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Forcing faces in plane bipartite graphs

Zhongyuan Che[a], Zhibo Chen[b,*]

[a]*Department of Mathematics, Penn State University, Beaver Campus, Monaca, PA 15061, USA*
[b]*Department of Mathematics, Penn State University, McKeesport Campus, McKeesport, PA 15132, USA*

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Abstract

Let Ω denote the class of connected plane bipartite graphs with no pendant edges. A finite face s of a graph $G \in \Omega$ is said to be a forcing face of G if the subgraph of G obtained by deleting all vertices of s together with their incident edges has exactly one perfect matching. This is a natural generalization of the concept of forcing hexagons in a hexagonal system introduced in Che and Chen [Forcing hexagons in hexagonal systems, MATCH Commun. Math. Comput. Chem. 56 (3) (2006) 649–668]. We prove that any connected plane bipartite graph with a forcing face is elementary. We also show that for any integers n and k with $n \geqslant 4$ and $n \geqslant k \geqslant 0$, there exists a plane elementary bipartite graph such that exactly k of the n finite faces of G are forcing. We then give a shorter proof for a recent result that a connected cubic plane bipartite graph G has at least two disjoint M-resonant faces for any perfect matching M of G, which is a main theorem in the paper [S. Bau, M.A. Henning, Matching transformation graphs of cubic bipartite plane graphs, Discrete Math. 262 (2003) 27–36]. As a corollary, any connected cubic plane bipartite graph has no forcing faces. Using the tool of Z-transformation graphs developed by Zhang et al. [Z-transformation graphs of perfect matchings of hexagonal systems, Discrete Math. 72 (1988) 405–415; Plane elementary bipartite graphs, Discrete Appl. Math. 105 (2000) 291–311], we characterize the plane elementary bipartite graphs whose finite faces are all forcing. We also obtain a necessary and sufficient condition for a finite face in a plane elementary bipartite graph to be forcing, which enables us to investigate the relationship between the existence of a forcing edge and the existence of a forcing face in a plane elementary bipartite graph, and find out that the former implies the latter but not vice versa. Moreover, we characterize the plane bipartite graphs that can be turned to have all finite faces forcing by subdivisions.

Keywords: Perfect matching; Plane bipartite graph; Plane elementary bipartite graph; Forcing edge; Forcing face; Reducible face decomposition; Z-transformation graph

1. Introduction

Stimulated by some chemical and physical problems, Harary et al. [4] introduced the concept of forcing edges in a hexagonal system (which is a special case of a 2-connected plane bipartite graph where every finite face is a hexagon.) An edge of a hexagonal system H is called a *forcing edge* if it is contained in exactly one perfect matching of H. Hansen and Zheng [3], and Zhang and Li [10], independently characterized the hexagonal systems that have a forcing edge. Motivated by their work, we introduced in [2] the concept of forcing hexagons for hexagonal systems. A hexagon h of a hexagonal system H is called a *forcing hexagon* of H if the subgraph of H obtained by deleting all vertices of h

* Corresponding author.
E-mail address: ZXC4@psu.edu (Z. Chen).

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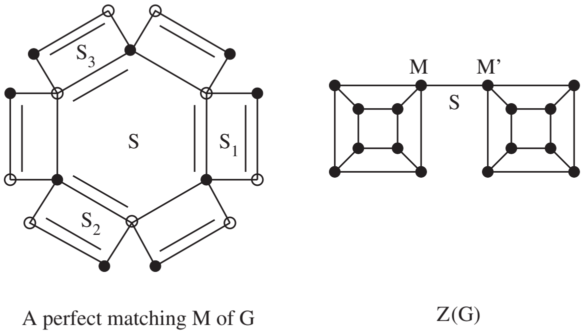

Fig. 1. G has a forcing face s but no forcing edges.

together with their incident edges has exactly one perfect matching. We proved that a linear hexagonal chain has all its hexagons forcing, and other hexagonal systems H may have 0, 1 or 2 forcing hexagons. We presented structural characterizations for the hexagonal systems with a given number of forcing hexagons. We also proved the co-existence property of forcing hexagons and forcing edges in a hexagonal system (see [2]). In order to extend various studies on hexagonal systems, Zhang and Zhang [12] conducted an extensive study on plane elementary bipartite graphs so that many important known results in hexagonal systems can be treated in a unified way. In particular, they extended the concept of forcing edges from hexagonal systems to forcing edges of connected plane bipartite graphs and got interesting results. Parallel to their work, in the present paper we generalize the concept of forcing hexagons from hexagonal systems to forcing faces of connected plane bipartite graphs. Recall that a *perfect matching* (or 1-*factor*) of G is a set of pairwise disjoint edges of G covering all vertices of G. Clearly, all pendant edges must belong to every perfect matching, and so for our purpose we can delete them with no concern. Hence, without loss of generality, we assume that throughout the paper the plane bipartite graph G in consideration has no pendant edges. In other words, we always assume that G is a connected plane bipartite graph with the minimum vertex degree $\delta(G) \geqslant 2$. The class of such graphs is denoted by Ω.

A graph with a perfect matching is said to be *elementary* if the union of all perfect matchings forms a connected subgraph. Note that plane elementary bipartite graphs with more than two vertices are 2-connected, and so all of them are included in Ω.

Definition 1.1. A finite face s of a graph $G \in \Omega$ is said to be a forcing face of G if $G - s$ has exactly one perfect matching, where $G - s$ is meant to be the subgraph of G obtained by deleting all vertices of s together with their incident edges.

For example, the finite face s of graph G in Fig. 1 is the only forcing face of G.

Note: (1) If $G \in \Omega$ has exactly one finite face s, then s is a forcing face because the empty graph is assumed to have exactly one perfect matching by convention.

(2) If $G \in \Omega$ has a forcing face, then G itself must have at least two perfect matchings. It is because G, as a bipartite graph, contains only cycles of even length.

(3) Let $n(>0)$ be the number of finite faces of G. From [2] we already know that for a hexagonal system the number of forcing faces may be 0, 1, 2 and n. In Section 3 we will further show that the number of forcing faces of $G \in \Omega$ can be any integer between 0 and n when $n \geqslant 4$.

In Section 2 we introduce needed terminologies and known results. Our new results are presented in Section 3. We prove that any connected plane bipartite graph with a forcing face is elementary. We also show that for any integers n and k with $n \geqslant 4$ and $n \geqslant k \geqslant 0$, there exists a plane elementary bipartite graph such that exactly k of the n finite faces of G are forcing. We then give a shorter proof for a recent result that any perfect matching of a connected cubic plane bipartite graph has at least two disjoint M-resonant faces, which is a main result in the paper [1]. As a corollary, any connected cubic plane bipartite graph has no forcing faces. Using the tool of Z-transformation graphs developed by Zhang et al. [9,12] (the reader is referred to [11] for a detailed survey on this topic), we characterize the plane elementary bipartite graphs whose finite faces are all forcing. We also obtain a necessary and sufficient condition for a finite face in a plane elementary bipartite graph to be forcing, which enables us to investigate the relationship between the existence of a forcing edge and the existence of a forcing face in a plane elementary bipartite graph, and find out that the former implies the latter but not vice versa. Moreover, we characterize the plane bipartite graphs that can be turned to have all finite faces forcing by subdivisions.

