its dual $(M^\bullet)^\vee$ such that each term M^i of M^\bullet is a $\Delta(P)$ -projective and p -permutation module as an $\mathcal{O}[G \times G']$ -module, where $C^b(A\text{-mod}\text{-}A')$ is the category of bounded complexes of finitely generated (A, A') -bimodules.

For a positive integer n , \mathfrak{A}_n and \mathfrak{S}_n denote the alternating and symmetric group on n letters, M_n denotes the Mathieu group, and C_n, D_n and SD_n denote the cyclic group, the dihedral group and the semi-dihedral group of order n , respectively. For a subgroup E of $\text{Aut}(G)$, $G \rtimes E$ denotes a semi-direct product such that G is normal in $G \rtimes E$ and E acts on G canonically. For $g \in G$ and a subset S of G , we denote $g^{-1}Sg$ by S^g , and similarly, $x^g = g^{-1}xg$ for $x \in G$. For non-empty subsets S and T of G , we write $S =_G T$ if $T = S^g$ for an element $g \in G$.

For other notation and terminology, see the books of Nagao and Tsushima [45] and Thévenaz [56].

2. Preliminaries

In this section we list many lemmas, some of which are theorems due to other people. These lemmas are so useful and powerful to prove our main results.

2.1. Lemma. (See [22, (1.1) Lemma].) Let A be a finite-dimensional algebra over a field and X an A -module. Assume that Y is a non-zero uniserial A -submodule of X with Loewy layers

$$\text{rad}^{i-1}(Y)/\text{rad}^i(Y) \cong S_i \quad \text{for } i = 1, \dots, n$$

where S_i is a simple A -module. Set $\tilde{X} = X/Y$. Then, we get the following:

- (i) For each $j = 1, \dots, j(X)$, $\text{rad}^{j-1}(X)/\text{rad}^j(X) \cong \text{rad}^{j-1}(\tilde{X})/\text{rad}^j(\tilde{X})$ or $\text{rad}^{j-1}(X)/\text{rad}^j(X) \cong \text{rad}^{j-1}(\tilde{X})/\text{rad}^j(\tilde{X}) \oplus S_i$ for some S_i .
- (ii) For each $i = 1, \dots, n$, there is a positive integer m_i such that $m_1 < m_2 < \dots < m_n$ and that $\text{rad}^{m_i-1}(X)/\text{rad}^{m_i}(X) \cong (\text{rad}^{m_i-1}(\tilde{X})/\text{rad}^{m_i}(\tilde{X})) \oplus S_i$.

2.2. Lemma. (See Okuyama [47, Lemma 2.2].) Let S be a simple kG -module with vertex P , and let f be the Green correspondence with respect to $(G, P, N_G(P))$. If S is a trivial source module, then its Green correspondent $f(S)$ is again simple as $kN_G(P)$ -module.

2.3. Lemma. (See Scott [33, II Theorem 12.4 and I Proposition 14.8] and [5, Corollary 3.11.4].)

- (i) If M is a trivial source kG -module, then M uniquely (up to isomorphism) lifts to a trivial source $\mathcal{O}G$ -lattice \hat{M} .
- (ii) If M and N are both trivial source kG -modules, then $[M, N]^G = (\chi_{\hat{M}}, \chi_{\hat{N}})^G$.

2.4. Lemma (Fong–Reynolds). Let H be a normal subgroup of G , and let A and B be block algebras of $\mathcal{O}G$ and $\mathcal{O}H$, respectively, such that A covers B . Let $T = T_G(B)$ be the inertial subgroup (stabiliser) of B in G . Then, there is a block algebra \tilde{A} of $\mathcal{O}T$ such that \tilde{A} covers B , $1_A 1_{\tilde{A}} = 1_{\tilde{A}} 1_A = 1_{\tilde{A}}$, $A = \tilde{A}^G$ (block induction), and the block algebras A and \tilde{A} are Morita equivalent via a pair $(1_A \cdot \mathcal{O}G \cdot 1_{\tilde{A}}, 1_{\tilde{A}} \cdot \mathcal{O}G \cdot 1_A)$, that is, the Morita equivalence is a Puig equivalence and induces a bijection

$$\text{Irr}(\tilde{A}) \rightarrow \text{Irr}(A), \quad \tilde{\chi} \mapsto \tilde{\chi} \uparrow^G; \quad \text{Irr}(A) \rightarrow \text{Irr}(\tilde{A}), \quad \chi \mapsto \chi \downarrow_T \cdot 1_{\tilde{A}}$$

between $\text{Irr}(\tilde{A})$ and $\text{Irr}(A)$, and a bijection

$$\text{IBr}(\tilde{A}) \rightarrow \text{IBr}(A), \quad \tilde{\phi} \mapsto \tilde{\phi} \uparrow^G; \quad \text{IBr}(A) \rightarrow \text{IBr}(\tilde{A}), \quad \phi \mapsto \phi \downarrow_T \cdot 1_{\tilde{A}}$$

between $\text{IBr}(\tilde{A})$ and $\text{IBr}(A)$.

Proof. See [28, 1.5 Theorem] and [45, Chapter 5, Theorem 5.10]. \square

2.5. Lemma. Let A be a block algebra of $\mathcal{O}G$ with a defect group P , let $N = N_G(P)$, and let A_N be a block algebra of $\mathcal{O}N$ which is the Brauer correspondent of A . Moreover, let (P, e) be a maximal A -Brauer pair, $H = N_G(P, e)$, the normaliser of (P, e) in $N_G(P)$, and let B be a block algebra of $\mathcal{O}H$ which is the Fong–Reynolds correspondent of A_N , see 2.4. Then, $A \downarrow_{G \times H}^{G \times G} \cdot 1_B = 1_A \cdot \mathcal{O}G \cdot 1_B$, as a right $\mathcal{O}[G \times H]$ -module, has a unique (up to isomorphism) indecomposable direct summand with vertex ΔP .

Proof. See [29, Lemma 2.4] and [45, Chapter 5, Theorem 5.10]. \square

2.6. Lemma. Assume that $G \geq H$, and let A and B respectively be block algebras of $\mathcal{O}G$ and $\mathcal{O}H$ with a common defect group P , and hence $P \leq H$. Suppose, moreover, that a pair (M, M^\vee) induces a splendid stable equivalence of Morita type between A and B , where M is an (A, B) -bimodule such that $M \mid 1_A \cdot \mathcal{O}G \cdot 1_B$ as (A, B) -bimodules.

- (i) If X is a non-projective trivial source kG -module in A , then $(X \otimes_A M)_B = Y \oplus (\text{proj})$ for a non-projective indecomposable kH -module Y such that Y has a trivial source.
- (ii) If X is a non-projective indecomposable kG -module in A , then $(X \otimes_A M)_B = Y \oplus (\text{proj})$ for a non-projective indecomposable kH -module Y such that there is a p -subgroup Q of H such that Q is a common vertex of X and Y .

Proof. See [29, Lemma 2.7]. \square

2.7. Lemma. Let k be a field, and let A be a finite-dimensional symmetric k -algebra. Moreover, suppose that S is a simple A -module and M is a projective-free A -module. Then, we have $\underline{\text{Hom}}_A(S, M) \cong \text{Hom}_A(S, M)$ and $\underline{\text{Hom}}_A(M, S) \cong \text{Hom}_A(M, S)$ as k -spaces.

Proof. Follows by [12, (3.2), (3.2*), (3.3)], see [33, II, Lemma 2.7, Corollary 2.8]. \square

2.8. Lemma. Let k be an algebraically closed field, and let A and B be finite-dimensional symmetric k -algebras. Suppose that M is an (A, B) -bimodule such that ${}_A M$ and M_B are both projective modules. Then a functor $F: \text{mod-}A \rightarrow \text{mod-}B$ defined by $F(X') = X' \otimes_A M$ for X'_A , is additive and exact. Assume, furthermore, that F induces a stable equivalence between A and B .

- (i) Let X be a projective-free A -module such that X has a simple A -submodule S . Set $T = F(S)$. Then, we can write $F(X) = Y \oplus R$ for a projective-free B -module Y and a projective B -module R . Now, if T is a simple B -module, then we may assume that Y contains T and that $F(X/S) = Y/T \oplus (\text{proj})$.
- (ii) (Dual of (i).) Let X be a projective-free A -module such that X has an A -submodule X' satisfying that X/X' is simple. Set $T = F(X/X')$. Then, we can write $F(X) = Y \oplus R$ for a projective-free B -module Y and a projective B -module R . Now, if T is a simple B -module, then we may assume that T is an epimorphic image of Y and that $\text{Ker}(F(X) \rightarrow T) = \text{Ker}(Y \rightarrow T) \oplus (\text{proj})$.

Proof. We get (i) from 2.7 and [28, 1.11 Lemma], just as in the proof of [28, 3.25 Lemma and 3.26 Lemma], see [32, Proposition 2.2]. (ii) is just the dual of (i). \square

2.9. Lemma. (See Linckelmann [34, Theorem 2.1(ii)].) Let A and B be finite-dimensional k -algebras for a field k such that A and B are both self-injective and indecomposable as algebras, and none of them are simple algebras. Suppose that there is an indecomposable (A, B) -bimodule M such that a pair (M, M^\vee) induces a stable equivalence between A and B . If S is a simple A -module, then $(S \otimes_A M)_B$ is a non-projective indecomposable B -module.

The next lemma is a new result due to Kunugi and the first author. This is actually so useful and convenient when we want to apply so-called ‘‘Rouquier’s gluing’’ to our inductive argument in order to get a stable equivalence between two block algebras which we are looking at.

2.10. Lemma. (See Koshitani and Kunugi [26, Theorem 1.2].) Let A be a block algebra of $\mathcal{O}G$ with a cyclic defect group $P \neq 1$. Let $H = N_G(P)$, and let B be a block algebra of $\mathcal{O}H$ such that B is the Brauer correspondent of A . Then, we get the following:

- (i) The following (1) and (2) are equivalent:
 - (1) The Brauer tree of A is a star with exceptional vertex in the centre, and there exists a non-exceptional irreducible ordinary character χ of G in A such that $\chi(u) > 0$ for any element $u \in P$.
 - (2) The block algebras A and B are Puig equivalent.

- (ii) If one of the conditions (1) and (2) in (i) holds (and hence both hold), then all simple kG -modules in A are trivial source modules.
- (iii) If one of the conditions (1) and (2) in (i) holds (and hence both hold), then there is an indecomposable (A, B) -bimodule \mathfrak{M} such that $1_A \cdot \mathcal{O}G \cdot 1_B = \mathfrak{M} \oplus (\text{proj})$ and \mathfrak{M} , as an $\mathcal{O}[G \times H]$ -module, has ΔP as its vertex, and \mathfrak{M} realises a Puig equivalence between A and B .

2.11. Lemma. Let A be a block algebra of $\mathcal{O}G$ with defect group P . Set $H = N_G(P)$, and let B be a block algebra of $\mathcal{O}H$ such that B is the Brauer correspondent of A . Assume that Q is a subgroup of P with $Q \subseteq Z(G)$. Set $\bar{G} = G/Q$, $\bar{H} = H/Q$ and $\bar{P} = P/Q$. It is well known that there exist block algebras \bar{A} and \bar{B} of $\mathcal{O}\bar{G}$ and $\mathcal{O}\bar{H}$, respectively, such that \bar{A} and \bar{B} dominate A and B , namely $\text{Irr}(\bar{A}) \subseteq \text{Irr}(A)$ and $\text{Irr}(\bar{B}) \subseteq \text{Irr}(B)$, and that both \bar{A} and \bar{B} have \bar{P} as defect groups, see [45, Chapter 5, Theorems 8.10 and 8.11].

- (i) It holds that $\bar{H} = N_{\bar{G}}(\bar{P})$ and that \bar{B} is the Brauer correspondent of \bar{A} .
In the rest of the lemma, assume in addition that P is elementary abelian of order p^2 , namely, $P = Q \times R$ with $Q \cong R \cong C_p$.
- (ii) It holds that

$$\bar{A} \otimes_{\mathcal{O}\bar{H}} \bar{B} = \bar{A}(\bar{A} \cdot 1_{\bar{B}})_{\bar{B}} = \bar{A}X_{\bar{B}} \oplus (\text{proj})$$

for an indecomposable (\bar{A}, \bar{B}) -bimodule X with vertex $\Delta\bar{P}$.

- (iii) In particular, if X realises a Morita equivalence between \bar{A} and \bar{B} , then there exists an (A, B) -bimodule M such that M is an indecomposable direct summand of ${}_A(A \cdot 1_B)_B$ with vertex ΔP , and hence M induces a Puig equivalence between A and B .

Proof. (i) The first part is easy. The second part follows from [46, (3.2) Lemma], see [38, *l.* 10 on p. 1314].

(ii) This follows by [35, Proposition 6.1] since $\bar{P} \cong C_p$.

(iii) This is obtained from (ii) and [25, Theorem], see [38, *l.* $-7 \sim \ell. -4$ on p. 1314] and [37, Theorem 4.1]. \square

2.12. Lemma. Suppose that $p = 3$ and $G = \mathfrak{A}_9$.

- (i) There uniquely exists a non-principal block algebra A of $\mathcal{O}G$ with defect group $P \cong C_3$. In addition we can write $\text{Irr}(A) = \{\chi_5, \chi_{17}, \chi_{18}\}$ such that $\chi_5(1) = 27$, $\chi_{17}(1) = 189$, $\chi_{18}(1) = 216$ and $\chi_5(u) = \chi_{17}(u) = 9$ for any element $u \in P - \{1\}$, and that a part of the 3-decomposition matrix is

	27	189
χ_5	1	0
χ_{17}	0	1
χ_{18}	1	1

where the indices of χ_i are the same as in [9, p. 37]. (In the following, we use the notation A and P as in (i).)

- (ii) Set $H = N_G(P)$. Then $H = (P \times \mathfrak{A}_6).C_2$, where the action on $P \times \mathfrak{A}_6$ by C_2 is the diagonal one, extending \mathfrak{A}_6 to \mathfrak{S}_6 .
- (iii) Let H be as in (ii), and let B be a block algebra of $\mathcal{O}H$, which is the Brauer correspondent of A . Then, A and B are Morita equivalent via an (A, B) -bimodule M such that M is (up to isomorphism) the unique indecomposable direct summand of ${}_A(A \cdot 1_B)_B$ with vertex ΔP , and hence it holds that M induces a Puig equivalence between A and B , and that the simples 27 and 189 in A are both trivial source kG -modules.

