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Turán theorems and convexity invariants for directed graphs

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Abstract

This paper is motivated by the desire to evaluate certain classical convexity invariants (specifically, the Helly and Radon numbers) in the context of transitive closure of arcs in the complete digraph. To do so, it is necessary to establish several new Turán type results for digraphs and characterize the associated extremal digraphs. In the case of the Radon number, we establish the following analogue for transitive closure in digraphs of Radon's classical convexity theorem [J. Radon, Mengen konvexer Körper, die einer gemeinsamen Punkt enthalten, Math. Ann. 83 (1921) 113–115]: in a complete digraph on $n \geqslant 7$ vertices with $> n^2/4$ arcs, it is possible to partition the arc set into two subsets whose transitive closures have an arc in common.
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1. Introduction

In this paper X will denote a finite directed graph. Loops are not allowed, and parallel edges are allowed only if they go in opposite directions. Our goal is to solve Turán type problems for two classes of forbidden subgraphs: bipaths and transitive cycles. A *bipath* consists of two internally disjoint directed paths from a source s to a target t. If one of these paths is a single edge, the bipath is a *transitive cycle*.

Bipath-free arc sets were previously studied for strong digraphs in [7,16] where they are called "unipathic". A digraph with no transitive cycle is *TC-free*. The *complete directed graph* (*digraph*) $\overrightarrow{K}_n$ has n vertices and all $n(n-1)$ arcs between them. In this paper we do the following:

- Characterize all TC-free digraphs (Theorem 1) and all unipathic digraphs (Theorem 8).
- Determine the maximum number of arcs in a TC-free (Theorem 10) [resp., unipathic (Corollary 11)] digraph in $\overrightarrow{K}_n$.
- Apply these results to evaluate some classical abstract convexity invariants for transitive closure in $\overrightarrow{K}_n$.

A *block* in an undirected graph G is a maximal connected subgraph of G that has no cut-vertex [21]. A connected undirected graph in which all blocks are cycles is a *cactus*. Directing all cycles of a cactus cyclically results in a *directed cactus*. In a partial order $(P, \preccurlyeq)$, y *covers* x iff $x \preccurlyeq y$ and for all z,

$$x \preccurlyeq z \preccurlyeq y \Longrightarrow x = z \quad \text{or} \quad z = y.$$

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A digraph D is a *cover digraph* iff there is a partial order $\preccurlyeq$ on the vertices of D such that vw is an arc iff v covers w. A *directed composition* over a directed graph D is obtained by replacing each vertex v of D by a directed graph B_v. If vw is an arc of D, then exactly one arc is to be drawn from B_v to B_w. If vw is not an arc of D, then there is no arc from B_v to B_w.

Our main characterization can be expressed as follows:

Theorem 1. *A digraph X is TC-free iff it is a directed composition of directed cacti over a cover digraph.*

It is a simple matter to obtain a characterization (Theorem 8) of unipathic diagraphs from this result. Moreover, solving the Turán problem becomes a matter of maximizing the number of edges in TC-free and unipathic digraphs. Three types of digraph play a role:

Double Tree: A directed cactus in which all cycles are 2-cycles.

Biclique: Let A and B be disjoint sets of vertices in $\overrightarrow{K}_n$. Take all arcs in $\overrightarrow{K}_n$ from A to B. This is a *biclique* and is denoted by $\langle A \to B\rangle$. The biclique is *balanced* iff $||A| - |B|| \leqslant 1$.

Triptych: Let A, B and C be three disjoint sets of vertices in $\overrightarrow{K}_n$. Take all arcs from A to B and all arcs from B to C. This is a *triptych* and is denoted by $\langle A \to B \to C\rangle$. The triptych is *balanced* iff $||A \cup C| - |B|| \leqslant 1$.

The authors of [7] show that the only 'bicliques' in a strong unipathic digraph are claws. Their result on claws can be deduced from our Theorem 4.

We will use the following notation and terminology:

- X is *strong* iff there is a directed path from any node of X to any other node of X;
- $|X|$ denotes the number of vertices of X;
- $\Gamma(X)$ denotes the underlying undirected graph of X.

2. Characterization of TC-free digraphs

We begin by showing that the strong components of a TC-free digraph are directed cacti. Two technical lemmata will be useful.

