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Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups

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Abstract

In this paper, we show strong convergence theorems for nonexpansive mappings and nonexpansive semigroups in Hilbert spaces by the hybrid method in the mathematical programming.

Keywords: Nonexpansive; Nonexpansive semigroup; Strong convergence; Metric projection; Resolvent

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space and let T be a nonexpansive mapping from C into itself, that is, $\|Tx - Ty\| \leqslant \|x - y\|$ holds for every $x, y \in C$. We denote by $\mathbf{N}$ the set of all positive integers. Halpern [3] introduced an iteration procedure as follows:

$$x_0 = x \in C, \qquad x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n$$

for each $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, 1]$. Wittmann [12] proved that $\{x_n\}$ converges strongly to $P_{F(T)}(x_0)$ when $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, where $F(T) = \{z \in C \mid Tz = z\}$ and $P_{F(T)}(\cdot)$ is the metric projection onto $F(T)$.

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The purpose of this paper is to make another method of strong convergence. Motivated by Solodov and Svaiter [10], we consider the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C \mid \|y_n - z\| \leqslant \|x_n - z\|\}, \\ Q_n = \{z \in C \mid (x_n - z, x_0 - x_n) \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1}$$

for each $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. Then, we show that $\{x_n\}$ converges strongly to $P_{F(T)}(x_0)$ by the hybrid method in the mathematical programming. By this method, we also study the proximal point algorithm [4,5,7,12]. Finally, we obtain a strong convergence theorem for a family of nonexpansive mappings in a Hilbert space.

2. Preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ will symbolize strong convergence. We know that H satisfies Opial's condition [6], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ holds for every $y \in H$ with $y \neq x$. We also know that for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, $\|x\| \leqslant \liminf_{n\to\infty} \|x_n\|$ holds. Further, let $\{x_n\}$ be a sequence of H with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then, there holds $x_n \to x$. Let C be a nonempty closed convex subset of H. We denote by $P_C(\cdot)$ the metric projection onto C. It is known that $z = P_C(x)$ is equivalent to $(z - y, x - z) \geqslant 0$ for every $y \in C$. Let T be a nonexpansive mapping from C into itself. It is known that $F(T)$ is closed and convex. A family $\mathcal{S} = \{T(s) \mid 0 \leqslant s < \infty\}$ of mappings from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geqslant 0$;
(iii) $\|T(s)x - T(s)y\| \leqslant \|x - y\|$ for all $x, y \in C$ and $s \geqslant 0$;
(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \bigcap_{0\leqslant s<\infty} F(T(s))$. It is known that $F(\mathcal{S})$ is closed and convex. An operator $A \subset H \times H$ is said to be monotone if $(x_1 - x_2, y_1 - y_2) \geqslant 0$ whenever $y_1 \in Ax_1$ and $y_2 \in Ax_2$. A monotone operator A is said to be maximal if the graph of A is not properly contained in the graph of any other monotone operator. Let A be a monotone operator. It is known that A is maximal iff $R(I + rA) = H$ for every $r > 0$, where $R(I + rA) = \bigcup\{z + rAz \mid z \in H, Az \neq \emptyset\}$. It is also known that A is maximal iff for $(u, v) \in H \times H$, $(x - u, y - v) \geqslant 0$ for every $(x, y) \in A$ implies $v \in Au$. For a maximal monotone operator A, we know that $A^{-1}0 = \{x \in H \mid 0 \in Ax\}$ is closed and convex. If A is monotone, then we can define, for each $r > 0$, a nonexpansive mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$, where $D(A) = \{z \in H \mid Az \neq \emptyset\}$. J_r is called a resolvent of A. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$. We also have $F(J_r) = A^{-1}0$ for each $r > 0$, where $F(J_r) = \{z \in D(A) \mid J_r z = z\}$; see [11] for more details.

The following lemma was proved by Shimizu and Takahashi [8]; see also [1,2,9].

Lemma 2.1. *Let C be a nonempty bounded closed convex subset of H and let $\mathcal{S} = \{T(s) \mid 0 \leqslant s < \infty\}$ be a nonexpansive semigroup on C. Then, for any $h \geqslant 0$,*

$$\lim_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)x\,ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x\,ds\right) \right\| = 0.$$

3. Strong convergence theorems for nonexpansive mappings

Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty. We consider the sequence $\{x_n\}$ generated by (1).

