A second-order accurate finite difference scheme for a class of nonlocal parabolic equations with natural boundary conditions

Zhi-Zhong Sun

Department of Mathematics and Mechanics, Southeast University, Nanjing 210018, PR China

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Abstract

A difference scheme is derived for a class of nonlocal parabolic equations with natural boundary conditions by the method of reduction of order. It is shown that the scheme is uniquely solvable and unconditionally convergent with the convergence rate of order $O(h^2 + \varepsilon^2)$. A numerical example with some comparisons is presented.

Keywords: Finite difference; Parabolic; Convergence; Solvability; Nonlocal

AMS classification: 65M06, 65M12, 65M15, 65F05

1. Introduction

Lin and Tait [1] consider the finite difference solution to a class of nonlocal parabolic equation given by

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} + \varepsilon \int_a^b \rho \frac{\partial u}{\partial t}(\rho, t) d\rho, \quad 0 < a < x < b, \quad 0 < t \leq T \]  

subject to suitable initial and boundary conditions, where $\varepsilon$ is a parameter. The physical derivation of (1a) is described in detail in [2]. If the boundary/initial conditions are of the form

\[ u(a, t) = f(t), \quad u(b, t) = g(t), \quad 0 < t \leq T, \]

\[ u(x, 0) = \phi(x), \quad a \leq x \leq b, \]  

1 Supported by national natural science foundation of China.
a backward Euler scheme and a Crank–Nicolson scheme are presented in [1], with the former giving rise to an error $O(t^2 + h^2)$ and the latter to an error $O(t^2 + h^2)$. If the natural boundary conditions

\begin{align}
&u(a,t) = f(t), \quad \partial_x u(b,t) + u(b,t) = g(t), \quad 0 < t \leq T, \\
u(x,0) = \phi(x), \quad a \leq x \leq b
\end{align}

are imposed, a difference scheme whose convergence order is only one in space and in time is also presented. In this paper, we will develop a second-order accurate finite difference scheme for (1a) and (1c) by the method of reduction of order [3, 4]. The method is applicable to various mixed initial boundary value problems.

For generality, we consider, instead of (1a), the inhomogeneous equation

\begin{equation}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} + \varepsilon \int_a^b \frac{\partial u}{\partial t}(\rho, t) d\rho + \Phi(x,t), \quad 0 < a < x < b, \quad 0 < t \leq T.
\end{equation}

Let $M$ and $K$ be two integers and $h = (b - a)/M$ and $\tau = T/K$. Denote

$$\Omega_h = \{x_i \mid x_i = a + ih, \ 0 \leq i \leq M\}, \quad \Omega_\tau = \{t_k \mid t_k = kt, \ 0 \leq k \leq K\}.$$

If $u = \{u_i \mid 0 \leq i \leq M\}$ and $v = \{v_i \mid 0 \leq i \leq M\}$ are two mesh functions on $\Omega_h$, take the notations

$$x_{i-1/2} = (x_i + x_{i-1})/2, \quad t_{k-1/2} = (t_k + t_{k-1})/2,$$

$$u_{i-1/2} = (u_i + u_{i-1})/2, \quad \delta_x u_{i-1/2} = (u_i - u_{i-1})/h,$$

and define the inner product

$$\langle u, v \rangle = h \sum_{i=1}^M x_{i-1/2} u_{i-1/2} v_{i-1/2},$$

and the norm

$$\|u\| = \sqrt{\langle u, u \rangle}.$$

If $w = \{w_k \mid 0 \leq k \leq K\}$ is a mesh function on $\Omega_\tau$, denote

$$w_k^{k-1/2} = (w^k + w^{k-1})/2, \quad \delta_t w_k^{k-1/2} = (w^k - w^{k-1})/\tau.$$

