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The Largest Parity Demigenus of a Simple Graph*

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A graph Γ is *parity embedded* in a surface if a closed path in the graph is orientation preserving or reversing according as its length is even or odd. The *parity demigenus* of Γ is the minimum of $2-\chi(S)$ (where χ is Euler characteristic) over all surfaces S in which Γ can be parity embedded. We calculate the maximum parity demigenus over all loopless graphs of order n. As a corollary we strengthen the calculation by Jungerman, Stahl, and White of the genus of $K_{n,n}$ with a perfect matching removed. We conclude by discussing numerous related problems.

1. INTRODUCTION AND THEOREMS

Suppose we have a graph Γ without loops, embedded in a surface so that every odd polygon (the graph of a simple closed path of odd length), regarded as a path in the surface, reverses orientation while every even polygon preserves it. What is the smallest surface in which this is possible? That is, what is the minimum *demigenus* $d(S)=2-\chi(S)$ over all embedding surfaces S? We call this kind of embedding *parity embedding*[1] and the smallest $d(S)$ the *parity demigenus* of Γ, written $d(-\Gamma)$. Euler's polyhedral formula, together with the obvious fact that a face boundary must (with trivial exceptions) have length at least 4, implies that

$$d(-\Gamma) \geqslant \left\lceil \frac{m}{2} \right\rceil - n + 2 \qquad (1)$$

if Γ is connected and has no multiple edges and $m \geqslant 2$, where $n=|V|$, the order of Γ, and $m=|E|$, the number of edges. Contrariwise, the obvious upper bound on $d(-\Gamma)$ in terms of the order is $d(-K_n)$, as multiple edges do not affect parity embeddability. Here we establish the value of this

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[1] The name and concept were, as far as I know, introduced by Lins [4].

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upper bound by proving that, if $n \geqslant 6$, $d(-K_n)$ equals the lower bound imposed by Euler's polyhedral formula.

THEOREM. *For every simple graph Γ of order n,*

$$d(-\Gamma) \leqslant d(-K_n) = \begin{cases} 0, & \text{if} \quad n = 1, 2, \\ 3, & \text{if} \quad n = 5, \quad \text{and} \\ \lceil \frac{1}{4} n(n-5) \rceil + 2, & \text{if} \quad n \geqslant 3 \quad \text{and} \quad n \neq 5. \end{cases} \tag{2}$$

One can interpret this as a theorem about symmetric embedding in orientable surfaces. Let $\tilde{\Gamma}$ be a bipartite graph with bipartition $V = V_1 \cup V_2$, embedded in T_g (the sphere with g handles) so that an involutory autohomeomorphism τ of T_g whose quotient is U_{g+1} (the sphere with $g+1$ crosscaps) carries $\tilde{\Gamma}$ to itself while interchanging the two independent vertex sets V_1 and V_2. The minimum possible g—call it the *antipodal genus* of $\tilde{\Gamma}$ with respect to the automorphism induced by τ—equals $d(-\Gamma) - 1$, where Γ is the quotient of $\tilde{\Gamma}$ by τ, provided that $\tilde{\Gamma}$ is connected. Now suppose that $\tilde{\Gamma}$ is $K_{n,n}$ with a perfect matching M_n removed and with the automorphism α *associated to* M_n, that is, which exchanges the endpoints of each deleted edge. Then the antipodal genus is $d(-K_n) - 1$, if $n \geqslant 3$. How does this compare with the genus of $K_{n,n} \backslash M_n$, which is the minimum g when the embedding is unrestricted? It turns out, rather surprisingly, that they are almost always the same.

COROLLARY 1. *Let $n \geqslant 1$, let M_n be a perfect matching in $K_{n,n}$, and let α be the associated automorphism of $K_{n,n} \backslash M_n$. Then the genus of $K_{n,n} \backslash M_n$ and its antipodal genus with respect to α both equal $\lceil \frac{1}{4}(n-1)(n-4) \rceil$, except that the antipodal genus is 2 when $n = 5$.*

The genus was previously evaluated by Jungerman, Stahl, and White by a different method in [3] (or see [8, Section 13–7]). However, since their proof for the case $n \equiv 1 \pmod 4$ (the only case they published) does not provide an antipodal embedding, their work does not yield the antipodal genus.

Since in parity embedding, multiple edges have no effect but loops, on the contrary, can alter the minimal surface, one would naturally ask also the largest parity demigenus of an arbitrary graph of order n, not necessarily simple. It obviously is $d(-K_n^\circ)$, where K_n° denotes K_n with a loop at every vertex. That quantity equals $\lceil \frac{1}{4} n(n-3) \rceil + 2$ for $n \geqslant 6$ (as we showed in the companion paper [11]), larger by about $\frac{1}{2} n$ than the loop-free upper bound $d(-K_n)$.

Before the proof, let us see how parity embedding fits into the more general scheme of orientation embedding of signed graphs. A *signed graph* (a graph with signed edges) is said to be *orientation embedded* in a surface if it is embedded so that a closed path preserves orientation if and only if its sign product is positive. Parity embedding is therefore the same as orientation embedding of $-\Gamma$, the all-negative signing of Γ. Let us call the *demigenus* $d(\Sigma)$ of a signed graph Σ the smallest demigenus of any surface in which it orientation embeds. I propose that $d(-K_n)$ is the greatest value attained by $d(\Gamma)$ over all signed simple graphs of order n. Equivalently,

Conjecture 1. $d(-K_n)$ is the largest demigenus of any signed K_n.

I discuss the meaning and the plausibility of this conjecture in Section 4a.

2. THE PROOF OF THE THEOREM

For the proof we need two things: a lower bound on the parity demigenus of K_n and a minimal parity embedding. Let ε_n denote the right-hand side of (2).

Proof that $d(-K_n) \geqslant \varepsilon_n$. From (1) we get Euler's lower bound,

$$d(-K_n) \geqslant \left\lceil \frac{n(n-5)}{4} \right\rceil + 2 \tag{3}$$

when $n \geqslant 3$. This takes care of all nontrivial cases except $n=5$, where the right side of (3) falls short of the actual parity demigenus ε_5. We need to prove there is no parity embedding of K_5 in U_2. We do so by demonstrating that any such embedding must have every face a quadrilateral. Since K_5 has only one quadrilateral embedding, which is orientable[2] so it is in T_1 rather than U_2, it follows that $d(-K_5) \geqslant 3$.

Suppose K_5 did have a parity embedding in U_2. Being minimal the embedding would be *cellular*: every face would be an open 2-cell. Let f_i denote the number of faces whose boundaries have length i. (We think of the boundary ∂F of a face F as a walk in the graph. If an edge appears twice on a face boundary we therefore count it as two edges in ∂F.) Then $f_i = 0$ unless $i = 4, 6, 8, \ldots$ because a complete walk around a face boundary must preserve orientation. Consequently $2m = 4f_4 + 6f_6 + \cdots = 4f + 2(f_6 + 2f_8 + \cdots)$. Since by Euler's formula $f = m - n + \chi(U_2) = 5$, all faces must be quadrilaterals.

Proof that $d(-K_n) \leqslant \varepsilon_n$. Figures 1 to 11 show the existence of a parity embedding of K_n in U_{ε_n} for all $n \geqslant 3$ by exhibiting embeddings for $3 \leqslant n \leqslant 8$

[2] This is well known. For the proof one can consult, e.g., [2, pp. 188–189].

and inductive enlargements to handle $n \geqslant 9$, one series for odd order and another for even order. We begin with instructions for reading the diagrams.

A *crosscap* is a hole whose opposite boundary points are identified. It is drawn as a circle or oval with a twiddly curve inside. In Fig. 1 the outer circular boundary also denotes a crosscap; that is, opposite points on it are identified.

