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Operator Monotone Functions which Are Defined Implicitly and Operator Inequalities

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monotone functions are known so far. We will systematically seek operator monotone functions which are defined implicitly. This investigation is new, and our method seems to be powerful. We will actually find a family of operator monotone functions which includes t^{α} $(0<\alpha<1)$. Moreover, by constructing one-parameter families of operator monotone functions, we will get many operator inequalities; especially, we will extend the Furuta inequality and the exponential inequality of Ando. © 2000 Academic Press

Key Words: operator monotone function; Pick function; Löwner theory; Löwner–Heinz inequality; Furuta inequality.

1. INTRODUCTION

Throughout this paper, A and B stand for bounded selfadjoint operators on a Hilbert space and $sp(X)$ for the spectrum of an operator X. A real valued function $f(t)$ is called an *operator monotone function* on an interval I in R^1 if, for A, B with $sp(A), sp(B) \subset I$,

$$A \geqslant B \qquad \text{implies} \quad f(A) \geqslant f(B).$$

Clearly a composite function of operator monotone functions is operator monotone too, provided it is well defined. A holomorphic function which maps the open upper half plane Π_+ into itself is called a *Pick function*. By Löwner's theorem [13], $f(t)$ is an operator monotone function on an open interval (a, b) if and only if $f(t)$ has an analytic continuation $f(z)$ to $\Pi_+ \cup (a, b)$ so that $f(z)$ is a Pick function; therefore $f(t)$ is analytically extended to the open lower half plane by reflection. Thus if $f(t) \geqslant 0$ and

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$g(t) \geqslant 0$ are operator monotone, then so is $f(t)^{\mu} g(t)^{\lambda}$ for $0 \leqslant \mu,\ \lambda \leqslant 1$, $\mu + \lambda \leqslant 1$. If $f(t)$ is operator monotone on (a, b) and if $f(t)$ is continuous on $[a, b)$, then $f(t)$ is operator monotone on $[a, b)$. It is known that t^{α} $(0 < \alpha \leqslant 1)$, $\log(1+t)$, and $\frac{t}{t+\lambda}$ $(\lambda > 0)$ are operator monotone on $[0, \infty)$. Thus,

$$A \geqslant B \geqslant 0 \qquad \text{implies} \quad A^{\alpha} \geqslant B^{\alpha} \qquad \text{for} \quad 0 < \alpha < 1, \tag{1}$$

which is called the *Löwner–Heinz inequality* [12, 13]. But $A \geqslant B \geqslant 0$ does not generally imply $A^2 \geqslant B^2$. We have shown that if A, $B \geqslant 0$ and $(A + tB^n)^2 \geqslant A^2$ for every $t > 0$ and $n = 1, 2, ...,$ then $AB = BA$ [16]. See [1, 3, 5, 9, 11, 14] for details about operator monotone functions.

Chan-Kwong [4] posed the following question:

$$\text{Does } A \geqslant B \geqslant 0 \text{ imply } (BA^2B)^{1/2} \geqslant B^2\,?$$

Furuta [7, 8] answered it affirmatively as follows:

$$A \geqslant B \geqslant 0 \qquad \text{implies} \quad \begin{cases} (B^{r/2}A^pB^{r/2})^{1/q} \geqslant (B^{r/2}B^pB^{r/2})^{1/q}, \\ (A^{r/2}A^pA^{r/2})^{1/q} \geqslant (A^{r/2}B^pA^{r/2})^{1/q}, \end{cases} \tag{2}$$

where $r, p \geqslant 0$ and $q \geqslant 1$ with $(1+r)\, q \geqslant p + r$. This is called the *Furuta inequality*. In this inequality, the case $p \leqslant 1$ is a deformation of the Löwner–Heinz inequality; further, the case $(1+r)\, q > p + r$ follows from the case $(1+r)\ q = p + r$ by the Löwner–Heinz inequality again. So the essentially important part of (2) is the case $p > 1$ and $(1+r)\ q = p + r$. One obtains the second inequality of (2) from the first one by taking inverses. Tanahashi [15] showed that the exponential condition $(1+r)\ q \geqslant p + r$ is the best possible condition for (2). Ando [2] obtained the related inequality: for $t > 0$,

$$A \geqslant B \qquad \text{implies} \quad \begin{cases} (e^{(t/2)B}e^{tA}e^{(t/2)B})^{1/2} \geqslant e^{tB}, \\ e^{tA} \geqslant (e^{(t/2)A}e^{tB}e^{(t/2)A})^{1/2}. \end{cases}$$

This was improved by use of the inequality itself and (2), by Fujii and Kamei [6] as follows: for $p \geqslant 0$, $r \geqslant s \geqslant 0$,

$$A \geqslant B \qquad \text{implies} \quad \begin{cases} (e^{(r/2)B}e^{pA}e^{(r/2)B})^{s/(r+p)} \geqslant e^{sB}, \\ e^{sA} \geqslant (e^{(r/2)A}e^{pB}e^{(r/2)A})^{s/(r+p)}. \end{cases} \tag{3}$$

It is evident that the essentially important part of this inequality is the case $s = r$. Recently, by making use of only (2), we [18] got a simple proof of (3).

Now we give a simple example that motivated us to investigate operator monotone functions which are defined implicitly,

$$A, B \geqslant 0 \qquad \text{and} \qquad A^2 \geqslant B^2 \qquad \text{imply} \quad (A+1)^2 \geqslant (B+1)^2,$$

because $A \geqslant B$ follows from $A^2 \geqslant B^2$. But we can easily construct 2×2 matrices A, B such that $(A+1)^2 \geqslant (B+1)^2$, but $A^2 \ngeqslant B^2$; for example,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1.94 \end{pmatrix}.$$

The above results mean that $\phi(t) = (t^{1/2}+1)^2$ is operator monotone on $[0, \infty)$, but $\psi(t) = (t^{1/2}-1)^2$ is not on $[1, \infty)$. We may say that ϕ and ψ are implicitly defined by $\phi(t^2) = (t+1)^2$ $(t \geqslant 0)$ and $\psi((t+1)^2) = t^2$ $(t \geqslant 0)$.