Proof. (i) This follows from [9, p. 37], [61, $A_9 \pmod{3}$] and [18].

(ii) Easy by inspection.

(iii) This is obtained from (i) and 2.10. \square

2.13. Lemma. Let A and B be finite-dimensional k -algebras. Assume that there exists a functor $F : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}B$ realising a stable equivalence between A and B . Assume, in addition, that there is a simple A -module S_0 such that S_0 is sent to a simple B -module T_0 , namely, $F(S_0) = T_0$. Then, for any simple A -module S with $S \not\cong S_0$, it holds $[F(S), T_0]^B = [T_0, F(S)]^B = 0$.

Proof. We get by 2.7 and the assumptions that

$$\begin{aligned} 0 &= \text{Hom}_A(S, S_0) \cong \underline{\text{Hom}}_A(S, S_0) \\ &\cong \underline{\text{Hom}}_B(F(S), F(S_0)) = \underline{\text{Hom}}_B(F(S), T_0) \\ &\cong \text{Hom}_B(F(S), T_0). \end{aligned}$$

Hence $[F(S), T_0]^B = 0$. The rest is similar. \square

2.14. Lemma. Let A be a finite-dimensional k -algebra, and assume that X is an A -module satisfying that $(I_1 \oplus I_2) \mid X$ where I_1 and I_2 are A -submodules of X with $I_1 \cong I_2 \cong I(S)$ for a simple A -module S . If Z is an A -submodule of X such that $\text{soc}(Z)$ is a simple A -module, then X/Z has a direct summand isomorphic to $I(S)$.

Proof. Set $S_i = \text{soc}(I_i)$ for $i = 1, 2$. Then, $\text{soc}(I_1 \oplus I_2) = S_1 \oplus S_2 \cong S \oplus S$. Hence $S_1 \oplus S_2 \not\subseteq Z$, so that $S_1 \not\subseteq Z$ or $S_2 \not\subseteq Z$. Thus we may assume $S_1 \not\subseteq Z$. Then, $I_1 \cap Z = 0$ since I_1 has a unique minimal A -submodule S_1 . This implies that $X/Z \supseteq (I_1 + Z)/Z \cong I_1/(I_1 \cap Z) \cong I_1$. Hence, $I_1 \hookrightarrow X/Z$. Since I_1 is injective, we finally have $I_1 \mid X/Z$. \square

2.15. Lemma. Let G, H and L be finite groups such that all of them contain a common subgroup P , namely, $P \subseteq G \cap H \cap L$. Let M be a $k[G \times H]$ -module such that $M \mid k_{\Delta P} \uparrow^{G \times H}$, and let N be a $k[H \times L]$ -module such that $N \mid k_{\Delta P} \uparrow^{H \times L}$. Then, it follows that $M \otimes_{kH} N \mid k_{\Delta P} \uparrow^{G \times L}$.

Proof. This is a special case of [15, 2.5 Proposition]. \square

2.16. Lemma. Let A be a finite-dimensional k -algebra, and assume that X is an indecomposable non-simple A -module. Then, it holds $\text{soc}(X) \subseteq \text{rad}(X)$.

Proof. Assume that $\text{soc}(X) \not\subseteq \text{rad}(X)$. Then, X has a simple A -submodule S with $S \not\subseteq \text{rad}(X)$. Hence, X has a maximal A -submodule M with $S \not\subseteq M$. These imply that $S \cap M = 0$ and $S + M = X$. Namely, $X = S \oplus M$. Since M is indecomposable, $X = S$. This is a contradiction. \square

3. 3-Local structure for HN

3.1. Notation and assumption. From now on, we assume that G is the Harada–Norton simple group HN, and hence $|G| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19 \cong 2.7 \times 10^{14}$, see [9, pp. 164–166] and [13].

3.2. Lemma.

- (i) In order to prove Broué’s abelian defect group conjecture for $G = \text{HN}$, it suffices to prove it for the case $p = 3$.
- (ii) There exists a unique 3-block A with non-cyclic abelian defect group P , and P is elementary abelian of order 9.
- (iii) P is the Sylow 3-subgroup of the second largest maximal subgroup 2.HS.2 of G , a two-fold cover of the automorphism group of the Higman–Sims simple group HS.

Proof. (i) We may assume $p \in \{2, 3, 5\}$ by 3.1 just as in the proof of [29, Lemma 3.2]. Assume that $p = 2$. Then, G has only two 2-blocks B_0 and B_1 with positive defect by [61], where B_0 is the principal 2-block. Then, the non-principal 2-block B_1 has a defect group D with $D \cong SD_{16}$, see [4, Lemma 4.2(c)]. Thus, B_0 and B_1 both have non-abelian defect groups. Next, suppose $p = 5$. By [61], G has only a unique 5-block B_0 which has defect ≥ 2 , and hence B_0 is the principal 5-block. Then, B_0 has non-abelian defect group $5_+^{1+4}.5$ by [9, pp. 164–166].

(ii) Finally, assume $p = 3$. Sylow 3-subgroups of G are non-abelian by [9, pp. 164–166]. Thus, G has a unique non-principal 3-block A such that A has a defect group P with $|P| \geq 3^2$, and actually $P \cong C_3 \times C_3$, see [4, Lemma 4.2(b)].

(iii) Using the character table of G , calculations with GAP [11] show that the conjugacy class $2A$ of G is a defect class of A , where we follow the notation in [9, pp. 164–166]. Hence P is a Sylow 3-subgroup of the centraliser $C_G(2A) \cong 2.HS.2$. \square

3.3. Notation. From now on, we assume $p = 3$, and we use the notation A and P as in 3.2, namely, A is a block algebra of kG with defect group $P \cong C_3 \times C_3$. Set $H = N_G(P)$, and let B be a block algebra of kH that is the Brauer correspondent of A . Let (P, e) be a maximal A -Brauer pair in G , that is, e is a block idempotent of $kC_G(P)$ such that $\text{Br}_P(1_A) \cdot e = e$, see [2,8] and [56, §40]. Set $\tilde{H} = N_G(P, e)$, namely, $\tilde{H} = \{g \in N_G(P) \mid e^g = e\}$, where $e^g = g^{-1}eg$. Finally set $E = H/C_G(P)$, and let Q be a subgroup of P of order 3.

3.4. Lemma. *It holds the following:*

- (i) $H = \tilde{H} = (P \times \mathfrak{A}_6).SD_{16}$.
- (ii) $C_G(P) = C_H(P) = P \times \mathfrak{A}_6$.
- (iii) $E = \tilde{H}/C_G(P) \cong SD_{16}$, where the action of E on P is given by the embedding of SD_{16} as a Sylow 2-subgroup of $\text{Aut}(P) \cong GL_2(3)$.
- (iv) All elements in $P - \{1\}$ are conjugate in H , and hence in G .
- (v) $P - \{1\} \subseteq 3A$, where $3A$ is a conjugacy class of G following the notation in [9, pp. 164–166].
- (vi) All subgroups of P of order 3 are conjugate in H , and hence in G .
- (vii) Recall the subgroup Q of P in 3.3. Then, we have $C_G(Q) = Q \times \mathfrak{A}_9$, $N_G(Q) = (Q \times \mathfrak{A}_9).2 \leq \mathfrak{A}_{12}$, $C_H(Q) = (P \times \mathfrak{A}_6).2$, and $N_H(Q) = (P \times \mathfrak{A}_6).2^2$.
- (viii) $C_G(Q)/Q \cong \mathfrak{A}_9$, $C_H(Q)/Q \cong (C_3 \times \mathfrak{A}_6).2$, $N_G(Q)/Q \cong \mathfrak{A}_9.2$, and $N_H(Q)/Q \cong (C_3 \times \mathfrak{A}_6).2^2$.

Proof. This is found using explicit computation with GAP [11]. The starting point is the smallest faithful permutation representation of G on 1 140 000 points, available in terms of so-called standard generators [59] in [62]. The associated one-point stabiliser is the largest maximal subgroup \mathfrak{A}_{12} of G , which hence can be found explicitly by a randomised Schreier-Sims technique. Having completed that, all the following computations can be done using this permutation representation of G .

Actually, one of the standard generators is an element of the $2A$ conjugacy class of G , where we use the notation in [9, pp. 164–166]. Hence the second largest maximal subgroup $2.HS.2 \cong C_G(2A)$ can be found by a centraliser computation. In turn, by 3.2(iii) P can be computed explicitly as a Sylow 3-subgroup of $2.HS.2$.

(i)–(ii) The normaliser $H = N_G(P)$ and the centraliser $C_G(P)$ of P can be computed explicitly, and as these are fairly small groups their structure is easily revealed.

(iii) It follows from [61, $A_6 \pmod{3}$] and [18] that \mathfrak{A}_6 has exactly two 3-blocks. Let β be the non-principal block algebra of $k\mathfrak{A}_6$, and hence β is of defect zero. Then, $e = 1_\beta$. Since β is a unique block algebra of $k\mathfrak{A}_6$ of defect zero, this shows $H = \tilde{H}$.

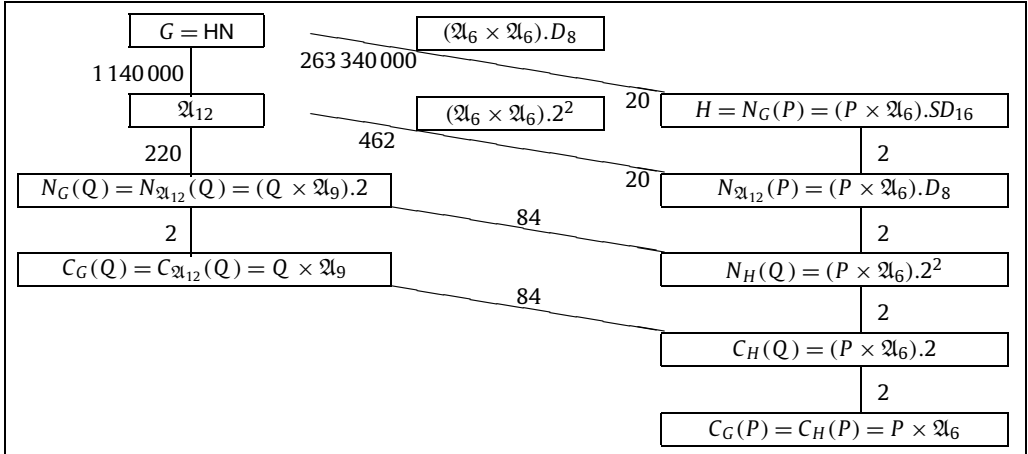
(iv) Easy by (iii) and inspection.

(v) We use the notation $3A$ and $3B$ as in [9, pp. 164–166]. By (iv), $P - \{1\} \subseteq 3A$ or $3B$. Assume $P - \{1\} \subseteq 3B$. Then, $\chi(u) = 0$ for any $\chi \in \text{Irr}(A)$ and any $u \in 3A$ by [45, Chapter 5 Corollary 1.10(i)]. But we know that $\chi_8 \in \text{Irr}(A)$ by [4, Lemma 4.2(b)], see also 4.1, and that $\chi_8(u) = 27$ for any $u \in 3A$. This is a contradiction.

(vi) Easy by (iv).

(vii)–(viii) It is easy to see that $N_G(Q) < \mathfrak{A}_{12}$, the largest maximal subgroup of G , which is the one-point stabiliser in the given permutation representation of G . Hence again the normaliser $N_{\mathfrak{A}_{12}}(Q)$ and the centraliser $C_{\mathfrak{A}_{12}}(P)$ of P can be computed explicitly and their structure determined. \square

3.5. Lemma. We get the following diagram:



where the numbers between two boxes are indices between the two corresponding groups.

Proof. This follows from [9, pp. 164–166], 3.4 and calculations with GAP [11]. \square

3.6. Lemma. The following holds:

- (i) $B \cong \text{Mat}_9(\mathcal{O}[P \rtimes SD_{16}])$ as \mathcal{O} -algebras.
- (ii) The block algebra B has a source algebra $jBj \cong \mathcal{O}[P \rtimes SD_{16}]$, as interior P -algebras, where j is a source idempotent of B with respect to P , namely, j is a primitive idempotent of B^P such that $\text{Br}_P(j) \neq 0$ for the Brauer homomorphism Br_P for P , see [56, §§19 and 27].
- (iii) We can write

$$\text{Irr}(B) = \{\chi_{9a}, \chi_{9b}, \chi_{9c}, \chi_{9d}, \chi_{18a}, \chi_{18b}, \chi_{18c}, \chi_{72a}, \chi_{72b}\}$$

and

$$\text{lBr}(B) = \{9a, 9b, 9c, 9d, 18a, 18b, 18c\},$$

where the numbers mean the degrees of characters and the dimensions of simples, respectively. Note that χ_{18b} and χ_{18c} are dual each other, and so are $18b$ and $18c$. The other characters and simples are self-dual.