A handle is drawn as a pair of circular holes (called its *ends*) whose boundaries are identified with each other in opposite senses or in the same sense. In the former case the handle preserves orientation with respect to the plane of the figure, which means that a closed path passing through that handle and no other handle or crosscap is orientation-preserving. Such a handle is a *prohandle*. In the latter case the handle reverses orientation; it is an *antihandle*. (Usually only one of the two ends is depicted—this is the case for the "outside handles" to be defined later.)

A *crossed quad* is a union of two closed quadrilateral faces that is homeomorphic to the surface figure consisting of a quadrilateral region, bounded, say, by a polygon *wxyzw*, that contains one crosscap and edges

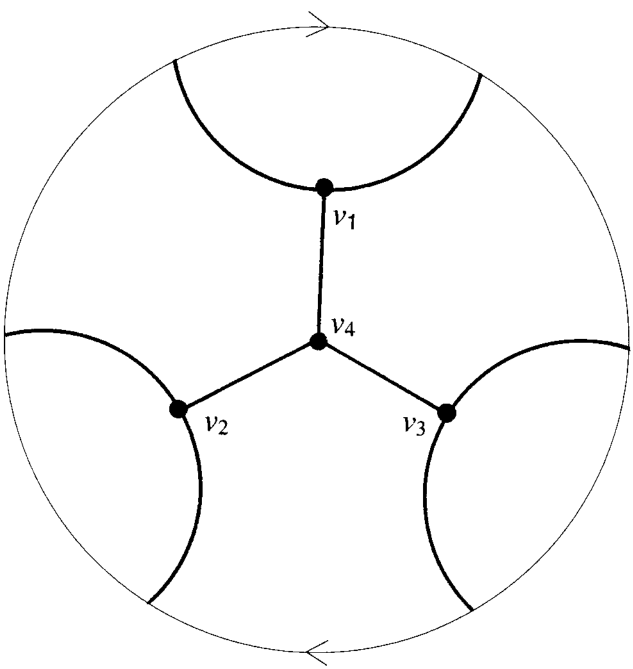

FIG. 1. A parity embedding of K_4 in the projective plane U_1, containing a parity embedding of K_3.

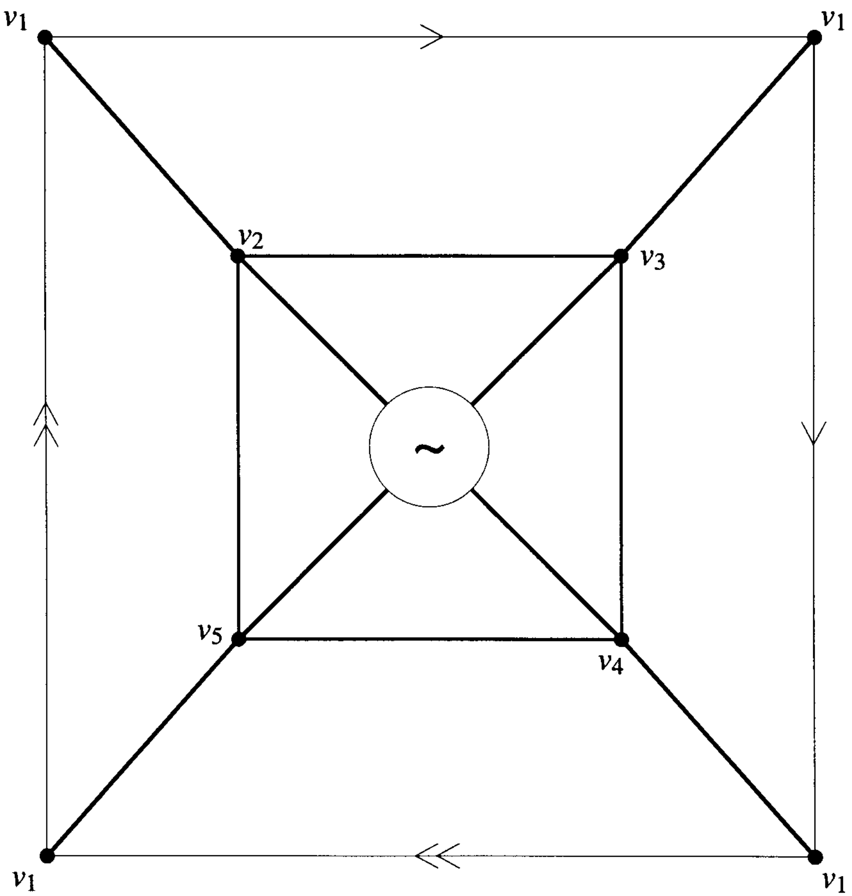

FIG. 2. A parity embedding of K_5 in U_3. (This figure is atypical in having a vertex on the boundary. That makes it harder to verify its correctness, but the drawing has pleasing symmetry.)

wy and *xz* that pass through the crosscap. These edges divide the region into two faces bounded by the polygons *wxzyw* and *wyxzw*. (See Fig. 7a for an example. The faces in a crossed quad are convenient to use as attachment faces—to be defined later—because two vertices that are adjacent along the crossed quad's boundary are diagonally opposite in one of its constituent faces.)

Any path has a sign, calculated by starting with a + sign and negating once for each crosscap or antihandle through which the path passes. Thus a closed path has positive sign if and only if it preserves orientation. Also, the sign of a concatenation of two paths (say from P to Q and Q to R) equals the product of the signs of the paths. In particular, the sign of a

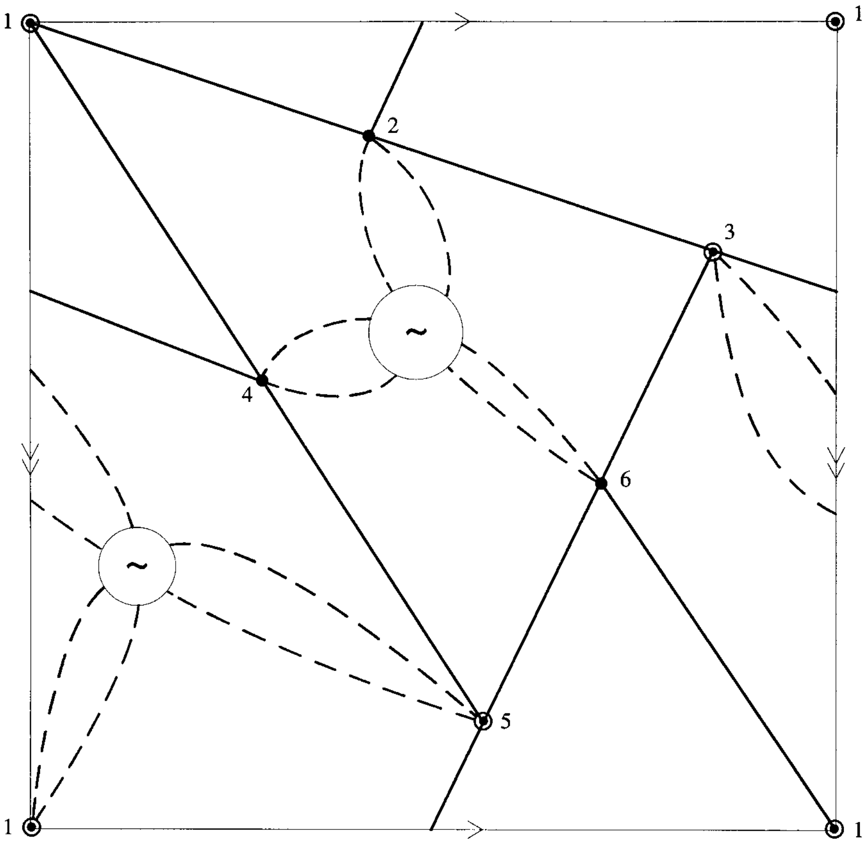

FIG. 3. A parity embedding of K_6 in U_4. It is presented as an orientation embedding of $-K_6^{\{2,4,6\}}$, which consists of $+K_{3,3}$ and two $-K_3$'s. (In Figs. 3 to 10, positive edges are solid and negative ones are dashed.)

polygon of length l in the graph is $(-1)^l$ if it has evenly many positive edges. Thus we have a parity embedding if we make every edge negative, but more generally, also, if we arrange things so that the positive edges form a complete bipartite graph on $V = V(K_n)$ and, as a consequence, the negative edges form two complete subgraphs on complementary vertex subsets X and $Y = V \backslash X$. Such a signing is called *antibalanced.* We denote it by $-K_n^X$ or, equivalently, $-K_n^Y$. It is this kind of signing that I employ in Figures 3 to 10 to keep track of the edge signs. The significant vertices are bicolored, by solid or hollow circles. An edge should be negative, as calculated by the path sign rule above, precisely when its endpoints have the same color. (The coloring is not consistent from diagram to diagram; its purpose is solely to help verify the correctness of the edge signs within each figure.)