2. Preliminaries

A plane graph G is a graph in the plane where any two edges are either disjoint or meet only at a common end vertex. If the vertices and edges of a plane graph G are removed from the plane, the remainder falls into connected components (in the plane topology), called faces. Clearly, each plane graph has exactly one unbounded face that will be called the infinite face. The other faces are all bounded and called finite faces. A finite face may also be simply called a face for brevity, when no confusion could occur. When G is 2-connected, the boundary of any face of G is a cycle. The boundary of a finite face s of G is denoted by ∂s. The boundary of the infinite face of G is denoted by ∂G, which is referred to as the *periphery* (or *boundary*) of G. A finite face s of G is called a *peripheral face* (or *boundary face*) of G if ∂s and ∂G have edges in common. Two finite faces s_1 and s_2 are said to be *adjacent* if their boundaries ∂s_1 and ∂s_2 have at least one edge in common.

Let M be a perfect matching of G. An edge of G is called an *M-double bond* if it belongs to M, and an *M-single bond* otherwise. An *M-alternating cycle* (resp. *M-alternating path*) of G is a cycle (resp. path) of G whose edges are alternately in M and $E(G) - M$. A face of G (including the infinite face) is said to be *M-resonant* if its boundary is an M-alternating cycle for some perfect matching M of G, and we say the face is *resonant* briefly if there is no need to specify the perfect matching. We say a cycle is an *(M_1, M_2)-alternating cycle* if the edges of the cycle appear alternatively in two matchings M_1 and M_2. An edge of G is said to be a *fixed single bond* (resp. *fixed double bond*) if it belongs to none (resp. all) of the perfect matchings of G. An edge of G is called a *fixed bond* if it is either a fixed single bond or a fixed double bond. It is well known that the symmetric difference of two perfect matchings $M_1 \oplus M_2 = (M_1 \cup M_2)\backslash(M_1 \cap M_2)$ of G is a union of disjoint (M_1, M_2)-alternating cycles of G. In this paper, we assume that all vertices of a plane bipartite graph G are colored white and black such that adjacent vertices received distinct colors.

Lemma 2.1 (*Shiu et al. [8]*). *Let G be a plane bipartite graph with a perfect matching. If all vertices with degree one of G are of the same color and lie on the boundary of G or if $\delta(G) \geqslant 2$, then for any perfect matching M of G, there is an M-resonant finite face in G.*

From Lemma 2.1, we can see that if a plane bipartite graph in Ω has a perfect matching, then it has at least two perfect matchings. A finite face s of a plane bipartite graph $G \in \Omega$ is forcing if and only if G has exactly two different perfect matchings M_i, $1 \leqslant i \leqslant 2$, such that s is an M_i-resonant face. It is clear that the symmetric difference $M_1 \oplus M_2 = \partial s$.

It is well known [7] that an elementary bipartite graph G with more than two vertices is 2-connected. It is also known [8] that a bipartite graph G is elementary if and only if it is connected and each edge of G belongs to a perfect matching of G; if and only if G is connected and has no fixed single bonds. Let G be a connected bipartite graph with a perfect matching. The connected components of the subgraph of G formed by all non-fixed bonds are elementary and thus called *elementary components* of G. The following lemma can be derived directly from Corollary 3.4 in [8].

Lemma 2.2. *Let G be a connected plane bipartite graph with a perfect matching and $\delta(G) \geqslant 2$. If G is not elementary, then it has at least two elementary components, each of which has more than two vertices.*

It was shown [12] that if G is a connected plane bipartite graph with more than two vertices, then G is elementary if and only if each face (including the infinite face) of G is resonant. In particular, if all the interior vertices of G are of the same degree, then G is elementary if and only if the infinite face of G is resonant. The following lemmas provide more properties of a plane elementary bipartite graph in terms of resonant faces.

Lemma 2.3 (*Zhang and Zhang [12]*). *Let G be a plane elementary bipartite graph with a perfect matching M. If there exist three distinct M-resonant finite faces, then there are two of them whose boundaries are disjoint.*

Lemma 2.4 (*Zhang and Zhang [12]*). *Let G be a plane elementary bipartite graph with a perfect matching M and let C be an M-alternating cycle. Then there exists an M-resonant face in the interior of C.*

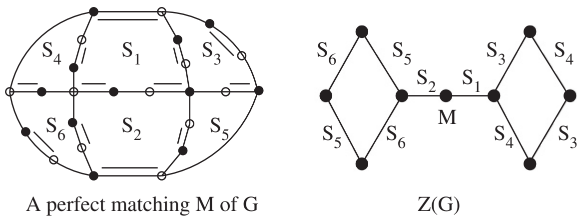

Fig. 2. A plane elementary bipartite graph G and its Z-transformation graph $Z(G)$.

It is well known [6,7] that an elementary bipartite graph has an "ear decomposition" as described below. Start from an edge e, and join its two end vertices by a path P_1 of odd length (called the "first ear"). Then proceed inductively to build a sequence of bipartite graphs as follows: if $G_i = e + P_1 + \cdots + P_i$ has already been constructed, add the $(i+1)$th ear P_{i+1} of odd length by joining any two vertices in different colors of G_i such that P_{i+1} has no internal vertices in common with the vertices of G_i. The decomposition $G = G_n = e + P_1 + P_2 + \cdots + P_n$ is called a *bipartite ear decomposition* of G. It was shown [6,7] that a bipartite graph is elementary if and only if it has a bipartite ear decomposition.

As defined in [12], a bipartite ear decomposition of a plane elementary bipartite graph G is called a *reducible face decomposition* (abbreviated RFD) if G_1 is the boundary of a finite face (s_1) of G, and the $(i+1)$th ear P_{i+1} lies in the exterior of G_i such that P_{i+1} and a part of the periphery of G_i surround a finite face (s_{i+1}) of G for all $1 \leqslant i < n$. So, the RFD $(G_1, G_2, \ldots, G_n(=G))$ is associated with a unique face sequence $s_1, s_2, \ldots, s_n$. A useful property of the RFD is that $\partial G_i \oplus \partial G_{i+1} = \partial s_{i+1}$ for all $1 \leqslant i < n$.

Lemma 2.5 (*Zhang and Zhang [12]*). *Let G be a plane bipartite graph with more than two vertices. Then G is elementary if and only if it has a reducible face decomposition.*

For example, the plane elementary bipartite graph G in Fig. 2 has an RFD $(G_1, G_2, \ldots, G_6)$ associated with the face sequence $s_1, s_2, \ldots, s_6$.

Let G be a plane bipartite graph with a perfect matching. The *Z-transformation graph* of G, denoted by $Z(G)$, is the graph whose vertices are the perfect matchings of G where two vertices M_1 and M_2 are adjacent if and only if their symmetric difference $M_1 \oplus M_2$ is the boundary of some finite face of G. For example, in the Z-transformation graph $Z(G)$ of G in Fig. 2, where an edge between two vertices in $Z(G)$ is marked by the finite face whose boundary is the symmetric difference of the two perfect matchings corresponding to the two vertices.

