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Endomorphism Rings of Torsionless Modules*

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INTRODUCTION

In a celebrated theorem, A. W. Goldie determined necessary and sufficient conditions that a ring have a left quotient ring which is semisimple with minimum condition [*4*]. Our first main result (Theorem 2.2) gives the following module-theoretic extension of Goldie's theorem.

Let M be a left module over a semiprime ring R and suppose that

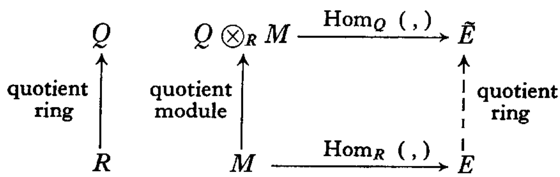
(i) M is *torsionless* (i.e., M is contained in a direct product of copies of R),

(ii) M is *finite-dimensional* (i.e., contains no infinite direct sums of submodules),

(iii) M is *nonsingular* (i.e., no element of M is annihilated by an essential left ideal of R).

Then the ring E of endomorphisms of M has a semisimple left quotient ring (endomorphisms being written on the right). Furthermore, this quotient ring can be obtained as the ring of endomorphisms of the injective hull of M.

The case where R itself has a semisimple (simple) left quotient ring, Q is of particular interest. (By semisimple we always mean semisimple with minimum condition.) Here Condition (iii) becomes redundant. Our result (Theorem 2.3) states that if M is any finite-dimensional torsionless R-module, then $E = \mathrm{Hom}_R(M, M)$ has a semisimple (simple) left quotient ring isomorphic to the ring $\tilde{E}$ of Q-endomorphisms of the "quotient module" $Q \otimes_R M$. The situation is described by the diagram below:



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If, additionally, Q is a right quotient ring of R, then $\tilde{E}$ is also a right quotient ring of E (Theorem 3.3).

The special case where Q is simple and M is a uniform left ideal of R (hence E is a domain and $\tilde{E}$ a division ring) formed an important part of Goldie's proof of his structure theorem for R [3; p. 596]. When Q is a two-sided quotient ring of R, Corollary 3.4 provides a generalization of a theorem of Feller and Swokowski [2], who treated the case where Q is a division ring and M is a finitely generated torsion-free R-module.

In Section 4 we produce a series of examples which illustrate the sensitivity of the hypothesis that M be contained in a direct product of copies of R (and also show several possible one-sided generalizations of Feller and Swokowski's theorem to be false). Taking M to be a properly selected two-generator left R-submodule of Q (Q the left quotient division ring of a domain R), we construct examples where (1) E is a domain with a two-sided quotient division ring strictly smaller than $\tilde{E}$, and (2) E is a domain which has no left or right quotient ring.

1. Preliminaries

Throughout this paper, unless otherwise indicated, all modules will be left modules and all homomorphisms will be written on the right. We begin with some elementary definitions.

Recall that a ring R is *semiprime* if it contains no nonzero nilpotent left ideals (equivalently, if $aRa \neq 0$ for every nonzero element a of R). R is said to be *prime* if $aRb \neq 0$ whenever a and b are nonzero elements of R. A prime ring is certainly semiprime.

A module M' is an *essential extension* of a submodule M in case every nonzero submodule of M' intersects M nontrivially. One then says that M is *essential* in M'.

Lemma 1.1. *Let M' be an essential extension of a torsionless module M over a semiprime ring R. Then for every homomorphism $f : M' \to M'$ and every $m \in M$ satisfying $mf \neq 0$, there is a homomorphism $\alpha : M \to Rm$ and an element b of R such that $(bm)f\alpha f$ is defined and nonzero, and $(M)\alpha f \subseteq M$.*

Proof. We may write elements of M as $\{r_\lambda\}_{\lambda\in\Lambda}$ for some fixed index set Λ, with each $r_\lambda \in R$. If $0 \neq mf \in M'$, then since M' is an essential extension of M, there exists $b \in R$ with $0 \neq b(mf) \in M$; write $bmf = \{b_\lambda\}$, where each $b_\lambda \in R$ and, say, $b_\nu \neq 0$. Since R is semiprime, there exists $a \in R$ such that $b_\nu a b_\nu \neq 0$. Now define $\alpha : M \to Rm$ by $\{r_\lambda\}\alpha = r_\nu abm$. Then $(bm)f\alpha f = \{b_\lambda\}\alpha f = (b_\nu abm)f = b_\nu a\{b_\lambda\} \neq 0$ because $b_\nu ab_\nu \neq 0$, and $M\alpha f \subseteq (Rabm)f \subseteq M$.

PROPOSITION 1.2. *Let M be a torsionless module over a semiprime ring R. Then*

(i) $E = \mathrm{Hom}_R(M, M)$ *is semiprime. If R is prime, so is E.*

(ii) *For each nonzero submodule N of M,* $\mathrm{Hom}_R(M, N) \neq 0$.

(iii) *For each essential submodule N of M,* $\mathrm{Hom}_R(M, N)$ *is an essential left ideal of E.*

(iv) *For each essential left ideal A of E, MA is an essential submodule of M.*

Proof. (ii) and the fact that E is semiprime are immediate consequences of the preceeding lemma. Next, suppose that R is prime and let β and γ be any nonzero endomorphisms of M. Pick $m, n \in M$ with $m\beta \neq 0$, $n\gamma \neq 0$; and write $m\beta = \{b_\lambda\}_{\lambda \in \Lambda}$, $n\gamma = \{c_\lambda\}_{\lambda \in \Lambda}$, where $b_\lambda, c_\lambda \in R$ with, say, $b_\eta \neq 0$, $c_\mu \neq 0$ for some $\eta, \mu \in \Lambda$. Choose $a \in R$ such that $b_\eta a c_\mu \neq 0$ (this can be done because R is prime), and define $\alpha \in E$ by $\{r_\lambda\}\,\alpha = r_\eta a n$. Then $m(\beta\alpha\gamma) = \{b_\lambda\}\,\alpha\gamma = (b_\eta a n)\,\gamma = b_\eta a\{c_\lambda\} \neq 0$. Hence E is prime.

