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Indiscernible sequences for extenders, and the singular cardinal hypothesis

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Abstract

We prove several results giving lower bounds for the large cardinal strength of a failure of the singular cardinal hypothesis. The main result is the following theorem:

Theorem. *Suppose κ is a singular strong limit cardinal and $2^\kappa \geqslant \lambda$ where λ is not the successor of a cardinal of cofinality at most κ. If $\mathrm{cf}(\kappa) > \omega$ then it follows that $o(\kappa) \geqslant \lambda$, and if $\mathrm{cf}(\kappa) = \omega$ then either $o(\kappa) \geqslant \lambda$ or $\{\alpha : K \models o(\alpha) \geqslant \alpha^{+n}\}$ is confinal in κ for each $n \in \omega$.*

We also prove several results which extend or are related to this result, notably

Theorem. *If $2^\omega < \aleph_\omega$ and $2^{\aleph_\omega} > \aleph_{\omega 1}$ then there is a sharp for a model with a strong cardinal.*

In order to prove these theorems we give a detailed analysis of the sequences of indiscernibles which come from applying the covering lemma to nonoverlapping sequences of extenders.

The covering lemma asserts, roughly, that for any uncountable set x of ordinals there is a set $y \supset x$ such that $|y| = |x|$ and $y \in K[C]$ where C is some sequence of indiscernibles. In many applications of the covering lemma, such as in the proof that $\lambda^+ = (\lambda^+)^K$ whenever λ is a singular cardinal, the indiscernibles do not pose a problem: the covering lemma is used in an interval where there are no measurable cardinals in K, and thus there are no indiscernibles. For other applications, such as the singular cardinal hypothesis, this is not possible. If κ is singular then the covering lemma asserts, in effect, that the number of subsets of κ is determined by the number of cofinal

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sequences of indiscernibles in κ. Thus giving an upper limit to 2^κ entails giving an upper limit on the number of sequences, of indiscernibles, which requires a detailed understanding of these sequences. It is this second class of applications which we will be considering in this paper.

Work on this class of problems began with the work of Dodd and Jensen on the model $L[\mu]$. Their ideas were extended to models for sequences of measurable cardinals by Mitchell [15, 16] and Gitik [6]. In this paper we extend this analysis to models containing sequences of nonoverlapping extenders, including models up to a strong cardinal. Our main application is the following result (Theorem 3.1 in Section 3).

Theorem 1. *Suppose that κ is a strong limit cardinal with $\mathrm{cf}(\kappa) = \delta < \kappa$, and that $2^\kappa \geqslant \lambda > \kappa^+$, where λ is not the successor of a cardinal of confinality less than κ.*

1. *If $\delta > \omega_1$ then $o(\kappa) \geqslant \lambda + \delta$.*
2. *If $\delta = \omega_1$ then $o(\kappa) \geqslant \lambda$.*
3. *If $\delta = \omega$ then either $o(\kappa) \geqslant \lambda$ or else $\{\alpha : K \models o(\alpha) \geqslant \alpha^{+n}\}$ is cofinal in κ for each $n < \omega$.*

Woodin (see [1]) has constructed models of $2^\kappa = \lambda$ and $\mathrm{cf}(\kappa) = \delta > \omega$ from a model of $o(\kappa) = \lambda + \delta$, so clause (1) cannot be strengthened. Another approach to the same conclusion has been taken by Segal in [20]. For $\delta = \omega$, Gitik and Magidor [4, 9] show that the condition $o(\kappa) \geqslant \lambda$ cannot be improved in clause (3), and recent work of Gitik [7] makes it unlikely that second condition in clause (3) can be eliminated. We will also show that if there is an n such that $\{\alpha : o(\alpha) \geqslant \alpha^{+n}\}$ is bounded in κ then the conclusion to clause (2) can be strengthened to match clause (1), but it is not known whether this is true without the added hypothesis.

If we assume that the GCH holds below κ then we can get slightly more:

Corollary 2. *Suppose that $n > 0$ and κ is a cardinal of cofinality ω such that $2^\kappa \geqslant \kappa^{+(n+2)}$ while $2^\alpha = \alpha^+$ for all $\alpha < \kappa$, and assume that there is an $m < \omega$ such that $\{\alpha : K \models o(\alpha) \geqslant \alpha^{+m}\}$ is bounded in κ. Then $o(\kappa) \geqslant \kappa^{n+2} + 1$.*

The above result is Corollary 3.23 in Section 3.