Eqs. (1d) and (1c) may be rewritten as

\begin{align}
&x \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x}\right) + x \cdot \varepsilon \int_a^b \frac{\partial u}{\partial t}(\rho, t) d\rho + x\Phi(x,t), \quad 0 < a < x < b, \quad 0 < t \leq T, \\
u(a,t) = f(t), \quad b \frac{\partial u}{\partial x}(b,t) + bu(b,t) = bg(t), \quad 0 < t \leq T, \\
u(x,0) = \phi(x), \quad a \leq x \leq b.
\end{align}
Our finite difference scheme for (2) is as follows:

\[
\frac{1}{2} \left( x_{i-1/2} \delta_i u_{k-1/2}^{k-1/2} + x_{i+1/2} \delta_i u_{k+1/2}^{k-1/2} \right)
\]

\[
= \delta_x \left( x_i \delta_x u_i^{k-1/2} \right) + x_i \partial h \sum_{j=1}^{M} x_{j-1/2} \delta_j u_{j-1/2}^{k-1/2}
\]

\[
+ \frac{1}{2} \left( x_{i-1/2} \Phi_{i-1/2}^{k-1/2} + x_{i+1/2} \Phi_{i+1/2}^{k-1/2} \right), \quad 1 \leq i \leq M - 1, \quad 1 \leq k \leq K,
\]

(3a)

\[
u_0^{k-1/2} = \frac{1}{2} \left( f(t_k) + f(t_{k-1}) \right), \quad 1 \leq k \leq K,
\]

(3b)

\[
x_{M-1/2} \delta_i u_{M-1/2}^{k-1/2} = \frac{b}{h} \left\{ \frac{1}{2} \left( g(t_k) + g(t_{k-1}) \right) - x_{M-1/2} \delta_x u_{M-1/2}^{k-1/2} - b u_{M-1}^{k-1/2} \right\}
\]

\[
+ x_{M-1/2} \partial h \sum_{j=1}^{M} x_{j-1/2} \delta_j u_{j-1/2}^{k-1/2} + x_{M-1/2} \Phi_{M-1/2}^{k-1/2}, \quad 1 \leq k \leq K,
\]

(3c)

\[
u_i^0 = \phi(x_i), \quad 0 \leq i \leq M,
\]

(3d)

where \( \Phi_{i-1/2}^{k-1/2} = \Phi(x_{i-1/2}, t_{k-1/2}) \). If \( Q = (a, b) \times (0, T) \), we denote

\[C^{4,3}(Q) = \left\{ u | u, \frac{\partial^n u}{\partial x^n}, \frac{\partial^m u}{\partial t^m} \in C(Q), \quad n \leq 4, \quad m \leq 3 \right\}.
\]

The main result of this paper is Theorem 2 proved in Section 3.

2. The derivation of the difference scheme

Let

\[v = x (\partial u / \partial x),\]

then (2) is equivalent to

\[
x \frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} + x e \int_a^b \rho \frac{\partial u}{\partial t}(\rho, t) d\rho + x \Phi(x, t), \quad a < x < b, \quad 0 < t \leq T,
\]

\[
\frac{1}{x} v = \frac{\partial u}{\partial x}, \quad a < x < b, \quad 0 < t \leq T,
\]

\[
u(x, 0) = \phi(x), \quad a \leq x \leq b,
\]

\[
u(a, t) = f(t), \quad v(b, t) + bu(b, t) = bg(t), \quad 0 \leq t \leq T.
\]

(4)
Here no derivative boundary conditions occur explicitly. We construct a difference scheme for (4) as follows:

\[
x_{i-1/2} \delta_{t} u_{i-1/2}^{k-1/2} = \delta_{x} v_{i-1/2}^{k-1/2} + x_{i-1/2} \varepsilon h \sum_{j=1}^{M} x_{j-1/2} \delta_{t} u_{j-1/2}^{k-1/2} + x_{i-1/2} \Phi_{i-1/2}^{k-1/2},
\]

\[1 \leq i \leq M, \ 1 \leq k \leq K,
\]

\[
\frac{1}{x_{i-1/2}} v_{i-1/2}^{k-1/2} = \delta_{x} u_{i-1/2}^{k-1/2}, \quad 1 \leq i \leq M, \ 1 \leq k \leq K,
\]

\[u_{i}^{0} = \phi(x_{i}), \quad 0 \leq i \leq M,
\]

\[u_{0}^{k-1/2} = \frac{1}{2} (f(t_{k}) + f(t_{k-1})), \quad v_{M}^{k-1/2} + b u_{M}^{k-1/2} = \frac{b}{2} (g(t_{k}) + g(t_{k-1})), \quad 1 \leq k \leq K.
\]

At the \(k\)th time level, (5) is regarded as a system of linear algebraic equations with respect to \(\{u_{i}^{k}, v_{i}^{k-1/2}, 0 \leq i \leq M\}\). We have the following Theorem.