Now suppose that N is an essential submodule of M, and let β be an arbitrary nonzero element of E. Then $M\beta \cap N \neq 0$, so we can choose $m \in M$ such that $0 \neq m\beta \in N$. By the lemma, we can define $\alpha : M \to Rm$ such that $(m)\,\beta\alpha\beta \neq 0$. Thus $0 \neq \alpha\beta \in \mathrm{Hom}_R(M, N)$. This proves (iii).

For (iv), let A be any essential left ideal of E and m any nonzero element of M. By (ii), $\mathrm{Hom}_R(M, Rm)$ is a nonzero left ideal of E, and hence $0 \neq B = A \cap \mathrm{Hom}_R(M, Rm)$. Since E is semiprime, $B^2 \neq 0$, which implies that $0 \neq MB^2$. But $MB^2 \subseteq RmB$, so $mB \neq 0$; i.e., there exists $\beta \in B$ and $r \in R$ such that $0 \neq m\beta = rm$. Since $B \subseteq A$ we have $0 \neq rm \in MA$, which proves that MA is essential in M.

Let M be a submodule of an R-module M'. For any $x \in M'$, set $(M : x) = \{r \in R : rx \in M\}$. It is an easy matter to prove that if M is essential in M' then $(M : x)$ is an essential left ideal of R for every $x \in M'$. The *singular submodule* $Z_R(M)$ of M is defined to be $\{m \in M : (0 : m)$ is an essential left ideal of $R\}$. It can be seen that $Z_R(M)$ is indeed a submodule and that $Z_R(M) = Z_R(M') \cap M$ for any R-module M' containing M. Thus if M' is an essential extension of M, $Z_R(M) = 0$ if and only if $Z_R(M') = 0$. Recall that we have defined a module to be nonsingular if its singular submodule equals zero. A ring will be called nonsingular if its left regular representation is nonsingular. We remark that if a ring S is an essential extension of a ring R, then S is a nonsingular ring if and only if R is. The following lemma will be used frequently.

LEMMA 1.3. *In a nonsingular module, the only endomorphism whose kernel is an essential submodule is the zero endomorphism.*

Proof. Suppose that $N\alpha = 0$ where α is an endomorphism of M and

N is an essential submodule of M. Then for any $x \in M$, $(N : x)$ is an essential left ideal and $(N : x)\, x\alpha = 0$. Since M is nonsingular, $x\alpha = 0$; and since x was arbitrary, $\alpha = 0$.

We conclude this section with one more lemma needed in the sequel. It is known that if M is a finite dimensional R-module then there exists an integer n such that every direct sum of submodules of M has $\leqslant n$ nonzero summands [*4*; p. 202]. When this is the case we will let $d_R(M)$ equal the least such integer n; otherwise $d_R(M) = \infty$. We will follow the practice of calling a ring finite dimensional if its left regular representation is finite dimensional.

LEMMA 1.4. *Suppose that R is a nonsingular ring, and S is a ring containing R as an essential R-submodule. Then $d_S(S) = d_R(R)$.*

Proof. Intersecting any direct sum of nonzero left ideals of S term by term with R, we get a direct sum of nonzero left ideals of R. To complete the proof, it suffices to show that if $I \cap J = 0$ where I and J are left ideals of R, then $SI \cap SJ = 0$.

So suppose that

$$\sum_{i=1}^{h} a_i x_i = \sum_{j=1}^{k} b_j y_j$$

where a_i, $b_j \in S$, $x_i \in I$, $y_j \in J$. Set

$$K = \bigcap_{i=1}^{h} (R : a_i) \cap \bigcap_{j=1}^{k} (R : b_j).$$

Then K is an essential left ideal of R; and for any $u \in K$, $u(\sum_i a_i x_i) = \sum_i (ua_i)\, x_i = \sum_j (ub_j)\, y_j \in I \cap J = 0$. Since $Z_R(S) = 0$, $\sum_{i=1}^{h} a_i x_i = 0$.

2. LEFT QUOTIENT RINGS

Recall that a ring Q with identity containing a ring R is the (classical) *left quotient ring* of R if every non-zero-divisor of R is invertible in Q and every element of Q is of the form $a^{-1}b$ with $a, b \in R$. Q is a *two-sided* quotient ring of R if, additionally, every element of Q is of the form cd^{-1} with $c, d \in R$. Goldie's theorem states that a necessary and sufficient condition that R have a left quotient ring which is semisimple (simple) is that R be a finite-dimensional nonsingular semiprime (prime) ring (see Theorems (4.1) and (4.4) of [*4*] as well as the discussion at the top of p. 206).

It is known that if R is a nonsingular ring, its injective hull $\hat{R}$ can be made uniquely into a ring in a manner which extends the module action of R

([7]; see also [1; p. 77]). The uniqueness of this multiplication also follows from the proof of the next lemma.

LEMMA 2.1. *Suppose S is a left self-injective ring which is an essential extension of a nonsingular ring R. Then S is the injective hull of R.*

Proof. Since S is an essential extension of R we may assume that $S \subseteq \tilde{R}$, the injective hull of R. Let $\times$ denote the multiplication of S and $\#$ denote the multiplication of $\tilde{R}$. We claim that $s \times t = s \# t$ for all $s, t \in S$. Set $I = (R : s) = \{r \in R : rs \in R\}$, and note that for any $x \in I$, $x(s \times t - s \# t) = xs \times t - xs \# t = (xs)\, t - (xs)\, t = 0$. Since $\tilde{R}$ is a nonsingular R-module and I is an essential left ideal of R we have that $s \times t = s \# t$. Hence $\tilde{R}$ may be considered as an S-module; and, in fact, $\tilde{R}$ (as an S-module) is an essential extension of S. Since S is left self-injective, $S = \tilde{R}$.

THEOREM 2.2. *Let M be a nonsingular torsionless module over a semiprime ring R.*

(i) *$E = \mathrm{Hom}_R(M, M)$ is semiprime and $Z_E(E) = 0$. If R is prime, so is E.*

(ii) *$\tilde{E}$, the injective hull of E, is a left self-injective and regular ring; and it can be obtained as the ring of endomorphisms of the injective hull $\tilde{M}$ of M.*

(iii) *$d_R(M) = d_E(E)$.*

(iv) *If M is finite-dimensional, then $\tilde{E}$ is the left quotient ring of E and is semisimple with minimum condition.*

(v) *$\tilde{E}$ is a division ring if and only if $d_R(M) = 1$.*

Proof. Let α be any element of E with $A = (0 : \alpha)$ an essential left ideal of E. From Proposition 1.2 we know that MA is an essential submodule of M. Since $(MA)\,\alpha = 0$, we conclude from Lemma 1.3 that $\alpha = 0$. Thus $Z_E(E) = 0$.