Lemma 2. *If X is strong and TC-free, then X is unipathic.*

Proof. If not, then X contains two internally disjoint paths P and Q directed from some vertex s to another vertex t. Let vt be the last arc on Q. If $v = s$, then we have a transitive cycle, and we are done. If $v \neq s$, then because X is strong, there is a path P^* from v to s. Let w be the first vertex in P^* that is also in P. Then the portion of P^* beginning at v and ending at w, joined with that portion of P going from w to t, together with the arc vt is a transitive cycle. □

The proof of the next lemma is routine and hence omitted.

Lemma 3. *If X contains two distinct directed cycles whose arc sets have nonempty intersection, then X contains a bipath.*

By a *block* of X, we mean a set of arcs which correspond to a block of the underlying undirected graph $\Gamma(X)$ of X.

Theorem 4. *Let X be strong. Then X is TC-free iff the blocks of X are precisely the directed cycles of X.*

Proof. Suppose first that X contains a transitive cycle C. The underlying graph $\Gamma(C)$ of C is 2-connected and hence C lies in a block of X. If s is the source of C, it has outdegree two in C, and hence outdegree at least two in the block of X containing C. Hence this block cannot be a directed cycle.

Now, suppose B is a block of X that is not a directed cycle. We must show that B contains a transitive cycle.

First we show that every arc ij of B lies in a unique directed cycle in B. Since X is strong, there is a directed path from j to i. This path followed by ij is a directed cycle C in X. Moreover, C lies in B since B is a block and $ij \in B \cap C$.

Hence every arc lies in a directed cycle of B. If arc ij lies in two (or more) directed cycles of B , then Lemmata 2 and 3 imply there is a transitive cycle and we are done. Thus we may assume that each arc of B lies in a unique directed cycle of B.

Let C be any directed cycle in B. Since B is not a directed cycle but $\Gamma(B)$ is connected, there is an (undirected) edge in $\Gamma(B)$ joining a node $v \in C$ to a node $w \notin C$. We will consider the case that the arc is directed as vw. The case that the arc is wv can be handled analogously. As $\Gamma(B)$ is 2-connected, removing v leaves the underlying undirected graph $\Gamma(B/v)$ connected. In particular, there is an undirected path Q in $\Gamma(B/v)$ from w to some first vertex, say y, in C.

The undirected path Q may not correspond to a directed path in B from w to y. However, every backward directed arc in Q is contained in a directed cycle which is distinct from and therefore arc disjoint from C. Hence replacing each such backwards arc by its complement in its directed cycle, we get a directed walk in B from w to y which is arc disjoint from C. This replacement could introduce new vertex intersections with C, so let z be the first vertex of C along this walk. The union of this walk with the arc vw must contain a directed path from v to z. By construction, this path is vertex disjoint from C except at v and w. This path together with the directed portion of C from v to w is a bipath. Since X is strong, Lemma 2 implies X also contains a transitive cycle. □

We now wish to extend our results to general TC-free digraphs. The *reduced* digraph X^* of X has the strong components of X as vertices and an arc between vertices in X^* iff there is an arc in X in the same direction between the corresponding strong components. An arc between strong components will be called a *transition arc*. Strong components, both as sets and as vertices of the reduced graph, will be denoted by capital letters.

Lemma 5. *Suppose X is TC-free. Then each pair of distinct strong components A and B of X is connected by at most one arc.*

Proof. There cannot be arcs directed from A to B and from B to A because then $A \cup B$ would be strong, violating the maximality of A and B. Thus suppose there are two arcs ab and $a'b'$ going the same way: from A to B. Since A is strong there is a directed path P from a to a'. Similarly since B is strong there is a directed path Q from b' to b. If a and a' are not distinct, then P is a single vertex path and similarly for Q. As ab and $a'b'$ are distinct arcs, however, at least one of the two paths is not degenerate. But then concatenating P, $a'b'$, and Q, we obtain a directed path of length at least two from a to b which together with the arc ab forms a transitive cycle in X. □

Suppose P is a path in X. Let P^* be the path in X^* obtained by collapsing the vertices in a single strong component to a point. The following observation on the relationship between directed paths in X and X^* will be useful.