**Theorem 1.** The difference scheme (5) is equivalent to (3a) – (3d) and

\[
v_{i}^{k-1/2} = x_{i+1/2} \delta_{x} u_{i+1/2}^{k-1/2} - \frac{1}{2} h x_{i+1/2} \left( \delta_{x} u_{i+1/2}^{k-1/2} - \varepsilon h \sum_{j=1}^{M} x_{j-1/2} \delta_{t} u_{j-1/2}^{k-1/2} - \Phi_{i+1/2}^{k-1/2} \right),
\]

\[0 \leq i \leq M - 1, 1 \leq k \leq K,
\]

\[
v_{M}^{k-1/2} = \frac{b}{2} (g(t_{k}) + g(t_{k-1})) - b u_{M}^{k-1/2}, \quad 1 \leq k \leq K.
\]

**Proof.** For convenience, denote

\[s^{k-1/2} = \varepsilon h \sum_{j=1}^{M} x_{j-1/2} \delta_{t} u_{j-1/2}^{k-1/2}.
\]

It follows from (5a) that

\[\frac{1}{2} h \delta_{x} v_{i-1/2}^{k-1/2} = \frac{1}{2} h \left( x_{i-1/2} \delta_{x} u_{i-1/2}^{k-1/2} - x_{i-1/2} s^{k-1/2} - x_{i-1/2} \Phi_{i-1/2}^{k-1/2} \right), \quad 1 \leq i \leq M, \ 1 \leq k \leq K,
\]

and from (5b) that

\[v_{i-1/2}^{k-1/2} = x_{i-1/2} \delta_{x} u_{i-1/2}^{k-1/2}, \quad 1 \leq i \leq M, \ 1 \leq k \leq K.
\]

Adding (7a) and (7b) one arrives at

\[
v_{i}^{k-1/2} = x_{i-1/2} \delta_{x} u_{i-1/2}^{k-1/2} + \frac{1}{2} h \left( x_{i-1/2} \delta_{x} u_{i-1/2}^{k-1/2} - x_{i-1/2} s^{k-1/2} - x_{i-1/2} \Phi_{i-1/2}^{k-1/2} \right),
\]

\[1 \leq i \leq M, \ 1 \leq k \leq K.
\]
Subtracting (7a) from (7b) arrives at
\[
\begin{align*}
\frac{1}{\sqrt{2}} & \approx c_0 \approx \frac{1}{\sqrt{2}} \left( x_{i+1/2} \delta_x u_{i+1/2}^{k-1/2} - x_{i+1/2} s^{k-1/2} - x_{i+1/2} \Phi_{i+1/2}^{k-1/2} \right), \\
0 & \leq i \leq M - 1, \quad 1 \leq k \leq K.
\end{align*}
\]
Therefore, from Eqs (8a) and (8b) for \( i \) from 1 to \( M - 1 \), we have
\[
\begin{align*}
x_{i-1/2} \delta_x u_{i-1/2}^{k-1/2} & + \frac{1}{2} h \left( x_{i-1/2} \delta_x u_{i-1/2}^{k-1/2} - x_{i-1/2} s^{k-1/2} - x_{i-1/2} \Phi_{i-1/2}^{k-1/2} \right) \\
= x_{i+1/2} \delta_x u_{i+1/2}^{k-1/2} & - \frac{1}{2} h \left( x_{i+1/2} \delta_x u_{i+1/2}^{k-1/2} - x_{i+1/2} s^{k-1/2} - x_{i+1/2} \Phi_{i+1/2}^{k-1/2} \right),
\end{align*}
\]
or
\[
\begin{align*}
\frac{1}{2} \left( x_{i-1/2} \delta_x u_{i-1/2}^{k-1/2} + x_{i+1/2} \delta_x u_{i+1/2}^{k-1/2} \right) & = \delta_x \left( x_{i} \delta_x u_{i}^{k-1/2} \right) + x_{i} s^{k-1/2} \\
+ \frac{1}{2} \left( x_{i-1/2} \Phi_{i-1/2}^{k-1/2} + x_{i+1/2} \Phi_{i+1/2}^{k-1/2} \right), & \quad 1 \leq i \leq M - 1, \quad 1 \leq k \leq K.
\end{align*}
\]
This is (3a). Noticing the second equation of (5d), we know Eq. (8a) for \( i = M \) is equivalent to
\[
\begin{align*}
x_{M-1/2} \delta_x u_{M-1/2}^{k-1/2} & + \frac{1}{2} h \left( x_{M-1/2} \delta_x u_{M-1/2}^{k-1/2} - x_{M-1/2} s^{k-1/2} - x_{M-1/2} \Phi_{M-1/2}^{k-1/2} \right) + b u_{M}^{k-1/2} \\
= b & \left( g(t_k) + g(t_{k-1}) \right), \quad 1 \leq k \leq K,
\end{align*}
\]
or
\[
\begin{align*}
x_{M-1/2} \delta_x u_{M-1/2}^{k-1/2} & = \frac{2}{h} \left[ b \left( g(t_k) + g(t_{k-1}) \right) - x_{M-1/2} \delta_x u_{M-1/2}^{k-1/2} - b u_{M}^{k-1/2} \right] \\
+ x_{M-1/2} s^{k-1/2} + x_{M-1/2} \Phi_{M-1/2}^{k-1/2}, & \quad 1 \leq k \leq K.
\end{align*}
\]
This is (3c).