Let $\tilde{M}$ be the injective hull of M, and set $\Lambda = \mathrm{Hom}_R(\tilde{M}, \tilde{M})$. We claim that every element of E extends to a unique element of Λ. For, given $\alpha \in E$, α certainly has an extension $\alpha_1 \in \mathrm{Hom}_R(\tilde{M}, \tilde{M})$ since $\tilde{M}$ is injective. If $\alpha_2 \in \mathrm{Hom}_R(\tilde{M}, \tilde{M})$ is another extension of α, then $M(\alpha_1 - \alpha_2) = 0$. Then by Lemma 1.3, we know that $\alpha_1 = \alpha_2$. Henceforth we will assume $E \subseteq \Lambda$. From Lemma 1.1 we know that E is in fact an essential E-submodule of Λ.

Now Λ is the endomorphism ring of an injective module. For any such ring Λ, Utumi [9; p. 19] has computed the Jacobson radical of Λ to be equal to $\{\lambda \in \Lambda : \text{kernel } \lambda \text{ is an essential submodule}\}$; and in [7] it is proved that if the Jacobson radical of Λ is zero, then Λ is left self-injective and regular. (For a simple proof of both of these facts, see [1; p. 52].) Applying these results to our situation, we have that Λ is a left self-injective essential extension of E, and hence $\Lambda = \tilde{E}$ by the previous lemma.

Note that by Proposition 1.2, every direct sum $\sum_{i\in I} \oplus N_i$ of nonzero submodules of M induces a corresponding direct sum $\sum_{i\in I} \oplus \mathrm{Hom}_R(M, N_i)$ of nonzero left ideals of E. Hence $d_R(M) \leqslant d_E(E)$. Now $d_E(E) = d_{\tilde{E}}(\tilde{E})$ by Lemma 1.4, and so we will have proved the reverse inequality provided we can show that every direct sum of left ideals of $\tilde{E}$ induces a direct sum of R-submodules of $\tilde{M}$.

Every direct sum of t nonzero left ideals of $\tilde{E}$ contains a direct sum of t nonzero principal left ideals of $\tilde{E}$. Since $\tilde{E}$ is a regular ring, such a direct sum of t principal left ideals induces a collection of orthogonal idempotents $e_1, ..., e_t$. It follows that the sum $\sum_{i=1}^{t} \tilde{M}e_i$ is direct. This completes the proof of (iii).

If M is finite-dimensional, then E is a finite-dimensional nonsingular semiprime ring, and hence by Goldie's theorem has a left quotient ring Q which is semisimple with minimum condition. Q is then certainly a left self-injective essential extension of E. Thus $Q = \tilde{E}$.

(v) is a triviality; for $d_R(M) = d_E(E) = d_{\tilde{E}}(\tilde{E})$, and $d_{\tilde{E}}(\tilde{E}) = 1$ if and only if $\tilde{E}$ is a division ring.

Note that if R is a ring with a left quotient ring Q, then every regular element (i.e., non-zero-divisor) of R generates an essential left ideal. (Let d be regular in R and set $I = Rd$. Then $1 = d^{-1}d$, so $QI = Q$. Thus every nonzero element r of R can be written in the form $(a^{-1}b)\,x$ with $a, b \in R$, $x \in I$. Hence $0 \neq ar = bx \in I \cap R$.)

If R has a semisimple left quotient ring Q, then every essential left ideal I of R contains a regular element. (For then $QI = Q$, so $1 = \sum_{i=1}^{t} q_i x_i$ for some $q_i \in Q$, $x_i \in I$. Write $q_i = d^{-1}r_i$ with $d, r_i \in R$. Then $d = \sum_{i=1}^{t} r_i x_i \in I$.)

THEOREM 2.3. *Let M be a finite-dimensional torsionless module over a ring R which has a semisimple left quotient ring Q. Then $E = \mathrm{Hom}_R(M, M)$ has a semisimple left quotient ring isomorphic to* $\mathrm{Hom}_Q(Q \otimes_R M, Q \otimes_R M)$.

Proof. M is nonsingular because it is a torsionless module over a nonsingular ring. Thus E has a semisimple left quotient ring; and it remains for us to prove that $Q \otimes_R M$ is the injective hull of M and that

$$\mathrm{Hom}_Q(Q \otimes_R M, Q \otimes_R M) = \mathrm{Hom}_R(Q \otimes_R M, Q \otimes_R M).$$

It is not a difficult matter to show that for a nonsingular module M, the homomorphism: $M \to Q \otimes_R M$, defined by $m \to 1 \otimes m$, is a monomorphism. (See Proposition 1.5 of [8], and note that, by the remarks preceeding this theorem, M is nonsingular if and only if it is torsion-free as defined in [8].) Thus we may assume that $M \subseteq Q \otimes_R M$; indeed, $Q \otimes_R M$ is an essential extension of M.

Choose an injective hull $\tilde{M}$ of M such that $M \subseteq Q \otimes_R M \subseteq \tilde{M}$. Repeating the argument in the previous paragraph, we know that $Q \otimes_R \tilde{M}$ is an essential extension of $\tilde{M}$, and hence $\tilde{M} = Q \otimes_R \tilde{M}$ (under the identification $m \leftrightarrow 1 \otimes m$, $m \in \tilde{M}$). Since Q is semisimple, $Q \otimes_R M$ is certainly an injective Q-module, and hence is a Q-direct summand of $\tilde{M} = Q \otimes_R \tilde{M}$. It follows that $Q \otimes_R M = \tilde{M}$.

To see that $\mathrm{Hom}_Q(Q \otimes_R M, Q \otimes_R M) = \mathrm{Hom}_R(Q \otimes_R M, Q \otimes_R M)$, we will prove the more general result that if N is any nonsingular Q-module, then $\mathrm{Hom}_Q(N, N) = \mathrm{Hom}_R(N, N)$. Certainly $\mathrm{Hom}_Q(N, N) \subseteq \mathrm{Hom}_R(N, N)$. For the reverse inclusion let $f \in \mathrm{Hom}_R(N, N)$ and let q be any element of Q and n any element of N. We have to show that $qf(n) = f(qn)$. Write $q = d^{-1}r$ with $d, r \in R$, and note that $d(qf(n) - f(qn)) = rf(n) - f(rn) = 0$. Multiplying by d^{-1} we have $qf(n) - f(qn) = 0$.