Results in [8] show that $o(\kappa) = \kappa^{n+2} + 1$ is sufficient. The restriction to $n > 0$ is necessary here since by the results of Woodin and Gitik [4] $o(\kappa) = \kappa^{++}$ is enough to obtain a model of GCH with $\mathrm{cf}(\kappa) = \omega$ and $2^\kappa = \kappa^{++}$.

For the case $\kappa = \omega_\omega$ we have the following result (Theorem 3.24 of Section 3):

Theorem 3. *If $2^\omega < \aleph_\omega$ and $2^{\aleph_\omega} > \aleph_{\omega_1}$ then there is a sharp for a model with a strong cardinal.*

The results concerning sequences of indiscernibles are much more difficult to state. The Dodd–Jensen covering lemma for $L[\mu]$ asserts that if $L[\mu]$ exists, but $0^\dagger$ does not exist, then either every uncountable set x of ordinals is contained in a set in

$L[\mu]$ of the same cardinality as x, or else there is a sequence C which is Prikry generic over $L[\mu]$ such that every uncountable set x of ordinals is contained in a set in $L[\mu, U]$ of the same cardinality as x. Furthermore, the sequence C is unique except for finite segments. Uniqueness may fail if there are more measures in the core model: starting from a model with inaccessibly many measurable cardinals it is possible [14] to construct a model in which each of the measurable cardinals of K has a Prikry sequence and hence is singular, but there is no single system of indiscernibles for all of the cardinals. A weaker uniqueness property is established in [15, 16], however. It is shown there that for each uncountable set x of ordinals there is a function $h \in K$, a "next indiscernible" function n, and an ordinal ρ of cardinality at most $|x|^{\omega}$ such that x is contained in the smallest set $X_{\rho,h,n}$ containing ρ and closed under the functions h and n. The function n is somewhat complicated. If $o(\kappa) \leqslant 1$ for all κ then $n(\alpha, \gamma)$ is just the least indiscernible larger than γ for the measure $\mathscr{E}_{\alpha}$, but for larger cardinals it also must generate certain limits of indiscernibles. It is shown in [15, 16] that the function n is unique in the sense that for any other choice ρ', h', n' there is an ordinal $\eta < \sup x$ such that $n'(\alpha, \gamma) = n(\alpha, \gamma)$ whenever $\alpha, \gamma \in X_{\rho,h,n} \cap X_{\rho',h',n'}$ and $\gamma \geqslant \eta$. In this paper we extend these results up to a strong cardinal, in the case $\mathrm{cf}(\sup x) > \omega$, and to cardinals $\kappa = \sup(x)$ such that $\{\alpha < \kappa : o(\alpha) \geqslant \alpha^{+n}\}$ is bounded in κ for some $n < \omega$ in the case $\mathrm{cf}(\kappa) = \omega$.

Section 1 is a brief introduction to the core model K for sequences of extenders and to its covering lemma. It is intended to describe the notation used in the rest of the paper as well as to establish some basic results concerning indiscernibles relative to a particular covering set. Most of the arguments which require a detailed reference to the proof of the covering lemma have been gathered into this section, so that with a few exceptions (mainly in Subsection 3.2) the rest of the paper can be read in a black box fashion, referring to results form Section 1 rather than to basic core model theory external to this paper.

Section 2 covers results concerning sequences of indiscernibles. The basic result is that such sequences are, except on a bounded set, independent of the particular covering set used to obtain the indiscernibles. The applications to the singular cardinal hypothesis are given in Section 3, and some open problems are stated in Section 4.

1. Introduction and notation

We assume throughout this paper that there is no sharp for an inner model with a strong cardinal, so that a core model is guaranteed to exist. Expositions of these models include [10, 19, 23]. The first of these uses a somewhat different notation, and the latter two are primarily concerned with larger cardinals and hence involve complications which are, from our point of view, unnecessary. Fortunately, our arguments will not make serious use of fine structure and hence are not heavily dependent on the exact construction of the core model.