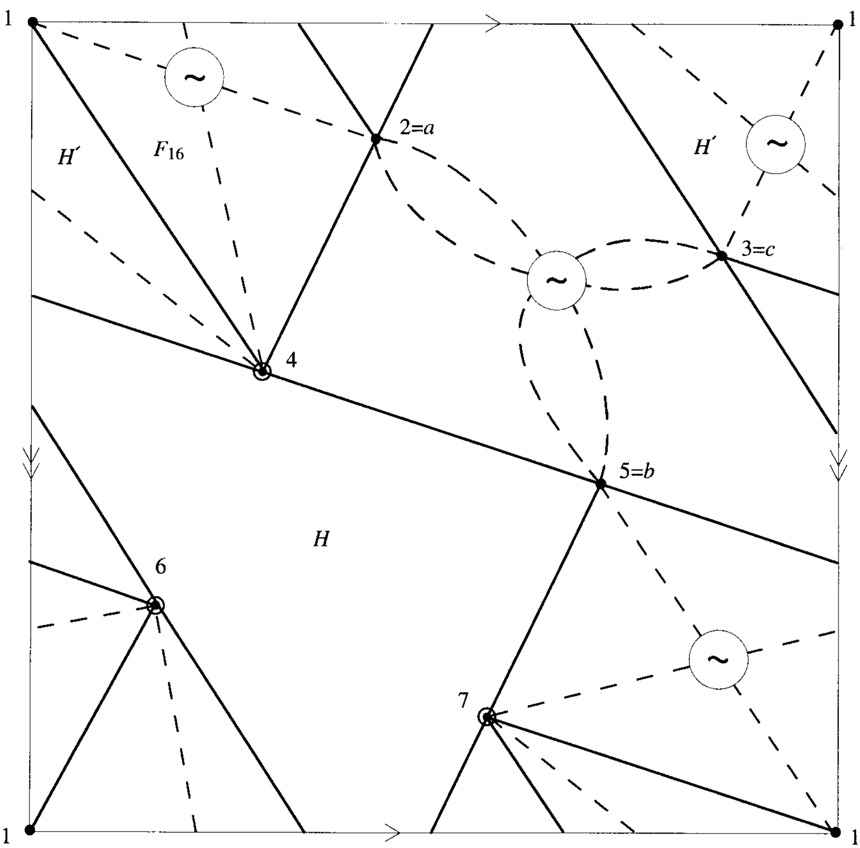

FIG. 4. K_7, parity embedded in U_6. It is presented as $-K_7^{\{1,2,3,5\}}$, i.e., $+K_{4,3}$ and a $-K_4$ and (in the face 245637 of the $+K_{4,3}$) a $-K_3$. The initial pairing is 16; F_{16} is the attachment face. The letters a, b, c indicate the alignment of the first odd gadget.

Now we describe the minimal parity embeddings. Embeddings for $3 \leqslant n \leqslant 8$ are shown in Figs. 1 to 5. We construct embeddings for higher order inductively. In outline, starting from an embedding of K_7 or K_8 we repeatedly add a two-vertex "gadget" to get embeddings for all larger odd or even orders. As the induction proceeds we alternate between two types of gadget: a "hex" gadget, which requires a hexagonal face, is added to K_n when $n \equiv 7$ or 10 (mod 4) (see Figs. 6, 8, and 10), and a "quad" gadget, which needs only quadrilateral faces, is added when $n \equiv 9$ or 8 (in which cases there is no hexagonal face anyway) (Figures 7 and 9). In each embedding the parity property is assured by making the edge signing antibalanced. Note that the odd hex gadget has a special form when $n = 7$ (Fig. 6).

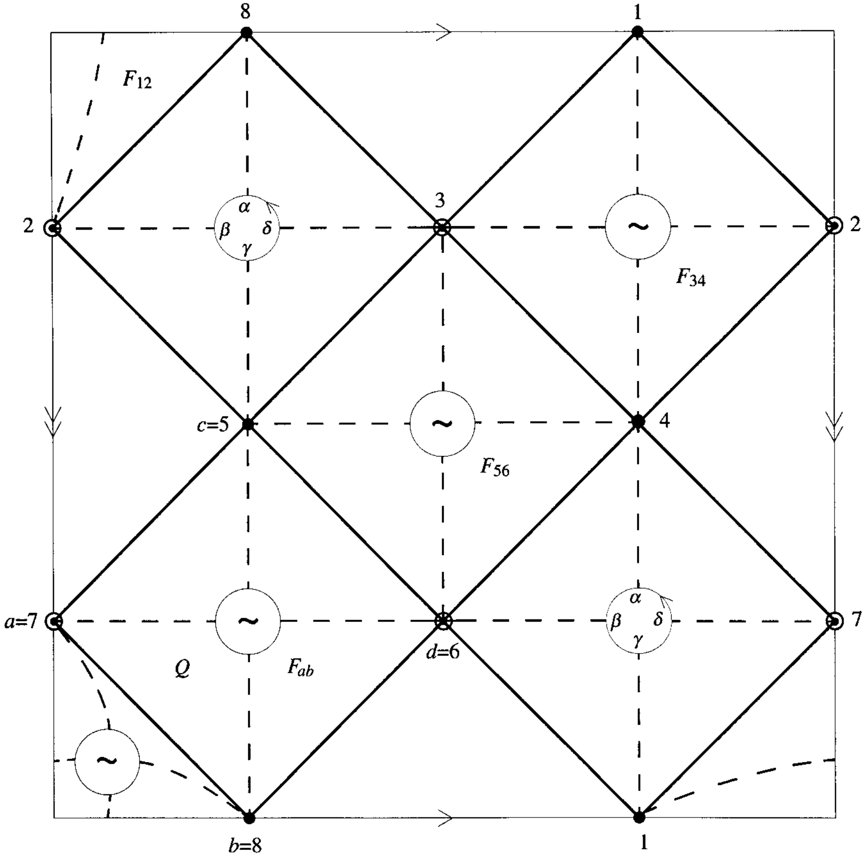

FIG. 5. A parity embedding of K_8. It is presented as an orientation embedding of $-K_8^{\{1,4,5,8\}}$. The embedding begins with $+K_{4,4}$ in the torus (solid lines). Crosscaps and an antihandle are added to accomodate the two $-K_4$'s that complete $-K_8^{\{1,4,5,8\}}$. The initial pairing is 12, 34, 56, with corresponding attachment faces F_{ij}. The environment for the initial even quad gadget is the crossed quad $Q \cup F_{ab}$. The gadget is attached in Q to vertices $a=7$ and $b=8$.