3. Two-Sided Quotient Rings

Our object now is to show that if, in Theorem 2.3, R has a two-sided quotient ring, then so has E (Theorem 3.3).

Lemma 3.1. *Let R be a ring with a semisimple left quotient ring Q, and let I be any essential left ideal of R. Then Q is a left quotient ring of the ring I.*

Proof. No regular element a of I is a right zero divisor in R. [Suppose $ra = 0$ for some $r \in R$. $(I : r)$ is an essential left ideal and so contains a regular element b. Then $br \in I$ and $(br)\,a = 0$. Since a is regular in I, $br = 0$, and then $r = 0$.] Hence every regular element of I is invertible in Q [*8*; Lemma 3.7].

Now suppose $0 \neq q \in Q$. As above, $(I : q)$ contains a regular element b of R. Rb is then an essential left ideal and so is $I \cap Rb$. Choose $r \in R$ such that rb is a regular element contained in $I \cap Rb$. Then $(rb)\,q \in I$, so q can be written as a left quotient of elements of I. This proves that Q is a left quotient ring of I.

It is a simple exercise to show that if R is a ring with a left quotient ring Q, and R has a right quotient ring, then Q is also a right quotient ring of R [*1*; p. 105]. We will use this fact implicitly in the sequel.

The next proposition gives us a criterion for deciding when the endomorphism ring of a module has a two-sided quotient ring.

Proposition 3.2. *Suppose that M is a nonsingular torsionless module over a semiprime ring R such that $E(M) = \mathrm{Hom}_R(M, M)$ has a semisimple left quotient ring $\tilde{E}(M)$. If there exists an essential submodule N of M such that $E(N) = \mathrm{Hom}_R(N, N)$ has a semisimple right quotient ring $\tilde{E}(N)$, then $\tilde{E}(M)$ is also a right quotient ring of $E(M)$, and $\tilde{E}(M) = \tilde{E}(N)$.*

Proof. Let N be an essential submodule of M satisfying the above hypothesis. Set $I = \mathrm{Hom}_R(M, N)$; I is an essential left ideal of $E(M)$ by Proposition 1.2.

Consider the restriction homomorphism: $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(N, N)$. Since N is essential and $Z_R(M) = 0$, this mapping is a monomorphism (Lemma 1.3), and so we may consider I as a right ideal of $E(N)$. We claim that I is in fact an essential right ideal of $E(N)$ because I, being an essential left ideal of $E(M)$, contains a regular element γ of $E(M)$. γ is then invertible in $\tilde{E}(M)$; and $\tilde{E}(M) = E(\tilde{M})$. [To see this, note that M is finite-dimensional by Theorem 2.2(iii), and then $\tilde{E}(M) = E(\tilde{M})$ by (iv) of the same theorem.] It follows that γ (restricted to N) must be a regular element of $E(N)$. Hence $\gamma E(N)$ is an essential right ideal of $E(N)$, and, a fortiori, so is I. By two applications of the previous lemma, $\tilde{E}(N)$ is a right quotient ring of I, and $\tilde{E}(M)$ is a left quotient ring of I. Therefore $\tilde{E}(M) = \tilde{E}(N)$, and it follows that $\tilde{E}(M)$ is also a right quotient ring of $E(M)$.

THEOREM 3.3. *Let R be a ring with a semisimple two-sided quotient ring Q, and let M be any finite-dimensional torsionless R-module. Then* $\mathrm{Hom}_R(M, M)$ *has a semisimple two-sided quotient ring.*

Proof. $E = \mathrm{Hom}_R(M, M)$ has a semisimple left quotient ring $\tilde{E} = \mathrm{Hom}_Q(Q \otimes_R M, Q \otimes_R M)$ by Theorem 2.3. It remains for us to show that $\tilde{E}$ is also a right quotient ring of E. We claim that it suffices to prove that $\tilde{E}$ is a right essential extension of E. For then E would be a right nonsingular ring since $\tilde{E}$ is; and this together with Lemma 1.4 and Goldie's theorem would imply that $\tilde{E}$ is a right quotient ring of E.

We first suppose that M is finitely generated. M is torsionless: say $M \subseteq \prod_{\alpha\in A} R^{(\alpha)}$ for some index set A with each $R^{(\alpha)} \simeq R$. Then $Q \otimes_R M \subseteq \prod_{\alpha\in A} Q^{(\alpha)}$ (making an obvious identification). Now let $0 \neq f \in \tilde{E}$ be given. Then $mf \neq 0$ for some $m \in M$; write $mf = \{q_\alpha\}$ where each $q_\alpha \in Q$, with say $q_\beta \neq 0$. Let π denote the projection homomorphism of $Q \otimes_R M$ into $Q^{(\beta)}$. Then $Mf\pi$ is a finitely generated R-submodule of Q. Hence there exists $a \in R$ such that $(Mf\pi)\, a \subseteq R$. (Write $Mf\pi = \sum_{i=1}^{n} Rp_i$ where each $p_i \in Q$. Then choose $a, r_1, \dots, r_n \in R$ such that $p_i = r_i a^{-1}$.). Thus $q_\beta a \in R$, and since R is semiprime we may choose $b \in R$ so that $(q_\beta a)\, b(q_\beta a) \neq 0$. Finally, define $\alpha : Q \otimes_R M \to Qm$ by $(x)\,\alpha = (x\pi)\, abm$, $x \in Q \otimes_R M$. Then $M\pi \subseteq R$, so $M\alpha \subseteq M$, $Mf\alpha = (Mf)\,\pi abm \subseteq Rm$, and $(m) f\alpha f = ((mf)\,\pi abm) f = q_\beta ab(mf) = q_\beta ab\{q_\alpha\} \neq 0$. Hence $\alpha \in E$ and $0 \neq f\alpha \in E$, which proves that $\tilde{E}$ is a right essential extension of E. Thus we have the theorem for finitely generated modules.

Next suppose that M is any finite dimensional torsionless R-module. M certainly contains a finitely generated essential submodule N; for example, take any longest direct sum $Rm_1 \oplus \cdots \oplus Rm_t$. By the last paragraph, $\mathrm{Hom}_R(N, N)$ has a semisimple two-sided quotient ring. Hence by the previous proposition, so does E.