Lemma 3.1. *$\{x_n\}$ is well defined and $F(T) \subset C_n \cap Q_n$ for every $n \in \mathbf{N} \cup \{0\}$.*

Proof. It is obvious that C_n is closed and Q_n is closed and convex for every $n \in \mathbf{N} \cup \{0\}$. It follows that C_n is convex for every $n \in \mathbf{N} \cup \{0\}$ because $\|y_n - z\| \leqslant \|x_n - z\|$ is equivalent to

$$\|y_n - x_n\|^2 + 2(y_n - x_n, x_n - z) \leqslant 0.$$

So, $C_n \cap Q_n$ is closed and convex for every $n \in \mathbf{N} \cup \{0\}$. Let $u \in F(T)$. Then from

$$\begin{aligned}\|y_n - u\| &= \big\|\alpha_n x_n + (1-\alpha_n)Tx_n - u\big\| \\ &\leqslant \alpha_n\|x_n - u\| + (1-\alpha_n)\|Tx_n - u\| \leqslant \|x_n - u\|\end{aligned}$$

we have $u \in C_n$ for each $n \in \mathbf{N} \cup \{0\}$. So, we have $F(T) \subset C_n$ for all $n \in \mathbf{N} \cup \{0\}$.

Next, we show by mathematical induction that $\{x_n\}$ is well defined and $F(T) \subset C_n \cap Q_n$ for each $n \in \mathbf{N} \cup \{0\}$. For $n = 0$, we have $x_0 = x \in C$ and $Q_0 = C$, and hence $F(T) \subset C_0 \cap Q_0$. Suppose that x_k is given and $F(T) \subset C_k \cap Q_k$ for some $k \in \mathbf{N} \cup \{0\}$. There exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x_0)$. From $x_{k+1} = P_{C_k \cap Q_k}(x_0)$, there holds

$$(x_{k+1} - z, x_0 - x_{k+1}) \geqslant 0$$

for each $z \in C_k \cap Q_k$. Since $F(T) \subset C_k \cap Q_k$, we get $F(T) \subset Q_{k+1}$. Therefore we have $F(T) \subset C_{k+1} \cap Q_{k+1}$. This completes the proof. □

Lemma 3.2. *$\{x_n\}$ is bounded.*

Proof. Since $F(T)$ is a nonempty closed convex subset of C, there exists a unique element $z_0 \in F(T)$ such that $z_0 = P_{F(T)}(x_0)$. From $x_{n+1} = P_{C_n \cap Q_n}(x_0)$, we have

$$\|x_{n+1} - x_0\| \leqslant \|z - x_0\|$$

for every $z \in C_n \cap Q_n$. As $z_0 \in F(T) \subset C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leqslant \|z_0 - x_0\| \tag{2}$$

for each $n \in \mathbf{N} \cup \{0\}$. This implies that $\{x_n\}$ is bounded. □

Lemma 3.3. $\|x_{n+1} - x_n\| \to 0$.

Proof. As $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n}(x_0)$, we have

$$\|x_{n+1} - x_0\| \geqslant \|x_n - x_0\|$$

for every $n \in \mathbf{N} \cup \{0\}$. Therefore, by Lemma 3.2 the sequence $\{\|x_n - x_0\|\}$ is bounded and nondecreasing. So there exists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+1} \in Q_n$, we have $(x_n - x_{n+1}, x_0 - x_n) \geqslant 0$ and hence

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \big\|(x_n - x_0) - (x_{n+1} - x_0)\big\|^2 \\
&= \|x_n - x_0\|^2 - 2(x_n - x_0, x_{n+1} - x_0) + \|x_{n+1} - x_0\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2(x_n - x_{n+1}, x_0 - x_n) \\
&\leqslant \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2
\end{aligned}$$

for every $n \in \mathbf{N} \cup \{0\}$. This implies that $\|x_{n+1} - x_n\| \to 0$. □

Theorem 3.4. $x_n \to z_0$, *where* $z_0 = P_{F(T)}(x_0)$.