All equivalent relations are given below:
\[
\begin{align*}
(5a) \iff (7a) \iff \{ (8a) \} \equiv \{ (8b) \} \iff \{ (8a)(i = M) \} \iff (3c) \\
(5b) \iff (7b) \iff \{ (8a) \} \equiv \{ (8b) \} \iff \{ (8b) = (6a) \}
\end{align*}
\]
(5c) = (3d), \quad (5d) = \{ (3b) \}
\]
This completes the proof. \( \square \)

3. Convergence and solvability

**Theorem 2.** Let \( u \) be the solution of (2) and \( \{ u^k \} \) be the solution of (3) with \( u \in C^{4,3}(Q) \) and suppose \( \frac{1}{2} \varepsilon(b^2 - a^2) < 1 \). Then there exists a \( c_0 > 0 \) depending only on \( u \) such that
\[
\| \tilde{u}^k \| \leq c_0(h^2 + \tau^2), \quad 0 \leq k \leq K,
\]
where
\[ \hat{u}_i^k = u(x_i, t_k) - u_i^k, \quad 0 \leq i \leq M, \ 0 \leq k \leq K \]
and
\[ \hat{u}^k = (\hat{u}_0^k, \hat{u}_1^k, \ldots, \hat{u}_M^k). \]

**Proof.** From Theorem 1, it suffices to prove that the solution \( \{u_i^k\} \) of (5) convergences to the solution \( \{u(x_i, t_k)\} \) of (4) with the convergence order \( O(h^2 + \tau^2) \).

Denote
\[ \hat{v}_i^k = v(x_i, t_k) - v_i^k. \]
According to Taylor expansion and the theory of numerical integration, we may obtain the error equations of the difference scheme (5) as follows:

\[ x_{i-1/2} \delta_i \hat{u}_{i-1/2}^k = \delta_i \hat{u}_{i-1/2}^k + x_{i-1/2} \varepsilon h \sum_{j=1}^{M} x_{j-1/2} \delta_j \hat{u}_{j-1/2}^k + e_i^k, \quad 1 \leq i \leq M, \ 1 \leq k \leq K, \quad (9a) \]
\[ \frac{1}{x_{i-1/2}} \hat{v}_{i-1/2}^k = \delta_i \hat{u}_{i-1/2}^k + s_i^k, \quad 1 \leq i \leq M, \ 1 \leq k \leq K, \quad (9b) \]
\[ \hat{u}_i^0 = 0, \quad 0 \leq i \leq M, \quad (9c) \]
\[ \hat{u}_0^k = 0, \quad \hat{v}_M^k = 0, \quad 1 \leq k \leq K, \quad (9d) \]
where \( e_i^k \) and \( s_i^k \) are the truncation errors and there exists a constant \( c_1 \), dependent on \( \partial^4 u/\partial x^4 \) and \( \partial^3 u/\partial t^3 \) but independent of \( h \) and \( \tau \), such that
\[ |e_i^k| \leq c_1(h^2 + \tau^2), \quad |s_i^k| \leq c_1(h^2 + \tau^2). \quad (10) \]