One aim of this paper is to seek operator monotone functions which are defined implicitly; this investigation is new, and we will actually find a family of operator monotone functions which includes t^α $(0<\alpha<1)$. This means that we can get not merely an extension of (1) but also another proof of (1). The other aim is to extend simultaneously (2) and (3), by making use of a one-parameter family of operator monotone functions.

2. THE CONSTRUCTION OF NEW OPERATOR MONOTONE FUNCTIONS

Let us define a non-negative increasing function $u(t)$ on $[-a_1, \infty)$ by

$$u(t) = \prod_{i=1}^{k} (t+a_i)^{\gamma_i} \qquad (a_1 < a_2 < \cdots < a_k,\ 1 \leqslant \gamma_1,\ 0 < \gamma_i). \tag{4}$$

THEOREM 2.1. *Let the function $s = u(t)$ be defined by* (4). *Then the inverse function $u^{-1}(s)$ is operator monotone on $[0, \infty)$.*

Proof. Since $u^{-1}(s)$ is continuous on $[0, \infty)$, we have to show that $u^{-1}(s)$ is operator monotone on $(0, \infty)$. We may assume that $a_1 = 0$; for, setting $v(t) = u(t-a_1)$ we have $u^{-1}(s) = v^{-1}(s) - a_1$; hence the operator monotonicity of $u^{-1}(s)$ follows from that of $v^{-1}(s)$. Set $D = \mathbf{C} \backslash (-\infty, 0]$, and restrict the argument by $-\pi < \arg z < \pi$ for $z \in D$. For $\gamma > 0$ define the single valued holomorphic function z^γ on D by

$$z^\gamma = \exp \gamma(\log |z| + i \arg z),$$

which is the principal branch of the analytic function $\exp(\gamma \log z)$. Using this we define a holomorphic function $u(z)$ on D by

$$u(z) = \prod_{i=1}^{k} (z+a_i)^{\gamma_i}, \qquad 0 = a_1 < a_2 < \cdots < a_k$$

which is an extension of $u(t)$. Since

$$u'(z)=\left\{\prod_{i=1}^{k}(z+a_i)^{\gamma_i}\right\}\left(\sum_{j=1}^{k}\frac{\gamma_j}{z+a_j}\right),$$

it is necessary and sufficient for $u'(z)=0$ in D that $\sum_{j=1}^{k}\gamma_j/(z+a_j)=0$. Since $\gamma_j>0$ and $a_j\geqslant 0$, the roots of $\sum_{j=1}^{k}\gamma_j/(z+a_j)=0$ are all in $(-\infty,0)$. Therefore, $u'(z)$ does not vanish in D. Let us consider the function $w=u(z)$ as a mapping from the z-plane to the w-plane. We denote D in the z-plane by D_z and D in the w-plane by D_w. Take a $t_0>0$ and set $s_0=u(t_0)$. Since $u'(t_0)\neq 0$, by the inverse mapping theorem, there is a univalent holomorphic function $g_0(w)$ from a disk $\Delta(s_0)$ with center s_0 onto an open set including t_0 such that $u(g_0(w))=w$ for $w\in\Delta(s_0)$. We show that for an arbitrary point w_0 in D_w and for an arbitrary path C in D_w from s_0 to w_0, the function element $(g_0,\Delta(s_0))$ admits an analytic continuation $(g_i,\Delta(\zeta_i))_{0\leqslant i\leqslant n}$ with $\zeta_0=s_0$ along C, which satisfies the condition

$$(\bigstar)\qquad \begin{cases} g_i(w) \text{ is univalent from } \Delta(\zeta_i) \text{ into } D_z,\\ u(g_i(w))=w \qquad \text{for} \quad w\in\Delta(\zeta_i).\end{cases}$$

For $\zeta\in C$ let us denote the subpath of C from s_0 to ζ by C_ζ, and let E be the set of points ζ in C such that $(g_0,\Delta(s_0))$ admits an analytic continuation satisfying $(\bigstar)$ along C_ζ. Since E includes s_0 and is a relatively open subset of C, if E is closed in C, then $w_0\in E$. Thus we need to show the closedness of E; actually we show that if $C_\zeta\setminus\{\zeta\}$ is included in E, so is ζ. Take a sequence $\{\zeta_n\}$ in $C_\zeta\setminus\{\zeta\}$ which converges to ζ, and construct a family $\{(g_n,\Delta(\zeta_n))\}$ so that $\{(g_i,\Delta(\zeta_i))\}_{1\leqslant i\leqslant n}$ is the analytic continuation of $(g_0,\Delta(s_0))$ along C_{ζ_n} satisfying $(\bigstar)$; $C_\zeta\setminus\{\zeta\}$ may be covered by finite numbers of $\Delta(\zeta_i)$, but even in this case we can construct an infinite number of $\Delta(\zeta_i)$ as above. If an infinite number of the radii of disks $\Delta(\zeta_n)$ are larger than a positive constant, then ζ is in some $\Delta(\zeta_n)$ and hence in E. Therefore, we assume that the sequence of radii of $\Delta(\zeta_n)$ converges to 0. The sequence of $z_n:=g_n(\zeta_n)$ is bounded in D_z, which is obvious from the form of the function $u(z)$ and the boundedness of the sequence of $\zeta_n=u(g_n(\zeta_n))$. Hence it contains a convergent subsequence $\{z_{n_i}\}$, whose limit we denote by z_0. We prove, by contradiction, that z_0 is in D_z. Assume that $z_0=0$; then from the definition of $u(z)$, $\zeta_{n_i}=u(z_{n_i})\to 0$; this implies $\zeta=0$, which contradicts $C_\zeta\subset D_w$. Assume that $\arg z_{n_i}\uparrow\pi$; then, because $\gamma_1\geqslant 1$ and $a_1=0$, $\lim\arg\zeta_{n_i}=\lim\arg u(z_{n_i})\geqslant\pi$ or $\lim u(z_{n_i})=0$; this implies that C_ζ intersects $(-\infty,0]$, which contradicts $C_\zeta\subset D_w$. Similarly assume that $\arg z_{n_i}\downarrow -\pi$; then C_ζ intersects $(-\infty,0]$, which contradicts $C_\zeta\subset D_w$. Therefore, z_0 is in D_z. Thus $u(z)$ is continuous at z_0. Hence $u(z_0)=\lim u(z_{n_i})=\lim\zeta_{n_i}=\zeta$. Since $u'(z_0)\neq 0$, by the inverse mapping theorem, there is a disk $\Delta(\zeta)$ and a holomorphic inverse function g_ζ from $\Delta(\zeta)$ into D_z such that $g_\zeta(\zeta)=z_0$ and $w=u(g_\zeta(w))$ for $w\in\Delta(\zeta)$. Since $\zeta_n\to\zeta$ and since the radii of the disks $\Delta(\zeta_n)$ tend to 0, $\Delta(\zeta)\supseteq\Delta(\zeta_n)$ for $n>N$. Therefore, by ($\bigstar$), we have $g_\zeta(w)=g_n(w)$ for $n>N$ and for $w\in\Delta(\zeta_n)$. This implies $z_n\to z_0$; in fact, for $n>N$ $z_n=g_n(\zeta_n)=g_\zeta(\zeta_n)$ which converges to $g_\zeta(\zeta)=z_0$.