Lemma 2.6 (*Zhang and Zhang [12]*). *Let G be a plane elementary bipartite graph. Then*

(i) $Z(G)$ *is a connected bipartite graph*,

(ii) $Z(G)$ *has at most two vertices of degree one*, *and*

(iii) *if* $Z(G)$ *has a vertex of degree* $\geqslant 3$, *then the girth of* $Z(G)$ *is* 4; *otherwise*, $Z(G)$ *is a path*.

Let M be a perfect matching of G. Then it is easy to see that the degree of M in $Z(G)$ is the number of M-resonant finite faces in G. Therefore, a perfect matching M has degree one in $Z(G)$ if and only if G has exactly one M-resonant finite face.

Lemma 2.7 (*Zhang and Zhang [12]*). *Let G be a plane elementary bipartite graph. Then the following statements are equivalent.*

(i) $Z(G)$ *has a vertex M of degree one*.

(ii) *G has an RFD* $(G_1, G_2, \ldots, G_n(=G))$ *such that each ear* P_i *starts with a black vertex and ends with a white vertex or each ear* P_i *starts with a white vertex and ends with a black vertex with respect to the clockwise orientation of the periphery of* G_i, $2 \leqslant i \leqslant n$.

(iii) *G has an RFD* $(G_1, G_2, \ldots, G_n(=G))$ *such that the periphery of each* G_i $(1 \leqslant i \leqslant n)$ *is an M-alternating cycle*.

Note that when (i) or (ii) holds, G_1 is the unique M-resonant finite face.

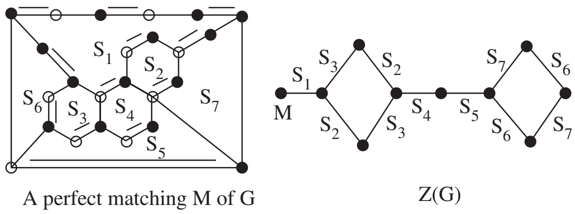

Fig. 3. A perfect matching M of G with degree one in $Z(G)$.

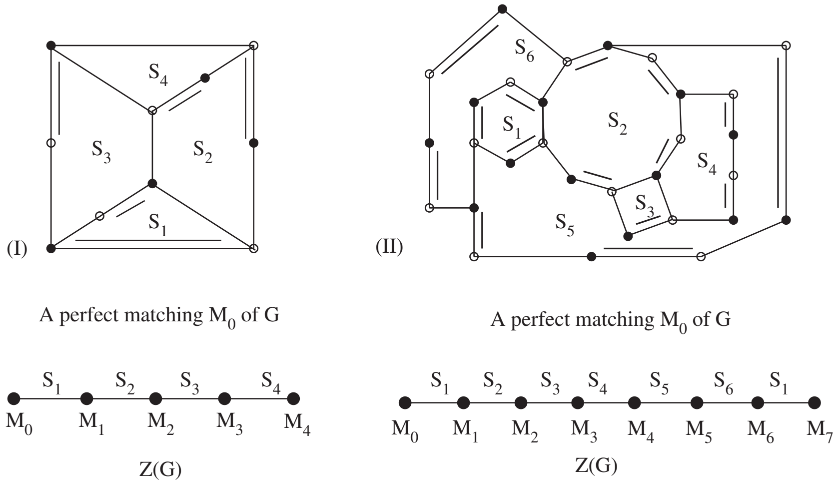

Fig. 4. Plane bipartite graphs whose $Z(G)$ is a path.

For example, the plane elementary bipartite graph G in Fig. 3 has a perfect matching M with degree one in $Z(G)$ and it has an RFD $(G_1, G_2, \ldots, G_7)$ associated with the face sequence $s_1, s_2, \ldots, s_7$ and ear sequence $P_1, P_2, \ldots, P_7$ such that each ear P_i starts with a white vertex and ends with a black vertex w.r.t. the clockwise orientation of the periphery of G_i, for $2 \leqslant i \leqslant 7$. But the plane elementary bipartite graph G in Fig. 2 does not have a perfect matching M with degree one in $Z(G)$.

Lemma 2.8 (*Zhang and Zhang [12]*). *Let G be a plane elementary bipartite graph with more than two vertices. Then $Z(G)$ is a path if and only if G has an RFD $(G_1, G_2, \ldots, G_n(=G))$ associated with the face sequence $s_1, s_2, \ldots, s_n$ and the ear sequence $P_1, P_2, \ldots, P_n$ such that*

(i) *the P_i's start with black (resp. white) vertices and end with white (resp. black) vertices w.r.t. the clockwise orientation of the boundaries of the G_i's*;

(ii) *s_i and s_{i+1} have edges in common for all i*; *and*

(iii) *s_1 is a periphery face of $G_n(=G)$ or G_{n-1}.*

For example, $Z(G)$ is a path for each plane elementary bipartite graph G in Fig. 4. The graph G in Fig. 4(I) has an RFD (G_1, G_2, G_3, G_4) associated with the face sequence s_1, s_2, s_3, s_4 and satisfying the three conditions in Lemma 2.8, where s_1 is a periphery face of $G_4(=G)$ in condition (iii). The graph G in Fig. 4(II) has an RFD $(G_1, G_2, G_3, G_4, G_5, G_6)$ associated with the face sequence $s_1, s_2, s_3, s_4, s_5, s_6$ and satisfying the three conditions in Lemma 2.8, where s_1 is a periphery face of G_5 but not of $G_6(=G)$ in condition (iii).

An edge of a plane bipartite graph G is called a *forcing edge* if it is contained in exactly one perfect matching of G. It was shown [12] that if a plane bipartite graph in Ω has a forcing edge, then it is elementary. The following two lemmas give characterizations of a plane bipartite graph with forcing edges.

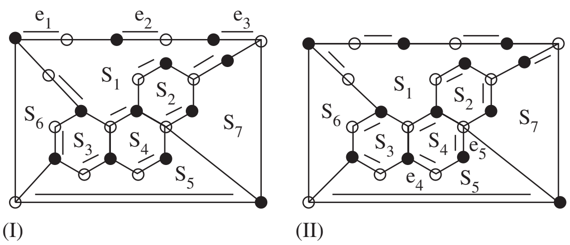

Fig. 5. Forcing edges of a plane bipartite graph G. (I) A perfect matching M_1 of G. (II) A perfect matching M_2 of G.

Lemma 2.9 (*Zhang and Zhang [12]*). *Let G be a plane bipartite graph with a perfect matching M. Then $e \in M$ is a forcing edge if and only if each M-alternating cycle passes through the edge e.*

Lemma 2.10 (*Zhang and Zhang [12]*). *Let G be a plane elementary bipartite graph with more than two vertices. Then G has a forcing edge if and only if one of the following statements holds*:

(i) *$Z(G)$ has a vertex M of degree one such that the unique M-resonant finite face s is a periphery face of G. See Fig.* 5(I).

(ii) *$Z(G)$ has a vertex M of degree two such that the two M-resonant finite faces of G have a path in common and the periphery of G is not an M-alternating cycle. See Fig.* 5 (II).

Remarks. (1) When (i) holds, if P is a maximal common path of ∂s and ∂G, then P is an M-alternating path with two end edges in M, and the edges of P belonging to M are the forcing edges of G. Furthermore, the periphery of G is an M-alternating cycle by Lemma 2.7.