COROLLARY 3.4. *Let M be a finitely generated nonsingular module over a ring R which has a semisimple two-sided quotient ring. Then* $\mathrm{Hom}_R(M, M)$ *has a semisimple two-sided quotient ring.*

Proof. Note that the two-sided quotient ring Q of R has the property that every finitely generated module is contained in a free module (since it is semisimple). In Theorem 5.2 of [*8*] it is proved that under these conditions every finitely generated nonsingular R-module can be embedded in a free module. The corollary is now immediate.

In Theorem 3.3, we may consider M as a right module over

$$E = \mathrm{Hom}_R(M, M),$$

and ask how the endomorphism ring of M over E is related to R. We conclude this section with a discussion of this relationship.

Let R, Q and M satisfy the hypotheses of Theorem 3.3, where M is now assumed to be a faithful R-module. Recall that $\tilde{E} = \mathrm{Hom}_Q(Q \otimes_R M, Q \otimes_R M)$ is the two-sided quotient ring of $E = \mathrm{Hom}_R(M, M)$. Note that

(1) M is a nonsingular E-module (since every essential right ideal of E contains a regular element), and

(2) $Q \otimes_R M \cong (Q \otimes_R M) \otimes_E \tilde{E} \cong Q \otimes_R (M \otimes_E \tilde{E}) \tilde{5}\, M \otimes_E \tilde{E}$ as R–E bimodules. (The verification of these natural isomorphisms can be safely left to the reader.) Since M is a nonsingular E-module, the computation performed in the proof of Theorem 2.3 shows that $M \otimes_E \tilde{E}$ is the E-injective hull of M. This, together with (2) and the fact that

$$Q = \mathrm{Hom}_{\tilde{E}}(Q \otimes_R M, Q \otimes_R M) = \mathrm{Hom}_E(Q \otimes_R M, Q \otimes_R M),$$

implies that $R \subseteq \mathrm{Hom}_E(M, M) \subseteq Q$. Thus $\mathrm{Hom}_E(M, M)$ has the same quotient ring as R.

This reciprocity between R and the second centralizer of M seems to depend strongly on the hypothesis that Q be a two-sided quotient ring of R. The author has not been able to determine how much of this reciprocity remains valid when Q is only a left quotient ring of R.

4. EXAMPLES AND COUNTEREXAMPLES

In this section, among other things, we produce examples which show that the results of Section 2 for torsionless modules cannot be extended to include all finitely generated nonsingular modules unless R has a two-sided quotient ring as in Corollary 3.4.

Let D be an integral domain with identity element (D may be noncommutative), and let Δ be a division ring containing D together with a monomorphism $\sigma : \Delta \to \Delta$ such that $D^\sigma \subseteq D$.

Then let Q be the "twisted" Laurent series ring whose elements are $\sum_{i=-n}^{\infty} \delta_i t^i$ ($\delta_i \in \Delta$), and whose multiplication is given by $t\delta = \delta^\sigma t$; and let R be the subring of Q whose elements are $d_0 + \sum_{i=1}^{\infty} \delta_i t^i$ with $d_0 \in D$, all other $\delta_i \in \Delta$. The diagram below may help the reader to keep track of the definitions.

$$\begin{array}{ccll}
\Delta & \text{———} & Q = \left\{ \sum_{i=-n}^{\infty} \delta_i t^i \right\} & (t\delta = \delta^\sigma t) \\
| & & | & \\
D & \text{———} & R = \left\{ d_0 + \sum_{i=1}^{\infty} \delta_i t^i \right\} & (d_0 \in D)
\end{array}$$

Finally, let a be an element of D. Our examples will deal with the ring of endomorphisms of the two-generator R-submodule $M = Rt^{-1}a + Rt^{-1}$ of Q, and the relation of this ring to the ring of endomorphisms of the Q-module $Q \otimes_R M\ (= Q)$.

PROPOSITION 4.1. *Q is the left quotient division ring for the domain R; and furthermore, Q is the right quotient ring of R if and only if $\Delta^\sigma = \Delta$.*

In order to exhibit our examples promptly we will temporarily delay the proof of this and the succeeding propositions.

Suppose that S and T are rings with $S \subseteq T$, and let x be an element of T. We say that x is *not quadratic* over S if and only if $s_2 x^2 + s_1 x + s_0 = 0$ with $s_i \in S$ implies that $s_0 = s_1 = s_2 = 0$.

PROPOSITION 4.2. *Suppose that there exists an element a in the center of D such that*

(i) *$a \notin \Delta^\sigma$, and*

(ii) *a is not quadratic over D^σ.*

Then $M = Rt^{-1}a + Rt^{-1}$ is a left R-submodule of Q containing R, and $\mathrm{Hom}_R(M, M) = D^\sigma$ *(i.e., the endomorphisms of M are given by right multiplications by elements of D^σ).*

Example 4.3. Take $D = \Delta = F$ a field with a monomorphism σ mapping F properly into itself, and an element $a \in F$ satisfying (i) and (ii). By the above proposition we have a two-generator nonsingular R-module M such that $\mathrm{Hom}_R(M, M) = F^\sigma$, which certainly is not an order in

$$\mathrm{Hom}_Q(Q \otimes_R M, Q \otimes_R M) = Q.$$

(Such situations abound; for example, take $F = \mathbf{C}(X)$, the rational functions in one variable X over the complex numbers, with σ defined by $X^\sigma = X^3$ and $a = X$.)

Note however that $\mathrm{Hom}_R(M, M)$ is its own quotient ring in this case. We want an example of a finitely generated nonsingular module M over an integral domain R which has a left quotient ring such that $\mathrm{Hom}_R(M, M)$ satisfies neither left nor right quotient conditions. It suffices to produce an integral domain D which has neither a left nor a right quotient ring, and such that $D \subseteq$ a division ring Δ with a monomorphism $\sigma : \Delta \to \Delta$ satisfying $D^\sigma \subseteq D$, and with an element $a \in D$ satisfying the hypotheses of Proposition 4.2.