In greater detail, a gadget added to an embedded K_n is inserted in a suitable environment, which is a portion of the embedding having a certain shape (which, with an exception at $n=7$, is constant within each residue class of n modulo 4). The environment is a union of closed faces of the embedded graph; its boundary therefore consists of some edges and vertices of K_n and its interior consists of open faces and edges (but no vertices, as it happens). The environments are shown in Figs. 4, 5, 7, 8, 9, and 10. Notice that some vertices are unlabelled: they play no part in adding a

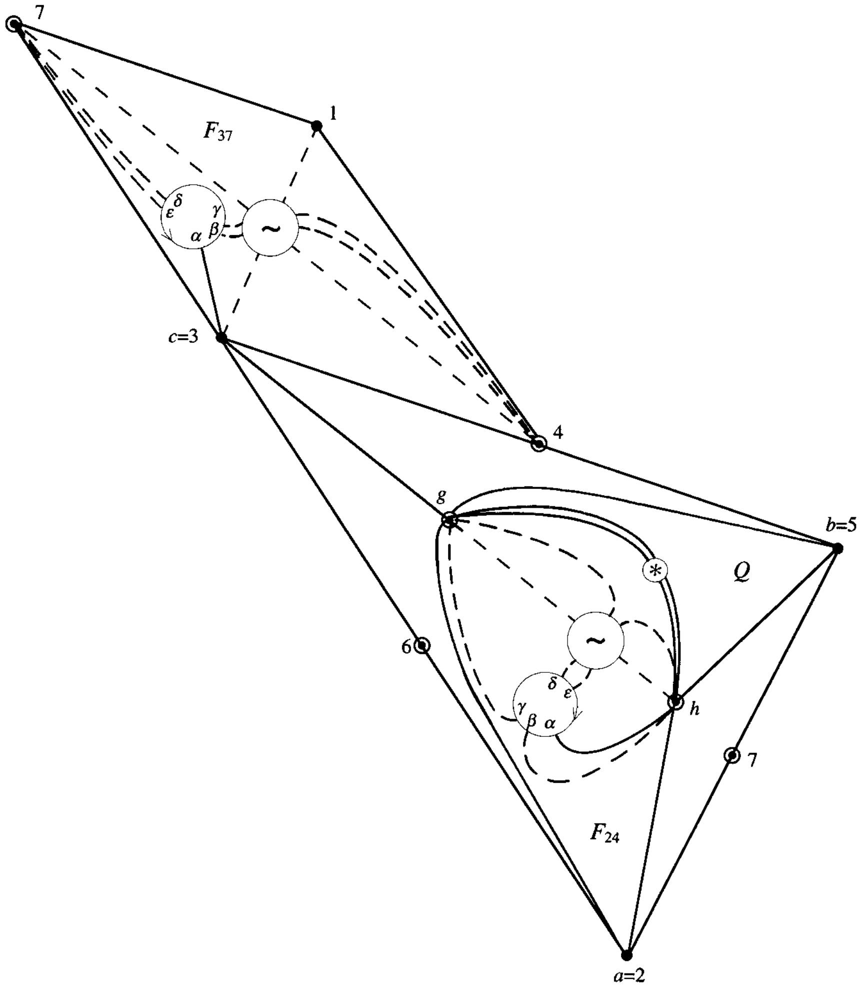

FIG. 6. The special odd hex gadget, embedded in its environment in K_7. The initial pairing is 16 with attachment face F_{16}. At the next induction step, 24 and 37 will be paired with attachment faces F_{24} and F_{37}.

gadget, so they can be anything. On the other vertices, different labels signify that the vertices are actually different.

Gadget insertion can be viewed as a two-step process. First, the interior of the environment is removed and replaced by a surface which contains a signed K_2 and edges joining the new vertices to some of those on the boundary of the environment. (We call these latter vertices *direct*. The

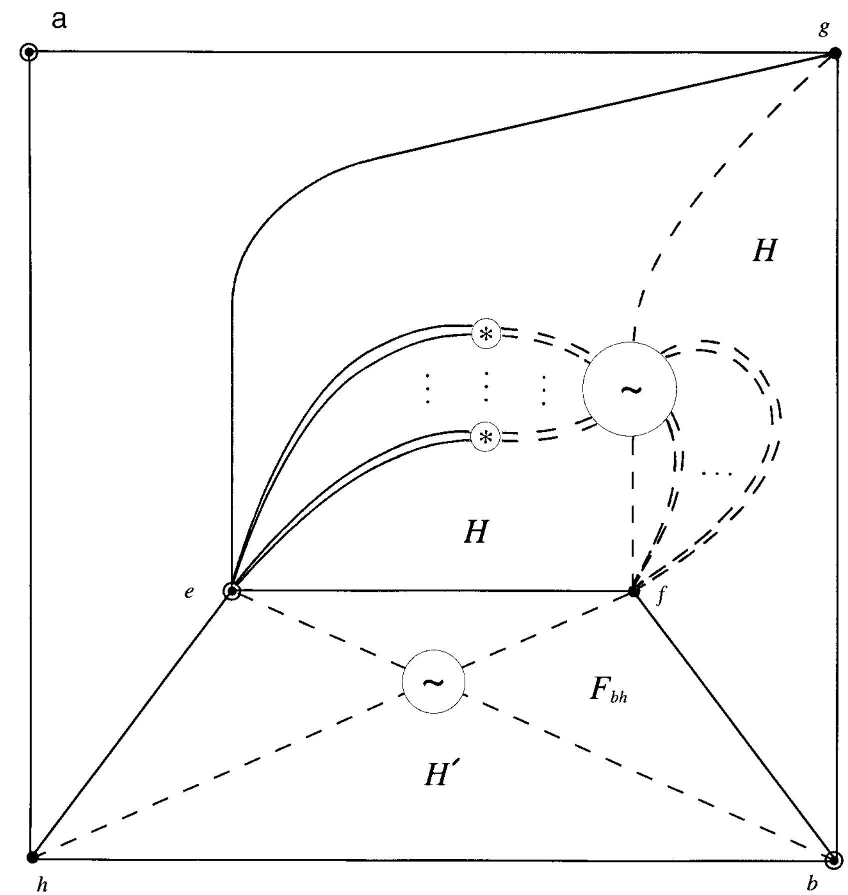

FIG. 7. (a) The odd quad gadget to be inserted in the parity embedding of K_{9+4s}. Here the gadget is in its environment, the latter being the face of K_{9+4s} that is surrounded by the outer boundary of this picture. The environment for the next odd hex gadget is $H \cup H' \cup F_{bh}$. The pairing for that gadget is the same as for this one. At the subsequent step, when the next odd quad gadget is added, F_{bh} will be the attachment face for the pair *bh*. (b) The environment for the odd hex gadget, from Fig. 7a, redrawn to resemble Fig. 8a. The asterisked letters correspond to the letter labels in Fig. 8a.

remaining old vertices are called *indirect*.) This *replacement surface* contains crosscaps and possibly a handle to permit all the connecting edges to be drawn with appropriate sign and no crossings.