Lemma 6. *Let X be TC-free. Suppose $\Pi : C_0, C_1, \ldots, C_{k-1}, C_k$ is a path in X^*. Suppose $a \in C_0$ and $b \in C_k$. Then there is a unique path P in X from a to b with $P^* = \Pi$.*

Proof. Since $C_0, C_1, \ldots, C_{k-1}, C_k$ is a path in X^*, for each $1 \leqslant i \leqslant k$ there are vertices s_i and t_i in C_i such that $s_{i-1}t_i$ is an arc in X. Now starting at a, use the strongness of C_0 to traverse a directed path in C_0 from a to s_0. Now take arc s_0t_1 into C_1. Traverse a directed path from t_1 to s_1 in C_1. Continue in this way until b is reached in C_k. This gives the path P in X. Uniqueness follows from the fact that in a directed cactus, any two points are joined by a unique directed path. □

Lemma 7. *If X is TC-free, then X^* is TC-free.*

Proof. Suppose X^* contains a transitive cycle. That is, suppose there is an arc AB in X^* from some strong component A of X to another strong component B of X and a nontrivial path Π in X^* from A to B but not using AB. Since AB is an arc in X^*, there is an arc ab in X from A to B. Lemma 6 gives a path P in X from a to b with $P^* = \Pi$. Adjoining the arc ab to P produces a transitive cycle in X, a contradiction. □

Proof of Theorem 1. Let X be TC-free. By Theorem 4 the strong components of X are directed cacti. Now Lemma 5 insures that X is a directed composition of its strong components over X^*. It is well known that contracting the strong components of any digraph to points results in an acyclic digraph which defines a partial order. The required cover property is equivalent to X^* being TC-free. Thus by Lemma 7, X^* is a cover digraph, and X has the requisite form.

Conversely, suppose X is of the form given in Theorem 1. Directed cacti are certainly TC-free, so any transitive cycle T in X cannot stay inside a single strong component. An argument like that in Lemma 7 would allow T to be lifted to a transitive cycle of X^*, a contradiction. □

We now give the promised characterization of unipathic digraphs. Let $x \preccurlyeq y$ be elements of a partially ordered set P. Define the interval $I(x, y)$ as $\{p \in P : x \preccurlyeq p \preccurlyeq y\}$. A *chain* in P is a set of elements of P any two of which are comparable in the order on P.

Theorem 8. *A digraph X is unipathic iff it is a directed composition of directed cacti over the cover digraph of a partial order in which all intervals are chains.*

Proof. We must decide which TC-free sets contain bipaths and exclude them. Directed cacti clearly contain no bipaths, so we need only work at the level of the composition's cover digraph. By the lifting Lemma 6, if X is TC-free, then X will be unipathic iff the cover digraph X^* is unipathic. It is easy to see that unipathic translates into the chain-interval condition for cover digraphs of ordered sets. □

3. TC-free sets with the maximum number of arcs

The following lemma is helpful in determining the structure of TC-free digraphs with the maximum number of arcs. Although it is required only in the case that X is acyclic, we will prove it when only directed 4-cycles are excluded since very little extra effort is necessary. The situation becomes more complex when directed 4-cycles are allowed.

Lemma 9. *Let X be TC-free with no directed* 4*-cycles. Suppose the underlying undirected graph $\Gamma(X)$ is simple and complete bipartite. Then X is a triptych or a biclique.*

Proof. Let the bipartition be into black and white vertices. Since $\Gamma(X)$ is simple, it contains no 2-cycle. So if X contains a directed cycle $C : v_1, v_2, v_3, \ldots, v_n$, it must alternate between black and white vertices and hence have length of at least 4. Suppose that $n \geqslant 6$. Since v_1 and v_4 have opposite colors, there is an arc between them. If it is v_1v_4, then $v_1v_2, v_2v_3, v_3v_4, v_1v_4$ is a transitive cycle. If the arc is v_4v_1, then traversing the cycle C in the other direction from v_4 to v_1 is a transitive cycle. Thus X contains no directed cycles, and hence the arcs of X induce a partial order on the vertices.

The maximal elements are sources, and the minimal elements are targets. Let A be the set of sources and C the set of targets. Let B be the remaining set of vertices which must have arcs going both in and out. If A, B or C is empty, then we have a biclique. Thus we can assume A, B and C are nonempty.

Suppose A contains a white node w and C contains a black node x. Since w and x are opposite colors, there is an arc between them. It must be wx since w is a source. Suppose B contains a black node y. Then there is an arc between w and y since they have opposite colors, and it must be wy since w is a source. Since $y \in B$, there are arcs coming in and out of y. Hence there is a node z such that yz is an arc. Since y is black, z must be white. But that means, there is an arc from z to x since they have opposite colors and x is a target. But now we have a transitive cycle $\{wy, yz, zx, wx\}$, a contradiction. The dual argument shows B contains no white nodes, contradicting that B is non-empty.