Let us join $(g_\zeta, \Delta(\zeta))$ to $\{(g_i, \Delta(\zeta_i))\}_{1\leqslant i\leqslant N}$. Then this new family is an analytic continuation of (g_0, s_0) satisfying ($\bigstar$). Hence $\zeta\in E$. Thus we have shown that an analytic element (g_0, s_0) has an analytic continuation satisfying ($\bigstar$) along every path in D_w. By the monodromy theorem, this analytic continuation is a single valued holomorphic function. We denote it by $g(w)$. Then $g(w)$ is a holomorphic function from D_w into D_z such that

$$u(g(w))=w \quad (w\in D_w) \qquad \text{and} \qquad g(s)=u^{-1}(s) \quad (0<s<\infty).$$

We finally show that $g(w)$ is a Pick function. We denote the open lower half plane by Π_-. Set $\Gamma=\sum_{i=1}^n \gamma_i$. Since $g(w)$ is continuous, there is a neighbourhood W of s_0 such that

$$g(W)\subseteq V:=\{z\colon -\pi/\Gamma<\arg z<\pi/\Gamma\}.$$

Here we note that

$$u(V\cap\Pi_+)\subset\Pi_+, \qquad u(V\cap\Pi_-)\subset\Pi_-, \qquad \text{and} \qquad u((0,\infty))=(0,\infty).$$

In fact, to see the first inclusion, take $z\in V\cap\Pi_+$; since $0=a_1<a_i$ for $i>1$, $z+a_i\in V\cap\Pi_+$, and hence $0<\arg(\prod_{i=1}^k (z+a_i)^{\gamma_i})<\pi$, which means that $u(V\cap\Pi_+)\subset\Pi_+$; similarly we can see the second inclusion, and the last equality is clear. From these inclusions, it follows that

$$g(W\cap\Pi_+)\subseteq\Pi_+.$$

In fact, take an arbitrary $w\in W\cap\Pi_+$; then $g(w)\in V$. Assume $g(w)\notin\Pi_+$; then, by the above argument, we have $w=u(g(w))\notin\Pi_+$; this is a contradiction.

Because $u((0,\infty))=(0,\infty)$ and $u(g(w))=w$ for $w\in D_w$ it follows that $g(\Pi_+)\cap(0,\infty)=\varnothing$. This and the connectedness of $g(\Pi_+)$ in D_z, together with the inclusion $\varnothing\neq g(W\cap\Pi_+)\subset\Pi_+$, show that $g(\Pi_+)\subseteq\Pi_+$. Hence g is a Pick function. ∎

For $0<\alpha<1$, a function $u(t)=t^{1/\alpha}$ satisfies (4). Hence the above theorem says $u^{-1}(s)=s^\alpha$ is operator monotone on $[0,\infty)$: which is (1).

In the above proof we used the condition $\gamma_1\geqslant 1$. To see that we cannot weaken this condition to $\sum_i r_i\geqslant 1$, we give a

COUNTEREXAMPLE. Set $u(t)=t^{1/2}(t+1)$. Then $u'(t)=\frac{1}{2}t^{-1/2}(3t+1)$ and $u''(t)=\frac{1}{4}t^{-3/2}(3t-1)$. Therefore $u''(t)<0$ $(0<t<1/3)$ hence $(u^{-1})''(s)>0$ $(0<s<\frac{4}{3\sqrt{3}})$.

Since an operator monotone function on $[0,\infty)$ is concave, this implies that $u^{-1}(s)$ is not operator monotone on $[0,\infty)$.

THEOREM 2.2. *Define a function $v(t)$ by*

$$v(t)=\prod_{j=1}^{l}(t+b_j)^{\lambda_j}\,(t\geqslant -b_1), \qquad b_1<b_2<\cdots<b_l, \quad 0<\lambda_j. \tag{5}$$

Then, for $u(t)$ represented by (4), *if the conditions*

$$\begin{cases} a_1\leqslant b_1, \\ \sum_{b_j<t}\lambda_j\leqslant\sum_{a_i<t}\gamma_i \qquad \textit{for every} \quad t\in\mathbf{R} \end{cases} \tag{6}$$

are satisfied, the function ϕ defined on $[0,\infty)$ by

$$\phi(u(t))=v(t) \quad (-a_1\leqslant t), \qquad \textit{that is,} \quad \phi(s)=v(u^{-1}(s)) \quad (0\leqslant s)$$

is an operator monotone function on $[0,\infty)$.

Proof. The notation of the preceding proof is retained. Set

$$\Gamma_i=\gamma_1+\cdots+\gamma_i, \qquad \Lambda_i=\sum_{b_j<a_{i+1}}\lambda_j, \qquad \text{where} \quad a_{k+1}=\infty.$$

Then the second condition of (6) is equivalent to

$$\Lambda_i\leqslant\Gamma_i \qquad (1\leqslant i\leqslant k).$$

We have seen that $u^{-1}(s)$ admits an analytic continuation $g(w)$ which is a Pick function. Let us define a holomorphic function $v(z)$ on D_z by

$$v(z)=\prod_{j=1}^{l}(z+b_j)^{\lambda_j}$$

in the same way that we defined $u(z)$ in the preceding proof. Then $v(g(w))$ on D_w is an analytic continuation of $v(u^{-1}(s))$. We need to show that $v(g(w))$ is a Pick function.