Proof. Since $\{x_n\}$ is bounded, we assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to w_0. It follows from $x_{n+1} \in C_n$ that

$$\begin{aligned}
\|Tx_n - x_n\| &= \frac{1}{1-\alpha_n}\|y_n - x_n\| \leqslant \frac{1}{1-\alpha_n}\big(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|\big) \\
&\leqslant \frac{2}{1-\alpha_n}\|x_{n+1} - x_n\|
\end{aligned}$$

for every $n \in \mathbf{N} \cup \{0\}$. By Lemma 3.3, we get

$$\|Tx_n - x_n\| \to 0. \tag{3}$$

Suppose that $w_0 \neq Tw_0$. From Opial's condition and (3), we have

$$\begin{aligned}
\liminf_{i\to\infty}\|x_{n_i} - w_0\| &< \liminf_{i\to\infty}\|x_{n_i} - Tw_0\| \leqslant \liminf_{i\to\infty}\big(\|x_{n_i} - Tx_{n_i}\| + \|x_{n_i} - w_0\|\big) \\
&= \liminf_{i\to\infty}\|x_{n_i} - w_0\|.
\end{aligned}$$

This is a contradiction. Hence, we get

$$w_0 \in F(T). \tag{4}$$

If $z_0 = P_{F(T)}(x_0)$, it follows from (2), (4) and the lower semicontinuity of the norm that

$$\|x_0 - z_0\| \leqslant \|x_0 - w_0\| \leqslant \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \leqslant \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \leqslant \|x_0 - z_0\|.$$

Thus, we obtain $\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|x_0 - w_0\| = \|x_0 - z_0\|$. This implies

$$x_{n_i} \to w_0 = z_0.$$

Therefore, we have $x_n \to z_0$. $\square$

We apply this method to the proximal point algorithm [4,5,7,12] and get the following theorem.

Theorem 3.5. *Let $A \subset H \times H$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$ and let J_r be the resolvent of A, where $r > 0$. Define a sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = J_{r_n}(x_n + f_n), \\ C_n = \{z \in H \mid \|y_n - z\| \leqslant \|x_n + f_n - z\|\}, \\ Q_n = \{z \in H \mid (x_n - z, x_0 - x_n) \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{5}$$

for every $n \in \mathbf{N} \cup \{0\}$, where $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} \|f_n\| = 0$. Then, $x_n \to z_0 = P_{A^{-1}0}(x_0)$.

Proof. As in the proof of Lemma 3.1, $\{x_n\}$ is well defined and $A^{-1}0 \subset C_n \cap Q_n$ for every $n \in \mathbf{N} \cup \{0\}$ because J_{r_n} is nonexpansive and $A^{-1}0 = \{z \in H \mid J_{r_n}z = z\}$ for every $n \in \mathbf{N} \cup \{0\}$. Results in Lemmas 3.2 and 3.3 hold because $A^{-1}0$ is nonempty, closed and convex. We also have from $\lim_{n\to\infty} \|f_n\| = 0$ that $\{y_n\}$ is bounded. Next, we suppose that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to w_0. It follows from $x_{n+1} \in C_n$ that

$$\begin{aligned} \|y_n - x_n\| &\leqslant \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leqslant \|x_n + f_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leqslant 2\|x_{n+1} - x_n\| + \|f_n\| \end{aligned}$$

for every $n \in \mathbf{N} \cup \{0\}$. From $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|f_n\| = 0$, we obtain $\|y_n - x_n\| \to 0$. This implies that

$$y_{n_i} \rightharpoonup w_0. \tag{6}$$

On the other hand, since A is monotone, we have, for every $i \in \mathbf{N}$ and $(u, v) \in A$,

$$\left(y_{n_i} - u, \frac{1}{r_{n_i}}(x_{n_i} + f_{n_i} - y_{n_i}) - v \right) \geqslant 0$$

and hence

$$(y_{n_i} - u, -v) \geqslant -\frac{1}{r_{n_i}} \|y_{n_i} - u\| \cdot \big\| y_{n_i} - (x_{n_i} + f_{n_i}) \big\|.$$

By the boundedness of $\{(1/r_{n_i})\|y_{n_i} - u\|\}$, $\|y_{n_i} - (x_{n_i} + f_{n_i})\| \to 0$ and (6), we have $(w_0 - u, -v) \geqslant 0$ for every $(u, v) \in A$. Therefore, we get $w_0 \in A^{-1}0$ as A is maximal. If $z_0 = P_{A^{-1}0}(x_0)$, as in the proof of Theorem 3.4, we have

$$\|z_0 - x_0\| \leqslant \|w_0 - x_0\| \leqslant \liminf_{i\to\infty} \|x_{n_i} - x_0\| \leqslant \limsup_{i\to\infty} \|x_{n_i} - x_0\| \leqslant \|z_0 - x_0\|.$$