In the second step one adds *outside handles* to carry edges from the new vertices to the indirect old ones. Each such handle carries edges to two indirect vertices. The two indirect vertices, say p and q, must therefore be on a common face F_{pq}, called their *attachment face*, which the handle

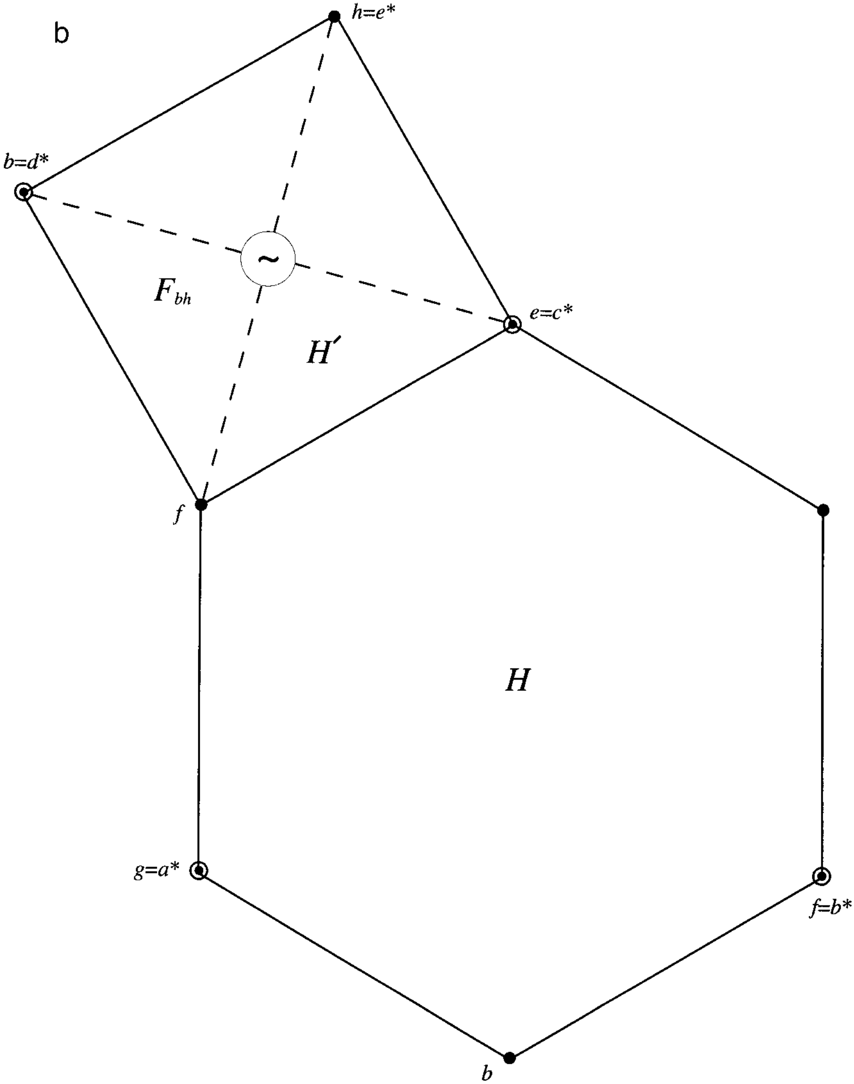

FIG. 7—*Continued*

reaches from a suitable face of the replacement surface. To make all this possible one wants in advance, besides the environment, a pairing of the indirect vertices such that all the attachment faces are distinct. In order to guarantee suitable edge signs we choose an F_{pq} in which p and q are diagonally opposite. We place in F_{pq} one end of the handle carrying edges to p and q (see Fig. 11); from there the handle edges can be distributed to

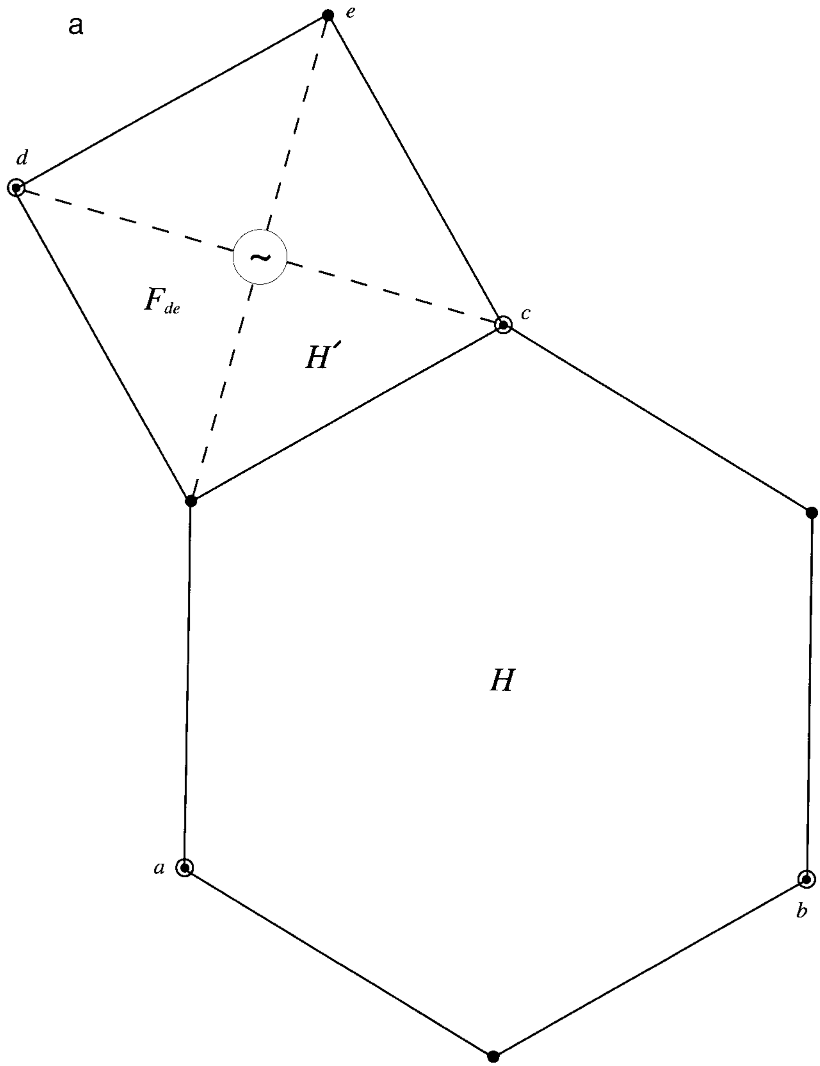

FIG. 8. (a) The environment in K_{11+4s} for the general odd hex gadget. It consists of the hexagonal face H and the crossed quad $H' \cup F_{de}$. The two parts of the environment share some vertices and edges, but only the common vertex c plays any role in the induction. (b) The odd hex gadget, embedded in its environment. At the next induction step, ac and de will be paired with F_{ac} and F_{de} as attachment faces. The next gadget, an odd quad gadget, will be inserted in face Q; its direct vertices will be the unpaired vertices g, b, h.