(2) When (ii) holds, if P is a maximal common path of ∂s and $\partial s'$, then P is an M-alternating path with two end edges in M, and the edges of P belonging to M are the forcing edges of G.

For example, the plane bipartite graph G in Fig. 5 has forcing edges e_1, e_2, e_3, e_4, e_5. In Fig. 5(I), the perfect matching M_1 of G has degree one in $Z(G)$ where s_1 is the unique M_1-resonant finite face, and edges e_1, e_2, e_3 belonging to M_1 are on a common path of ∂s_1 and ∂G. Furthermore, ∂G is an M_1-alternating cycle. In Fig. 5(II), the perfect matching M_2 of G has degree two in $Z(G)$ where s_4, s_5 are the two adjacent M_2-resonant finite faces, and edges e_4, e_5 belonging to M_2 are on a common path of ∂s_4 and ∂s_5. Furthermore, ∂G is not an M_2-alternating cycle.

3. Main results

Theorem 3.1. *If a plane bipartite graph $G \in \Omega$ has a forcing face, then G is elementary.*

Proof. It is trivial when G has exactly one finite face. Assume that G has at least two finite faces. Suppose that G is not elementary. Then it has at least two elementary components each of which has more than two vertices by Lemma 2.2. Let s be a forcing face of G. Then s must be in some elementary component of G. Recall that each elementary component of G with more than two vertices is 2-connected, and so has at least two perfect matchings by Lemma 2.1. It follows that $G - s$ has at least two perfect matchings, which is a contradiction. □

Recall that any plane elementary bipartite graph with more than two vertices is in Ω. By Theorem 3.1, only plane elementary bipartite graphs with more than two vertices can have forcing faces. On the other hand, it needs to be pointed out that not every plane elementary bipartite graph with more than two vertices has a forcing face. In fact, we have proved in [2] the following results on the forcing hexagons in a hexagonal system H.

(i) A hexagon h of H is forcing if and only if h is a periphery hexagon of H and there is perfect matching M of H such that h is an M-alternating hexagon and M has degree one in $Z(H)$.

(ii) A hexagon h in H is forcing if and only if h contains a forcing edge.

(iii) A linear hexagonal chain has all its hexagons forcing, and other hexagonal systems may have 0, 1 or 2 forcing hexagons. (See Figs. 6–9.)

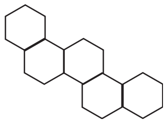

Fig. 6. A hexagonal system with no forcing hexagons.

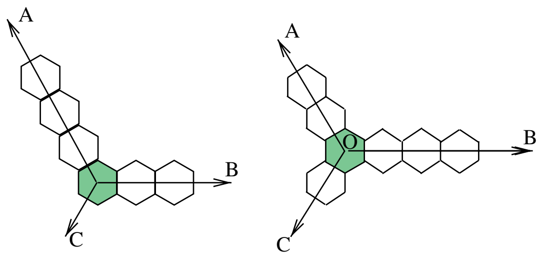

Fig. 7. A hexagonal system with exactly one forcing hexagon.

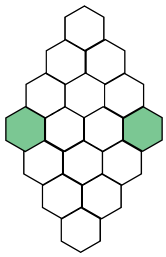

Fig. 8. A hexagonal system with exactly two forcing hexagons.

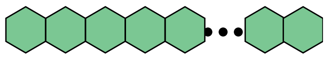

Fig. 9. A linear hexagonal chain with all hexagons forcing.

Note that these properties of the forcing hexagons in a hexagonal system cannot be extended to the forcing faces in a general plane bipartite graph.

For example, in Fig. 1, the finite face s is a forcing face of the plane bipartite graph G, but s is not a periphery face of G, and there are no perfect matchings with degree one in $Z(G)$ at all. Also G does not have any forcing edges. In Fig. 5, the plane bipartite graph G has three forcing faces s_1, s_4, s_5 out of seven finite faces of G.

However, for general plane bipartite graphs, we have the following theorem on the number of forcing faces.

Theorem 3.2. *For any integers n and k with $n \geqslant 4$ and $0 \leqslant k \leqslant n$, there exists a plane elementary bipartite graph G such that exactly k of the n finite faces of G are forcing.*

Proof. Let $n \geqslant 4$. First, we can easily see the theorem holds for the cases $k = 0$, 1 and n. (See Figs. 6, 7 and 9.)

Now we prove the theorem for $n \geqslant 4$ and $2 \leqslant k \leqslant n - 1$ by constructing a desired graph G as in Fig. 10, where $j = n - k + 1$. Since $2 \leqslant k \leqslant n - 1$ and $n \geqslant 4$, then $2 \leqslant j \leqslant n - 1$ and hence the graph can be constructed. It is easy to see that G has an RFD($G_1, G_2, \ldots, G_n (= G)$) associated with the face sequence $s_1, s_2, \ldots, s_n$. By Lemma 2.5, G is a plane elementary bipartite graph. By direct verification we can see that exactly k of the n finite faces of G, $s_j, s_{j+1}, \ldots, s_n$, are forcing. $\square$

Note: Let k denote the number of forcing faces in a plane elementary bipartite graph G with $n (> 0)$ finite faces and a perfect matching. Theorem 3.2 shows that k can take any integer from 0 to n when $n \geqslant 4$. Here we point out that it is

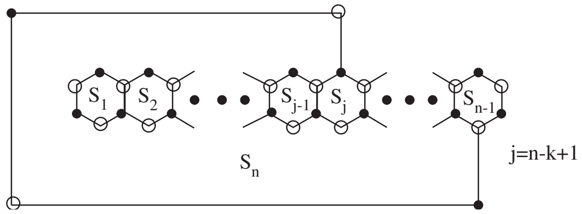

Fig. 10. A plane bipartite graph with n finite faces and k forcing faces.

not the case for $0 < n < 4$, since then we must have $k > 0$. It is not difficult to see that k must be equal to n when $n = 1$ or 2, and k can take any integer from 1 to n when $n = 3$.

Concerning cubic plane bipartite graphs, we have the following result, which is a direct corollary of a main theorem in the recent paper [1] by Bau and Henning.

Corollary 3.3. *Any connected cubic plane bipartite graph has no forcing faces.*

The original proof for the theorem of Bau and Henning is quite long. Here we give a shorter proof below.

Theorem 3.4 (*Bau and Henning*). *If G is a connected cubic plane bipartite graph and M is any perfect matching of G, then G has at least two disjoint M-resonant faces* (*one of which could be the infinite face*).

Proof. By Lemma 2.1, G has an M-resonant finite face s. Then the restriction of M on $G_1 = G - s$ is a perfect matching of G_1. It implies that there are no isolated vertices in G_1. Hence each vertex in G_1 is adjacent to at most two vertices on the boundary of s. We distinguish two cases.

Case 1: There is a vertex u_1 in G_1 that is adjacent to two vertices, say v and w, on the boundary of s. Since G is bipartite, v and w are of the same color and cannot be adjacent. Then they divide the boundary of s into two v–w paths of even length. One of the two paths, say P_1, together with the path vu_1w encloses a plane region R of G outside s. It is easy to see that R is not finite a face of G. The vertices on the boundary of R are vertices from P_1 and u_1. We may further assume that there is no vertex in R that is adjacent to two inner vertices of P_1. Let x be the third vertex of G that is adjacent to u_1. Then u_1x must be an M-double bond.