Multiplying (9a) by \( 2\hat{u}_{i-1/2}^{k-1/2} \) and multiplying (9b) by \( 2\hat{v}_{i-1/2}^{k-1/2} \), then adding the results, we obtain
\[
\frac{1}{\tau} \left[ (\hat{u}_{i-1/2}^k)^2 - (\hat{u}_{i-1/2}^{k-1})^2 \right] + \frac{2}{x_{i-1/2}} (\hat{v}_{i-1/2}^{k-1/2})^2 \]
\[
= \frac{2}{h} \left( \hat{u}_i^{k-1/2} \hat{v}_i^{k-1/2} - \hat{u}_i^{k-1/2} \hat{v}_i^{k-1/2} \right) + 2x_{i-1/2} \hat{u}_{i-1/2}^{k-1/2} \left( \varepsilon h \sum_{j=1}^{M} x_{j-1/2} \delta_j \hat{u}_{j-1/2}^{k-1/2} \right) \]
\[
+ 2\hat{u}_{i-1/2}^{k-1/2} e_i^{k-1/2} + 2\hat{v}_{i-1/2}^{k-1/2} s_i^{k-1/2} \]
\[ \leq \frac{2}{h} \left( \hat{u}_i^{k-1/2} \hat{v}_i^{k-1/2} - \hat{u}_i^{k-1/2} \hat{v}_i^{k-1/2} \right) + 2x_{i-1/2} \hat{u}_{i-1/2}^{k-1/2} \left( \varepsilon h \sum_{j=1}^{M} x_{j-1/2} \delta_j \hat{u}_{j-1/2}^{k-1/2} \right) \]
\[ + x_{i-1/2} (\hat{u}_{i-1/2}^{k-1/2})^2 + \frac{1}{x_{i-1/2}} (e_i^{k-1/2})^2 + \frac{2}{x_{i-1/2}} (\hat{v}_{i-1/2}^{k-1/2})^2 + \frac{x_{i-1/2}}{2} (s_i^{k-1/2})^2, \quad 1 \leq k \leq K. \]
Summing up for $i$ and noticing (9d) and (10), we obtain
\[
\left(\left\| \dot{u}^k \right\|^2 - \left\| \dot{u}^{k-1} \right\|^2 \right) / \tau \leq 2(\dot{u}^{k-1/2}v^{k-1/2}_m - \dot{u}^{k-1/2}v^{k-1/2}_0)
\]
\[+ \epsilon \left[ (\dot{u}^k, 1)^2 - (\dot{u}^{k-1}, 1)^2 \right] / \tau + \left\| \dot{u}^{k-1/2} \right\|^2 + h \sum_{i=1}^{M} \left[ \frac{1}{x_i-1/2} (e_i^{k-1/2})^2 + \frac{1}{2} x_i-1/2(s_i^{k-1/2})^2 \right]
\]
\[
\leq \epsilon [(\dot{u}^k, 1)^2 - (\dot{u}^{k-1}, 1)^2] / \tau + \left\| \dot{u}^{k-1/2} \right\|^2 + \left( \frac{b^2 - a^2}{4} + \ln \frac{b}{a} \right) [c_1(h^2 + \tau^2)]^2, \quad 1 \leq k \leq K
\]
where we have used
\[
h \sum_{i=1}^{M} x_{i-1/2} = \int_a^b x \, dx = \frac{b^2 - a^2}{2}
\]
and
\[
h \sum_{i=1}^{M} \frac{1}{x_{i-1/2}} \leq \int_a^b \frac{1}{x} \, dx = \ln \frac{b}{a}.
\]
Then, summing up for $k$ from $1 \leq k \leq m$ we arrive at
\[
\left(\left\| \dot{u}^m \right\|^2 - \left\| \dot{u}^0 \right\|^2 \right) / \tau \leq \epsilon [\left(\dot{u}^m, 1\right)^2 - (\dot{u}^0, 1)^2] / \tau
\]
\[+ \sum_{k=1}^{m} \left\| \dot{u}^{k-1/2} \right\|^2 + \left( \frac{b^2 - a^2}{4} + \ln \frac{b}{a} \right) [c_1(h^2 + \tau^2)]^2 m, \quad 1 \leq m \leq K.
\]
Noticing (9c), we have
\[
\left\| \dot{u}^m \right\|^2 - \epsilon (\dot{u}^m, 1)^2 \leq \frac{1}{2} \sum_{k=1}^{m} \left\| \dot{u}^{k-1/2} \right\|^2 + \left( \frac{b^2 - a^2}{4} + \ln \frac{b}{a} \right) [c_1(h^2 + \tau^2)]^2 T.
\]
Since
\[
(\dot{u}^m, 1)^2 \leq \left\| \dot{u}^m \right\|^2 \|1\|^2 \leq \frac{1}{2} (b^2 - a^2) \left\| \dot{u}^m \right\|^2,
\]
we have
\[
\left[ 1 - \frac{1}{2} \epsilon (b^2 - a^2) \right] \left\| \dot{u}^m \right\|^2 \leq \frac{1}{2} \left\| \dot{u}^m \right\|^2 + \tau \sum_{k=1}^{m-1} \left\| \dot{u}^k \right\|^2 + \left( \frac{b^2 - a^2}{4} + \ln \frac{b}{a} \right) [c_1(h^2 + \tau^2)]^2 T.
\]
Noticing $c_2 \equiv 1 - \frac{1}{2} \epsilon (b^2 - a^2) > 0$, when $\tau \leq c_2$, we obtain
\[
\left\| \dot{u}^m \right\|^2 \leq \frac{2}{c_2} \sum_{k=1}^{m-1} \left\| \dot{u}^k \right\|^2 + \frac{2}{c_2} \left( \frac{b^2 - a^2}{4} + \ln \frac{b}{a} \right) [c_1(h^2 + \tau^2)]^2 T, \quad 1 \leq m \leq K.
\]
Using Gronwall inequality gets
\[
\left\| \dot{u}^m \right\|^2 \leq \exp \left( \frac{2}{c_2} m \tau \right) \cdot \frac{2}{c_2} \left( \frac{b^2 - a^2}{4} + \ln \frac{b}{a} \right) [c_1(h^2 + \tau^2)]^2 T, \quad 1 \leq m \leq K,
\]
or