Hence the nodes in A and C have the same color, say white, and there are no arcs between them. Now suppose B contains a white node w. Since nodes in B have arcs both coming in and going out, there are arcs xw and wy where x and y are black. Hence letting z be any (necessarily white) node in A, the arcs zx and zy are forced. Thus zx, xw, wy, zy is a transitive cycle, contradicting independence.

It follows that all nodes in B are black. Thus there are no arcs within A, B or C and all arcs from A to B and B to C are forced. B cannot be empty, but A or C, but not both, could be empty. If all three sets are nonempty, X is a triptych; if A or C is empty, then X is a biclique. □

Let $\varphi(n)$ denote the maximum number of arcs in a TC-free digraph of order n. A digraph which achieves this value is a *maximum TC-free* digraph.

Theorem 10.

- *For* $1 \leqslant n \leqslant 6$, $\varphi(n) = 2n - 2$. *The maximum TC-free digraphs are double trees.*
- $\varphi(7) = 12$ *and the maximum TC-free digraphs are double trees, balanced bicliques, and balanced triptychs.*
- *For* $n \geqslant 8$, $\varphi(n) = \lfloor n^2/4 \rfloor$. *The maximum TC-free digraphs are balanced bicliques and balanced triptychs.*

Proof. Let X be maximum TC-free. Let $A_1, A_2, \ldots, A_m$ be the strong components of X, let $n_i = |A_i|$ and let $b_1, \ldots, b_m$ be the number of blocks in each. By Theorem 4, the blocks of A_i are directed cycles. By removing an arc from each directed cycle (block), we obtain a (directed) tree. Hence the number of arcs of A_i is $n_i - 1 + b_i$. The maximum value for b_i is achieved when each directed cycle is as small as possible, namely a 2-cycle. In this case, the strong component A_i is a double tree or a single vertex.[1] We then have $b_i = n_i - 1$, so A_i has $2(n_i - 1)$ arcs. This counts the arcs internal to the strong components.

Let us now bound the external arcs between strong components. There is a one-to-one correspondence between external arcs in X and arcs in X^*. By construction, every strong component of X^* is a single vertex. Since X^* is acyclic, it contains no directed triangles. Moreover by Lemma 7, X^* contains no transitive triangles. Thus $\Gamma(X)$ is simple and triangle free. Hence from the Mantel–Turán Theorem [15,20], the number of external arcs in X^* is at most $\lfloor m^2/4 \rfloor$. Letting

$$B(n,m) := \sum_{i=1}^{m} 2(n_i - 1) + \left\lfloor \frac{m^2}{4} \right\rfloor = 2n - 2m + \left\lfloor \frac{m^2}{4} \right\rfloor,$$

we have that $\varphi(n) = \max\{B(n,m) : 1 \leqslant m \leqslant n\}$.

For $n \leqslant 6$ direct computation shows that a maximum of $\varphi(n) = 2(n-1)$ occurs uniquely at $m = 1$. This is achieved uniquely by spanning double trees.

For $n = 7$, the maximum occurs at both $m = 1$ and $m = 7$. The case $m = 1$ corresponds to a spanning double tree. In the case $m = 7$, there are seven strong components and seven vertices. Hence the vertices are the strong components in this case and $X^* = X$. To achieve $\lfloor \frac{7^2}{4} \rfloor = 12$ edges, $\Gamma(X)$ must be $K_{3,4}$. Thus by Lemma 9, X is a triptych or a biclique. Because the underlying undirected graph is $K_{3,4}$, the parts must be evenly divided, so X is balanced.

Fix $n \geqslant 8$. We want to show that the maximum of $B(n,m)$ occurs uniquely at $m = n$. Consider the function $f(x) := 8n - 8x + x^2$; it is a convex function, so its maximum on the interval $[1, n-1]$ is achieved at one or both of the endpoints. Define $g(x) := f(x-1) - f(1) = x^2 - 10x + 16$. Now $g(8) = 0$ and $g'(x) = 2x - 10 > 0$ for $n \geqslant 8$. Thus $g(x) \geqslant 0$ for $x \geqslant 8$, so for $n \geqslant 8$, f assumes its maximum of $[1, n-1]$ at $n-1$. Note that $f(n) - f(n-1) = 2n - 9 \geqslant 7$ if $n \geqslant 8$. Since f assumes its maximum on $[1, n-1]$ at $n-1$, we have $f(m) + 7 \leqslant f(n-1) + 7 \leqslant f(n)$ for $1 \leqslant m \leqslant n-1$. Then

$$B(n,n) = \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{f(n)}{4} \right\rfloor \geqslant \left\lfloor \frac{f(m)+7}{4} \right\rfloor > \left\lfloor \frac{f(m)}{4} \right\rfloor = B(n,m).$$

Thus $m = n$ again. The same argument as in the case $n = 7$ shows that $B(n,n)$ is attained by balanced triptychs and bicliques. □

Since balanced triptychs are the only maximum TC-free digraphs that contain bipaths, we get the following companion result for unipathic digraphs:

Corollary 11.