- (iv) The 3-decomposition matrix and the Cartan matrix of B are the following:

	9a	9b	9c	9d	18a	18b	18c
χ_{9a}	1
χ_{9b}	.	1
χ_{9c}	.	.	1
χ_{9d}	.	.	.	1	.	.	.
χ_{18a}	1	.	.
χ_{18b}	1	.
χ_{18c}	1
χ_{72a}	1	1	.	.	1	1	1
χ_{72b}	.	.	1	1	1	1	1

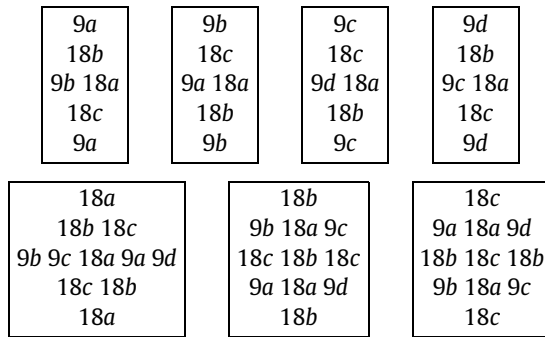
	$P(9a)$	$P(9b)$	$P(9c)$	$P(9d)$	$P(18a)$	$P(18b)$	$P(18c)$
9a	2	1	0	0	1	1	1
9b	1	2	0	0	1	1	1
9c	0	0	2	1	1	1	1
9d	0	0	1	2	1	1	1
18a	1	1	1	1	3	2	2
18b	1	1	1	1	2	3	2
18c	1	1	1	1	2	2	3

(v) There are unique conjugacy classes 4A and 4B of H , consisting of elements of order 4, and having centralisers of order 40 and 48, respectively. A part of the character table of $\text{Irr}(B)$ then is the following:

Conjugacy class	4A	4B	12A
Centraliser	40	48	24
χ_{9a}	1	-1	-1
χ_{9b}	-1	-1	-1
χ_{9c}	1	1	1
χ_{9d}	-1	1	1
χ_{18a}	0	0	0
χ_{18b}	0	0	0
χ_{18c}	0	0	0
χ_{72a}	0	-2	1
χ_{72b}	0	2	-1

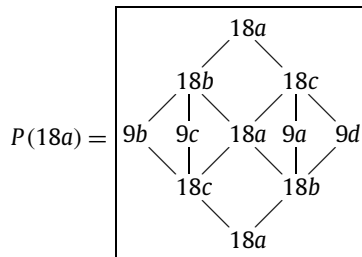
Note that this identifies the characters $\chi_{9a}, \chi_{9b}, \chi_{9c}, \chi_{9d}, \chi_{72a}$ and χ_{72b} uniquely.

(vi) The radical and socle series of PIMs in B are the following:



Note that this identifies the simples 18b and 18c uniquely.

(vii) An Alperin diagram of the PIM $P(18a)$ is given as follows:



Proof. This again relies on computations with GAP [11]. Starting with the explicit restriction of the permutation representation of G to H obtained in 3.4, we find a faithful permutation representation of H on a small number of points. This then is used to compute the conjugacy classes of H , and its ordinary character table using the Dixon–Schneider algorithm.

(i) Since the Schur multiplier of SD_{16} is trivial, see e.g. [23, Proof of Lemma 1.3], we get the assertion by 3.4(i)–(iii) [31, A Theorem].

(ii) This follows by a result of Puig [51, Proposition 14.6] and (i), see [3, Theorem 13] and [56, (45.12) Theorem].

(iii)–(v) Easy from the character table of H .

(vi) The radical and socle series have been determined in [57].

(vii) To find the structure of $P(18a)$, we have used the MeatAxe [55] to construct $P(18a)$ explicitly as a matrix representation, from the permutation representation of H obtained above, and subsequently we have used the method described in [39] to compute the whole submodule lattice of $P(18a)$, from which the result follows easily. \square

3.7. Notation. We use the notation $\chi_{9a}, \chi_{9b}, \chi_{9c}, \chi_{9d}, \chi_{18a}, \chi_{18b}, \chi_{18c}, \chi_{72a}, \chi_{72b}, 9a, 9b, 9c, 9d, 18a, 18b, 18c$, and also the source idempotent j as in 3.6.

3.8. Lemma. *The block algebra B and its source algebra $k[P \rtimes SD_{16}]$ have exactly 18 trivial source modules. In fact, it holds the following:*

(i) Seven PIMs: $P(9a), P(9b), P(9c), P(9d), P(18a), P(18b), P(18c)$.

(ii) Seven trivial source modules with a vertex P : $9a, 9b, 9c, 9d, 18a, 18b, 18c$.

(iii) Four trivial source modules with vertex $Q \cong C_3$:

$$\chi_{9a} + \chi_{9b} + \chi_{18a} + \chi_{72a} \leftrightarrow V_1 = \begin{array}{|c|} \hline \begin{array}{ccc} 9a & & 18a & & 9b \\ & \diagdown & & \diagup & \\ & 18b & & 18c & \\ & \diagup & & \diagdown & \\ 9b & & 18a & & 9a \end{array} \\ \hline \end{array}$$

$$\chi_{9c} + \chi_{9d} + \chi_{18a} + \chi_{72b} \leftrightarrow V_2 = \begin{array}{|c|} \hline \begin{array}{ccc} 9c & & 18a & & 9d \\ & \diagdown & & \diagup & \\ & 18c & & 18b & \\ & \diagup & & \diagdown & \\ 9d & & 18a & & 9c \end{array} \\ \hline \end{array}$$

$$\chi_{18b} + \chi_{18c} + \chi_{72a} \leftrightarrow V_3 = \begin{array}{|c|} \hline \begin{array}{ccc} & 18b & & 18c & \\ & \diagdown & & \diagup & \\ 9b & & 18a & & 9a \\ & \diagup & & \diagdown & \\ & 18c & & 18b & \end{array} \\ \hline \end{array}$$

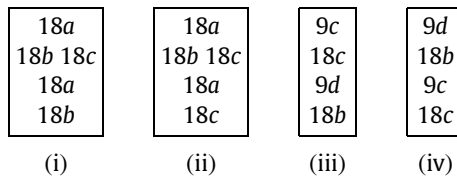
$$\chi_{18b} + \chi_{18c} + \chi_{72b} \leftrightarrow V_4 = \begin{array}{|c|} \hline \begin{array}{ccc} & 18b & & 18c & \\ & \diagdown & & \diagup & \\ 9c & & 18a & & 9d \\ & \diagup & & \diagdown & \\ & 18c & & 18b & \end{array} \\ \hline \end{array}$$

and all characters χ_{V_i} realised by V_i has values $\chi_{V_i}(u) = 27$ for any $u \in 3A$, where $3A$ is the unique conjugacy class of H of elements of order 3 on which χ_{V_i} does not vanish, see [45, Chapter 5, Corollary 1.10(i)].

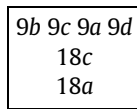
Proof. These follow from 3.4, a theorem of Green [45, Chapter 4, Problem 10, p. 302] As for (iii), starting again with the permutation representation of H , using GAP [11] we compute $N_H(Q)$, use the MeatAxe [55] and the methods described in [40] to find the PIMs of $N_H(Q)/Q$ as direct summands of its regular representation, induce them to H , and find the submodule structure of the induced modules using the methods described in [39]. \square

3.9. Notation. We use the notation V_1, V_2, V_3, V_4 as in 3.8.

3.10. Lemma. *There are no kH -modules in B whose radical and socle series are the same and which have the following structure:*



Proof. (i) Assume that such a kH -module, which we call M , exists. There is an epimorphism $\pi : P(18a) \twoheadrightarrow M$. Set $K = \text{Ker}(\pi)$. Then, 3.6(vi) and 1.1 imply that K has radical and socle series



Since there does not exist a kH -module $\begin{bmatrix} 9a \\ 18c \end{bmatrix}$ by 3.6(vi), we have a contradiction.

(ii) Similar to (i).

(iii) Assume that such a kH -module, which we call M , exists. There is an epimorphism $\pi : P(9c) \twoheadrightarrow M$. Set $K = \text{Ker}(\pi)$. Then, by 3.6(vi) we get $K = \begin{bmatrix} 18a \\ 9c \end{bmatrix}$. This contradicts the structure of $P(9c)$ in 3.6(vi).

(iv) Similar to (iii). \square

4. 3-Modular representations of HN

4.1. Theorem. (See Hiss, Müller, Noeske and Thackray [16].) *The 3-decomposition matrix and the Cartan matrix of A are the following:*

Degree	[9, pp. 164–166]	S_1	S_2	S_3	S_4	S_5	S_6	S_7
8910	χ_8	1
16929	χ_{10}	.	1
270864	χ_{19}	.	.	1
1185030	χ_{32}	1	1	.	1	.	.	.
1354320	χ_{33}	1	.	.	.	1	1	.
1575936	χ_{37}	.	.	1	.	.	1	.
2784375	χ_{43}	1	.	1	1	1	1	.
4561920	χ_{49}	.	.	.	1	1	.	1
4809375	χ_{50}	.	1	1	1	.	.	1

	$P(S_1)$	$P(S_2)$	$P(S_3)$	$P(S_4)$	$P(S_5)$	$P(S_6)$	$P(S_7)$
S_1	4	1	1	2	2	2	0
S_2	1	3	1	2	0	0	1
S_3	1	1	4	2	1	2	1
S_4	2	2	2	4	2	1	2
S_5	2	0	1	2	3	2	1
S_6	2	0	2	1	2	3	0
S_7	0	1	1	2	1	0	2

where S_1, \dots, S_7 are non-isomorphic simple kG -modules in A whose degrees respectively are 8910, 16929, 270864, 1 159 191, 40 338, 1 305 072, 3 362 391.

4.2. Notation. We use the notation $\chi_8, \chi_{10}, \chi_{19}, \chi_{32}, \chi_{33}, \chi_{37}, \chi_{43}, \chi_{49}, \chi_{50}$ and S_1, \dots, S_7 as in 4.1.

4.3. Lemma.

- (i) All simples S_1, \dots, S_7 are self-dual.
- (ii) (Knörr) All simples S_1, \dots, S_7 have P as vertices.

Proof. (i) Easy from 4.1.

(ii) This is a result of Knörr [20, 3.7 Corollary]. \square

4.4. Lemma.

- (i) The heart $\mathcal{H}(P(S_i)) = \text{rad}(P(S_i))/\text{soc}(P(S_i))$ is indecomposable as a kG -module for any $i = 1, \dots, 7$.
- (ii) $\text{Ext}_{kG}^1(S_i, S_j) = 0$ for any pair $(i, j) \in \{(1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 7), (3, 1), (3, 2), (3, 5), (3, 7), (4, 6), (5, 3), (5, 5), (5, 7), (6, 4), (6, 6), (7, 2), (7, 3), (7, 5), (7, 7)\}$.

Proof. (i) This follows by the Cartan matrix of A in 4.1 and results of Erdmann and Kawata, see [10, Theorem 1], [19, Theorem 1.5] and [27, 1.9 Lemma].

(ii) If $\text{Ext}_{kG}^1(S_1, S_2) \neq 0$, then $S_2 \mid \mathcal{H}(P(S_1))$ since $c_{12} = 1$ by 4.1, which contradicts to (i). Similar for the others. \square

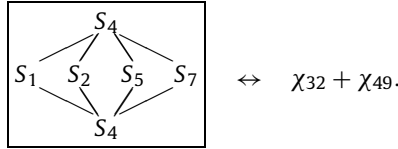
4.5. Lemma.

- (i) The simple S_1 is a trivial source module with $S_1 \leftrightarrow \chi_8$.
- (ii) The simple S_2 is a trivial source module with $S_2 \leftrightarrow \chi_{10}$.
- (iii) The simple S_3 is a trivial source module with $S_3 \leftrightarrow \chi_{19}$.

Proof. (i)–(ii) Let $M = 2.\text{HS}.2$, where HS is the Higman–Sims simple group, be the second largest maximal subgroups of G , see by [9, pp. 164–166]. Now, a calculation with GAP [11], using the character tables of M and G , shows that $1_M \uparrow^G \cdot 1_A = \chi_8 + \chi_{10}$. Set $X = k_M \uparrow^G \cdot 1_A$. We then get $X = S_1 + S_2$ (as composition factors) by 4.1. Since X, S_1, S_2 are all self-dual by 4.3(i), we obtain $X = S_1 \oplus S_2$.

(iii) Let M be the same as above. There uniquely exists a non-trivial linear character χ of M . Then, a calculation with GAP [11] shows that $\chi \uparrow_M^G \cdot 1_A = \chi_{19}$. Hence, by 4.1, S_3 is a trivial source module. \square

4.6. Lemma. *There is a trivial source kG -module in A whose vertex is Q and whose structure is*

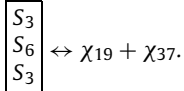


Proof. It follows from [9, pp. 164–166] that the fourth largest maximal subgroup of G is of the form $M = 2_+^{1+8} \cdot (\mathfrak{A}_5 \times \mathfrak{A}_5)$. 2. Let $P_M \in \text{Syl}_3(M)$. Then $P_M \cong C_3 \times C_3$, but a calculation with GAP [11], using the character tables of G and M , shows that P_M contains elements belonging to the $3B$ conjugacy class of G , hence $P_M \neq_G P$ by 3.4(v). Clearly, there is a non-trivial kM -module T with $\dim_k T = 1$. Set $X = T \uparrow_M^G \cdot 1_A$. Then, X is a direct sum of trivial source kG -modules, and a calculation with GAP [11] shows that $X \leftrightarrow \chi_{32} + \chi_{49}$. Since P is a defect group of A , any indecomposable kG -module Y with $Y \mid X$ does not have P as its vertex.

Suppose that X is decomposable. Then, 2.3(i) implies that $X = Y \oplus Z$ such that $Y \leftrightarrow \chi_{32}$ and $Z \leftrightarrow \chi_{49}$. Hence, 4.1 yields that $Y = S_1 + S_2 + S_4$ (as composition factors). We know by 4.3(ii) and 4.1 that S_1, S_2 , and Y are all self-dual. If $[Y, S_1]^G \neq 0$, then the self-dualities imply $S_1 \mid Y$, and hence $0 \neq [S_1, Y]^G = (\chi_8, \chi_{32})^G$ from 2.3(ii) and 4.5(i), a contradiction. Hence, $[Y, S_1]^G = [S_1, Y]^G = 0$. Similarly, we obtain $[Y, S_2]^G = [S_2, Y]^G = 0$ by 2.3(ii) and 4.5(ii). This is a contradiction since Y has only three composition factors S_1, S_2 and S_4 .