\[ \|\hat{u}^m\| \leq \exp \left( \frac{1}{c_2} T \right) c_1 \sqrt{\frac{2}{c_2} \left( \frac{b^2 - a^2}{4} + \ln \frac{b}{a} \right) T (h^2 + \tau^2) - \frac{1}{2}} \quad 1 \leq m \leq K. \]

This completes the proof. \( \square \)

**Theorem 3.** Difference scheme (3) is uniquely solvable.

**Proof.** From Theorem 1, it suffices to prove that the homogeneous equations of (5) has trivial solution. The homogeneous equations of (5) are as follows:

\[
\begin{align*}
    x_{i-1/2} u_{i-1/2}^{k-1/2} + x_{i-1/2} \frac{1}{h} \varepsilon \sum_{j=1}^{M} x_{j-1/2} u_{j-1/2}^k / \tau, & \quad 1 \leq i \leq M, \quad 1 \leq k \leq K \quad (11a) \\
    \frac{1}{x_{i-1/2}} v_{i-1/2}^{k-1/2} = \frac{1}{2} \delta_x u_{i-1/2}^k, & \quad 1 \leq i \leq M, \quad 1 \leq k \leq K \quad (11b) \\
    u_0^k = 0, \quad v_M^{k-1/2} + \frac{b}{2} u_M^k = 0, & \quad 1 \leq k \leq K. \quad (11c)
\end{align*}
\]

Multiplying (11a) by \( u_{i-1/2}^k \) and multiplying (11b) by \( v_{i-1/2}^{k-1/2} \), then similar to the proof of Theorem 2, we can obtain

\[ \|\hat{u}^k\| = 0, \quad \|v^{k-1/2}\| = 0, \]

which together with (11c) give

\[ u_i^k = 0, \quad v_i^{k-1/2} = 0, \quad 0 \leq i \leq M. \]

This completes the proof. \( \square \)