We obtain $\lim_{i\to\infty} x_{n_i} = w_0 = z_0$. Therefore, we get $\lim_{n\to\infty} x_n = z_0$. □

4. Strong convergence theorem for nonexpansive semigroups

Let C be a nonempty closed convex subset of H and $\mathcal{S} = \{T(s) \mid 0 \leqslant s < \infty\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Note that $F(\mathcal{S})$ is closed and convex. Consider a sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds, \\ C_n = \{z \in C \mid \|y_n - z\| \leqslant \|x_n - z\|\}, \\ Q_n = \{z \in C \mid (x_n - z, x_0 - x_n) \geqslant 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0) \end{cases} \tag{7}$$

for every $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\}$ is a sequence in $[0, a]$ for some $a \in [0, 1)$ and $\{t_n\}$ is a positive real divergent sequence. Using Lemma 2.1, we get the following theorem.

Theorem 4.1. $x_n \to z_0 = P_{F(\mathcal{S})}(x_0)$.

Proof. Since we have, for every $u \in F(\mathcal{S})$ and $n \in \mathbf{N} \cup \{0\}$,

$$\begin{aligned} \|y_n - u\| &\leqslant \alpha_n\|x_n - u\| + (1-\alpha_n)\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - u\right\| \\ &\leqslant \alpha_n\|x_n - u\| + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} \left\|T(s)x_n - u\right\|ds \\ &\leqslant \alpha_n\|x_n - u\| + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} \|x_n - u\|\,ds \\ &= \alpha_n\|x_n - u\| + (1-\alpha_n)\|x_n - u\| = \|x_n - u\|, \end{aligned}$$

it follows that $F(\mathcal{S}) \subset C_n$ for every $n \in \mathbf{N} \cup \{0\}$. As in the proof of Lemma 3.1, we get that $\{x_n\}$ is well defined and $F(\mathcal{S}) \subset C_n \cap Q_n$ for each $n \in \mathbf{N} \cup \{0\}$. Since $F(\mathcal{S})$ is nonempty and $z_0 = P_{F(\mathcal{S})}(x_0)$, as in the proofs of Lemmas 3.2 and 3.3, we get that $\|x_{n+1} - x_0\| \leqslant \|z_0 - x_0\|$ for each $n \in \mathbf{N} \cup \{0\}$, $\{x_n\}$ is bounded and $\|x_{n+1} - x_n\| \to 0$. We assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to w_0. We have

$$\begin{aligned}
\|T(s)x_n - x_n\| &\leqslant \left\| T(s)x_n - T(s)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right)\right\| \\
&\quad + \left\| T(s)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right) - \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right\| \\
&\quad + \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - x_n\right\| \\
&\leqslant 2\left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - x_n\right\| \\
&\quad + \left\| T(s)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right) - \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right\| \qquad (8)
\end{aligned}$$

for every $0 \leqslant s < \infty$ and $n \in \mathbf{N} \cup \{0\}$. On the other hand, from $x_{n+1} \in C_n$, we have that

$$\begin{aligned}
\left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - x_n\right\| &= \frac{1}{1-\alpha_n}\|y_n - x_n\| \\
&\leqslant \frac{1}{1-\alpha_n}\big(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|\big) \leqslant \frac{2}{1-\alpha_n}\|x_{n+1} - x_n\| \qquad (9)
\end{aligned}$$

for every $n \in \mathbf{N} \cup \{0\}$. Let $X = \{z \in C \mid \|z - z_0\| \leqslant 2\|z_0 - x_0\|\}$. Then, X is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $s \in [0, \infty)$ and contains $\{x_n\}$. By Lemma 2.1, we get

$$\lim_{n\to\infty}\left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right)\right\| = 0 \qquad (10)$$

for every $h \in [0, \infty)$. By (8)–(10) and $\|x_{n+1} - x_n\| \to 0$, we obtain

$$\|T(s)x_n - x_n\| \to 0$$

for each $0 \leqslant s < \infty$. This implies that

$$w_0 \in F(\mathcal{S}) \qquad (11)$$

by Opial's condition. As in the proof of Theorem 3.4, we have $x_{n_i} \to w_0 = z_0$. Therefore, we get $x_n \to z_0$. □

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