Thus, X is indecomposable. By the decomposition matrix of A in 4.1, X is not a PIM. Thus, the order of a vertex of X is 3, and hence Q is a vertex of X by 3.4(vi). Clearly, X is a trivial source kG -module in A . We know by 4.1 that $X = S_1 + S_2 + 2 \times S_4 + S_5 + S_7$ (as composition factors). Note that $X, S_1, S_2, S_4, S_5, S_7$ are all self-dual from 4.3(i). Then, $[S_i, X]^G = [X, S_i]^G = 0$ for any $i = 1, 2, 5, 7$ since X is indecomposable. Thus, $X/\text{rad}(X) \cong \text{Soc}(X) \cong S_4$. Therefore, again by the self-dualities, it holds that $\text{rad}(X)/\text{Soc}(X) \cong S_1 \oplus S_2 \oplus S_5 \oplus S_7$. \square

4.7. Lemma. *There is a trivial source kG -module in A which has Q as a vertex and has radical and socle series*



(Note: We can prove that this module has Q as its vertex, but only later on in 7.2(ii).)

Proof. First, the third largest maximal subgroup of G is of shape $M = U_3(8).3$, see [9, pp. 164–166]. Then, a calculation with GAP [11], using the character tables of G and M , shows that

$$1_M \uparrow^G \cdot 1_A = \chi_{19} + \chi_{32} + \chi_{37} + \chi_{49}. \tag{1}$$

Set $X = k_M \uparrow^G \cdot 1_A$, hence X is self-dual and is a direct sum of trivial source kG -modules. Then, by the decomposition matrix in 4.1, we know

$$X = S_1 + S_2 + 2 \times S_3 + 2 \times S_4 + S_5 + S_6 + S_7 \quad (\text{as composition factors}). \tag{2}$$

If $[X, S_1]^G \neq 0$, then 2.3(ii) and 4.5(i) imply that $(\chi_{\hat{X}}, \chi_8)^G = [X, S_1]^G \neq 0$ where $\chi_{\hat{X}}$ is a character afforded by X (see 2.3(i)), which is a contradiction by (1). Hence, it holds $[X, S_1]^G = [S_1, X]^G = 0$ by the self-dualities in 4.3(i). Similarly, by 4.5(ii)–(iii) and 2.3(ii), we know also $[X, S_2]^G = [S_2, X]^G = 0$ and $[X, S_3]^G = [S_3, X]^G = 1$. If $[X, S_5]^G \neq 0$, then (2) and the self-dualities imply that $S_5 \mid X$, and hence S_5 is liftable by 2.3(i), which contradicts 4.1. Hence, $[X, S_5]^G = [S_5, X]^G = 0$ by the self-dualities. Similarly, it holds also that $[X, S_i]^G = [S_i, X]^G = 0$ for $i = 6, 7$. If $[X, S_4]^G = 2$, then it follows from (2) and the self-dualities that $(S_4 \oplus S_4) \mid X$, and hence S_4 is liftable by 2.3(i), which contradicts 4.1. This shows $[X, S_4]^G = [S_4, X]^G \neq 2$. Namely,

$$[S_3, X]^G = [X, S_3]^G = 1, \tag{3}$$

$$[S_4, X]^G = [X, S_4]^G \neq 2, \tag{4}$$

$$[S_i, X]^G = [X, S_i]^G = 0 \text{ for } i = 1, 2, 5, 6, 7. \tag{5}$$

Now, 4.6 says that there is a trivial source kG -module Y that has radical and socle series

$$Y = \begin{array}{c} \boxed{\begin{array}{ccccc} & & S_4 & & \\ & \swarrow & & \searrow & \\ S_1 & & S_2 & & S_5 & S_7 \\ & \swarrow & & \searrow & \\ & & S_4 & & \end{array}} \leftrightarrow \chi_{32} + \chi_{49} \tag{6} \end{array}$$

in A . Then, by (1), (6) and 2.3(ii), we have

$$[Y, X]^G = [X, Y]^G = 2. \tag{7}$$

Then, by (4) and (2), we know that

$$[S_4, X]^G = [X, S_4]^G \leq 1. \tag{8}$$

Next, we want to claim that there is a homomorphism $\varphi \in \text{Hom}_{kG}(Y, X)$ with $0 \neq \text{Im}(\varphi) \cong S_4$. Suppose that any non-zero $\varphi \in \text{Hom}_{kG}(Y, X)$ satisfies that $\text{Im}(\varphi) \cong S_4$. By (7), let $\{\varphi_1, \varphi_2\}$ be a k -basis of $\text{Hom}_{kG}(Y, X)$. Then, it follows from Schur’s lemma that $\text{Im}(\varphi_1) \neq \text{Im}(\varphi_2)$, and hence that there exists a direct sum $\text{Im}(\varphi_1) \oplus \text{Im}(\varphi_2) \subseteq X$. This means that $[S_4, X]^G \geq 2$, contradicting (8).

Therefore, there is a homomorphism $\varphi \in \text{Hom}_{kG}(Y, X)$ with $0 \neq \text{Im}(\varphi) \cong S_4$. Then, by (6), we know $\text{Ker}(\varphi) = 0$ since $S_i \nmid \text{soc}(X)$ for $i = 1, 2, 5, 7$ by (5). That is, there is a monomorphism $\varphi: Y \hookrightarrow X$ of kG -modules.

Then, just by the dual argument, we know also that there is an epimorphism $\psi: X \twoheadrightarrow Y$ of kG -modules. It follows then by (2) and (6) that there is a direct sum $\text{Im}(\varphi) \oplus \text{Ker}(\psi) \subseteq X$, and hence $\text{Im}(\varphi) \oplus \text{Ker}(\psi) = X$. Set $Z = \text{Ker}(\psi)$. We can write $X = Y \oplus Z$. Since $Z = 2 \times S_3 + S_6$ (as composition

factors), we get by (5) that $Z = \begin{array}{c} \boxed{\begin{array}{c} S_3 \\ S_6 \\ S_3 \end{array}} \end{array}$. Hence, it is easy to know from (1) and (6) that Z is a trivial source kG -module with $Z \leftrightarrow \chi_{19} + \chi_{37}$. \square

4.8. Lemma. *There is a trivial source kG -module in A whose structure is*

$$\begin{array}{c} \boxed{\begin{array}{ccc} S_1 & & S_2 \\ & \swarrow & \searrow \\ & S_4 & \\ & \swarrow & \searrow \\ S_1 & & S_2 \end{array}} \leftrightarrow \chi_8 + \chi_{10} + \chi_{32}. \end{array}$$

(Note: We can prove that this module has Q as its vertex, but only later on in 7.2(i).)

Proof. By [9, p. 91], we have $1_{\mathfrak{A}_{11}} \uparrow^{\mathfrak{A}_{12}} = 1_{\mathfrak{A}_{12}} + \tilde{\chi}_{11}$, where $\tilde{\chi}_{11} \in \text{Irr}(\mathfrak{A}_{12})$ is of degree 11. It follows from the 3-decomposition matrix of \mathfrak{A}_{12} in [61, $A_{12} \pmod{3}$] and [18] that

$$k_{\mathfrak{A}_{11}} \uparrow^{\mathfrak{A}_{12}} = \begin{array}{c} \boxed{\begin{array}{c} k \\ 10 \\ k \end{array}} \leftrightarrow 1_{\mathfrak{A}_{12}} + \tilde{\chi}_{11}, \tag{9} \end{array}$$

where 10 is a simple $k\mathfrak{A}_{12}$ -module of dimension 10 . Set $X = (k\mathfrak{A}_{11} \uparrow^{\mathfrak{A}_{12}}) \uparrow^G \cdot 1_A$. Note that X is a direct sum of trivial source kG -modules. Then, we know from a calculation with GAP [11], using the character tables of G and \mathfrak{A}_{12} , that

$$X \leftrightarrow \chi_8 + \chi_{10} + \chi_{32} \tag{10}$$

and

$$X = 2 \times S_1 + 2 \times S_2 + S_4 \quad (\text{as composition factors}). \tag{11}$$

By (7), 2.3(ii) and 4.5(ii), we obtain $[X, S_1]^G = (\chi_{\tilde{X}}, \chi_8)^G = 1$. Hence, $[X, S_1]^G = [S_1, X]^G = 1$ by the self-dualities. Similarly, we have $[X, S_2]^G = [S_2, X]^G = 1$. Since S_4 is not liftable by 4.1, S_4 is not a trivial source module by 2.3(i). This implies that $[X, S_4]^G = [S_4, X]^G = 0$ by (8). These yield

$$X/\text{rad}(X) \cong \text{Soc}(X) \cong S_1 \oplus S_2. \tag{12}$$

Next, we want to claim that X is indecomposable. Suppose that X is decomposable. By (12), we can write $X = X_1 \oplus X_2$ for A -submodules X_1 and X_2 of X with $\text{soc}(X_i) \cong S_i$ for $i = 1, 2$. If $X_1/\text{rad}(X_1) \not\cong S_1$, then (12) shows that $X_1/\text{rad}(X_1) \cong S_2$, and hence we get by (12) and (11) that

$$X = X_1 \oplus X_2 = \begin{bmatrix} S_2 \\ S_4 \\ S_1 \end{bmatrix} \oplus \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \text{ or } X = X_1 \oplus X_2 = \begin{bmatrix} S_2 \\ S_1 \end{bmatrix} \oplus \begin{bmatrix} S_1 \\ S_4 \\ S_2 \end{bmatrix},$$

which is a contradiction by the self-dualities of X and each S_i in 4.4(i). This means that $X_i/\text{rad}(X_i) \cong S_i$ for $i = 1, 2$ by (12). If X_1 is simple, then we get by (12) that X_2 has radical and socle series which is one of the following three cases:

$$\begin{bmatrix} S_2 \\ S_1 \\ S_4 \\ S_2 \end{bmatrix} \quad \begin{bmatrix} S_2 \\ S_1 \\ S_4 \\ S_2 \end{bmatrix} \quad \begin{bmatrix} S_2 \\ S_4 \\ S_1 \\ S_2 \end{bmatrix}$$

So we have a contradiction by 4.4(ii). Thus, X_1 is not simple. Similarly, we know that X_2 is not

simple. Hence, 2.16 yields that $\text{soc}(X_i) \subseteq \text{rad}(X_i)$ for $i = 1, 2$. Thus, $X = X_1 \oplus X_2 = \begin{bmatrix} S_1 \\ S_4 \\ S_1 \end{bmatrix} \oplus \begin{bmatrix} S_2 \\ S_2 \end{bmatrix}$ or

$$X = X_1 \oplus X_2 = \begin{bmatrix} S_1 \\ S_1 \end{bmatrix} \oplus \begin{bmatrix} S_2 \\ S_4 \\ S_2 \end{bmatrix}. \text{ This is a contradiction by (10), 2.3(i) and 4.1.}$$

Therefore X is indecomposable. Hence, we get by (11), (12) and 2.16 that $\text{soc}(X) \subseteq \text{rad}(X)$. Thus we get the structure of X as desired. \square

4.9. Notation. In the rest of paper let f be the Green correspondence from G to H with respect to P , see [45, Chapter 4, §4].

4.10. Lemma. It holds that $f(S_1) = 9a$.

Proof. It follows from 4.5(i), 4.3(ii) and 2.1 that $f(S_1)$ is a simple kH -module in B , see 3.4(i). Using the ordinary characters afforded by the trivial source kH -modules in B , see 3.8, we get the following possible decompositions of $S_1 \downarrow_H \cdot 1_B$, by a calculation with GAP [11] using the character tables of G and H :

$$S_1 \downarrow_H \cdot 1_B = 9a \oplus (7 \times P(9a) \oplus 7 \times P(9b) \oplus 5 \times P(18a) \oplus P(18b) \oplus P(18c))$$

or

$$S_1 \downarrow_H \cdot 1_B = 9b \oplus (8 \times P(9a) \oplus 6 \times P(9b) \oplus 5 \times P(18a) \oplus P(18b) \oplus P(18c)).$$

In particular, $f(S_1) = 9a$ or $f(S_1) = 9b$, and we have to decide which case actually occurs.

To this end, let $M = 2.HS.2$ be the second largest maximal subgroup of G , see 4.5. By [61, HS (mod 3)] and [18], let A^- be the block algebra of $\mathcal{O}M$ containing the unique non-trivial linear character χ of M . Hence letting A^+ and A' , see 5.1, be the principal block algebras of $\mathcal{O}M$ and of $\mathcal{O}HS$, respectively, we have $A^+ \cong A'$ and an isomorphism $-\otimes \chi : A^+ \rightarrow A^-$. Moreover, P being a Sylow 3-subgroup of M , it is the block defect group of A^- , and hence let B^- be the Brauer correspondent of A^- in $N_M(P)$.

Using the smallest faithful permutation representation of M on 1408 points, available in [62], the normaliser $N_M(P)$ and the centraliser $C_M(P)$ of the Sylow 3-subgroup P are easily computed explicitly with GAP [11] and their structure determined, we find $N_M(P) = (P \times D_8).SD_{16}$ and $C_M(P) = P \times D_8$. Now the conjugacy classes of $N_M(P)$ can be computed, its ordinary character table is found using the Dixon–Schneider algorithm, from which its blocks are determined and B^- is identified.

Then a computation with GAP [11], using the character tables of G and M , shows that $S_1 \downarrow_M \cdot 1_{A^-} = 22^-$, where the latter denotes the unique simple A^- -module of that dimension. Moreover, using the character tables of M and $N_M(P)$, GAP [11] shows that $(22^-) \downarrow_{N_M(P)} \cdot 1_{B^-} = \lambda$, where λ is a certain linear character; actually, λ is the Green correspondent of 22^- with respect to $(M, P, N_M(P))$, which must be linear in view of 5.7. Hence $\lambda = (S_1 \downarrow_M \cdot 1_{A^-}) \downarrow_{N_M(P)} \cdot 1_{B^-}$ is a direct summand of

$$\begin{aligned} (S_1 \downarrow_H) \downarrow_{N_M(P)} \cdot 1_{B^-} &= (f(S_1) \oplus (\mathcal{R}\text{-proj})) \downarrow_{N_M(P)} \cdot 1_{B^-} \\ &= f(S_1) \downarrow_{N_M(P)} \cdot 1_{B^-} \oplus (Q\text{-proj}) \oplus (\text{proj}), \end{aligned}$$

where \mathcal{R} consists of elementary abelian 3-subgroups of H , of order at most 9, not H -conjugate to P , and $Q \cong C_3$ is as in 3.3. Since λ has P as a vertex, we conclude that λ is a direct summand of $f(S_1) \downarrow_{N_M(P)} \cdot 1_{B^-}$.