Let S be any ring. $S[X_1, X_2, \ldots, X_n]$ will denote the polynomial ring over S on n noncommuting indeterminates $X_1, X_2, \ldots, X_n$ which commute with the elements of S. Note that for $n > 1$, $S[X_1, X_2, \ldots, X_n]$ satisfies neither the right nor the left common multiple property (e.g., X_1 and X_2 do not have a common multiple on either side), and hence has neither a left nor a right quotient ring.

If T is a ring containing S, and R is a subring of T, we say that R satisfies a *polynomial identity* over S if, for some integer n, there exists $0 \neq f[X_1, \ldots, X_n] \in S[X_1, \ldots, X_n]$ such that $f[r_1, \ldots, r_n] = 0$ for all choices of $r_1, \ldots, r_n \in R$.

PROPOSITION 4.4. *Let F be a division ring with a monomorphism $\tau : F \to F$, and let F' be any division ring containing F such that*

(i) *τ extends to a monomorphism τ' of F', and*

(ii) *$\Omega = \{x \in F' : x^{\tau'} = x\}$ satisfies no polynomial identities over F.*

Then $D = F[X, Y]$ can be embedded in a division ring Δ together with a monomorphism $\sigma : \Delta \to \Delta$ such that $D^\sigma \subseteq D$. Moreover, if there exists an element a in the center of F such that

(iii) *a is not quadratic over F^τ, and*

(iv) *$a \notin (F')^{\tau'}$,*

then $a1 \in D$ satisfies the hypotheses of Proposition 4.2.

Example 4.5. We thus have the desired example provided we can satisfy the hypotheses of this last proposition. Fortunately this is possible, although somewhat tedious.

For an example, define ring monomorphisms τ and μ on $\mathbf{C}(X)$ as follows:

τ is the identity on $\mathbf{C}$, and $X^\tau = X^3$.

μ is any monomorphism of $\mathbf{C}$ properly into itself, and $X^\mu = X$.

Note that $\tau\mu = \mu\tau$ on $\mathbf{C}(X)$. Next, set $F = \mathbf{Q}(X)$ ($\mathbf{Q}$ = rationals) and $F' = \{\sum_{i=-n}^{\infty} f_i t^i : f_i \in \mathbf{C}(X)\}$; with the multiplication of F' given by $tf = f^\mu t$, $f \in \mathbf{C}(X)$. τ is certainly a monomorphism of F, and we can extend it to a monomorphism of F' by defining $(\sum_i f_i t^i)^{\tau'} = \sum_i f_i^\tau t^i$. ($\tau'$ is multiplicative because τ and μ commute.)

Now $\Omega = \{\omega \in F' : \omega^{\tau'} = \omega\} \supseteq \Omega_0$ where $\Omega_0 = \{\sum_{i=-n}^{\infty} \rho_i t^i : \rho_i \in \mathbf{C}\}$. Let us suppose that Ω satisfies a polynomial identity over $F = \mathbf{Q}(X)$. Then so does Ω_0, and it follows that Ω_0 satisfies a polynomial identity over $\mathbf{Q}$. By a well-known theorem [*6*; p. 226], Ω_0 must then be finite-dimensional over its center. But by direct computation, the center of Ω_0 equals $\{\rho \in \mathbf{C} : \rho^\mu = \rho\}$, over which Ω_0 is patently infinite-dimensional. We thus conclude that Ω does not satisfy a polynomial identity over F.

Finally, choose $a = X$, and note that X is not quadratic over $F^\tau = \mathbf{Q}(X^3)$, and also $X \notin (F')^{\tau'} = \{\sum f_i t^i : f_i \in \mathbf{C}(X^3)\}$.

Example 4.6. The author has not been able to determine whether Corollary 3.4 can be extended to include all finite-dimensional nonsingular modules. It is however a comparative triviality to give an example of such a module whose endomorphism ring is not an order in the endomorphism ring of its injective hull $\tilde{M}$.

For example, take $M = \mathbf{Z} \oplus \mathbf{Q}$ where $\mathbf{Z}$ = integers and $\mathbf{Q}$ = rational numbers. Then $\tilde{M} = \mathbf{Q} \oplus \mathbf{Q}$, and

$$\mathrm{Hom}_{\mathbf{Z}}(M, M) = \begin{bmatrix} \mathbf{Z} & \mathbf{Q} \\ O & \mathbf{Q} \end{bmatrix},$$

which is easily seen not to be a right or left order in

$$\mathrm{Hom}_{\mathbf{Z}}(\tilde{M}, \tilde{M}) = \begin{bmatrix} \mathbf{Q} & \mathbf{Q} \\ \mathbf{Q} & \mathbf{Q} \end{bmatrix}.$$

The rub, of course, is that $\mathrm{Hom}_{\mathbf{Z}}(M, M)$ has a two-sided quotient ring equal to

$$\begin{bmatrix} \mathbf{Q} & \mathbf{Q} \\ O & \mathbf{Q} \end{bmatrix}.$$

Proof of Proposition 4.1. Recall that an integral domain has a left quotient division ring if and only if it satisfies the left common multiple property (i.e., for every pair of elements x and y there exist elements w and z such that $wx = zy$) [*6*; p. 262].

First note the following facts:

(1) Every element of R of the form $1 + \delta_1 t + \delta_2 t^2 + \cdots$ is invertible; in fact its inverse is $1 + \sum_{n=1}^{\infty} (-\delta_1 t - \delta_2 t^2 - \cdots)^n$.

(2) If $\varphi = \delta_n t^n + \delta_{n+1} t^{n+1} + \cdots \in R$ with $\delta_n \neq 0$, then $t^{n+1} \in R\varphi$. For,

$$((\delta_n{}^\sigma)^{-1}\, t)\, \varphi = t^{n+1} + (\delta_n{}^\sigma)^{-1}\, \delta_{n+1}^\sigma t^{n+2} + \cdots$$

$$= (1 + (\delta_n{}^\sigma)^{-1}\, \delta_{n+1}^\sigma t + \cdots)\, t^{n+1} \in R\varphi,$$

and from (1) we know that $(1 + (\delta_n{}^\sigma)^{-1}\, \delta_{n+1} t + \cdots)$ is a unit.

The fact that R satisfies the left common multiple property is now an immediate consequence of (2). We will next verify by a direct computation that R satisfies the right common multiple property if and only if $\Delta^\sigma = \Delta$.

If $\Delta^\sigma = \Delta$, then σ is actually an automorphism of Δ. The right symmetric version of (2) then implies that R satisfies the right common multiple property.