Subcase 1.1: The vertex x is outside R. Let H denote the part of $G_1 - u_1$ contained inside R. Then $\delta(H) \geqslant 2$. Note that the restriction of M on H is a perfect matching of H. By Lemma 2.1, there is an M-resonant finite face s_1 of H, which is also a finite face of G and disjoint from s. It is done.

Subcase 1.2: The vertex x is inside R. Let H denote the part of $G_1 - u_1 - x$ contained inside R. Then the restriction of M on H is a perfect matching of H. Furthermore, if there is a vertex z in H which is adjacent to both x and an inner vertex of P_1, then $\deg(z) = 1$ in H and z must be the colored differently from x and lie on the boundary of H; if there are no vertices of H that are adjacent to both x and some inner vertex of P_1, then $\delta(H) \geqslant 2$. By Lemma 2.1, there is an M-resonant finite face s_1 of H, which is also a finite face of G and disjoint from s. It is done.

Case 2: Each vertex in G_1 is adjacent to at most one vertex on the boundary of s. Then $\delta(G_1) \geqslant 2$. It is easy to see that the restriction of M on G_1 is a perfect matching of G_1. By Lemma 2.1, there is an M-resonant finite face s_1 of G_1, which is disjoint from s.

Subcase 2.1: If s_1 is also a finite face of G, then it is done.

Subcase 2.2: If the boundary of s_1 is the boundary of G, then s_1 is the infinite face of G, and it is done.

Subcase 2.3: If s_1 is neither a finite face nor the infinite face of G, then ∂s_1 is an M-alternating cycle in G different from ∂G and s is contained in the interior of s_1. Let G_2 be the subgraph of G obtained by deleting all vertices contained inside s_1 and on ∂s_1, together with all their incident edges. Then the restriction of M on G_2 is a perfect matching of G_2. It implies that there are no isolated vertices in G_2. Hence each vertex in G_2 is adjacent to at most two vertices on the boundary of s_1. Again, we can distinguish two cases:

(i) If there is a vertex u_2 in G_2 that is adjacent to two vertices, say v_1 and w_1 on the boundary of s_1, then similar to Case 1, we can get an M-resonant finite face s_2 of G, which is disjoint from s.

(ii) If each vertex in G_2 is adjacent to at most one vertex on the boundary of s_1, then there is an M-resonant finite face s_2 of G_2. If s_2 is also an M-resonant face of G (it is possible the infinite face of G) as subcases 2.1 and 2.2, then it is done. Otherwise, ∂s_2 is an M-alternating cycle in G which is different from ∂G and s_1 is contained in the interior of s_2. Let G_3 be the subgraph of G obtained by deleting all vertices contained inside s_2 and on ∂s_2, together with all their incident edges. Then the restriction of M on G_3 is a perfect matching of G_3. Continuing the process, either we get another M-resonant face s_i of G (which may be the infinite face of G), or we get a sequence of subgraphs $G_1, G_2, \ldots, G_n$ where $G_{i+1} \subseteq G_i$, the restriction of M on G_i is a perfect matching of G_i, and $\delta(G_i) \geqslant 2$ since each vertex of G_i is adjacent to at most one vertex on the boundary of s_{i-1}. Note that G is finite, this sequence must stop at some n and G_n a cycle, which is the boundary of G. Recall that the restriction of M on $G_n(=\partial G)$ is a perfect matching of $G_n(=\partial G)$. Then the infinite face of G is an M-resonant face, which is disjoint from s. □

Now we give a necessary and sufficient condition for a finite face in a plane elementary bipartite graph to be forcing.

Theorem 3.5. *Let G be a plane elementary bipartite graph. Then a finite face s of G is forcing if and only if there is a perfect matching M of G such that s is M-resonant and each M-alternating cycle of G has at least one edge in common with ∂s.*

Proof. It is trivial when s is the unique finite face of G. Let G be a plane elementary bipartite graph with at least two finite faces. Necessity is easily seen from the definition of a forcing face. We show the sufficiency by contradiction. If s is not a forcing face, then $G - s$ has at least two different perfect matchings. It implies that there is a perfect matching M' different from M such that s is an M'-resonant face and $M \oplus M'$ is different from ∂s. Recall that $M \oplus M'$ consists of mutually disjoint (M, M')-alternating cycles of G. It follows that there exists an M-alternating cycle which is disjoint from ∂s. This contradicts the hypothesis. □

Lemma 3.6. *Let G be a plane elementary bipartite graph. If $Z(G)$ is a path, then for any perfect matching M with degree two in $Z(G)$, the two M-resonant finite faces are adjacent.*

Proof. Let s_1, s_2 be the two M-resonant finite faces. If s_1 and s_2 are not adjacent, then ∂s_1 and ∂s_2 do not have common edges. Let $M_1 = M \oplus \partial s_1$, $M_2 = M_1 \oplus \partial s_2$ and $M_3 = M_2 \oplus \partial s_1$. Then $M = M_3 \oplus \partial s_2$. Hence, $MM_1M_2M_3M$ is a 4-cycle in $Z(G)$. It contradicts the hypothesis that $Z(G)$ is a path. □

For a hexagonal system H, we proved in [2] that if H has a perfect matching M such that M is of degree two in $Z(H)$ and the two M-resonant hexagons are adjacent, then $Z(H)$ is a path. We also showed that every hexagon of a hexagonal system H is forcing if and only if H is a linear hexagonal chain; if and only if $Z(H)$ is a path [2]. We have discovered that these properties cannot be extended to arbitrary plane bipartite graphs. For example, the perfect matching M of G in Fig. 2 has degree two in $Z(G)$ and G has two adjacent M-resonant finite faces s_1 and s_2, but $Z(G)$ is not a path. On the other hand, if G is a plane elementary bipartite graph whose $Z(G)$ is a path, then it is not necessary that each finite face of G is forcing. For example, it is easy to check that the finite face s_1 is not a forcing face of G in Fig. 4(II), though $Z(G)$ is a path. Below we will first investigate plane elementary bipartite graphs whose finite faces are all forcing.

Lemma 3.7. *Let G be a plane elementary bipartite graph. If each finite face of G is forcing, then $Z(G)$ is a path.*

Proof. By contradiction. If $Z(G)$ is not a path, then by Lemma 2.6(iii), there is a perfect matching M of G such that M has degree at least three in $Z(G)$. So G has least three distinct M-resonant finite faces, say, s_1, s_2 and s_3. By Lemma 2.3, there are two of them, say s_2 and s_3, whose boundaries are disjoint from each other. It implies that s_2, s_3 cannot be forcing faces of G, which contradicts the hypothesis that each finite face of G is forcing. Therefore, $Z(G)$ is a path. □

Lemma 3.8. *Let G be a plane elementary bipartite graph with more than two vertices. If $Z(G)$ is a path, then G has an RFD $(G_1, G_2, \ldots, G_n(=G))$ associated with the face sequence $s_1, s_2, \ldots, s_n$ satisfying the three conditions in Lemma* 2.8. *Furthermore*,

(1) *If s_1 is not a periphery face of G, then s_1 is not a forcing face.*
(2) *If s_1 is a periphery face of G, then each finite face of G is forcing.*

Proof. Since $Z(G)$ is a path, then by Lemma 2.8, G has an RFD $(G_1, G_2, \ldots, G_n(=G))$ associated with the finite face sequence $s_1, s_2, \ldots, s_n$ and the ear sequence $P_1, P_2, \ldots, P_n$ satisfying the three conditions in Lemma 2.8, namely, (i) the P_i's start with black (resp. white) vertices and end with white (resp. black) vertices w.r.t. the clockwise orientation of the boundaries of the G_i's; (ii) s_i and s_{i+1} have edges in common for all i; and (iii) s_1 is a periphery face of $G_n(=G)$ or G_{n-1}.