Now a computation with GAP [11], using the character tables of H and $N_M(P)$, shows that $f(S_1) \downarrow_{N_M(P)} \cdot 1_{B^-} = (9x) \downarrow_{N_M(P)} \cdot 1_{B^-}$, where $x \in \{a, b\}$, already is linear, where

$$(9a) \downarrow_{N_M(P)} \cdot 1_{B^-} = \lambda \neq (9b) \downarrow_{N_M(P)} \cdot 1_{B^-}.$$

This shows that $f(S_1) = 9a$. \square

4.11. Lemma. *It holds that $f(S_2) = 9b$.*

Proof. It follows from 4.5(i), 4.3(ii) and 2.1 that $f(S_1)$ is a simple kH -module in B , see 3.4(i). Using the ordinary characters afforded by the trivial source kH -modules in B , see 3.8, we get the following possible decompositions of $S_1 \downarrow_H \cdot 1_B$, by a calculation with GAP [11] using the character tables of G and H :

$$S_2 \downarrow_H \cdot 1_B = 9b \oplus (9 \times P(9a) \oplus 8 \times P(9b) \oplus 7 \times P(18a) \oplus 5 \times P(18b) \oplus 5 \times P(18c))$$

or

$$S_2 \downarrow_H \cdot 1_B = 9a \oplus (8 \times P(9a) \oplus 9 \times P(9b) \oplus 7 \times P(18a) \oplus 5 \times P(18b) \oplus 5 \times P(18c)).$$

In particular, $f(S_2) = 9b$ or $f(S_2) = 9a$, hence the assertion follows from 4.10. \square

4.12. Lemma. *It holds that $f(S_3) = 9c$.*

Proof. It follows from 4.5(i), 4.3(ii) and 2.1 that $f(S_1)$ is a simple kH -module in B , see 3.4(i). Using the ordinary characters afforded by the trivial source kH -modules in B , see 3.8, we get the following possible decompositions of $S_1 \downarrow_H \cdot 1_B$, by a calculation with GAP [11] using the character tables of G and H :

$$S_3 \downarrow_H \cdot 1_B = 9c \oplus (54 \times P(9a) \oplus 54 \times P(9b) \oplus 40 \times P(9c) \oplus 41 \times P(9d) \\ \oplus 94 \times P(18a) \oplus 95 \times P(18b) \oplus 95 \times P(18c)) \oplus V_3$$

or

$$S_3 \downarrow_H \cdot 1_B = 9c \oplus (54 \times P(9a) \oplus 54 \times P(9b) \oplus 39 \times P(9c) \oplus 40 \times P(9d) \\ \oplus 93 \times P(18a) \oplus 96 \times P(18b) \oplus 96 \times P(18c)) \oplus V_2$$

or

$$S_3 \downarrow_H \cdot 1_B = 9d \oplus (54 \times P(9a) \oplus 54 \times P(9b) \oplus 41 \times P(9c) \oplus 40 \times P(9d) \\ \oplus 94 \times P(18a) \oplus 95 \times P(18b) \oplus 95 \times P(18c)) \oplus V_3$$

or

$$S_3 \downarrow_H \cdot 1_B = 9d \oplus (54 \times P(9a) \oplus 54 \times P(9b) \oplus 40 \times P(9c) \oplus 39 \times P(9d) \\ \oplus 93 \times P(18a) \oplus 96 \times P(18b) \oplus 96 \times P(18c)) \oplus V_2,$$

where V_3 and V_2 are the trivial source kH -modules in B with vertex Q given in 3.8. In particular, $f(S_3) = 9c$ or $f(S_3) = 9d$, and we have to decide which case actually occurs.

Keeping the notation from 4.10, we by the proof of 4.5(iii) have $S_3 = \chi \uparrow^G \cdot 1_A$, hence $(\chi \uparrow^G \cdot 1_A) \downarrow_H \cdot 1_B = S_3 \downarrow_H \cdot 1_B = f(S_3) \oplus (Q\text{-proj}) \oplus (\text{proj})$. Hence $f(S_3)$ is a direct summand of

$$(\chi \uparrow^G) \downarrow_H \cdot 1_B = \bigoplus_g ((\chi^g) \downarrow_{M^g \cap H}) \uparrow^H \cdot 1_B,$$

where g runs through a set of representatives of the M - H double cosets in G . Since $f(S_3)$ has P as a vertex, and P is normal in H , we only have to look at summands coming from $g \in G$ such that $P \leq M^g \cap H$. But for these g we have $P, P^{g^{-1}} \leq M$, which since P is a Sylow 3-subgroup of M implies the existence of $m \in M$ such that $P^m = P^{g^{-1}}$, hence $h := mg \in H$, and thus $g = m^{-1}h \in MH$, that is, we may assume $g = 1$.

Thus we conclude that $f(S_3)$ is a direct summand of $(\chi \downarrow_{M \cap H}) \uparrow^H \cdot 1_B = (\chi \downarrow_{N_M(P)}) \uparrow^H \cdot 1_B$. Now a computation with GAP [11], using the character tables of $N_M(P)$ and H , shows that $(\chi \downarrow_{N_M(P)}) \uparrow^H \cdot 1_B = 9c$ is indecomposable, showing that $f(S_3) = 9c$. \square

4.13. Remark. We use the notation as in the proof of 4.12. We just remark that it is possible, using GAP [11] and specially tailored programs to deal efficiently with permutations on millions of points,

to construct the transitive permutation representation of G on 3078000 points, that is, the action of G on the cosets of $2.HS$ in G , where $2.HS$ is the derived subgroup of M , and to use the restriction of this representation to H to show that the first of the four possible decompositions of $S_3 \downarrow_H \cdot 1_B$ listed above actually occurs. But we will not need this fact.

5. Green correspondence for HS

5.1. Notation and assumption. In the rest of this paper, we use the following notation, too. Let G' be the Higman–Sims simple group HS. Since Sylow 3-subgroups of G' are isomorphic to $C_3 \times C_3$, we by abuse of notation let P denote a Sylow 3-subgroup of HS as well. There is exactly one conjugacy class of G' which contains elements of order 3, that is, P has exactly one G' -conjugacy class of subgroups of order 3, see [9, p. 81]. Let $H' = N_{G'}(P)$, and hence $H' = (P \rtimes SD_{16}) \times 2$, where the action of SD_{16} on P is given by the embedding of SD_{16} as a Sylow 2-subgroup of $\text{Aut}(P) \cong \text{GL}_2(3)$. Let A' and B' , respectively, be the principal block algebras of $\mathcal{O}G'$ and $\mathcal{O}H'$.

5.2. Lemma.

(i) *The character table of $P \rtimes SD_{16}$ is given as follows:*

Conjugacy class	1A	2A	2B	3A	4A	4B	6A	8A	8B
Centraliser	144	16	12	18	8	4	6	8	8
χ_{1a}	1	1	1	1	1	1	1	1	1
χ_{1b}	1	1	1	1	1	-1	1	-1	-1
χ_{1c}	1	1	-1	1	1	1	-1	-1	-1
χ_{1d}	1	1	-1	1	1	-1	-1	1	1
χ_{2a}	2	2	0	2	-2	0	0	0	0
χ_{2b}	2	-2	0	2	0	0	0	$\sqrt{-2}$	$-\sqrt{-2}$
χ_{2c}	2	-2	0	2	0	0	0	$-\sqrt{-2}$	$\sqrt{-2}$
χ_{8a}	8	0	2	-1	0	0	-1	0	0
χ_{8b}	8	0	-2	-1	0	0	1	0	0

Note that this identifies the characters $\chi_{1a}, \chi_{1b}, \chi_{1c}, \chi_{1d}, \chi_{8a}$, and χ_{8b} uniquely.

(ii) $B' \cong \mathcal{O}[P \rtimes SD_{16}]$, as interior P -algebras and hence k -algebras, and we can write that

$$\text{Irr}(B') = \{1_{H'} = \chi_{1a}, \chi_{1b}, \chi_{1c}, \chi_{1d}, \chi_{2a}, \chi_{2a}^\vee, \chi_{2c}, \chi_{2c}^\vee, \chi_{8a}, \chi_{8b}\},$$

$$\text{IBr}(B') = \{1a, 1b, 1c, 1d, 2a, 2b, 2c = 2b^\vee\},$$

where the numbers mean the degrees (dimensions) of characters (modules). In particular, all simple modules $1a, 1b, 1c, 1d, 2a$ in B' except $2b$ and $2c$ are self-dual.

(iii) *The 3-decomposition and the Cartan matrices of B' , respectively, are the following:*

	1a	1b	1c	1d	2a	2b	2c
χ_{1a}	1
χ_{1b}	.	1
χ_{1c}	.	.	1
χ_{1d}	.	.	.	1	.	.	.
χ_{2a}	1	.	.
χ_{2b}	1	.
χ_{2c}	1
χ_{8a}	1	1	.	.	1	1	1
χ_{8b}	.	.	1	1	1	1	1

	$P(1a)$	$P(1b)$	$P(1c)$	$P(1d)$	$P(2a)$	$P(2b)$	$P(2c)$
1a	2	1	0	0	1	1	1
1b	1	2	0	0	1	1	1
1c	0	0	2	1	1	1	1
1d	0	0	1	2	1	1	1
2a	1	1	1	1	3	2	2
2b	1	1	1	1	2	3	2
2c	1	1	1	1	2	2	3

Proof. (i) This is found using explicit computation with GAP [11]. Using the smallest faithful permutation representation of G' on 100 points, available in [62], P can be computed as a Sylow 3-subgroup of G' , and hence the normaliser $H' = N_{G'}(P)$ of P is easily determined explicitly. Now the conjugacy classes of H' can be computed, and its ordinary character table is found using the Dixon–Schneider algorithm. Note that there are unique conjugacy classes $2B$ and $4B$ consisting of elements of order 2 and 4, respectively, and having centralisers of order 12 and 4, respectively.

(ii)–(iii) Easy from the character table. \square

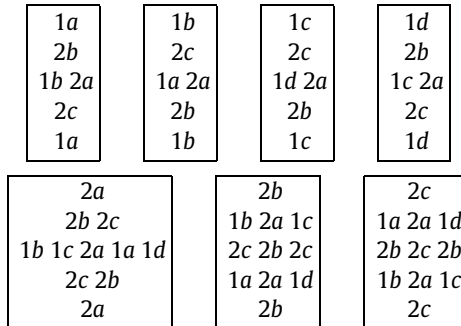
5.3. Notation. We use the notation $1_{H'} = \chi_{1a}, \chi_{1b}, \chi_{1c}, \chi_{1d}, \chi_{2a}, \chi_{2a}, \chi_{2c} = \chi_{2b}^\vee, \chi_{8a}, \chi_{8b}$ and $1a, 1b, 1c, 1d, 2a, 2b, 2c = 2b^\vee$, as in 5.2. Namely, we can write

$$\begin{aligned} \text{Irr}(B') &= \text{Irr}(H') = \{1_{H'} = \chi_{1a}, \chi_{1b}, \chi_{1c}, \chi_{1d}, \chi_{2a}, \chi_{2a}, \chi_{2c} = \chi_{2b}^\vee, \chi_{8a}, \chi_{8b}\}, \\ \text{IBr}(B') &= \text{IBr}(H') = \{1a, 1b, 1c, 1d, 2a, 2b, 2c = 2b^\vee\}. \end{aligned}$$

Let f' and g' be the Green correspondences with respect to (G', P, H') .

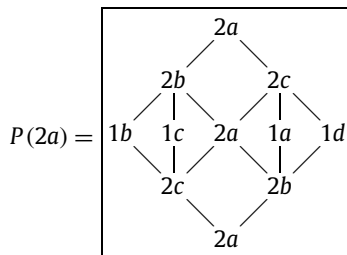
5.4. Lemma.

(i) *The radical and socle series of PIMs in B' are the following:*



Note that this identifies the simples 2b and 2c uniquely.

(ii) *An Alperin diagram of the PIM $P(2a)$ is given as follows:*



Proof. Using the faithful permutation representation of H' obtained in 5.2, we have used the MeatAxe [55] to construct the PIMs explicitly as matrix representations. Then we have used the method described in [42] to find the radical and socle series, and the method in [39] to compute the whole submodule lattice of $P(2a)$. \square

5.5. Lemma.

(i) We can write that

$$\text{Irr}(A') = \{\chi'_1 = 1_{G'}, \chi'_{154}, \chi'_{22}, \chi'_{1408}, \chi'_{1925}, \chi'_{770}, \chi'_{3200}, \chi'_{2750}, \chi'_{1750}\},$$

$$\text{IBr}(A') = \{1_{G'}, 154, 22, 1253, 1176, 748, 321\}.$$

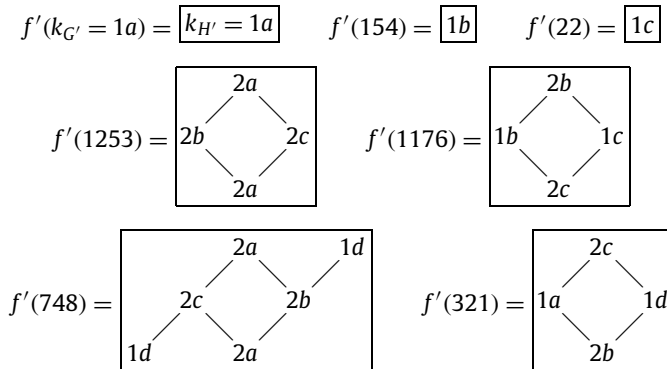
(ii) All simples $1_{G'}, 154, 22, 1253, 1176, 748, 321$ in A' are self-dual, and have P as their vertices.
 (iii) The simples $1_{G'}, 154, 22$ are trivial source kG' -modules.

Proof. (i) This was first calculated by Humphreys [17, p. 329]; see also [61, HS (mod 3)] and [18].
 (ii) This is obtained by a result of Knörr [20, 3.7 Corollary].