Conversely, suppose R satisfies the right common multiple property, and let α be an arbitrary nonzero element of Δ. There exist nonzero elements $\varphi = \sum_{i=0}^{\infty} \beta_i t^i$ and $\psi = \sum_{i=0}^{\infty} \gamma_i t^i$ of R such that $(\alpha t)\, \varphi = t\psi$. Then equating coefficients, for some $i \geqslant 0$ we must have $\alpha = \gamma_i{}^\sigma(\beta_i{}^\sigma)^{-1} = (\gamma_i \beta_i^{-1})^\sigma \in \Delta^\sigma$.

Finally, note that every element of Q can be written in the form $t^{-i}\varphi$ with $i \geqslant 0$ and $\varphi \in R$; e.g.,

$$\sum_{i=-n}^{\infty} \delta_i t^i = t^{-n-1} \left(\sum_{i=1}^{\infty} \delta^{\sigma^{n+1}}_{-n-1+i}\, t^i \right),$$

and also in the form $t^{-1}\gamma\psi$ with $i \geqslant 0$, $\gamma \in \Delta$, and ψ invertible in R; e.g.,

$$\sum_{i=-n}^{\infty} \delta_i t^i = t^{-n} \delta^{\sigma^n}_{-n} \left(1 + (\delta^{\sigma^n}_{-n})^{-1} \sum_{i=1}^{\infty} \delta^{\sigma^n}_{-n+i}\, t^i \right),$$

when $\delta_{-n} \neq 0$. The latter remark implies that Q is a division ring, and this together with the former remark proves that Q is indeed the left quotient ring of R.

Proof of Proposition 4.2. For convenience, set

$$R' = \Delta 1 + Rt = \left(\sum_{i=0}^{\infty} \delta_i t^i : \delta_i \in \Delta \right).$$

We first show that $M = t^{-1}(D^\sigma + D^\sigma a) + R'$.

Suppose $m \in M$; say,

$$m = \left(\sum_{i=0}^{\infty} \gamma_i t^i \right) t^{-1} a + \left(\sum_{i=0}^{\infty} \delta_i t^i \right) t^{-1},$$

with γ_0, $\delta_0 \in D$, the other γ_i, $\delta_i \in \Delta$. Then

$$m = t^{-1}(\gamma_0{}^\sigma a + \delta_0{}^\sigma) + \sum_{i=0}^{\infty} (\gamma_{i+1} a^{\sigma^i} + \delta_{i+1})\, t^i \in t^{-1}(D^\sigma + D^\sigma a) + R'.$$

Conversely, given

$$m = t^{-1}\gamma + \sum_{i=0}^{\infty} \gamma_i t^i$$

with $\gamma = \alpha^\sigma + \beta^\sigma a \in D^\sigma + D^\sigma a$ and $\gamma_i \in \Delta$, then

$$m = \left(\alpha + \sum_{i=1}^{\infty} \gamma_{i-1} t^i\right) t^{-1} + \beta t^{-1} a \in M.$$

Since M is a left R-submodule of Q containing R, each element f of $\mathrm{Hom}_R(M, M)$ can be effected by right multiplication by a unique element q_f of Q, and in fact the correspondence $f \to$ (right multiplication by q_f) is a ring monomorphism. Hence we can make the identification

$$\mathrm{Hom}_R(M, M) = \{q \in Q : Mq \subseteq M\}.$$

Let $q \in \mathrm{Hom}_R(M, M)$. Since $1 \in R \subseteq M$, q is an element of M; write $q = t^{-1}\gamma + \varphi$, $\gamma \in D^\sigma + D^\sigma a$, $\varphi \in R'$. Now $t^{-1}q \in M$; and writing out what this means in detail and equating coefficients we find that $\gamma = 0$ and $q = \varphi \in (D^\sigma + D^\sigma a) + tR'$. Write $q = \delta + t\theta$ where $\delta \in D^\sigma + D^\sigma a$ and $\theta \in R'$. Now also $t^{-1}aq \in M$, and this together with the fact that $a \notin \Delta^\sigma$ implies, in turn, that $\theta = 0$ and $q = \delta$ where $\delta \in D^\sigma + D^\sigma a$ and $a\delta \in D^\sigma + D^\sigma a$. Thus $q \in (D^\sigma + D^\sigma a) \cap a^{-1}(D^\sigma + D^\sigma a)$. Since q was arbitrary,

$$\mathrm{Hom}_R(M, M) \subseteq (D^\sigma + D^\sigma a) \cap a^{-1}(D^\sigma + D^\sigma a),$$

and the reverse inclusion is evident.

Finally, let $\alpha^\sigma + \beta^\sigma a = a^{-1}(\gamma^\sigma + \delta^\sigma a) \in (D^\sigma + D^\sigma a) \cap a^{-1}(D^\sigma + D^\sigma a)$, where $\alpha, \beta, \gamma, \delta \in D$. Since $a \in$ center of D, we have

$$\beta^\sigma a^2 + (\alpha^\sigma - \delta^\sigma)\, a - \gamma^\sigma = 0.$$

But a is not quadratic over D^σ. Hence $\beta^\sigma = 0$, and it follows that

$$\mathrm{Hom}_R(M, M) = (D^\sigma + D^\sigma a) \cap a^{-1}(D^\sigma + D^\sigma a) = D^\sigma. \qquad \text{Q.E.D.}$$

Before one can begin the proof of Proposition 4.4, it is necessary to review some basic facts about ultraproducts of rings.

A *filter* $\mathscr{F}$ of subsets of a set A is a family of subsets of A satisfying the following properties:

(i) $\emptyset$, the empty set, is not in $\mathscr{F}$.

(ii) $S_1, S_2 \in \mathscr{F}$ implies $S_1 \cap S_2 \in \mathscr{F}$.

(iii) $S \in \mathscr{F}$, $S \subseteq T \subseteq A$, implies $T \in \mathscr{F}$.

Suppose $\mathscr{S}$ is a family of subsets of a set A such that any finite intersection of elements of $\mathscr{S}$ is nonempty. Then there exists a filter $\mathscr{F}$ containing $\mathscr{S}$; e.g., take $\mathscr{F}$ to be the family of all subsets of A which contain a finite intersection $\bigcap_{i=1}^{n} S_i$, $S_i \in \mathscr{S}$. It should be evident what it means for one filter to be contained in another, and so we may define an *ultrafilter* to be a maximal filter. By Zorn's lemma, every filter is contained in an ultrafilter. One can give the following internal characterization of ultrafilters.