By Lemma 2.7, there is a perfect matching M_0 of G that has degree one in $Z(G)$ such that $s_1(=G_1)$ is the unique M_0-resonant finite face and each ∂G_i is an M_0-alternating cycle for $1 \leqslant i \leqslant n$.

(1) Assume that s_1 is not a periphery face of $G_n(=G)$. Then s_1 is not forcing by Theorem 3.5 since the M_0-alternating cycle ∂G does not have any edge in common with ∂s_1.

(2) Assume that s_1 is a periphery face of $G_n(=G)$. If s_1 is the unique finite face of G, then trivially s_1 is forcing. Hence we may assume that G has more than one finite face. Let $L_{i-1,i}$ be the part of ∂G_{i-1} that surrounds the finite face s_i with the ear P_i, where $2 \leqslant i \leqslant n$. Let $\overline{\partial G_{i-1}}$ be the path obtained from ∂G_{i-1} by removing all the edges and interior vertices of $L_{i-1,i}$. Recall that $s_1(=G_1)$ is the unique M_0-resonant finite face and each ∂G_i is an M_0-alternating cycle for $1 \leqslant i \leqslant n$. Then, for each i with $2 \leqslant i \leqslant n$, $L_{i-1,i}$, P_i and $\overline{\partial G_{i-1}}$ are three M_0-alternating paths, where the two end edges of $\overline{\partial G_{i-1}}$ are M_0-double bonds and all the two end edges of $L_{i-1,i}$ and P_i are M_0-single bonds. To show that each finite face of G is forcing, we first prove the following claim.

Claim. *Any perfect matching of G is either M_0 or can be written as $M_i = M_0 \oplus \partial G_i = M_0 \oplus \partial s_1 \oplus \partial s_2 \oplus \cdots \oplus \partial s_i$ for some $1 \leqslant i \leqslant n$.*

The claim can be proved as follows. For $1 \leqslant i \leqslant n$, let $M_i = M_0 \oplus \partial G_i$. It is easy to see that $M_i = M_0 \oplus \partial s_1 \oplus \partial s_2 \oplus \cdots \oplus \partial s_i$. Consider $M_n = M_0 \oplus \partial G_n(=M_0 \oplus \partial G)$. It is clear that $L_{n-1,n}$ is an M_n-alternating path whose two end edges are M_n-single bonds since $L_{n-1,n}$ has no edges on ∂G_n, and P_n is an M_n-alternating path whose two end edges are M_n-double bonds since P_n is on ∂G_n. Then, s_n is an M_n-resonant finite face since ∂s_n consists of $L_{n-1,n}$ and P_n. We will show that $\deg(M_n) = 1$ by contradiction. Suppose the contrary holds. Then since $Z(G)$ is a path, we must have $\deg(M_n) = 2$. Let s_j $(1 \leqslant j < n)$ be the other M_n-resonant finite face. Then s_j is a periphery face of G, since M_0 and M_n have no difference on any non-periphery face and all non-periphery faces are not M_0-resonant, and so all non-periphery faces are not M_n-resonant either. It is easy to see that s_1 cannot be an M_n-resonant facebecause s_1 is M_0-resonant and only a non-empty proper part (not all) of ∂s_1 is on ∂G since s_1 is a periphery face of G. Hence s_j cannot be s_1. Now $2 \leqslant j \leqslant n-1$. Recall that s_j is a periphery face of G. Then only a non-empty proper part of P_j is on ∂G since ∂s_j and ∂s_{j+1} have common edges. Consider each s_j for $2 \leqslant j \leqslant n-1$. Recall that P_j is an M_0-alternating path. Then P_j cannot be an M_n-alternating path, and so s_j cannot be an M_n-resonant face either for all $2 \leqslant j \leqslant n-1$. This is a contradiction. Thus, we have proved that $\deg(M_n) = 1$ and s_n is the unique M_n-resonant face.

Now, we have seen that both M_0 and M_n have degree one in the path $Z(G)$. Since $M_i = M_0 \oplus \partial G_i$ and $\partial G_i \oplus \partial G_{i+1} = \partial s_{i+1}$, we have $M_i \oplus M_{i+1} = \partial s_{i+1}$, namely, M_i is adjacent with M_{i+1} for each $i = 0, 1, \ldots, n-1$. It implies that M_i, $i = 0, 1, \ldots, n$ are all the perfect matchings of G. This completes the proof of the claim.

Next, we will show that each s_i is forcing for $1 \leqslant i \leqslant n$.

(2a) We first show that s_i is forcing when $i = 1, n$. Let M be a perfect matching of G such that s_1 is an M-resonant face. If $\deg(M) = 1$ in $Z(G)$, then $M = M_0$ and ∂G is an M-alternating cycle. Let P be a common path of ∂s_1 and ∂G. Then the edges in P belonging to M are forcing edges of G by Remark (1) following Lemma 2.10. It implies that any M-alternating cycle has at least one common edge with ∂s_1 by Lemma 2.9. If M has degree two in $Z(G)$, then $M = M_0 \oplus \partial s_1$ and the two M-resonant faces are s_1 and s_2. Note that ∂G is an M_0-alternating cycle and s_1 is a periphery face of G. It is easy to see that ∂G is not an M-alternating cycle. Let P' be a common path of ∂s_1 and ∂s_2. Then all the edges in P' belonging to M are forcing edges of G by Remark (2) following Lemma 2.10. It follows that each M-alternating cycle has at least one common edge with ∂s_1 by Lemma 2.9. Therefore, s_1 is forcing by Theorem 3.5. Similarly, we can show that s_n is forcing.

(2b) Now we prove that s_i is forcing for any given $1 < i < n$. Let M be a perfect matching of G such that s_i is an M-resonant face. By the proof of the Claim, either $M = M_{i-1} = M_0 \oplus \partial G_{i-1}$ or $M = M_i = M_0 \oplus \partial G_i$. Therefore, $\deg(M) = 2$ and ∂G is not an M-alternating cycle. If $M = M_{i-1}$, then s_{i-1} and s_i are the two M-resonant finite faces. Let P be a common path of ∂s_{i-1} and ∂s_i. If $M = M_i$, then s_i and s_{i+1} are the two M_i-resonant finite faces. Let P be a common path of ∂s_i and ∂s_{i+1}. Then all the edges of P that belong to M are forcing edges of G by Remark (2) following Lemma 2.10. It follows that each M-alternating cycle has at least one common edge with ∂s_i by Lemma 2.9. Therefore, s_i is forcing by Theorem 3.5. $\square$

Theorem 3.9. *Let G be a plane elementary bipartite graph with more than two vertices. Then each finite face of G is forcing if and only if G has an RFD $(G_1, G_2, \ldots, G_n(=G))$ associated with the face sequence $s_1, s_2, \ldots, s_n$ and the ear sequence $P_1, P_2, \ldots, P_n$ satisfying*: (i) *the P_i's start with black* (*resp. white*) *vertices and end with white* (*resp. black*) *vertices w.r.t. the clockwise orientation of the boundaries of the G_i's*; (ii) *s_i and s_{i+1} have edges in common for all i*; *and* (iii) *s_1 is a periphery face of $G_n(=G)$*.