4. The computation of the difference scheme

The actual computational algorithm can be simplified as follows. By introducing a new variable \( s_{k-1/2} = \varepsilon h \sum_{j=1}^{M} x_{j-1/2} \delta_x u_{j-1/2}^{k-1/2} \) as in the proof of Theorem 1, the difference scheme (3a)–(3d) is equivalent to

\[
\begin{align*}
    \frac{1}{2} (x_{i-1/2} \delta_x u_{i-1/2}^{k-1/2} + x_{i+1/2} \delta_x u_{i+1/2}^{k-1/2})
    = & \quad \delta_x (x_i \delta_x u_i^{k-1/2}) + x_i s^{k-1/2} + \frac{1}{2} \left[ x_i \Phi_{i-1/2}^{k-1/2} + x_{i+1/2} \Phi_{i+1/2}^{k-1/2} \right], \quad 1 \leq i \leq M - 1, \quad 1 \leq k \leq K \\
    u_0^{k-1/2} = & \quad \frac{1}{2} (f(t_k) + f(t_{k-1})), \quad 1 \leq k \leq K, \\
    x_{M-1/2} \delta_x u_{M-1/2}^{k-1/2} = & \quad \frac{2}{h} \left[ \frac{b}{2} (g(t_k) + g(t_{k-1})) - x_{M-1/2} \delta_x u_{M-1/2}^{k-1/2} - bu_M^{k-1/2} \right] \\
    & \quad + x_{M-1/2} s^{k-1/2} + x_{M-1/2} \Phi_{M-1/2}^{k-1/2}, \quad 1 \leq k \leq K.
\end{align*}
\]
\[ s^{k-1/2} = \varepsilon h \sum_{j=1}^{M} x_{j-1/2} \delta u_{j-1/2}^{k-1/2}, \quad 1 \leq k \leq K, \]

\[ u_i^0 = \phi(x_i), \quad 0 \leq i \leq M. \]

Eq. (12) can be rewritten in matrix-vector form:

\[
\begin{bmatrix}
  b_0 & c_0 & 0 & \cdots & 0 & 0 & q_0 \\
  a_1 & b_1 & c_1 & \cdots & 0 & 0 & q_1 \\
  0 & a_2 & b_2 & c_2 & \cdots & 0 & 0 & q_2 \\
  0 & 0 & a_3 & b_3 & \cdots & 0 & 0 & q_3 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & b_{M-1} & c_{M-1} & q_{M-1} \\
  0 & 0 & 0 & \cdots & a_M & b_M & q_M \\
  p_0 & p_1 & p_2 & \cdots & p_{M-1} & p_M & q_{M+1}
\end{bmatrix}
\begin{bmatrix}
  u_0^{k-1/2} \\
  u_1^{k-1/2} \\
  u_2^{k-1/2} \\
  \vdots \\
  u_{M-1}^{k-1/2} \\
  u_M^{k-1/2} \\
  s^{k-1/2}
\end{bmatrix}
= \begin{bmatrix}
  g_0^{k-1/2} \\
  g_1^{k-1/2} \\
  g_2^{k-1/2} \\
  \vdots \\
  g_{M-1}^{k-1/2} \\
  g_M^{k-1/2} \\
  s^{k-1/2}
\end{bmatrix},
\]

where \( a_i, b_i, c_i, p_i, q_i \) are constants independent of \( k \), \( g^{k-1/2}_M \) known. Nonzero are only diagonal, secondary diagonal, last row and last column elements of the coefficient matrix. If \( \{u_{k-1/2}^i, 0 \leq i \leq M\} \) is determined, then \( \{u_{k-1/2}^i = 2u_{k-1/2}^{i-1} - u_{k-1/2}^{i-2}, 0 \leq i \leq M\} \).

The first equation can be used to eliminate \( u_{k-1/2}^0 \) from the second equation and from the last equation; the new second equation used to eliminate \( u_{k-1/2}^i \) from the third equation and from the new last equation, and so on, until the new last but two equation can be used to eliminate \( u_{k-1/2}^{M-2} \) from the last but one equation and from the last new equation. At last, the new last but one equation used to eliminate \( u_{k-1/2}^{M-1} \) from the last but one equation and from the last new equation, giving one equation with one unknown, \( s^{k-1/2} \). The unknowns \( u_{M-1}, u_{k-1/2, M-1}, \ldots, u_1, u_0 \) can then be found in turn by back substitution. This only takes a total of \( 11M + 6 \) multiplication–division operations and \( 8M + 3 \) addition–subtraction operations. It will reduce the computational amount deeply to decompose the coefficient matrix into the product of two triangular matrixes since the coefficient matrix is independent of \( k \).