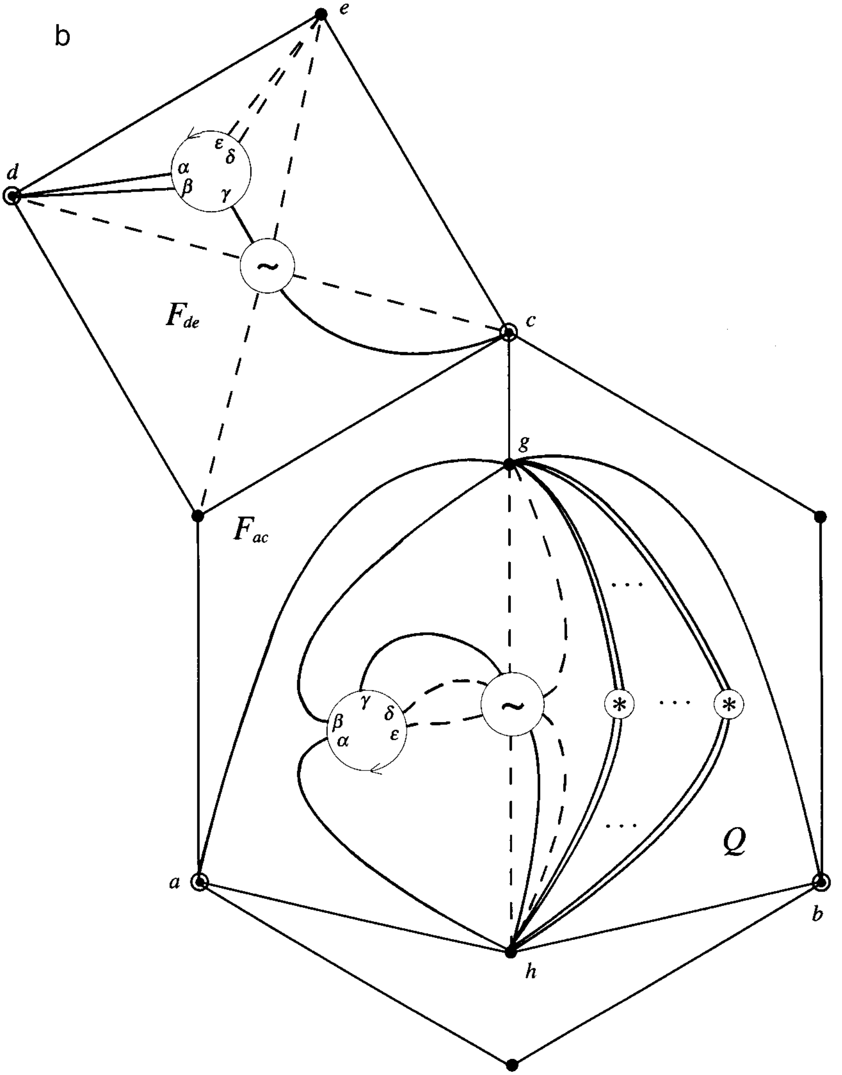

FIG. 8—*continued*

p and q so that the sign contribution to an edge at this end of the handle, when calculated within F_{pq}, will be the same for p and for q. At the gadget end of the handle we arrange things so that the two edges from the two new vertices (call them z_1 and z_2) to each of p and q have the same sign if z_1z_2 is negative, but opposite signs if z_1z_2 is positive. Thus the polygons pz_1z_2p and qz_1z_2q will be negative, just as parity embedding requires. (It

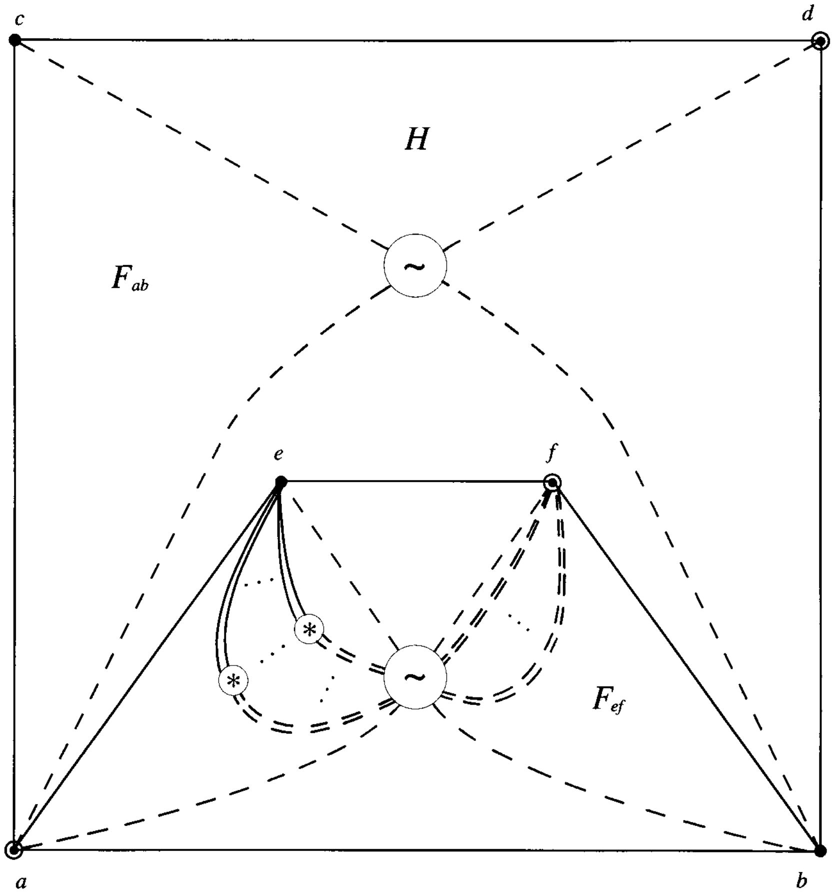

FIG. 9. The even quad gadget for K_{8+4s}, shown inserted in its environment (a crossed quad of K_{8+4s}) and attached at vertices a and b. The new vertices are $e = 9 + 4s$ and $f = 10 + 4s$. F_{ab} and F_{ef} will be the attachment faces for the pairs ab and ef, ab at the next step (inserting an even hex gadget in face H) and ef at the subsequent one (inserting the next even quad gadget).

makes no matter whether the handle itself is a pro- or antihandle, so in the drawings we can ignore the exact orientation type of the outside handles.)

Now we come to the last crucial point. In order to make induction possible, adding the gadget must produce an environment for the next gadget insertion: the hex gadget must generate the environment for the quad gadget, and *vice versa*. Furthermore, it must admit a suitable pairing of vertices which are outside the new environment. To explain how we meet these requirements we note that the new surface consists of two parts, the replacement surface and the remnant of the old surface, both modified by the addition of outside handles. The boundary between the two parts we

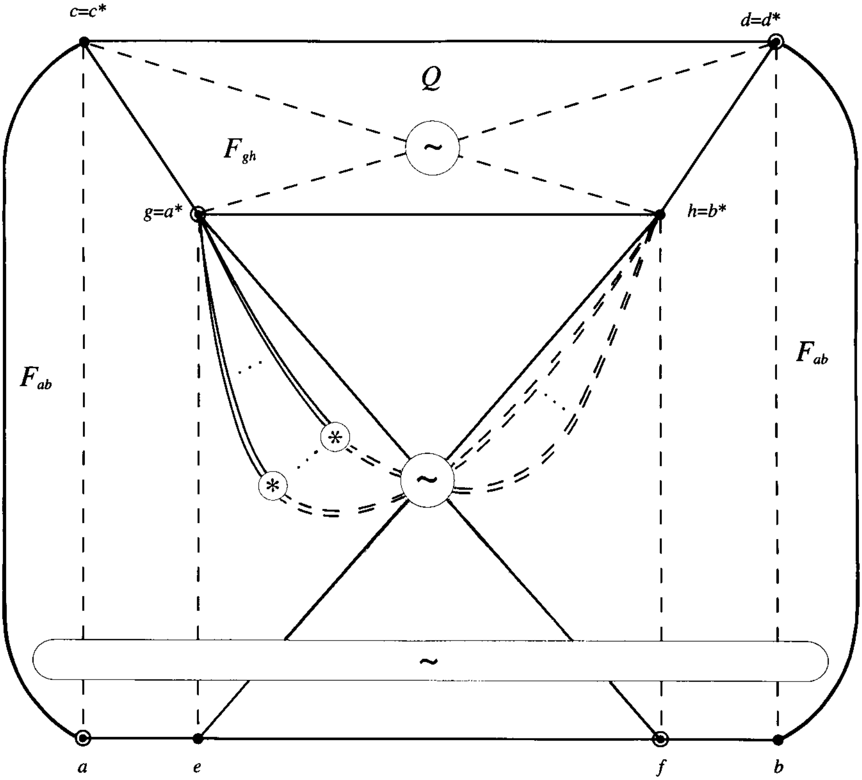

FIG. 10. The even hex gadget, shown inserted in the face $H = aefbcd$ of K_{10+4s}. (These a, b, c, d, e, f, as well as the face F_{ab}, are the same as in Fig. 9.) The next even quad gadget will go into the environment $Q \cup F_{gh}$; $a^* = g$ and $b^* = h$ will be the attaching vertices.

call the *border*. Bear in mind that each outside handle consists of two circular holes with identified boundaries, one hole in the remnant and one in the replacement surface; thus it is topologically a circle. The border therefore consists of the original environment's boundary and the outside handles. Note that the border is transverse to the embedded K_{n+2}; they intersect in a finite set of points, none a vertex.