- *For* $1 \leqslant n \leqslant 6$, *the maximum unipathic digraphs are double trees.*
- *For* $n = 7$, *the maximum unipathic digraphs are double trees and balanced bicliques.*
- *For* $n \geqslant 8$, *the maximum TC-free sets are balanced bicliques.*

[1] This happens iff $n_i = 1$ which forces $b_i = 0$ Note that the formulae still hold in this case.

4. Convexity invariants

Transitive closure on $\overrightarrow{K}_n$ can be viewed as an example of abstract convexity as developed in [1–5,9–14,19]. These papers represent just some of the investigations of analogues of classical Euclidean invariants in more abstract spaces. The most commonly studied in this setting are rank (independence number) [17], Radon number [6,18], and Helly number [6,8].

A set S is *independent* provided no point of S is in the closure of the remaining points of S. In the setting of transitive closure, this clearly corresponds to TC-free. Thus the *rank* (maximum cardinality of an independent set) in $\overrightarrow{K}_n$ is given by Theorem 10.

A *Radon partition* of a set S is a partition of S into two parts A and B whose closures have nonempty intersection.

Lemma 12. *In $\overrightarrow{K}_n$ a set S of arcs has a Radon partition iff S contains a bipath.*

Proof. Suppose $S = A \cup B$ is a Radon partition. Let vw be any arc in the intersection of the closures of A and B. Since vw is in the transitive closure of A, there is a directed path in A from v to w; likewise, for B. The union of these two paths contain a bipath.

Conversely, if S contains a bipath from v to w, extend these two path into a partition $S = A \cup B$. Then vw is in the intersection of the transitive closures of both A and B. $\square$

The *Radon number* r is the smallest integer r such that any set S with $|S| > r$ has a Radon partition. By Lemma 12 being Radon partitionable means *not* being unipathic. Hence the Radon number r is the cardinality of a maximum unipathic arc set. This number is given by Corollary 11.

The Helly number is classically defined as follows. A family $\mathscr{F}$ of sets satisfies the *h Intersection Property* (*h.I.P.*) iff every h sets in $\mathscr{F}$ has nonempty intersection. The *Helly number* $h(X)$ is the smallest integer h such that any (finite) family of closed sets with the h.I.P. has nonempty intersection.

This is somewhat cumbersome, but there is a more computationally efficient approach. Following Doignon et al. [3], we define $\operatorname{core}(S) := \bigcap\{c\ell(S\backslash\{x\})|x \in S\}$ where $c\ell(A)$ denotes the transitive closure of the arc set A. As shown in [3,11], the Helly number is the largest cardinality of a set with empty core. Clearly core is monotone, so if a set has nonempty core, so does every superset. In a bipath from v to w, the arc vw is easily seen to be in the core. Thus maximum sets with empty core are unipathic. Bicliques obviously have empty core.

Lemma 13. *Double trees have empty core.*

Proof. Let DT be a double tree with underlying tree T. Any arc ab in the core of DT must be in the transitive closure of DT and hence must connect two vertices a and b of T. Thus there is a unique (shortest) directed path in DT from a to b. Let cd be any arc on this path. Deleting cd and its reverse dc breaks DT into two strong components: A containing a and B containing b. Now consider the transitive closure of $DT\backslash\{cd\}$. Once cd is removed, there is no way to go from A to B on a directed path. Hence the transitive closure fails to contain any arcs from A to B. In particular, it fails to contain ab. Thus for every arc ab in $\overrightarrow{K}_n$, there is some arc cd of DT such that $ab \notin c\ell(DT\backslash\{cd\})$. Thus there can be no arcs in the core of DT. $\square$

We thus conclude with the application to convexity invariants of transitive closure in the complete digraph promised in Introduction.

Theorem 14. *For transitive closure on the arcs of $\overrightarrow{K}_n$, the rank, the Radon number, and the Helly number are all the same*: $2n-2$ *for* $1 \leqslant n \leqslant 7$ *and* $\lfloor n^2/4 \rfloor$ *for* $n \geqslant 7$.

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