Take w such that $0<\arg w<\pi$. Since, for $a_i\leqslant b_j<a_{i+1}$,

$$0<\arg(g(w)+b_j)\leqslant\arg(g(w)+a_i),$$

we have

$$
\begin{aligned}
0 &< \arg v(g(w)) \\
&= \sum_j \lambda_j \arg(g(w)+b_j) \\
&= \sum_i \sum_{a_i \leqslant b_j < a_{i+1}} \lambda_j \arg(g(w)+b_j) \\
&\leqslant \sum_i \sum_{a_i \leqslant b_j < a_{i+1}} \lambda_j \arg(g(w)+a_i) \\
&= \sum_i (\Lambda_i - \Lambda_{i-1}) \arg(g(w)+a_i) \qquad (\Lambda_0 = 0) \\
&= \sum_{i=1}^{k-1} \Lambda_i \{\arg(g(w)+a_i) - \arg(g(w)+a_{i+1})\} + \Lambda_k \arg(g(w)+a_k) \\
&\leqslant \sum_{i=1}^{k-1} \Gamma_i \{\arg(g(w)+a_i) - \arg(g(w)+a_{i+1})\} + \Gamma_k \arg(g(w)+a_k) \\
&= \sum_{i=1}^{k} (\Gamma_i - \Gamma_{i-1}) \arg(g(w)+a_i) \\
&= \sum_{i=1}^{k} \gamma_i \arg(g(w)+a_i) = \arg u(g(w)) = \arg w < \pi.
\end{aligned}
$$

This completes the proof. ∎

3. THE FURTHER CONSTRUCTION OF OPERATOR MONOTONE FUNCTIONS

This section is a continuation of the preceding section. We start with a simple lemma.

LEMMA 3.1. *Let f_n $(n = 1, 2, ...)$ be strictly increasing continuous functions on $[a, \infty)$ $(a \in \mathbf{R})$ with $f_n(a) = 0$, $f_n(\infty) = \infty$, and let $f_n(t) \leqslant f_{n+1}(t)$ for $t \in [a, \infty)$. If $f_n(t)$ converges pointwise to a strictly increasing continuous function $f(t)$, then $f_n^{-1}(s)$ converges uniformly to $f^{-1}(s)$ on every bounded closed interval $[0, b]$ $(0 < b < \infty)$. Furthermore, if a sequence $\{h_n\}$ of continuous functions on $[0, \infty)$ satisfies $h_n(t) \leqslant h_{n+1}(t)$ and converges to a continuous function $h(t)$, then $h_n(f_n^{-1}(s))$ converges uniformly to $h(f^{-1}(s))$ on $[0, b]$ as well.*

Proof. Since $f^{-1}(s) \leqslant f_{n+1}^{-1}(s) \leqslant f_n^{-1}(s)$, it is easy to see that $f_n^{-1}(s)$ converges pointwise to $f^{-1}(s)$. Therefore, by Dini's theorem the sequence converges uniformly on $[0, b]$. By making use of Dini's theorem again, $\{h_n(t)\}$ converges uniformly to $h(t)$ on $[a, f_1^{-1}(b)]$, and it is equicontinuous there. Since $a \leqslant f_n^{-1}(s) \leqslant f_1^{-1}(b)$ for $0 \leqslant s \leqslant b$, in virtue of

$$\begin{aligned} &h_n(f_n^{-1}(s)) - h(f^{-1}(s)) \\ &\quad = h_n(f_n^{-1}(s)) - h_n(f^{-1}(s)) + h_n(f^{-1}(s)) - h(f^{-1}(s)), \end{aligned}$$

we obtain the uniform convergence of $h_n(f_n^{-1}(s))$ on $[0, b]$. ∎

THEOREM 3.2. *Let $u(t)$, $v(t)$ be the functions defined by* (4), (5). *Suppose that condition* (6) *is satisfied. Then, if $0 \leqslant \beta \leqslant \alpha$, the function ϕ on $[0, \infty)$ defined by*

$$\phi(u(t)\, e^{\alpha t}) = v(t)\, e^{\beta t} \qquad (-a_1 \leqslant t < \infty)$$

is operator monotone on $[0, \infty)$.

Proof. We assume $\beta > 0$; the proof below is modified when $\beta = 0$. The two functions

$$\tilde{u}_n(t) = u(t) \left(t + \frac{n}{\alpha} \right)^n \qquad \text{and} \qquad \tilde{v}_n(t) = v(t) \left(t + \frac{n}{\beta} \right)^n$$

satisfy (6) of Theorem 2.2 for sufficiently large n. Thus the function $\tilde{\phi}_n$ defined on $[0, \infty)$ by $\tilde{\phi}_n(\tilde{u}_n(t)) = \tilde{v}_n(t)$ $(t \geqslant -a_1)$ is operator monotone. In general, if $\psi(t)$ is operator monotone on $[0, \infty)$, so is $c_1 \psi(c_2 t)$ $(c_1, c_2 > 0)$. The function ϕ_n defined by

$$\phi_n(u_n(t)) = v_n(t), \quad \text{where } u_n(t) = u(t) \left(1 + \frac{\alpha}{n} t \right)^n, \qquad v_n(t) = v(t) \left(1 + \frac{\beta}{n} t \right)^n$$

satisfies

$$\phi_n(s) = \left(\frac{\beta}{n} \right)^n \tilde{\phi}_n \left(\left(\frac{\alpha}{n} \right)^{-n} s \right),$$

so that it is operator monotone on $[0, \infty)$. By Lemma 3.1, $\phi_n(s) = v_n(u_n^{-1}(s))$ converges uniformly to $\phi(s)$ on every finite closed interval as $n \to \infty$. Hence the limit function ϕ is operator monotone on $[0, \infty)$. ∎

Using the above theorem we construct a one-parameter family of operator monotone functions.