The border may cut a face F of the new embedding into components, each of which is clearly a topological disk. A component is a *pseudopod* if its boundary consists of a path in the border and a path in ∂F and contains exactly one vertex. One can check in Figs. 6, 7a, 8b, 9, 10, and 11 that every face cut by the border has exactly one component that is not a pseudopod.

Figures 6, 7, 8b, 9, and 10 show the new environments, which lie within the replacement surface except, in the odd case, for a pseudopod. The new

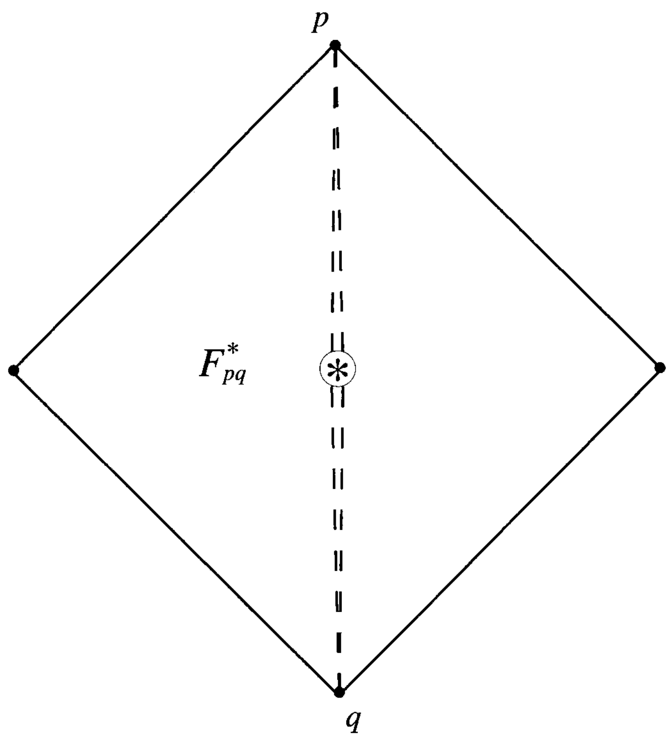

FIG. 11. An attachment face F_{pq}, showing the indirect vertices p and q, the end in F_{pq} of the outside handle, the four edges (dashed) that run through the handle to p and q, and the new attachment face F^*_{pq}. (The choice of F^*_{pq} is arbitrary: either half of F_{pq} will do equally well.)

pairing is chosen to be the old one, together with up to two new pairs tu whose attachment faces F_{tu} are in the replacement surface (again, except for pseudopods). An old attachment face F_{pq} is broken up by the outside handle and its edges but a new attachment face F^*_{pq} can be found by taking a part of F_{pq}, as split up. (See Fig. 11.) The new face F^*_{pq} is outside the replacement surface (aside from pseudopods). It follows that the new attachment faces are all distinct.

The final verification of all requirements is by the reader's inspection of Figs. 4 to 10.

We still should verify the demigenus of the surface resulting from addition of the gadget to a minimal parity embedding of K_n for $n \geqslant 7$. A quad gadget has one or two crosscaps and $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$ outside handles (one for each vertex pair), depending on whether n is even or odd. The total increment to $d(-K_n)$ is therefore $n-1$. Assuming $d(-K_n) = \varepsilon_n$, we have K_{n+2} embedded in demigenus $\varepsilon_n + (n-1) = \varepsilon_{n+2}$, as we wanted. A hex gadget contains 2 crosscaps and $\frac{1}{2}(n-4)$ outside handles if n is even, or 1 crosscap, 1 internal handle, and $\frac{1}{2}(n-5)$ outside handles if n is odd. The total demigenus increment is therefore $n-2$. Assuming $d(-K_n)=\varepsilon_n$, K_{n+2} is embedded in demigenus $\varepsilon_n+(n-2)=\varepsilon_{n+2}$. Thus in every case we have K_{n+2} parity embedded in the correct surface. ▌

3. THE PROOF OF COROLLARY 1

Assume $n \geqslant 3$, since $n=1$ and 2 are trivial. The antipodal genus follows from the theorem. The genus of $K_{n,n} \backslash M_n$ is bounded below by $\lceil \frac{1}{4}n(n-5) \rceil + 1$ because all faces have four or more sides, and, of course, it is not greater than the antipodal genus. That leaves to be proved only the possibility of embedding $K_{5,5} \backslash M_5$ in T_1, for which we refer to Fig. 12. ▌

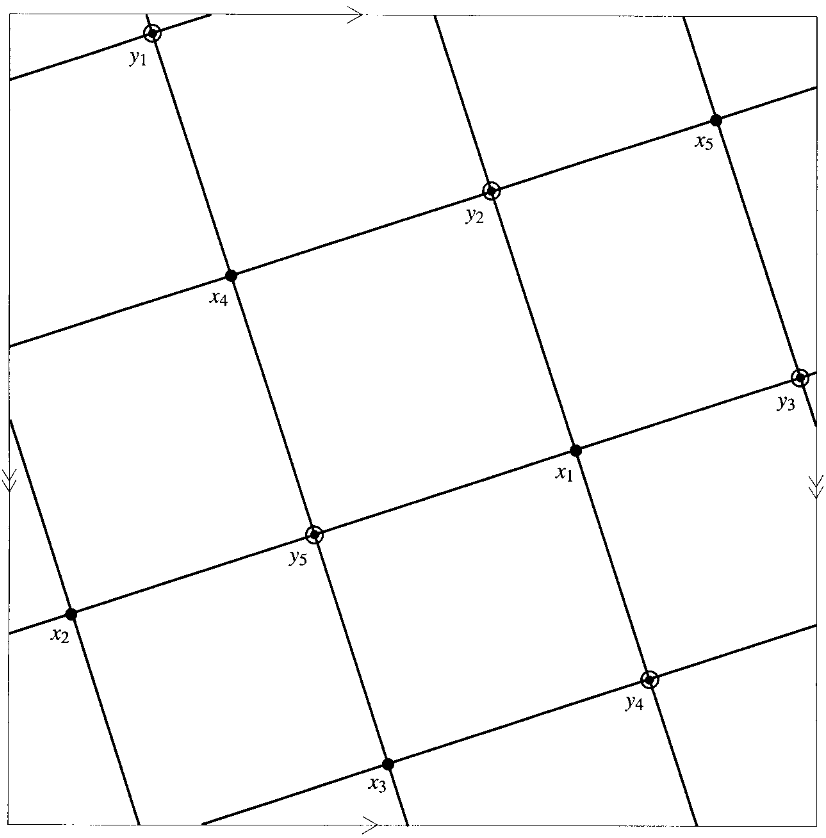

FIG. 12. A toroidal embedding of $K_{5,5} \backslash M_5$.

More is true: the quadrilateral embedding of $K_{5,5} \backslash M_5$ is unique, a fact that directly implies both the impossibility of an antipodal toroidal embedding and, as well, the uniqueness of the quadrilateral embedding of K_5. I intend to go more deeply elsewhere into this and other aspects of antipodal embedding, including a full proof of the basic fact that $d(\Sigma)-1$ is the minimum genus for antipodal embedding.

4. DISCUSSION

4a. *The Biggest Demigenus? Commentary on Conjecture 1*

First of all, what does Conjecture 1 mean? Say we want to embed a graph Γ in a surface so that certain polygons reverse orientation while the remainder preserve it. It is not hard to show that, if this is possible at all, then the orientation requirements amount to asking for an orientation embedding of a signing σ of Γ, whence the minimal surface has demigenus $d(\Gamma, \sigma)$. Conjecture 1 therefore means that $d(-K_n)$ is the smallest demigenus that suffices to meet every feasible prescription of polygon orientabilities in K_n.