PROPOSITION. *A filter $\mathscr{F}$ on a set A is an ultrafilter if and only if for all $T \subseteq A$, either $T \in \mathscr{F}$ or $A - T \in \mathscr{F}$.*

The proof of this, as well as the following remarks can be found in [5].

Suppose $\{R_\alpha : \alpha \in A\}$ is a collection of rings, and consider $\prod_{\alpha \in A} R_\alpha = \{f : f(\alpha) \in R_\alpha\}$. Let $\mathscr{F}$ be any filter on A, and define $f \equiv g(\text{mod } \mathscr{F})$ if and only if $\{\alpha : f(\alpha) = g(\alpha)\} \in \mathscr{F}$. Set $I_{\mathscr{F}} = \{f : f \equiv 0(\text{mod } \mathscr{F})\}$; and note that $I_{\mathscr{F}}$ is a two-sided ideal in $\prod_{\alpha \in A} R_\alpha$.

In view of the preceeding remark we may study the ring $\prod R_\alpha / I_{\mathscr{F}}$, which we will usually denote by $\prod R_\alpha I \mathscr{F}$. If $\mathscr{F}$ is an ultrafilter, $\prod R_\alpha / \mathscr{F}$ is called an *ultraproduct* of the rings R_α.

It is known that ultraproducts preserve "elementary properties" of the rings R_α. For our purpose, however, the following result will suffice.

PROPOSITION. *An ultraproduct of division rings is a division ring.*

Proof of Proposition 4.4. Let $A = \Omega \times \Omega$, and consider $\prod_A F'$ with elements written as $\{b_{\mu\nu}\} = \{b_{\mu\nu} : \mu, \nu \in \Omega\}$. We can define an "evaluation" homomorphism $e : D \to \prod_A F'$ by $f[X, Y]^e = \{f[\mu, \nu]\}$. e is a monomorphism because $f[X, Y]^e = 0$ would mean that $f[X, Y]$ is a polynomial identity for Ω with coefficients in F.

Now consider $\mathscr{S} = \{S_f : f = f[X, Y] \neq 0 \in D\}$ where

$$S_f = \{(\mu, \nu) \in A : f[\mu, \nu] \neq 0\}.$$

$S_{f_1} \cap S_{f_2} \cap \cdots \cap S_{f_n} = S_{f_1 f_2 \cdots f_n} \in \mathscr{S}$. Hence we may choose an ultrafilter $\mathscr{F}$ containing $\mathscr{S}$. Set $\Delta = \prod_A F' / \mathscr{F}$; Δ is a division ring by our previous remarks. Let π denote the natural projection mapping from $\prod_A F'$ onto Δ.

Suppose that $f = f[X, Y] \in D$ with $f[X, Y]^{e\pi} = 0$. Then $f[X, Y]^e = \{f[\mu, \nu]\} \equiv 0 \pmod{\mathscr{F}}$, so $A - S_f = \{(\mu, \nu) : f[\mu, \nu] = 0\}$ belongs to $\mathscr{F}$. On the other hand, by our choice of $\mathscr{F} \supseteq \mathscr{S}$, $S_f \in \mathscr{F}$ whenever $f \neq 0$. Hence, if $f \neq 0$, $\emptyset = (A - S_f) \cap S_f \in \mathscr{F}$, which is a contradiction. Therefore we must have $f[X, Y] = 0$. We have thus proved that the composed mapping $e\pi$ is a monomorphism of D into the division ring Δ.

Let us now turn to the monomorphism τ of F. We may extend τ to a monomorphism of D by defining $X^\tau = X$, $Y^\tau = Y$. Also we can extend τ' to a monomorphism of $\prod_A F'$ by $\{b_{\mu\nu}\}^{\tau'} = \{b_{\mu\nu}^{\tau'}\}$. It follows that

$$f[X, Y]^{e\tau'} = f[X, Y]^{\tau e}$$

for all $f[X, Y] \in D$; for,

$$f[X, Y]^{e\tau'} = \{f[\mu, \nu]\}^{\tau'} = \{f[\mu, \nu]^{\tau'}\} = \{f^\tau[\mu, \nu]\} = f[X, Y]^{\tau e}$$

(since $\mu, \nu \in \Omega$). *A fortiori*, $(D^e)^{\tau'} \subseteq D^e$.

In order to show that τ' induces a monomorphism $\sigma : \Delta \to \Delta$, it suffices to prove that $\ker \tau'\pi = \ker \pi$. Now $\{b_{\mu\nu}\} \in \ker \pi$ if and only if

$$\{(\mu, \nu) \in A : b_{\mu\nu} = 0\} \in \mathscr{F},$$

and since τ' is a monomorphism, $\{(\mu, \nu) : b_{\mu\nu} = 0\} = \{(\mu, \nu) : b_{\mu\nu}^{\tau'} = 0\}$. Therefore $\{b_{\mu\nu}\} \in \ker \pi$ if and only if $\{b_{\mu\nu}\}^{\tau'} = \{b_{\mu\nu}^{\tau'}\} \in \ker \pi$; i.e.,

$$\ker \tau'\pi = \ker \pi.$$

Note that σ is given by $(\{b_{\mu\nu}\}^\pi)^\sigma = \{b_{\mu\nu}^{\tau'}\}^\pi$. Henceforth we will identity D with its isomorphic image in Δ.

Finally, suppose that there exists an element $a \in$ center of F such that $a \notin (F')^{\tau'}$ and a is not quadratic over F^τ. Then $a = a1 \in$ (center of F) $1 =$ center of D, and a is not quadratic over D^σ because it is not quadratic over F^τ. Suppose that $a \in \Delta^\sigma$; say $a = \{b_{\mu\nu}\}^{\pi\sigma} = \{b_{\mu\nu}^{\tau'}\}^\pi$. By definition of π, this means that $\{(\mu, \nu) : b_{\mu\nu}^{\tau'} = a\} \in \mathscr{F}$. But $a \notin (F')^{\tau'}$, so this would imply that $\emptyset \in \mathscr{F}$, which is impossible. Thus we must have $a \notin \Delta^\sigma$.

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