Corollary 3.3. *Let $u(t)$, $v(t)$ be the functions given by* (4), (5). *Suppose that condition* (6) *is satisfied and that* $0 \leqslant \beta \leqslant \alpha$, $0 \leqslant c \leqslant 1$. *Then, for each* $r > 0$ *the function* $\phi_r(s)$ *on* $[0, \infty)$ *defined by*

$$\phi_r(u(t)\, e^{\alpha t}(v(t)\, e^{\beta t})^r) = (v(t)\, e^{\beta t})^{c+r} \qquad (-a_1 \leqslant t < \infty)$$

is operator monotone.

Proof. Let us represent $u(t)\, v(t)^r$ and $v(t)^{c+r}$ as (4) and (5), respectively. Then it is easy to see that their exponents satisfy (6). Since $(c+r)\, \beta \leqslant \alpha + \beta r$, all conditions in the theorem are satisfied. Hence operator monotonicity of ϕ_r follows from it. ∎

It is not difficult to derive the next corollary from Lemma 3.1 and Theorem 3.2.

Corollary 3.4. *Suppose that two infinite products*

$$\tilde{u}(t) := \prod_{i=1}^{\infty} (t + a_i)^{\gamma_i} \qquad (a_i < a_{i+1}, 1 \leqslant \gamma_1, 0 < \gamma_i)$$

and

$$\tilde{v}(t) := \prod_{j=1}^{\infty} (t + b_j)^{\lambda_j}, \qquad (b_j < b_{j+1}, 0 < \lambda_j)$$

are both convergent on $-a_1 \leqslant t < \infty$. *If condition* (6) *is satisfied and if* $0 \leqslant \beta \leqslant \alpha$, *then the function* ϕ *defined by*

$$\phi(\tilde{u}(t)\, e^{\alpha t}) = \tilde{v}(t)\, e^{\beta t} \qquad (-a_1 \leqslant t < \infty)$$

is operator monotone on $[0, \infty)$. *Moreover, if* $0 \leqslant c \leqslant 1$ *and* $r > 0$, *then the function* $\phi_r(s)$ *on* $[0, \infty)$ *defined by*

$$\phi_r(\tilde{u}(t)\, e^{\alpha t}(\tilde{v}(t)\, e^{\beta t})^r) = (\tilde{v}(t)\, e^{\beta t})^{c+r} \qquad (-a_1 \leqslant t < \infty)$$

is operator monotone.

We remark that each family $\{\phi_r\}$ of operator montone functions constructed above satisfies the following relation,

$$\phi_r(h(t)\, f(t)^r) = f(t)^{c+r} \qquad (r > 0),$$

where $h(t)$ and $f(t)$ are appropriate increasing functions.

4. AN ESSENTIAL INEQUALITY AND AN EXTENSION OF THE FURUTA INEQUALITY

The aim of this section is to give an essential inequality which leads us to extensions of (2) and (3). We need some tools from the theory of operator inequalities. We adopt the notion of a *connection* (or *mean*) that was introduced by Kubo and Ando [10]: the connection σ corresponding to an operator monotone function $\phi(t) \geqslant 0$ on $[0, \infty)$ is defined by

$$A\sigma B = A^{1/2}\phi(A^{-1/2}BA^{-1/2})\, A^{1/2}$$

if A is invertible, and $A\sigma B = \lim_{t \to +0}(A+t)\,\sigma B$ if A is not invertible. In this paper we need the following property:

$$A \geqslant C \qquad \text{and} \qquad B \geqslant D \qquad \text{imply} \quad A\sigma B \geqslant C\sigma D.$$

From now on, we assume that a function means a *continuous* function, I, J represent intervals (maybe unbounded) in the real line, and J^i the interior of J. To simplify future proofs, we make a preliminary remark.

Remark. Suppose that $sp(A) \subseteq [a, b] \subseteq J$, and that f is a function on the interval J. Then for each $\varepsilon > 0$ there is an affine function $p_\varepsilon(t) = ct + d$ such that $c > 0$, $p_\varepsilon(a) = a + \varepsilon$, $p_\varepsilon(b) = b - \varepsilon$ and $p_\varepsilon(t)$ converges uniformly to t on $[a, b]$ as $\varepsilon \to 0$. We have

$$\|f(p_\varepsilon(A)) - f(A)\| \to 0 \quad (\varepsilon \to 0), \qquad \text{and} \qquad sp(p_\varepsilon(A)) \subseteq [a+\varepsilon, b-\varepsilon].$$

Therefore, to show something about $f(A)$ under a condition $sp(A) \subseteq J$ we will often assume that $sp(A)$ is in the interior of J.

LEMMA 4.1. *Let $\phi(t) \geqslant 0$ be an operator monotone function on $[0, \infty)$. Let $k(t)$ be a non-negative and strictly increasing function on an interval $I \subseteq [0, \infty)$. Suppose*

$$\phi(k(t)\, t) = t^2 \qquad (t \in I).$$

Then

$$sp(A),\, sp(B) \subseteq I, \qquad A \geqslant B \Rightarrow \begin{cases} \phi(B^{1/2}k(A)\, B^{1/2}) \geqslant B^2, \\ A^2 \geqslant \phi(A^{1/2}k(B)\, A^{1/2}). \end{cases}$$

Proof. Let us assume that $sp(A)$, $sp(B)$ are in the interior of I, so A and B are invertible. By making use of the connection σ corresponding to ϕ, we have

$$\begin{aligned} B^{-1/2}\phi(B^{1/2}k(A)\,B^{1/2})\,B^{-1/2} &= B^{-1}\sigma k(A) \geqslant A^{-1}\sigma k(A) \\ &= A^{-1}\phi(Ak(A)) = A \geqslant B. \end{aligned}$$

Here we used $B^{-1} \geqslant A^{-1}$ and the property of the connection mentioned above. Thus we obtain the first inequality $\phi(B^{1/2}k(A)\,B^{1/2}) \geqslant B^2$. For general A, B, since $p_\varepsilon(A) \geqslant p_\varepsilon(B)$ for $p_\varepsilon(t)$ as in the Remark, we can apply the result that we have just shown to $p_\varepsilon(A)$ and $p_\varepsilon(B)$. By letting $\varepsilon \to 0$, we obtain the first inequality. We can similarly obtain the second inequality. ∎