(iii) It follows from [58] or [48, Example 4.8] that the Green correspondents $f'(k_{G'})$, $f'(22)$ and $f'(154)$ are $k_{H'} = 1a, 1b$ and $1c$, respectively, see 5.7 below. \square

5.6. Notation. We write $\chi'_1 = 1_{G'}, \chi'_{154}, \chi'_{22}, \chi'_{1408}, \chi'_{1925}, \chi'_{770}, \chi'_{3200}, \chi'_{2750}, \chi'_{1750}$, as well as $1_{G'}, 154, 22, 1253, 1176, 748, 321$ as in 5.5.

5.7. Lemma.



Proof. This follows from [58], see [48, Example 4.8, HS]. \square

5.8. Lemma. The Cartan matrix of A' is the following:

	$P(k_{G'})$	$P(154)$	$P(22)$	$P(1253)$	$P(1176)$	$P(748)$	$P(321)$
$k_{G'}$	4	1	1	2	2	2	0
154	1	3	1	2	0	0	1
22	1	1	4	2	1	2	1
1253	2	2	2	4	2	1	2
1176	2	0	1	2	3	2	1
748	2	0	2	1	2	3	0
321	0	1	1	2	1	0	2

Proof. This was first calculated by Humphreys [17, p. 329]; see also [61, HS (mod 3)] and [18]. \square

6. Stable equivalence between A and B for HN

6.1. Notation. First of all, recall the notation $G, A, P, H, B, e, Q, E, f$ as in 3.3 and 4.9. Let i and j respectively be source idempotents of A and B with respect to P . As remarked in [37, pp. 821–822], we can take i and j such that $\text{Br}_P(i) \cdot e = \text{Br}_P(i) \neq 0$ and that $\text{Br}_P(j) \cdot e = \text{Br}_P(j) \neq 0$. Set $G_P = C_G(P) = C_H(P) = H_P$, and set $G_Q = C_G(Q)$ and $H_Q = C_H(Q)$. By replacing e_Q and f_Q (if necessary), we may assume that e_Q and f_Q respectively are block idempotents of kG_Q and kH_Q such that e_Q and f_Q are determined by i and j , respectively. Namely, $\text{Br}_Q(i) \cdot e_Q = \text{Br}_Q(i)$ and $\text{Br}_Q(j) \cdot f_Q = \text{Br}_Q(j)$. Let $A_Q = kC_G(Q) \cdot e_Q$ and $B_Q = kC_H(Q) \cdot f_Q$, so that $e_Q = 1_{A_Q}$ and $f_Q = 1_{B_Q}$.

6.2. Lemma. Let \mathfrak{M}_Q be a unique (up to isomorphism) indecomposable direct summand of $A_Q \downarrow_{G_Q \times H_Q}^{G_Q \times G_Q} \cdot 1_{B_Q}$ with vertex ΔP (note that such an \mathfrak{M}_Q always exists by 2.5). Then, a pair $(\mathfrak{M}_Q, \mathfrak{M}_Q^\vee)$ induces a Puig equivalence between A_Q and B_Q .

Proof. This follows from 3.4(viii), 2.12(iii) and 2.11(iii). \square

6.3. Lemma.

- (i) The (A, B) -bimodule $1_A \cdot kG \cdot 1_B$ has a unique (up to isomorphism) indecomposable direct summand ${}_A\mathfrak{M}_B$ with vertex ΔP . Moreover, a functor $F : \text{mod-}A \rightarrow \text{mod-}B$ defined by $X_A \mapsto (X \otimes_A \mathfrak{M})_B$ induces a splendid stable equivalence of Morita type between A and B . We use the notation F below as well.
- (ii) If X is a non-projective trivial source kG -module in A , then $F(X) = Y \oplus (\text{proj})$ for a non-projective indecomposable kH -module Y in B such that Y is also a trivial source module, and X and Y have a common vertex.
- (iii) If X is a non-projective kG -module in A , then $F(\Omega X) = \Omega(F(X)) \oplus (\text{proj})$.

Proof. This follows just like in [29, Proof of Lemma 6.3]. Namely, we get the assertion by [2, Proposition 4.21] and [8, Theorem 1.8(i)] for the morphisms in the Brauer categories and also by [30, Theorem], 6.2, 3.4(vi) and [37, Theorem 3.1], see [38, Theorem A.1]. \square

6.4. Notation. We use the notation \mathfrak{M} and F as in 6.3.

7. Images of simples via the functor F

7.1. Lemma. $F(S_1) = 9a, F(S_2) = 9b, F(S_3) = 9c$.

Proof. These follow from 4.10, 4.11, 4.12, 2.9 and 6.3. \square

7.2. Lemma.

- (i) The trivial source kG -module in 4.8 has $Q \cong C_3$ as its vertex.
- (ii) The trivial source kG -module in 4.7 has $Q \cong C_3$ as its vertex.

Proof. (i) Let X be the trivial source kG -module in 4.8. We get by 6.3(ii) that $F(X) = Y \oplus (\text{proj})$ for a non-projective indecomposable B -module Y . Then, it follows from 2.7, 6.3(i) and 7.1 that

$$\begin{aligned} 0 \neq \text{Hom}_A(X, S_1) &\cong \underline{\text{Hom}}_A(X, S_1) \cong \underline{\text{Hom}}_B(F(X), F(S_1)) \\ &= \underline{\text{Hom}}_B(F(X), 9a) = \underline{\text{Hom}}_B(Y, 9a) \cong \text{Hom}_B(Y, 9a) \end{aligned}$$

as k -spaces. Clearly, Y is a trivial source kH -module in B by 6.3(ii).

Suppose that X has P as a vertex. Then, so does Y by 6.3(ii). This yields that $Y \in \{9a, 9b, 9c, 9d, 18a, 18b, 18c\}$ from 3.8, and hence $Y \in \{9d, 18a, 18b, 18c\}$ by 7.1. But, the above computation shows that $\text{Hom}_B(Y, 9a) \neq 0$, a contradiction.

Since X is non-projective, we know that Q is a vertex of X from 3.4(vi).

(ii) Let X' be the trivial source kG -module in 4.7. We get by 6.3(ii) that $F(X') = Y' \oplus (\text{proj})$ for a non-projective indecomposable B -module Y' . Then, it follows from 2.7, 6.3(i) and 7.1 that

$$0 \neq \text{Hom}_A(X', S_3) \cong \underline{\text{Hom}}_A(X', S_3) \cong \underline{\text{Hom}}_B(F(X'), F(S_3)) \\ = \underline{\text{Hom}}_B(F(X'), 9c) = \underline{\text{Hom}}_B(Y', 9c) \cong \text{Hom}_B(Y', 9c)$$

as k -spaces. Clearly, Y' is a trivial source kH -module in B by 6.3(ii).

Suppose that X' has P as a vertex. Then, so does Y' by 6.3(ii). This yields that $Y' \in \{9a, 9b, 9c, 9d, 18a, 18b, 18c\}$ from 3.8, and hence $Y' \in \{9d, 18a, 18b, 18c\}$ by 7.1. But, the above computation shows that $\text{Hom}_B(Y', 9c) \neq 0$, a contradiction.

Since X' is non-projective, we know that Q is a vertex of X' from 3.4(vi). \square

7.3. Lemma. Let X be the trivial source kG -module with vertex Q showing up in 4.8 and 7.2(i). Then, $F(X) = V_1 \oplus (\text{proj})$, where V_1 is the trivial source kH -module in B with vertex Q given in 3.8(iii). Namely,

$$F \left(\begin{array}{ccc} S_1 & & S_2 \\ & \searrow & / \\ & S_4 & \\ & / & \searrow \\ S_1 & & S_2 \end{array} \right) = \begin{array}{ccc} 9a & & 18a & & 9b \\ & \searrow & / & & \searrow & / \\ & & 18b & & 18c & \\ & / & \searrow & & / & \searrow \\ 9b & & 18a & & 9a \end{array} \oplus (\text{proj}).$$

Proof. We know from the proof of 7.2(i) that $[Y, 9a]^B \neq 0$. Hence we get the assertion by 3.8(iii). \square

7.4. Lemma. Let X' be the trivial source kG -module with vertex Q showing up in 4.7 and 7.2(ii). Then, $F(X') = V_2 \oplus (\text{proj})$, where V_2 is the trivial source kH -module in B with vertex Q given in 3.8(iii). Namely,

$$F \left(\begin{array}{c} S_3 \\ S_6 \\ S_3 \end{array} \right) = \begin{array}{ccc} 9c & & 18a & & 9d \\ & \searrow & / & & \searrow & / \\ & & 18c & & 18b & \\ & / & \searrow & & / & \searrow \\ 9d & & 18a & & 9c \end{array} \oplus (\text{proj}).$$

Proof. We know from the proof of 7.2(ii) that $[Y', 9c]^B \neq 0$. Hence we get the assertion by 3.8(iii). \square

7.5. Lemma. It holds that $F(S_4) =$

$$\begin{array}{ccc} & 18a & \\ & / & \searrow \\ 18b & & 18c \\ & \searrow & / \\ & 18a & \end{array}$$

Proof. Let X be the trivial source kG -module in A with vertex Q given in 4.8 and 7.2(i). By 7.3, we can write $F(X) = V_1 \oplus (\text{proj})$, where V_1 is the trivial source kH -module in B given in 3.8(iii). Then, since $F(S_1) = 9a$ by 7.1, it follows from 2.8 that

$$F \left(\begin{array}{ccc} S_1 & & S_2 \\ & \searrow & / \\ & S_4 & \\ & | & \\ & S_2 & \end{array} \right) = F(X/S_1) = (V_1/9a) \oplus (\text{proj}) = \begin{array}{ccc} 9a & & 18a & & 9b \\ & \searrow & / & & \searrow & / \\ & & 18b & & 18c & \\ & / & \searrow & & / & \searrow \\ 9b & & 18a & & 9a \end{array} \oplus (\text{proj}).$$

Similarly, we get by 2.8 that

$$F \left(\begin{array}{|c|} \hline S_2 \\ \hline S_4 \\ \hline S_2 \\ \hline \end{array} \right) = F \left(\text{Ker} \left[\begin{array}{ccc} S_1 & & S_2 \\ & \swarrow & \searrow \\ & S_4 & \\ & | & \\ & S_2 & \end{array} \right] \rightarrow S_1 \right)$$

$$\cong \text{Ker} \left(\begin{array}{ccc} 9a & 18a & 9b \\ & \swarrow & \searrow \\ & 18b & 18c \\ & \swarrow & \searrow \\ 9b & 18a & \end{array} \rightarrow 9a \right) \oplus (\text{proj}) = \begin{array}{ccc} & 18a & 9b \\ & \swarrow & \searrow \\ & 18b & 18c \\ & \swarrow & \searrow \\ 9b & 18a & \end{array} \oplus (\text{proj}).$$

Then, since $F(S_2) = 9b$ by 7.1, we similarly obtain by 2.8 that

$$F(S_4) = \begin{array}{ccc} & 18a & \\ & \swarrow & \searrow \\ 18b & & 18c \\ & \swarrow & \searrow \\ & 18a & \end{array} \oplus (\text{proj}).$$

Therefore, 6.3(i) and 2.9 imply the assertion. \square

7.6. Lemma. It holds that $F(S_6) = \begin{array}{ccc} & 18a & 9d \\ & \swarrow & \searrow \\ & 18c & 18b \\ & \swarrow & \searrow \\ 9d & 18a & \end{array}.$

Proof. Let $X' = \begin{array}{|c|} \hline S_3 \\ \hline S_6 \\ \hline S_3 \\ \hline \end{array}$ in 4.7, that is, X' is a trivial source kG -module in A with vertex Q . Then, 7.4 yields that

$$F(X') = \begin{array}{ccc} 9c & 18a & 9d \\ & \swarrow & \searrow \\ & 18c & 18b \\ & \swarrow & \searrow \\ 9d & 18a & 9c \end{array} \oplus (\text{proj}).$$

Since $F(S_3) = 9c$ by 7.1, we obtain the assertion from 6.3(i) and 2.9 just as in the proof of 7.5. \square

7.7. Notation. We use the notation $W = F(S_5) \oplus F(S_7)$ in the rest of this paper.

7.8. Lemma. We get the following.

- (i) The module W is self-dual.
- (ii) The module W is a direct sum of exactly two non-projective non-simple indecomposable B -modules, and both of them are self-dual.
- (iii) It holds that $F(S_5)$ and $F(S_7)$ are neither simple B -modules, and $2 \leq j(W) \leq 4$.
- (iv) $[9x, W]^B = [W, 9x]^B = 0$ for any $x \in \{a, b, c\}$.
- (v) $[18a, W]^B = [W, 18a]^B = 0$.

Proof. (i) This follows from 4.3(i) and 6.3(i).

(ii) This follows from 2.9, 6.3(i), 4.3(i), 2.3(i) and 4.1.

(iii) By (ii) and 3.6(vi), we get $j(W) \leq 4$. Assume that $F(S_5)$ is simple. Then, we know by 3.8(ii) and 6.3(ii) that S_5 is a trivial source module, and hence S_5 lifts to a trivial source $\mathcal{O}G$ -module by 2.3(i). This contradicts the 3-decomposition matrix in 4.1. Hence, $F(S_5)$ is not simple. Similarly, we know that $F(S_7)$ is not simple. These imply $j(W) \geq 2$.

(iv) This is obtained by 7.1 and 2.13.

(v) Set $X = F(S_4)$. By 7.5, there is an epimorphism $X \twoheadrightarrow 18a$. Hence, we get from 2.7 and 7.5 that

$$\begin{aligned} \text{Hom}_B(18a, W) &\cong \underline{\text{Hom}}_B(18a, W) \cong \underline{\text{Hom}}_A(F^{-1}(18a), F^{-1}(W)) \\ &= \underline{\text{Hom}}_A(F^{-1}(18a), S_5 \oplus S_7) \cong \text{Hom}_A(F^{-1}(18a), S_5 \oplus S_7) \\ &\subseteq \text{Hom}_A(F^{-1}(X), S_5 \oplus S_7) = \text{Hom}_A(S_4, S_5 \oplus S_7) = 0. \quad \square \end{aligned}$$

7.9. Notation. Let $M = \begin{array}{c} \boxed{\begin{array}{ccccc} & & S_4 & & \\ & \swarrow & & \searrow & \\ S_1 & & S_2 & & S_5 & & S_7 \\ & \swarrow & & \searrow & \\ & & S_4 & & \end{array}} \end{array}$ be the trivial source kG -module in A showing up in 4.6,

and set $\mathfrak{X}_B = F(M)$ and we use the notation \mathfrak{X} in the rest of this paper.