Proof. By Lemmas 3.7, 3.8 and 2.8. □

In the next theorem we will characterize the forcing faces in a plane elementary bipartite graph using the tool of Z-transformation graphs, which will enable us to find out whether the co-existence of forcing edges and forcing faces (i.e., forcing hexagons) in hexagonal systems is still valid for general plane bipartite graphs.

Theorem 3.10. *Let s be a finite face of a plane elementary bipartite graph G. Then s is a forcing face if and only if one of the following statements holds*:

(i) *$Z(G)$ has a vertex M of degree one such that s is the unique M-resonant finite face of G and s is a periphery face of G.* (*Note that the periphery of G must be an M-alternating cycle in this case by Lemma 2.7. For example, see Fig. 5*(I) *where* $\deg(M_1) = 1$ *in $Z(G)$ and the unique M_1-resonant finite face s_1 is the forcing face*.)

(ii) *$Z(G)$ has a vertex M of degree two such that s is one of the two M-resonant finite faces, and these two faces are adjacent. Furthermore, if the periphery of G is an M-alternating cycle, then s is periphery face of G.* (*For example, see Fig. 5*(II) *where* $\deg(M_2) = 2$ *in $Z(G)$ and the two M_2-resonant finite faces s_4 and s_5 are the forcing faces*.)

(iii) *$Z(G)$ has a vertex M of degree $n+1$ where $n \geqslant 2$ and s, s_i $(1 \leqslant i \leqslant n)$ are the M-resonant finite faces such that s is adjacent to each s_i for $1 \leqslant i \leqslant n$; s_i and s_j have disjoint boundaries whenever $1 \leqslant i \neq j \leqslant n$. Furthermore, if the periphery of G is an M-alternating cycle, then s is a periphery face of G.* (*For example, see Fig. 1 where* $\deg(M) = 4$ *in $Z(G)$ and exactly one of the M-resonant faces s is the forcing face*.)

Proof. *Necessity*: Let G be a plane elementary bipartite graph and s be a forcing face of G. Then s is an M-resonant finite face for some perfect matching M of G.

If $d(M) = 1$, then s is the unique M-resonant finite face. By Lemma 2.7, the periphery of G is an M-alternating cycle. By Theorem 3.5, s must be a periphery face of G.

If $d(M) = 2$, then the two M-resonant finite faces s, s' are adjacent by Theorem 3.5. If the periphery of G is an M-alternating cycle, then s is a periphery face by Theorem 3.5.

If $d(M) = n + 1 \geqslant 3$ and $s, s_1, s_2, \ldots, s_n$ are the M-resonant finite faces, then s is adjacent to each s_i for $1 \leqslant i \leqslant n$ by Theorem 3.5. Any two faces s_i, s_j $(1 \leqslant i \neq j \leqslant n)$ have disjoint boundaries since two of the three M-resonant faces s, s_i, s_j have disjoint boundaries by Lemma 2.3. If the periphery of G is an M-alternating cycle, then s is a periphery face by Theorem 3.5.

Sufficiency: (i) Assume that $Z(G)$ has a vertex M of degree one where the unique M-resonant finite face s is a periphery face of G. Let P be a common path of ∂s and ∂G. Let C be an arbitrary M-alternating cycle. Then there exists an M-resonant face in the interior of C by Lemma 2.4, which must be s since $d(M) = 1$. Hence C must pass through the path P, and so C has common edges with ∂s. Therefore, s is a forcing face by Theorem 3.5.

(ii) Assume that $Z(G)$ has a vertex M of degree two where the two M-resonant finite faces s, s' of G are adjacent.

Case 1: The periphery of G is not an M-alternating cycle. Let P be a common path of ∂s and $\partial s'$. By Remark (2) following Lemma 2.10, all the edges in P belonging to M are forcing edges of G. By Lemma 2.9, each M-alternating cycle C must pass through those forcing edges in P. Therefore, both s and s' are forcing faces by Theorem 3.5.

Case 2: The periphery of G is an M-alternating cycle. Then s is a periphery face of G by the hypothesis. We claim that each M-alternating cycle C must pass through either a common path of ∂s and ∂G, or a common path of ∂s and $\partial s'$. Otherwise, if there is an M-alternating cycle C that does not pass through any common path of ∂s and ∂G, or any common path of ∂s and $\partial s'$. Then neither s nor s' is contained in the interior of C. By Lemma 2.4, there is an M-resonant finite face different from s, s' in the interior of C. It is impossible since $\deg(M) = 2$. So, each M-alternating cycle C must pass through either a common path of ∂s and ∂G, or a common path of ∂s and $\partial s'$. Therefore, s is a forcing face of G by Theorem 3.5.

(iii) Assume that $Z(G)$ has a vertex M of degree $n + 1$ where $n \geqslant 2$ and s, s_i $(1 \leqslant i \leqslant n)$ are M-resonant finite faces such that s is adjacent to each s_i of G for $1 \leqslant i \leqslant n$, s_i and s_j have disjoint boundaries whenever $1 \leqslant i \neq j \leqslant n$.

Case 1: The periphery of G is not an M-alternating cycle. Then each M-alternating cycle C must pass through a common path of ∂s and ∂s_i for some $1 \leqslant i \leqslant n$. Otherwise, if there is an M-alternating cycle C does not pass through any common path of ∂s and ∂s_i for $1 \leqslant i \leqslant n$, then either both s and s_i $(1 \leqslant i \leqslant n)$ are all contained in the interior of C or none of them are contained in the interior of C. If none of them is contained in the interior of C, then there is an M-resonant finite face in the interior of C by Lemma 2.4, which is different from s and s_i $(1 \leqslant i \leqslant n)$. It is impossible since $\deg(M) = n + 1$. Hence, s and s_i $(1 \leqslant i \leqslant n)$ are all contained in the interior of C. Let $I[C]$ be the subgraph of G formed by C together with its interior. Then $I(C) \neq G$ since the periphery of G is not an M-alternating cycle. It is easy to see that there exists an M-alternating path P_1 in the exterior of C such that only its two end vertices belong to C. Note that the two end edges of P_1 must be M-single bonds, and so P_1 is of odd length. Let C_1 be the cycle formed by P_1 and part of C (denoted by P_C) and the interior of which lies in the exterior of C. Then P_C is also of odd length since C_1 must be an even cycle. By Lemma 2.4, C_1 cannot be an M-alternating cycle since s and s_i $(1 \leqslant i \leqslant n)$ are the only M-resonant finite faces of G. It implies that the two end edges of P_C are M-single bonds too. Hence, $C \oplus C_1$ is an M-alternating cycle. Also the interior of C is properly contained in the interior of $C \oplus C_1$. Continue this process, we finally have the conclusion that the periphery of G is an M-alternating cycle which contradicts the hypothesis. So, each M-alternating cycle C must pass through a common path of ∂s and ∂s_i for some $1 \leqslant i \leqslant n$. Therefore, s is a forcing face of G by Theorem 3.5.