LEMMA 4.2. *Let $\{\phi_r : r > 0\}$ be a one-parameter family of non-negative functions on $[0, \infty)$, and J an arbitrary interval. Let $f(t)$, $h(t)$ be non-negative strictly increasing functions on J. If, for a fixed real number $c : 0 \leqslant c \leqslant 1$, the condition*

$$\phi_r(h(t)\,f(t)^r) = f(t)^{c+r} \qquad (t \in J,\ r > 0) \tag{7}$$

is satisfied, then

$$\phi_{c+2r}(s\phi_r^{-1}(s)) = s^2 \qquad (s = f(t)^{c+r}).$$

Proof. Since $h(t)\,f(t)^r = \phi_r^{-1}(s)$, by (7) with $2r + c$ in place of r,

$$\begin{aligned} \phi_{c+2r}(s\phi_r^{-1}(s)) &= \phi_{c+2r}(f(t)^{c+r}\,h(t)\,f(t)^r) \\ &= \phi_{c+2r}(h(t)\,f(t)^{c+2r}) = f(t)^{2c+2r} = s^2. \end{aligned}$$

This completes the proof. ∎

We call the following inequality the *essential inequality*.

THEOREM 4.3. *Let $\{\phi_r : r > 0\}$ be a one-parameter family of non-negative operator monotone functions on $[0, \infty)$, and J an arbitrary interval. Let $f(t)$, $h(t)$ be non-negative strictly increasing functions on J. If condition (7) is satisfied for a fixed c: $0 \leqslant c \leqslant 1$, then*

$$\left.\begin{aligned} sp(A), sp(B) \subseteq J^i, \\ f(A) \geqslant f(B) \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\phi_r(f(B)^{r/2}\,h(A)\,f(B)^{r/2}) \geqslant f(B)^{c+r}, \\ &f(A)^{c+r} \geqslant \phi_r(f(A)^{r/2}\,h(B)\,f(A)^{r/2}). \end{aligned}\right. \tag{8}$$

Proof. We will only prove the first inequality of (8). Since $sp(A)$, $sp(B)$ are in the interior of J, $f(A)$ and $f(B)$ are invertible, because $f(t)$ is strictly increasing. We first obtain (8) in the case $0 < r \leqslant 1$. By making use of the connection σ corresponding to ϕ_r, we have

$$
\begin{aligned}
& f(B)^{-r/2}\,\phi_r(f(B)^{r/2}\,h(A)\,f(B)^{r/2})\,f(B)^{-r/2} \\
& \quad = f(B)^{-r}\,\sigma h(A) \geqslant f(A)^{-r}\,\sigma h(A) = f(A)^{-r}\,f(A)^{c+r} \\
& \quad = f(A)^{c} \geqslant f(B)^{c}.
\end{aligned}
$$

Thus (8) follows. We next assume (8) holds for all r such that $0 < r \leqslant n$. Take any fixed r such that $n < r \leqslant n+1$. Because $(r-c)/2 \leqslant n$, we have

$$
\phi_{(r-c)/2}(f(B)^{(r-c)/4}\,h(A)\,f(B)^{(r-c)/4}) \geqslant f(B)^{(r+c)/2}.
$$

Here we simply denote the left hand side by H and the right hand side by K; clearly $H \geqslant K$. Set $I := \{f(t)^{(r+c)/2} : t \in J\}$. Then $I \subseteq [0, \infty)$ and $sp(K) \subseteq I$. To see $sp(H) \subseteq I$, take a, b in J such that $a \leqslant A$, $B \leqslant b$. Since $h(a) \leqslant h(A) \leqslant h(b)$,

$$
h(a)\,f(a)^{(r-c)/2} \leqslant f(B)^{(r-c)/4}\,h(A)\,f(B)^{(r-c)/4} \leqslant h(b)\,f(b)^{(r-c)/2}.
$$

In conjunction with (7), this shows $sp(H) \subseteq I$. It follows from Lemma 4.2 that

$$
\phi_r(s\phi_{(r-c)/2}^{-1}(s)) = s^2 \qquad \text{for} \quad s \in I.
$$

Thus we can apply Lemma 4.1 to get

$$
\phi_r(K^{1/2}\phi_{(r-c)/2}^{-1}(H)\,K^{1/2}) \geqslant K^2,
$$

which means

$$
\phi_r(f(B)^{r/2}\,h(A)\,f(B)^{r/2}) \geqslant f(B)^{c+r}. \quad \blacksquare
$$

In the above proof, the strict condition $sp(A), sp(B) \subseteq J^i$ was necessary just to say that $f(A)$ and $f(B)$ are invertible. Even if we replace A and B by $p_\varepsilon(A)$ and $p_\varepsilon(B)$, respectively, $f(p_\varepsilon(A)) \geqslant f(p_\varepsilon(B))$ does not necessarily hold, so that we cannot weaken the condition to $sp(A), sp(B) \subseteq J$.

In addition to the conditions of the above theorem, let us assume that $f(t)$ is operator monotone. Then we get

THEOREM 4.4. *Let $\{\phi_r : r > 0\}$ be a one-parameter family of non-negative operator monotone functions on $[0, \infty)$, and J an arbitrary interval. Let $f(t)$, $h(t)$ be non-negative strictly increasing functions on J. If $f(t)$ is operator monotone, and if condition (7) is satisfied for a fixed $c : 0 \leqslant c \leqslant 1$, then*

$$
\left.\begin{array}{r} sp(A), sp(B) \subseteq J, \\ A \geqslant B \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \phi_r(f(B)^{r/2}\,h(A)\,f(B)^{r/2}) \geqslant f(B)^{c+r}, \\ f(A)^{c+r} \geqslant \phi_r(f(A)^{r/2}\,h(B)\,f(A)^{r/2}). \end{array}\right. \tag{9}
$$

Proof. Since $p_\varepsilon(A) \geqslant p_\varepsilon(B)$, and since $sp(p_\varepsilon(A))$, $sp(p_\varepsilon(B)) \subseteq J^i$, we may assume that $sp(A), sp(B) \subseteq J^i$. The operator monotonicity of $f(t)$ ensures that $f(A) \geqslant f(B)$. Hence (9) follows from (8). ∎