7.10. Lemma.

(i) The module \mathfrak{X} has a filtration

$$\mathfrak{X} = \begin{array}{c} \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \Big| 9a \oplus 9b \oplus W \Big| \begin{array}{c} \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \end{array}$$

namely, \mathfrak{X} has submodules $\mathfrak{X} \supseteq Y \supseteq Z$ such that $\mathfrak{X}/Y \cong Z \cong \begin{array}{c} \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \end{array}$ and $Y/Z \cong 9a \oplus 9b \oplus W$.

(ii) It holds $\mathfrak{X} = V \oplus P(18a)$ where $V \in \{V_3, V_4\}$.

Proof. (i) This follows from 4.6, 7.1 and 7.5.

(ii) We know by 6.3(ii) that $\mathfrak{X} = V \oplus L$ for an indecomposable kH -module V in B with vertex Q and a projective kH -module L in B . Note that $V_i \nmid \mathfrak{X}$ for $i = 1, 2$ by 7.3 and 7.4. Thus, $V \in \{V_3, V_4\}$ by 3.8(iii). Moreover, since $[V_3, 18a]^B = [V_4, 18a]^B = 0$ by 3.8(iii), we know that $[V, 18a]^B = [V, 18a]^B = 0$ again by 3.8(iii). Thus, we have $P(18a) \mid L$ by (i), and hence $P(18a) \mid \mathfrak{X}$.

Next, assume that $P(T) \mid L$ for a simple kH -module T in B with $T \neq 18a$. Since Z has a unique minimal submodule, and which is isomorphic to $18a$, we have that $P(T) \cap Z = 0$ in \mathfrak{X} , and hence that there is a direct sum $P(T) \oplus Z$ in \mathfrak{X} . Set $\tilde{\mathfrak{X}} = \mathfrak{X}/Z$. Clearly, $\tilde{\mathfrak{X}} \supseteq (P(T) \oplus Z)/Z \cong P(T)$. Since $P(T)$ is

injective, it holds $P(T) \mid \tilde{\mathfrak{X}}$. Set $U = (\tilde{\mathfrak{X}})^\vee$. Then, by the dualities, we know $P(T^\vee) \mid U$. Now, by the filtration of \mathfrak{X} , U has a filtration

$$U = 9a \oplus 9b \oplus W \quad \left| \begin{array}{c} \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \end{array} \right.$$

Namely, U has a submodule Z' such that

$$Z' \cong \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \quad \text{and} \quad U/Z' \cong 9a \oplus 9b \oplus W.$$

We have $T^\vee \not\cong 18a$ by 3.6(iii). Hence, we get $P(T^\vee) \cap Z' = 0$ in U , and hence there is a direct sum $P(T^\vee) \oplus Z' \subseteq U$. Then, we have

$$P(T^\vee) \cong (P(T^\vee) \oplus Z')/Z' \subseteq U/Z' \cong 9a \oplus 9b \oplus W.$$

Since $P(T^\vee)$ is injective, it holds that $P(T^\vee) \mid (9a \oplus 9b \oplus W)$, so that $P(T^\vee) \mid W$ by 3.6(vi). This is a contradiction by 7.8(ii).

Now, assume that $(P(18a) \oplus P(18a)) \mid \mathfrak{X}$. Then, since $\text{soc}(Z) \cong 18a$, it follows from 2.14 that

$$P(18a) \left| \mathfrak{X}/Z = \left(\begin{array}{c} \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \left| \begin{array}{c} 9a \oplus 9b \oplus W \\ Z \end{array} \right. \right).$$

Then, by taking its dual, we get also that

$$P(18a) \left| (\mathfrak{X}/Z)^\vee = \left(\begin{array}{c} 9a \oplus 9b \oplus W \\ Z \end{array} \right| \right)$$

where the right-hand side is a filtration, by using 7.8(i) and 3.6(iii). Set $N = (\mathfrak{X}/Z)^\vee$. Then, we may consider that N has a B -submodule Z such that $N/Z \cong 9a \oplus 9b \oplus W$ and $N = P(18a) \oplus N'$ for a B -submodule N' of N . Since $j(Z) = 3$, it holds $Z \subseteq \text{soc}_3(N) = \text{soc}_3(P(18a)) \oplus \text{soc}_3(N')$. This implies that there exists a B -epimorphism $\pi : N/Z \rightarrow N/\text{soc}_3(N)$. Clearly,

$$\begin{aligned} N/\text{soc}_3(N) &= [P(18a) \oplus N'] / [\text{soc}_3(P(18a)) \oplus \text{soc}_3(N')] \\ &\cong [P(18a)/\text{soc}_3(P(18a))] \oplus [N'/\text{soc}_3(N')]. \end{aligned}$$

Since $P(18a)/\text{soc}_3(P(18a)) = \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \end{array}}$ by 3.6(vi), we get that $18a \mid [(N/Z)/\text{rad}(N/Z)] \cong 9a \oplus 9b \oplus [W/\text{rad}(W)]$. This shows that $[W, 18a]^B \neq 0$, which is a contradiction by 7.8(v).

Thus, we know $[P(18a) \mid L]^B = 1$. Therefore, we get $L \cong P(18a)$. We are done. \square

7.11. Lemma. $W/\text{rad}(W) \cong \text{soc}(W) \cong 18b \oplus 18c$.

Proof. By 7.8(i) and 3.6(iii), it suffices to show only $W/\text{rad}(W) \cong 18b \oplus 18c$. By 7.10(ii), we have

$$\mathfrak{X} = V \oplus P(18a) = \begin{array}{|c|} \hline \begin{array}{ccc} & 18b & 18c \\ 9x & \swarrow & \searrow \\ & 18a & \\ & \swarrow & \searrow \\ & 18c & 18b \end{array} \\ \hline \end{array} \oplus P(18a), \quad \text{where } (x, y) \in \{(b, a), (c, d)\}. \quad (13)$$

By 7.10(i), \mathfrak{X} has a filtration

$$\mathfrak{X} = \begin{array}{|c|} \hline \begin{array}{ccc} & 18a & \\ 18b & \swarrow & \searrow \\ & 18c & \\ & \swarrow & \searrow \\ & 18a & \end{array} \\ \hline \end{array} \Big| 9a \oplus 9b \oplus W \Big| \begin{array}{|c|} \hline \begin{array}{ccc} & 18a & \\ 18b & \swarrow & \searrow \\ & 18c & \\ & \swarrow & \searrow \\ & 18a & \end{array} \\ \hline \end{array} \quad (14)$$

Set $L_i(W) = \text{rad}^i(W)/\text{rad}^{i+1}(W)$ for each $i = 0, 1, \dots$. Then (13) and (14) show that $(18b \oplus 18c) \mid L_1(W)$. Recall $(18b)^\vee \cong 18c$ by 3.6(iii).

Suppose that $(18b \oplus 18b) \mid L_1(W)$. Then, by (14), \mathfrak{X} has a factor module $\tilde{\mathfrak{X}}$ which has a filtration

$$\begin{array}{|c|} \hline \begin{array}{ccc} & 18a & \\ 18b & \swarrow & \searrow \\ & 18c & \\ & \swarrow & \searrow \\ & 18a & \end{array} \\ \hline \end{array} \Big| 18b \oplus 18b$$

Since $[\mathfrak{X}, 18b]^B = 1$ by (13), and since there do not exist modules of forms $\begin{array}{|c|} \hline 18b \\ \hline 18b \end{array}$ nor $\begin{array}{|c|} \hline 18c \\ \hline 18b \end{array}$ by 3.6(vi), there must be a kH -module having radical and socle series

$$\begin{array}{|c|} \hline 18a \\ 18b \ 18c \\ 18a \\ 18b \\ \hline \end{array}$$

But this is a contradiction by 3.10(i).

Similarly, we get a contradiction by using 3.10(ii) if $(18c \oplus 18c) \mid L_1(W)$.

Thus it holds that $[W, 18b]^B = [W, 18c]^B = 1$ and $[W, T]^B = 0$ for any $T \in \{9a, 9b, 9c, 18a\}$ by 3.6(iii) and 7.8(iv)–(v). However, we have to investigate for $9d$.

Assume, first, that the case $(x, y) = (b, a)$ happens in (13). Then, (14) and (13) imply that $W = 9a + 9b + 9c + 9d + 2 \times 18b + 2 \times 18c$, as composition factors.

Suppose that $9d \mid L_1(W)$. Then, since $c_W(9d) = 1$ and since W and $9d$ are both self-dual by 3.6(iii) and 7.8(i), we get that $9d \mid W = F(S_5) \oplus F(S_7)$. Recall that $F(S_5)$ and $F(S_7)$ are both non-projective indecomposable kH -modules by 2.9 and 6.3(i)–(ii). Since $9d$ is a trivial source kH -module by 3.8(ii),

we know by 6.3(ii) that S_5 or S_7 is a trivial source module, and hence that S_5 or S_7 lifts from k to \mathcal{O} by 2.3(i). This is a contradiction by the 3-decomposition matrix in 4.1.

Hence, $9d \nmid L_1(W)$. This yields $L_1(W) \cong 18b \oplus 18c$.

Next, assume that the case $(x, y) = (c, d)$ in (13) happens. Then, (13) and (14) imply that

$$W = 2 \times 9c + 2 \times 9d + 2 \times 18b + 2 \times 18c, \quad \text{as composition factors.} \tag{15}$$

Suppose that $(9d \oplus 9d) \mid L_1(W)$. Then, the self-dualities of $9d$ and W in 3.6(iii) and 7.8(i) imply that $(9d \oplus 9d) \mid W = F(S_5) \oplus F(S_7)$. Hence, $W \cong 9d \oplus 9d$ by 7.8(ii), contradicting 7.7 and 7.8.

Thus,

$$[W, 9d]^B \leq 1. \tag{16}$$

Assume, next, that $[W, 9d]^B = 1$. Hence, by the dualities in 3.6(iii), we have

$$L_1(W) \cong \text{soc}(W) \cong 18b \oplus 18c \oplus 9d. \tag{17}$$

We get by 7.8 that $W = W_1 \oplus W_2$ where W_i is a non-simple non-projective indecomposable self-dual B -module for $i = 1, 2$. Thus, by (17) and by interchanging W_1 and W_2 , we may assume that $L_1(W_1) \cong 18b, 18c$ or $9d$.

Case 1: $L_1(W_1) \cong 18b$. Then, $\text{soc}(W_1) \cong 18c$ since $(18b)^\vee \cong 18c$ by 3.6(iii) and since W_1 is self-dual. Hence, the structure of $P(18b)$ in 3.6(vi) yields that $W_1 = \begin{matrix} 18b \\ 9c \\ 18c \end{matrix}$. Hence, (15) and (17) imply that $L_1(W_2) \cong 18c \oplus 9d$ and $L_2(W_2) \cong 9c$. But this is a contradiction since $\text{Ext}_B^1(18c, 9c) = 0 = \text{Ext}_B^1(9d, 9c)$ by 3.6(vi).

Case 2: $L_1(W_1) \cong 18c$. As in Case 1, we know that $W_1 = \begin{matrix} 18c \\ 9c \\ 18b \end{matrix}$. Then we get a contradiction by 3.6(vi) as in Case 1.

Case 3: $L_1(W_1) \cong 9d$. By the self-dualities of W_1 in 7.8(ii) and simple B -modules in 3.6(iii), we get that $\text{soc}(W_1) \cong 9d$. It follows by 2.16 that $\text{soc}(W_1) \subseteq \text{rad}(W_1)$. Hence $c_{W_1}(9d) = 2$ by (15). Thus, the structure of $P(9d)$ in 3.6(vi) yields that $W_1 \cong P(9d)$, a contradiction.

Therefore $[W, 9d]^B \neq 1$, and hence $[W, 9d]^B = 0$ by (16). So that we have $L_1(W) \cong 18b \oplus 18c$. \square

7.12. Lemma. $\mathfrak{X} = V_3 \oplus P(18a)$.

Proof. Suppose that $\mathfrak{X} = V_4 \oplus P(18a)$. Then, we get by 7.10(i)–(ii) and 3.6(iv) that $W = 2 \times 9c + 2 \times 9d + 2 \times 18b + 2 \times 18c$, as composition factors. We use the same notation $L_i(W)$ as in the proof of 7.11. By 7.11, $L_1(W) \cong 18b \oplus 18c$. Since $c_W(9c) = 2$, it follows from 3.6(vi) and 7.8(iii) that $j(W) = 4$ and $9c \mid L_4(W)$. This means $9c \mid \text{soc}(W)$, contradicting 7.11. Therefore, we get the assertion by 7.10(ii). \square

7.13. Lemma. $W = \begin{matrix} & 18b & \\ 9b & / \quad \backslash & 9c \\ & 18c & \end{matrix} \oplus \begin{matrix} & 18c & \\ 9a & / \quad \backslash & 9d \\ & 18b & \end{matrix} = F(S_5) \oplus F(S_7).$

Namely, either one of the following two cases occurs:

$$\begin{array}{l}
 \text{Case (a): } F(S_5) = \begin{array}{|c|} \hline \begin{array}{ccc} & 18b & \\ 9b & & 9c \\ & 18c & \end{array} \\ \hline \end{array} \quad \text{and} \quad F(S_7) = \begin{array}{|c|} \hline \begin{array}{ccc} & 18c & \\ 9a & & 9d \\ & 18b & \end{array} \\ \hline \end{array} \\
 \\
 \text{Case (b): } F(S_5) = \begin{array}{|c|} \hline \begin{array}{ccc} & 18c & \\ 9a & & 9d \\ & 18b & \end{array} \\ \hline \end{array} \quad \text{and} \quad F(S_7) = \begin{array}{|c|} \hline \begin{array}{ccc} & 18b & \\ 9b & & 9c \\ & 18c & \end{array} \\ \hline \end{array}
 \end{array}$$

Proof. Here as well we use the notation $L_i(W)$ for $i = 1, 2, \dots$ just as in the proof of 7.11. It follows from 7.10(i) that \mathfrak{X} has a filtration

$$\mathfrak{X} = \begin{array}{|c|} \hline \begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array} \\ \hline \end{array} \Big| 9a \oplus 9b \oplus W \Big| \begin{array}{|c|} \hline \begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array} \\ \hline \end{array} \tag{18}$$

namely, \mathfrak{X} has submodules Y and Z such that $\mathfrak{X} \supsetneq Y \supsetneq Z$, $\mathfrak{X}/Y \cong Z \cong \begin{array}{|c|} \hline \begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array} \\ \hline \end{array}$ and $Y/Z \cong$

$9a \oplus 9b \oplus W$. On the other hand, 7.12 says that

$$\mathfrak{X} = \begin{array}{|c|} \hline \begin{array}{ccccc} & 18b & & 18c & \\ & \diagdown & & \diagup & \\ 9b & & 18a & & 9a \\ & \diagup & & \diagdown & \\ & 18c & & 18b & \end{array} \\ \hline \end{array} \oplus P(18a). \tag{19}$$

Then, we know by (18), (19) and 3.6(iv) that

$$W = 9a + 9b + 9c + 9d + 2 \times 18b + 2 \times 18c, \quad \text{as composition factors.} \tag{20}$$

By 7.11 and (20), we know $j(W) \geq 3$.