I have both philosophical and observational reasons for believing the conjecture. The philosophical reasons derive from the fact that $-K_n$ is the most unbalanced signing of K_n. A signed graph is *balanced* if every polygon has positive sign. (Topologically, this means that every polygon should be orientably embedded.) Two natural measures of imbalance are the *frustration index* (or *line index of imbalance*), $l(K_n, \sigma)$, which is the smallest number of edges whose deletion results in a balanced graph, and the number of unbalanced triangles, $u_3(K_n, \sigma)$. Intuitively, one would expect more negative edges to require more crosscaps and therefore a larger surface. Petersdorf proved that $-K_n$ maximizes $l(K_n, \sigma)$ [5]. As for unbalanced triangles, $-K_n$ maximizes $u_3(K_n, \sigma)$, indeed it has no positive triangles at all. Since a face boundary has to be positive, $-K_n$ can have no triangular faces, unlike any other signing of K_n; from which it is plausible to believe that a minimal orientation embedding of $-K_n$, as compared to any other signed K_n, has the fewest faces and so (by Euler's polyhedral formula) the largest demigenus. Of course, none of this is definitive.

I have observed, however, a remarkable fact: for $n \leqslant 7$, there is an *enveloping graph* for K_n: a signed graph Ω_n such that (i) $d(\Omega_n) \leqslant d(-K_n)$ and (ii) every signing (K_n, σ), possibly excepting $-K_n$, is essentially a subgraph of Ω_n. (To be exact, every (K_n, σ) is *switching isomorphic* to a subgraph of Ω_n or to $-K_n$. This means that, by *switching* σ, i.e., by reversing the signs of all edges between a vertex subset and its complement, one creates a sign-preserving isomorphism between (K_n, σ) and a subgraph of

Ω_n. Switching does not alter polygon signs, so it does not affect any orientation-embedding properties of (K_n, σ).) Evidently, the existence of Ω_n immediately implies Conjecture 1. I propose:

Conjecture 2. For every n, an enveloping graph Ω_n exists with properties (i) and (ii).

Just to add a little more weight to the conjectures, I found that the analogous properties of the Petersen graph P are true. That is, $d(-P)$ is the largest demigenus of any signing of P. Furthermore, P has an enveloping graph.

4*b*. *Complete-Graph Analogs*

A referee of [11] remarked that it is interesting that a simple induction suffices to establish the parity demigenus of K_n°—just as it does for K_n—yet no such proof is known for the genus of K_n. I believe one can understand this as analogous to the relative ease (as seen in [6]) of establishing the genus of $K_{r,s}$. There are several reasons for regarding both antibalanced and bipartite signed graphs as the nearest signed analogs of bipartite graphs. This would lead one to suspect that they are relatively easy to treat. The true signed analogs of the complete graph K_n are the *complete signed graph* $\pm K_n^\circ$, which has all possible positive and negative links (nonloop edges) and negative loops, and to a lesser extent the *complete signed link graph* $\pm K_n$, which is the same without the loops. I believe that these have demigenus equal or almost equal to the Eulerian lower bound, and that proving this will be hard.

Conjecture 3. If $n \geqslant 4$, then

$$d(\pm K_n^\circ) = \left\lceil \frac{n(n-3)}{3} \right\rceil + 2.$$

Conjecture 4. If $n \geqslant 2$, then

$$d(\pm K_n) = \left\lceil \frac{n(n-4)}{3} \right\rceil + 2 + \rho_n,$$

where $\rho_n = 0$ usually but in a few cases $\rho_n = 1$.

That the conjectured values (with $\rho_n = 0$) are lower bounds follows from Euler's polyhedral formula. I have proved Conjecture 3 for $n \leqslant 9$ by *ad hoc* drawings and for $n \equiv 3 \pmod{12}$ by building on the known minimal orientable embedding of K_{12s+3} (for which see [7]). Also, I proved Conjecture 4 for $n \leqslant 7$ (with $\rho_4 = \rho_5 = \rho_6 = 1$ and the other $\rho_n = 0$) and for $n \equiv 3$ (mod 12) (with $\rho_n = 0$). However, I have no general method.

On the other hand, I was able to solve $\pm K_{r,s}$, the *complete signed bipartite graph*, which consists of two sets of r and s vertices and all positive and negative links between them; so all the natural signed analogs of $K_{r,s}$, namely $-K_n^{\circ}$, $-K_n$, and $\pm K_{r,s}$, are now solved. (The proof is fairly simple—much more so than Ringel's proof of the genus of $K_{r,s}$, although of a similar nature. It will appear separately.)

4c. *Forbidden Minors*

We call $-K_n$ a *forbidden link minor* for orientation embedding (in $U_{d(-K_n)-1}$) if deleting any edge reduces the demigenus and contracting any edge does the same. (The *contraction* of $-K_n$ by an edge consists of $-K_{n-1}$, together with positive edges paralleling all the negative edges at one vertex. For general definitions of link contraction and forbidden link minors consult [9, Sections 1 and 10] or [10, Section 1].)

One at least of the all-negative complete graphs is known to be a forbidden link minor: $-K_5$ (according to [10, Section 12, Example 8]). Could any others be, as well? Euler's lower bound for $n \geqslant 6$, and a glance at the graphs with $n \leqslant 4$, show that the only possibilities are those of orders $n = 6, 7, 10, 11, 14, 15, \ldots$. I proved that $-K_6$ is not a forbidden link minor; which led me to doubt that any larger $-K_n$'s could be. However, $-K_6$ is the sole exception. Elizabeth Klipsch has found that $-K_n$ is after all a forbidden minor for all $n \equiv 6, 7 \pmod 4$ with $n \geqslant 7$. Moreover, $-K_n \backslash$edge is a forbidden link minor for $n = 6$ and for all $n \equiv 8, 9 \pmod 4$ with $n \geqslant 8$. The proofs are based largely on modifying the constructions used herein. The details will appear elsewhere.

4d. *Triangle-free Embedding*

The lower bound (3) depends only on the fact that every face has at least four sides. Thus we have a second corollary. Define $\tilde{\gamma}_4(\Gamma)$ to be the smallest positive number h such that Γ has an embedding in U_h in which every face has four or more sides.

COROLLARY 2. *If $n \geqslant 3$, then $\tilde{\gamma}_4(K_n) = d(-K_n)$.*

Proof. It is clear that $\tilde{\gamma}_4(\Gamma) \leqslant d(-\Gamma)$ as long as Γ has an odd polygon. Since $\tilde{\gamma}_4(-K_n)$ has the lower bound (3), our theorem implies that $\tilde{\gamma}_4(K_n) = d(-K_n)$ when $n \geqslant 3$ and $n \neq 5$. As for $n = 5$, we know that $2 \leqslant \tilde{\gamma}_4(K_5) \leqslant d(-K_5) = 3$. If $\tilde{\gamma}_4(K_5)$ were less than $d(-K_5)$, K_5 would have a nonorientable quadrilateral embedding; but it does not. So equality holds. ∎

In other words, relaxing the parity constraint on an embedding of K_n by requiring just that there be no triangular faces and that the embedding surface be nonorientable does not lower the number of necessary crosscaps.

Hartsfield and Ringel [2] and Hartsfield [1] have previously evaluated $\tilde{\gamma}_4(K_n)$ for $n = 4t + 1 \geqslant 13$ and $n = 4t \geqslant 8$ (the case $n = 4$ being well known) by providing current graphs for quadrilateral embeddings [2] or by inductive constructions [1]. Thus Corollary 2 extends that result to all $n \geqslant 3$. Their embeddings in [2] and (apparently, but I did not completely verify this) in [1] have the nice property, not shared by our parity embeddings, that no two faces have more than one edge in common; but with the sole exception of $n = 17$, they are not parity embeddings.

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Note added in proof. Elizabeth Klipsch has observed that another case of Conjecture 4, that in which $n = 12s + 7$ (with $\rho_n = 0$), follows from the minimal nonorientable embedding of K_n shown in [7].

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