We explain why the above theorem includes the Furuta Inequality. Let $p \geqslant 1$, and put

$$f(t) = t, \qquad h(t) = t^p \qquad (0 \leqslant t < \infty).$$

Define a one-parameter family of operator monotone functions $\{\phi_r : r > 0\}$ by

$$\phi_r(t) = t^{(1+r)/(p+r)} \qquad (0 \leqslant t < \infty).$$

Then

$$\phi_r(h(t)\, f(t)^r) = t^{1+r} = f(t)^{1+r}.$$

Thus (7) with $c = 1$ is satisfied. Therefore, from Theorem 4.4 it follows that

$$A \geqslant B \geqslant 0 \Rightarrow (B^{r/2} A^p B^{r/2})^{(1+r)/(p+r)} \geqslant B^{1+r}.$$

If $q(1+r) \geqslant p+r$, take λ such that

$$\frac{1}{q} = \lambda \frac{1+r}{p+r}.$$

Then $0 < \lambda \leqslant 1$, hence by the Löwner–Heinz inequality (1) we have

$$(B^{r/2} A^p B^{r/2})^{1/q} \geqslant B^{(p+r)/q}.$$

This is just the Furuta inequality.

Remark. In the above theorems, we assumed that condition (7) is satisfied for all $r > 0$. However, it is evident from the above proof that if we assume that (7) is satisfied for r in an interval $(0, \alpha)$, then (8) and (9) hold for $r \in (0, \alpha)$.

Equations (8) and (9) are abstract inequalities; however we can get concrete inequalities by using one-parameter families of non-negative operator monotone functions on $[0, \infty)$ in Corollary 3.3.

COROLLARY 4.5. *Under the conditions of Corollary* 3.3, *suppose* $A, B \geqslant -a_1$. *Then*

$$v(A)\, e^{\beta A} \geqslant v(B)\, e^{\beta B} \Rightarrow \phi_r((v(B)\, e^{\beta B})^{r/2}\, u(A)\, e^{aA} (v(B)\, e^{\beta B})^{r/2}) \geqslant (v(B)\, e^{\beta B})^{c+r}.$$

Proof. Set $J=[-a_1,\infty)$, $h(t)=u(t)\,e^{\alpha t}$ and $f(t)=v(t)\,e^{\beta t}$. Then the operator monotone function ϕ_r in Corollary 3.3 satisfies (7). Thus, if $sp(A), sp(B)\subseteq J^i$, we can apply (8). For general A, B, take an arbitrary $\varepsilon>0$. Since A is bounded and $f(t)$ is strictly increasing, there is $\delta>0$ so that

$$\delta\leqslant f(A+\varepsilon)-f(A).$$

Moreover, for this δ there is $\varepsilon'>0$ so that

$$0\leqslant f(B+\varepsilon')-f(B)\leqslant\delta.$$

Thus we obtain $f(A+\varepsilon)\geqslant f(B+\varepsilon')$. Since $sp(A+\varepsilon), sp(B+\varepsilon')\subseteq J^i$, we can apply (8), then let $\varepsilon\to 0$. ∎

Corollary 4.6. *Let $u(t)$, $v(t)$ be the functions given by* (4), (5). *Let us assume that $a_1\leqslant b_1$ and $\sum\lambda_j<1$. For fixed α, c such that $0\leqslant\alpha$, $0\leqslant c\leqslant 1$, define the function $\phi_r(s)$ on $[0,\infty)$ by*

$$\phi_r(u(t)\,e^{\alpha t}v(t)^r)=v(t)^{c+r}\qquad (r>0).$$

Then

$$A\geqslant B\geqslant -a_1\Rightarrow\phi_r(v(B)^{r/2}\,u(A)\,e^{aA}v(B)^{r/2})\geqslant v(B)^{c+r}.$$

Proof. The operator monotonicity of $v(t)$ on $[-a_1,\infty)$ is clear, and that of $\phi_r(s)$ on $[0,\infty)$ follows from Corollary 3.3 with $\beta=0$. Thus this corollary follows from Theorem 4.4. ∎

5. EXTENSIONS OF THE EXPONENTIAL TYPE OPERATOR INEQUALITY OF ANDO

In this section, we treat only an infinite interval with the right end point ∞, so we denote it by J_∞.

Recall the inequality (3): for $p\geqslant 0$, $r\geqslant s>0$

$$A\geqslant B\Rightarrow (e^{(r/2)\,B}e^{pA}e^{(r/2)\,B})^{s/(r+p)}\geqslant e^{sB}.$$

In this section we will obtain an extension. We consider (7) under the condition $c=0$, and denote the function by φ_r instead of ϕ_r. In addition to the conditions of Theorem 4.3 we assume that $\log f(t)$ is operator monotone. Then we have

THEOREM 5.1. *Let $f(t)$ and $h(t)$ be non-negative strictly increasing functions on an infinite interval J_∞, and let $\{\varphi_r : r>0\}$ be the one-parameter family of non-negative operator monotone functions on $[0,\infty)$ satisfying*

$$\varphi_r(h(t)\, f(t)^r) = f(t)^r \qquad (t \in J_\infty;\, r>0). \tag{10}$$

If $\log f(t)$ *is an operator monotone function in the interior of J_∞, then*

$$\left.\begin{aligned} sp(A), sp(B) \subseteq J_\infty, \\ A \geqslant B \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\varphi_r(f(B)^{r/2}\, h(A)\, f(B)^{r/2}) \geqslant f(B)^r \\ &f(A)^r \geqslant \varphi_r(f(A)^{r/2}\, h(B)\, f(A)^{r/2}). \end{aligned}\right. \tag{11}$$

Proof. We remark that $f(t)>0$ on the interior of J_∞, so that $\log f(t)$ is well-defined there. We may assume that $sp(A), sp(B) \subseteq (J_\infty)^i$. Suppose $A \geqslant B$. Then, by assumption, $\log f(A) \geqslant \log f(B)$. Take $\eta \in (J_\infty)^i$ so that $B \geqslant \eta$, and note that for every $\varepsilon>0$, $sp(A+\varepsilon) \subseteq (J_\infty)^i$. We claim that there is $a>0$ such that

$$f(A+\varepsilon)^a \geqslant f(B)^a. \tag{12}$$

Since an operator monotone function on J_∞ is concave, for

$$\delta := \varepsilon \left.\frac{d}{dt}\right|_{\|A\|+\varepsilon} \log f(t) > 0,$$

we have $\log f(t+\varepsilon) \geqslant \log f(t) + \delta$ ($\eta \leqslant t \leqslant \|A\|$) and hence

$$\log f(A+\varepsilon) \geqslant \log f(A) + \delta \geqslant \log f(B) + \delta.$$