Assume that $j(W) \geq 4$. Then, $j(W) = 4$ by 7.8(iii). Since $L_1(W) \cong 18b \oplus 18c$ by 7.11, we get by 3.6(vi) that

$$L_4(W) \Big| L_4(P(18b)) \oplus L_4(P(18c)) = (9a \oplus 18a \oplus 9d) \oplus (9b \oplus 18a \oplus 9c)$$

and

$$L_4(W) \Big| \text{soc}(W) = 18b \oplus 18c.$$

This is a contradiction.

Hence $j(W) = 3$. Thus, again by 7.11, (20) and 3.6(vi), we know that W has radical and socle series

$$\begin{array}{|c|c|} \hline 18b & 18c \\ \hline 9b & 9c & 9a & 9d \\ \hline 18c & 18b \\ \hline \end{array} \tag{21}$$

Now, as in the proof of 7.11, we get by 7.8 that $W = W_1 \oplus W_2$ where W_i is a non-simple non-projective indecomposable self-dual B -module for $i = 1, 2$. Then, by (21), we may assume that $L_1(W_1) \cong 18b$, $\text{soc}(W_1) \cong 18c$, $L_1(W_2) \cong 18c$ and $\text{soc}(W_2) \cong 18b$ since $(18b)^\vee \cong 18c$ by 3.6(iii). Hence the structures of $P(18b)$ and $P(18c)$ in 3.6(vi) yield that

$$W_1 = \begin{array}{|c|} \hline 18b \\ \hline 9b & 9c \\ \hline 18c \\ \hline \end{array} \quad \text{and} \quad W_2 = \begin{array}{|c|} \hline 18c \\ \hline 9a & 9d \\ \hline 18b \\ \hline \end{array} \quad \square$$

8. Proof of main results

8.1. Notation. We still keep the notation F, j, B', f' and g' , see 6.4, 3.7 and 5.2–5.4. Set $E = SD_{16}$, and let $P \rtimes E$ be the canonical semi-direct product such that E acts on P faithfully. Recall that $\text{Aut}(P) \cong GL_2(3)$ since $P = C_3 \times C_3$, and hence SD_{16} is a Sylow 2-subgroup of $GL_2(3)$.

8.2. Lemma. *The non-principal block algebra A of HN and the principal block algebra A' of HS are Puig equivalent.*

Proof. Let j be the same as in 3.6(ii). Since $jBj \cong \mathcal{O}[P \rtimes E] = B'$ as interior P -algebras by 3.6(ii), we can identify jBj and B' . Define a functor $F' : \text{mod-}B \rightarrow \text{mod-}B'$ via $F'(-) = - \otimes_B Bj$. By 3.6(ii), F' induces a Puig equivalence (which is stronger than a Morita equivalence) between B and B' . In the following we use the information on the structures of PIMs in B and B' described in 3.6(vi) and 5.2(iii), respectively, without quoting these statements.

Then, first of all, we know that $F'(18a) = 2a$ by looking at the PIMs $P(18a)$ and $P(2a)$. Similarly, we know at least that $\{F'(9a), F'(9b), F'(9c), F'(9d)\} = \{1a = k_{H'}, 1b, 1c, 1d\}$. It follows from 5.4 that $1x \otimes 1x = 1a$ for any $x \in \{a, b, c, d\}$ since they are just in $\text{Irr}(E)$. Hence a technique of self-Puig equivalence in [29, 2.8 Lemma] can be used just as in the proof of [29, 6.8 Lemma]. Namely, we can assume that $F'(9a) = 1a$. Hence, by comparing the second Loewy layers of $P(9a)$ and $P(1a)$, we get $F'(18b) = 2b$. Similarly, by looking at the third Loewy layers of $P(9a)$ and $P(1a)$, we have $F'(9b) = 1b$. If we look at the fourth Loewy layers of these PIMs, then we know $F'(18c) = 2c$. Thus, by looking at the second Loewy layers of $P(18c)$ and $P(2c)$, we know also that $F'(9d) = 1d$. These mean that $F'(9c) = 1c$. Namely, we can assume that

$$\begin{aligned} F'(9a) = 1a, & \quad F'(9b) = 1b, & \quad F'(9c) = 1c, & \quad F'(9d) = 1d, \\ F'(18a) = 2a, & \quad F'(18b) = 2b, & \quad F'(18c) = 2c. & \end{aligned} \tag{22}$$

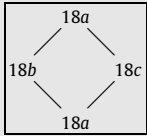
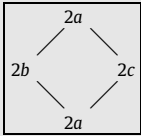
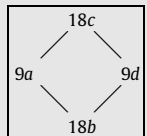
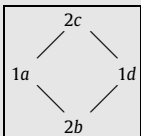
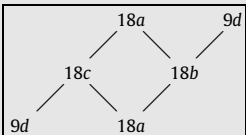
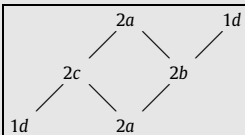
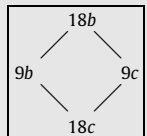
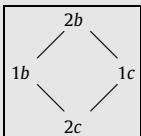
We know by 7.13 that Case (a) or Case (b) happens.

Assume, first, that Case (b) occurs. Then, by bunching up 2.2, 7.1, 7.5, 7.6, 7.13 and 5.7, we get the diagram shown in Table 1.

First, all the three functors above are given by bimodules which are p -permutation modules over $\mathcal{O}[G_1 \times H_1]$ for corresponding two finite groups G_1 and H_1 , which are ΔP -projective, and also which induce a stable equivalence of Morita type at each step, if we identify the source algebra jBj as $\mathcal{O}[P \rtimes E]$.

Secondly, it has to be noted that all non-simple modules in the above diagram are uniquely determined (up to isomorphism) by just the diagrams given in the above boxes: This is clear for $F(S_1)$,

Table 1
Case (b).

mod- A	\xrightarrow{F}	mod- B	$\xrightarrow{F'}$	mod- B'	$\xrightarrow{f'^{-1}}$	mod- A'
S_1	\mapsto	$\boxed{9a}$	\mapsto	$\boxed{1a}$	\mapsto	$k_{G'}$
S_2	\mapsto	$\boxed{9b}$	\mapsto	$\boxed{1b}$	\mapsto	154
S_3	\mapsto	$\boxed{9c}$	\mapsto	$\boxed{1c}$	\mapsto	22
S_4	\mapsto		\mapsto		\mapsto	1253
S_5	\mapsto		\mapsto		\mapsto	321
S_6	\mapsto		\mapsto		\mapsto	748
S_7	\mapsto		\mapsto		\mapsto	1176

$F(S_2)$, $F(S_1)$, $f'(k_{G'})$, $f'(154)$, and $f'(22)$ anyway, as well as for $F(S_4)$ and $f'(1253)$ by the structure of $P(18a)$ and $P(2a)$ given in 3.6(vi) and 5.2(iii).

To tackle $F(S_6)$, the structure of $P(18a)$ specified in 3.6(vii) shows that $P(18a)$ has a unique quotient with composition factors $9d + 2 \times 18a + 18b + 18c$. Moreover, $P(9d)$ has a unique quotient with composition factors $18a + 18b$. Since they both have a unique submodule with composition factors $18a + 18b$, the glueing to yield $F(S_6)$ also is uniquely defined, and thus $F(S_6)$ is uniquely determined by the diagram given. For $f'(748)$ we argue similarly using 5.2(iv).

We consider $F(S_7)$: Note first that for $P(18b)$ there is no Alperin diagram defined. By 3.6(vi), let X be the unique quotient module of $P(18b)$ having radical and socle series $\begin{bmatrix} 18b \\ 9b & 9c \end{bmatrix}$. By the structure of $P(18b)$ given in 3.6(vi) we have $[\Omega(X), 18a]^B = 1$, hence using 3.6(vii) there is a homomorphism $\varphi \in \text{Hom}_B(P(18a), \Omega(X))$ such that

$$\text{Im}(\varphi) = \begin{array}{c} \boxed{\begin{array}{ccccc} & & 18a & & \\ & & / & & \backslash \\ & 18c & & & 18b \\ & / & & & \backslash \\ 9a & & 9d & & 18a \\ & \backslash & & & / \\ & & 18b & & \end{array}} \end{array}$$

Table 2
Case (a).

mod- A	\xrightarrow{F}	mod- B	$\xrightarrow{F'}$	mod- B'	$\xrightarrow{f'^{-1}}$	mod- A'
S_1	\mapsto	$\boxed{9a}$	\mapsto	$\boxed{1a}$	\mapsto	$k_{C'}$
S_2	\mapsto	$\boxed{9b}$	\mapsto	$\boxed{1b}$	\mapsto	154
S_3	\mapsto	$\boxed{9c}$	\mapsto	$\boxed{1c}$	\mapsto	22
S_4	\mapsto	$\begin{array}{c} \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \end{array}$	\mapsto	$\begin{array}{c} \boxed{\begin{array}{ccc} & 2a & \\ 2b & & 2c \\ & 2a & \end{array}} \end{array}$	\mapsto	1253
S_5	\mapsto	$\begin{array}{c} \boxed{\begin{array}{ccc} & 18b & \\ 9b & & 9c \\ & 18c & \end{array}} \end{array}$	\mapsto	$\begin{array}{c} \boxed{\begin{array}{ccc} & 2b & \\ 1b & & 1c \\ & 2c & \end{array}} \end{array}$	\mapsto	1176
S_6	\mapsto	$\begin{array}{c} \boxed{\begin{array}{cccc} & 18a & & 9d \\ & & 18c & \\ 9d & & & 18b \\ & & 18a & \end{array}} \end{array}$	\mapsto	$\begin{array}{c} \boxed{\begin{array}{cccc} & 2a & & 1d \\ & & 2c & \\ 1d & & & 2b \\ & & 2a & \end{array}} \end{array}$	\mapsto	748
S_7	\mapsto	$\begin{array}{c} \boxed{\begin{array}{ccc} & 18c & \\ 9a & & 9d \\ & 18b & \end{array}} \end{array}$	\mapsto	$\begin{array}{c} \boxed{\begin{array}{ccc} & 2c & \\ 1a & & 1d \\ & 2b & \end{array}} \end{array}$	\mapsto	321

This implies $\Omega(X)/\text{Im}(\varphi) \cong 18c$. Since $18c$ occurs exactly twice as a composition factor of $\Omega(X)$, and also is a composition factor of $\text{Im}(\varphi)$, we conclude that $[\Omega(X), 18c]^B = 1$, thus $\dim_k[\text{Ext}_B^1(X, 18c)] = 1$.

Hence a module having radical and socle series $\begin{array}{c} \boxed{18b} \\ 9b \ 9c \\ \boxed{18c} \end{array}$ is uniquely defined. For $F(S_5)$, $f'(1176)$, and $f'(321)$ we argue similarly.

Then, it follows from 2.15 that A and A' are splendidly stable equivalent of Morita type, that is, A and A' are stable equivalent which is realised by an $\mathcal{O}[G \times C']$ -bimodule which is a p -permutation module and ΔP -projective. Hence, first of all, the stable equivalence actually gives a Morita equivalence by a result of Linckelmann [34, Theorem 2.1(iii)]. Then, if we look at the proof of [34, Theorem 2.1(iii)] which is actually given in [34, Remark 2.7], we know that the Morita equivalence between A and A' gives a bijection such as $S_5 \leftrightarrow 321$. Hence, we must have equalities between the corresponding Cartan invariants, namely, $c(S_5, S_5) = c(321, 321)$. However, we get that $c(S_5, S_5) = 3$ by 4.1, and on the other hand, that $c(321, 321) = 2$ by 5.8. This is a contradiction. Thus, Case (b) cannot happen.

This means that only Case (a) occurs, as is shown in Table 2. Then, again the same argument given above still works. Namely, we have a Morita equivalence between A and A' , and hence the Morita equivalence is a Puig equivalence by a result of Puig (and, independently, of Scott) [52, Remark 7.5], see [37, Theorem 4.1]. \square

8.3. Proofs of 1.3 and 1.4. Recall that a Puig equivalence lifts from k to \mathcal{O} by a result of Puig [50, 7.8 Lemma] (see [56, (38.8) Proposition]), and that so does a splendid Rickard equivalence by a result of Rickard [53, Theorem 5.2], see [14, P. 75, lines $-17 \sim -16$]. Thus, it is enough to consider blocks A , B , A' and B' only over k . Thus, we get 1.4 by 8.2.

By results of Okuyama [48, Example 4.8] and [49, Corollary 2], Conjectures 1.1 and 1.2 hold for A' . Namely, we get the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\text{Puig equiv.}} & A' \\ & & \downarrow \text{splendid Rickard equiv.} \\ B & \xleftarrow{\text{Puig equiv.}} & B' \end{array}$$

Therefore, we finally get that A and B are splendidly Rickard equivalent. That is, the proof of 1.3 is completed. \square

8.4. Proof of 1.5. We get 1.5 from 3.2 and 1.3. \square

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