Case 2. The periphery of G is an M-alternating cycle. Then s is a periphery face of G by the hypothesis. Similar to the proof for part (ii), we can show that each M-alternating cycle C must pass through either a common path of ∂s and ∂G or a common path of ∂s and ∂s_i for some $1 \leqslant i \leqslant n$. Therefore, s is a forcing face of G by Theorem 3.5. □

We have seen (from the example in Fig. 1) that a plane elementary bipartite graph with a forcing face does not necessarily have a forcing edge. By comparing Lemma 2.10 with Theorem 3.10(i) and (ii), we immediately get the following theorem.

Corollary 3.11. *Let G be a plane elementary bipartite graph with more than two vertices. If G has a forcing edge e, then each finite face of G containing e is a forcing face of G. On the other hand, G does not have to possess a forcing edge when it has a forcing face.*

We will conclude the paper by investigating when we can use subdivisions to transform a plane bipartite graph into a plane elementary bipartite graph such that every finite face is forcing. We will first introduce a concept called a *face decomposition* of a plane graph which generalizes the concept of reducible face decomposition of a plane bipartite graph: start from an edge e, and join its two end vertices by a path P_1 to get a finite face $s_1 (=G_1)$. Proceed inductively to build a sequence of plane graphs as follows: if $G_i = e + P_1 + P_2 + \cdots + P_i$ has already been constructed, add the $(i+1)$th path P_{i+1} by joining any two vertices on the boundary of G_i such that P_{i+1} lies in the exterior of G_i and has no internal vertices in common with the vertices of G_i. For all $1 \leqslant i < n$, P_{i+1} and a part of the periphery of G_i surround a finite face s_{i+1}. The decomposition $F(G_1, G_2, \ldots, G_n(=G))$ is called a face decomposition of G, which is associated with the path sequence $P_1, P_2, \ldots, P_n$ and the face sequence $s_1, s_2, \ldots, s_n$. Clearly, an RFD $(G_1, G_2, \ldots, G_n(=G))$ of G is a face decomposition of G, but not vice versa.

Theorem 3.12. *A plane bipartite graph G can be transformed by subdivisions to a plane elementary bipartite graph with every finite face forcing if and only if G has a face decomposition $F(G_1, G_2, \ldots, G_n(=G))$ associated with the face sequence $s_1, s_2, \ldots, s_n$ such that* (i) *s_1 is a periphery face of G and* (ii) *s_i and s_{i+1} have edges in common, for $1 \leqslant i \leqslant n-1$.*

Proof. The necessity is trivial by Theorem 3.9. So we only need to show the sufficiency. Assume that G has a face decomposition $F(G_1, G_2, \ldots, G_n(=G))$ associated with the face sequence $s_1, s_2, \ldots, s_n$ such that (i) s_1 is a periphery face of G and (ii) s_i, s_{i+1} have edges in common, for $1 \leqslant i \leqslant n-1$. Let $P_1, P_2, \ldots, P_n$ be the associated path sequence. Then the ith path P_i and a part of the periphery of G_{i-1} surround a finite face s_i for all $2 \leqslant i \leqslant n$. Let u_i (resp. v_i) be the starting vertex (resp. the ending vertex) of P_i along the clockwise orientation of the periphery of G_i.

Claim. *There is no vertex x that is both a starting vertex of a path P_j and an ending vertex of a path P_k for some $2 \leqslant j \neq k \leqslant n$.*

The claim can be proved by contradiction. Assume that there exists a vertex x that is both a starting vertex of a path P_j and an ending vertex of a path P_k. That is, $x = u_j = v_k$. Note that G_i is obtained from G_{i-1} by adding a path P_i that goes from u_i to v_i along the clockwise orientation of the periphery of G_i, and the two faces s_{i-1}, s_i have edges in common for each integer $2 \leqslant i \leqslant n$. If $2 \leqslant j < k \leqslant n$, then it is easy to see that s_1 is not a periphery face of G_k because part of $P_j \cup P_{j+1} \cup \cdots \cup P_{k-1} \cup P_k$ forms a cycle enclosing s_1 along clockwise orientation from P_j to P_k. So s_1 is not a periphery face of G. This contradicts the given condition (i). Thus we have $2 \leqslant k < j \leqslant n$. Then it is also easy to check that s_1 cannot be a periphery face of G_j because part of $P_k \cup P_{k+1} \cup \cdots \cup P_{j-1} \cup P_j$ forms a cycle enclosing s_1 along counterclockwise orientation from P_k to P_j. It follows that s_1 is not a periphery face of G, which is a contradiction. This completes the proof of the claim.

Now we can recolor the starting vertex u_i black and the ending vertex v_i white for each P_i, $2 \leqslant i \leqslant n$. Then we subdivide some edges of G which are selected as follows. For each $1 \leqslant i \leqslant n$, we take each path P that is a maximal common path of ∂s_i and ∂s_j where $i < j \leqslant n$. If s_i is a periphery face, we also take each path P that is a maximal common path of ∂s_i and ∂G. If P is of odd length and its two end vertices have the same color, or if P is of even length and its two end vertices have different colors, then subdivide an arbitrarily chosen edge on the path by adding one new vertex. Thus we get a new plane graph G^* whose finite faces s_i^* are the same as the finite faces s_i of G except that some edges on ∂s_i^* are obtained by subdivisions on some edges on ∂s_i satisfying the following: each maximal common path of ∂s_i^* and ∂s_j^* can be properly 2-colored if s_i^* and s_j^* have edges in common, and each maximal common path of ∂s_i^* and ∂G^* can be properly 2-colored if s_i^* is a periphery face of G^*. It follows that G^* is 2-colorable, i.e., G^* is a plane bipartite graph. Note that G has a face decomposition $F(G_1, G_2, \ldots, G_n(=G))$ associated with the face sequence $s_1, s_2, \ldots, s_n$ if and only if G^* has a face decomposition $F(G_1^*, G_2^*, \ldots, G_n^*(=G^*))$ associated with the face sequence $s_1^*, s_2^*, \ldots, s_n^*$; two faces s_i, s_{i+1} have edges in common in G if and only if s_i^*, s_{i+1}^* have edges in common in G^*; and s_1 is a periphery face of G if and only if s_1^* is a periphery face of G^*. Hence $F(G_1^*, G_2^*, \ldots, G_n^*(=G^*))$ is also a reducible face decomposition of G^* associated with the face sequence $s_1^*, s_2^*, \ldots, s_n^*$ and the ear sequence $P_1^*, P_2^*, \ldots, P_n^*$ satisfying the following three conditions: (i) the P_i^*'s start with black vertices and end with white vertices w.r.t. the clockwise orientation of the boundaries of the G_i^*'s; (ii) s_i^* and s_{i+1}^* have edges in common for all i; (iii) s_1^* is a periphery face of $G_n^*(=G^*)$. Then, by Theorem 3.9, every finite face of G^* is forcing. □

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