Now, we note that for every bounded selfadjoint operator X such that $X \geqslant \eta$ we have $0 < f(\eta) \leqslant f(X) \leqslant f(\|X\|)$, and hence

$$\left\| \frac{f(X)^\lambda - I}{\lambda} - \log f(X) \right\| \to 0 \qquad (\lambda \to +0).$$

Therefore, from the above it follows that

$$\frac{f(A+\varepsilon)^\lambda - I}{\lambda} \geqslant \frac{f(B)^\lambda - I}{\lambda}$$

for sufficiently small $\lambda>0$. Thus we have derived (12). Since

$$\varphi_{ar}(h(t)\, f(t)^{ar}) = f(t)^{ar} \qquad (t \in J_\infty,\, 0<r),$$

by setting $\tilde{\varphi}_r = \varphi_{ar}$, $\tilde{f}(t) = f(t)^a$ we have

$$\tilde{\varphi}_r(h(t)\, \tilde{f}(t)^r) = \tilde{f}(t)^r \qquad (t \in J_\infty,\, 0<r).$$

Therefore, condition (7) with $c=0$ is satisfied. Since $\tilde{f}(A+\varepsilon)=f(A+\varepsilon)^a \geqslant f(B)^a=\tilde{f}(B)$, and since $sp(A+\varepsilon)$, $sp(B) \subseteq (J_\infty)^i$, by Theorem 4.3 we have

$$\tilde{\varphi}_r((\tilde{f}(B)^{r/2}\, h(A+\varepsilon)\, \tilde{f}(B)^{r/2}) \geqslant \tilde{f}(B)^r.$$

This implies

$$\varphi_{ar}(f(B)^{ar/2}\, h(A+\varepsilon)\, f(B)^{ar/2}) \geqslant f(B)^{ar}.$$

Since r is arbitrary, for every r

$$\varphi_r(f(B)^{r/2}\, h(A+\varepsilon)\, f(B)^{r/2}) \geqslant f(B)^r,$$

and hence, by letting $\varepsilon \to 0$, we get (11). ∎

Now we explain why this theorem is an extension of (3). For $p, r>0$, put $\varphi_r(s)=s^{r/(p+r)}$ for $s \geqslant 0$, $f(t)=e^t$ and $h(t)=e^{pt}$ for $t \in J_\infty := (-\infty, \infty)$. Then (10) and all the other conditions of Theorem 5.1 are satisfied. Thus $A \geqslant B$ implies

$$(e^{(r/2)B} e^{pA} e^{(r/2)B})^{r/(r+p)} \geqslant e^{rB}.$$

By the Löwner–Heinz theorem, we get (3).

Since $\varphi_r(s)=s^{r/(p+r)}$ $(p, r>0)$ is operator monotone on $[0, \infty)$ and satisfies $\varphi_r(f(t)^p\, f(t)^r)=f(t)^r$ for every function $f(t)$, we can obtain

Corollary 5.2. *Let $0 \leqslant f(t)$ be a strictly increasing function on an infinite interval J_∞, and let $sp(A)$, $sp(B) \subseteq J_\infty$. If $\log f(t)$ is an operator monotone function in the interior of J_∞, then for $r>0$, $p>0$*

$$A \geqslant B \Rightarrow \begin{cases} (f(B)^{r/2}\, f(A)^p\, f(B)^{r/2})^{r/(p+r)} \geqslant f(B)^r \\ f(A)^r \geqslant (f(A)^{r/2}\, f(B)^p\, f(A)^{r/2})^{r/(p+r)}. \end{cases}$$

By using this we can get a concrete inequality: let us recall the function $u(t)$ defined by (4) in Section 2; since $\log(u(t)\, e^{\alpha t})$ is operator monotone on the interior of $J_\infty := [-a_1, \infty)$, we obtain

Corollary 5.3. *If α, $p, r>0$, then*

$$A \geqslant B \geqslant -a_1$$
$$\Rightarrow \begin{cases} [(u(B)\, e^{\alpha B})^{r/2}\, (u(A)\, e^{\alpha A})^p\, (u(B)\, e^{\alpha B})^{r/2}]^{r/(p+r)} \geqslant (u(B)\, e^{\alpha B})^r, \\ (u(A)\, e^{\alpha A})^r \geqslant [(u(A)\, e^{\alpha A})^{r/2}\, (u(B)\, e^{\alpha B})^p\, (u(A)\, e^{\alpha A})^{r/2}]^{r/(p+r)}. \end{cases}$$

By applying this inequality to $u(t)=1$, we get (3) again. We end this paper with a slightly complicated inequality:

Corollary 5.4. *Let* $u(t), v(t)$ *be the functions defined by* (4), (5), *and let* $a_1 \leqslant b_1$. *For fixed* $\alpha, \beta \geqslant 0$, *define* $\varphi_r(s)$ $(r>0)$ *on* $[0, \infty)$ *by*

$$\varphi_r(u(t)\, v(t)^r\, e^{(\alpha+\beta r)t}) = v(t)^r\, e^{\beta r t} \qquad (t \geqslant -a_1).$$

Then, for each $r>0$, $\varphi_r(s)$ *is operator monotone and*

$$A \geqslant B \geqslant -a_1$$

$$\Rightarrow \begin{cases} \varphi_r((v(B)\, e^{\beta B})^{r/2}\, (u(A)\, e^{\alpha A})(v(B)\, e^{\beta B})^{r/2}) \geqslant (v(B)\, e^{\beta B})^r, \\ (v(A)\, e^{\beta A})^r \geqslant \varphi_r((v(A)\, e^{\beta A})^{r/2}\, (u(B)\, e^{\alpha B})(v(A)\, e^{\beta A})^{r/2}). \end{cases}$$

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