4.1. Numerical example

Compute by difference scheme (3) the following problem:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} + 0.5 \int_1^2 \rho \frac{\partial u}{\partial t}(\rho, t) \, d\rho + \left( e^x - \frac{1}{2} e^2 \right) \cos t - e^t \left( 1 + \frac{1}{x} \right) \sin t,
\]

\[ 1 < x < 2, \quad 0 < t \leq 1, \]

\[ u(1, t) = e \sin t, \quad \frac{\partial u}{\partial x}(2, t) + u(2, t) = 2e^2 \sin t, \quad 0 < t \leq 1, \]

\[ u(x, 0) = 0, \quad 1 \leq x \leq 2, \]

whose exact solution is \( u(x, t) = e^x \sin t \). Some numerical solutions and the comparisons with the scheme in [1] are listed at Tables 1–3. It can be seen that as the mesh sizes decrease two times, the errors of scheme (3) in this paper decrease about four times and those of the scheme in [1] do only about two times, which coincides with the theoretical analysis.
Table 1
The solution with $h = \tau = 0.1 \ (M = K = 10)$

<table>
<thead>
<tr>
<th>$(x, t)$</th>
<th>Exact solution</th>
<th>Scheme in this paper</th>
<th>Scheme in [1]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Difference solution</td>
<td>Absolute error</td>
</tr>
<tr>
<td>$(0.2, 1.0)$</td>
<td>2.794</td>
<td>2.793</td>
<td>$-8.6e-4$</td>
</tr>
<tr>
<td>$(0.4, 1.0)$</td>
<td>3.412</td>
<td>3.411</td>
<td>$-1.1e-3$</td>
</tr>
<tr>
<td>$(0.6, 1.0)$</td>
<td>4.168</td>
<td>4.167</td>
<td>$-7.0e-4$</td>
</tr>
<tr>
<td>$(0.8, 1.0)$</td>
<td>5.091</td>
<td>5.091</td>
<td>$+3.4e-4$</td>
</tr>
</tbody>
</table>

Table 2
The solution with $h = \tau = 0.05 \ (M = K = 20)$

<table>
<thead>
<tr>
<th>$(x, t)$</th>
<th>Exact solution</th>
<th>Scheme in this paper</th>
<th>Scheme in [1]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Difference solution</td>
<td>Absolute error</td>
</tr>
<tr>
<td>$(0.2, 1.0)$</td>
<td>2.7938</td>
<td>2.7936</td>
<td>$-2.2e-4$</td>
</tr>
<tr>
<td>$(0.4, 1.0)$</td>
<td>4.0923</td>
<td>4.0922</td>
<td>$-3.8e-4$</td>
</tr>
<tr>
<td>$(0.6, 1.0)$</td>
<td>5.0906</td>
<td>5.0905</td>
<td>$-2.4e-5$</td>
</tr>
<tr>
<td>$(0.8, 1.0)$</td>
<td>6.0906</td>
<td>6.0906</td>
<td>$+6.8e-6$</td>
</tr>
</tbody>
</table>

Table 3
The solution with $h = \tau = 0.025 \ (M = K = 40)$

<table>
<thead>
<tr>
<th>$(x, t)$</th>
<th>Exact solution</th>
<th>Scheme in this paper</th>
<th>Scheme in [1]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Difference solution</td>
<td>Absolute error</td>
</tr>
<tr>
<td>$(0.2, 1.0)$</td>
<td>2.79378</td>
<td>2.79372</td>
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<td>3.41225</td>
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<tr>
<td>$(0.6, 1.0)$</td>
<td>4.16783</td>
<td>4.16777</td>
<td>$-6.4e-5$</td>
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<tr>
<td>$(0.8, 1.0)$</td>
<td>5.09060</td>
<td>5.09061</td>
<td>$+4.1e-6$</td>
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Acknowledgements

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References