The proof is heavily dependent on the covering lemma, and indeed on the proof of the covering lemma. We will begin this section with an outline of this proof, partly to orient the reader and partly to introduce the notation which will be used later in the paper. Most of our references to the proof of covering lemma will be concentrated in this section, so that a reader who is not fully comfortable with the details of the proof will be able to get something out of the rest of the paper.

1.1. Extenders and the core model

A κ,λ-extender E is a sequence of ultrafilters, $E = \{E_a : a \in [\lambda]^{<\omega}\}$, with E_a an ultrafilter on ${}^a\kappa$. An extender may be obtained from an embedding π by

$$E_a = \{x \subset {}^a\kappa : \dot{a} \in \pi(x)\},$$

where $\dot{a} = \pi^{-1} \restriction (\pi(a))$. We will frequently identify a finite function $\sigma \in {}^a\kappa$ with $\{\sigma(\xi) : \xi \in a\} \in [\kappa]^{|a|}$, so that the equation above could be written

$$E_a = \{x \subset [\kappa]^{|a|} : a \in \pi(x)\}.$$

Going the other direction, an embedding π can be generated from the extender E and a model M which is to be the domain of π:

$$\pi : M \to \text{ult}(M,E) = \{[a,f] : a \in [\lambda]^{<\omega} \text{ and } f \in M \text{ and } f : {}^a\kappa \to M\},$$

where $[a,f] = [a',f']$ if and only if $\{\sigma \in {}^{a\cup a'}\kappa : f(\sigma \restriction a) = f'(\sigma \restriction a')\} \in E_{a\cup a'}$. This will define an embedding on M provided that E_a is an ultrafilter on at least the subsets of $[\kappa]^{|a|}$ which are in M.

If E is a κ,λ-extender then we call κ the *critical point* of E, written crit(E). If $\eta \leqslant \lambda$ then we write $E \restriction \eta$ for the restriction of E to the support η, that is, $E \restriction \eta = (E_a : a \in [\eta]^{<\omega})$. The *natural length* of E, written len(E), is defined to be the least ordinal $\eta \geqslant \kappa^+$ such that $\text{ult}(M,E) = \text{ult}(M, E \restriction \eta)$ for any model M such that E is an extender on M.

The core model, K, is a model of the form $L[\mathscr{E}]$, where $\mathscr{E}$ is a sequence of extenders and partial extenders on $L[\mathscr{E}]$. Each member $\mathscr{E}_\gamma$ of the sequence $\mathscr{E}$ is an extender on $L(\mathscr{E} \restriction \gamma)$. The set $\mathscr{E}_\gamma$ may or may not be a full extender on all sets in $L[\mathscr{E}]$: this depends on whether there are any subsets of crit($\mathscr{E}_\gamma$) in $L[\mathscr{E}]$ which are constructed after $L_\gamma[\mathscr{E}]$ and hence are not measured by $\mathscr{E}_\gamma$. The ordinal γ is called the *index* of $E = \mathscr{E}_\gamma$, written $\gamma = \text{index}(E)$. It is defined by $\text{index}(E) = \text{len}(E)^+$ as evaluated in $L[\mathscr{E} \restriction \gamma]$.

The following theorem lists some of the properties of K which we shall need. The proof can be found in the references.

Theorem 1.1. *The core model $K = L[\mathscr{E}]$ is maximal among all iterable models $L[\mathscr{F}]$ in the following three senses:*

1. *If $\mathfrak{m}$ is a mouse which is coiterable with K and agrees with K up to the projectum of $\mathfrak{m}$ then $\mathfrak{m} \in K$.*

2. *If E is an extender such that $\mathscr{E} \upharpoonright \gamma^\frown E$ is good and* $\mathrm{ult}(K,E)$ *is iterable then* $E = \mathscr{E}_\gamma$.

3. *If $M = L_\nu[\mathscr{F}]$ or $M = L[\mathscr{F}]$ is iterable then there is an iterated ultrapower of K such that M is a (possibly proper) initial segment of the last model of the iteration. Furthermore, if there is any elementary embedding $j: L[\mathscr{E}] \to M$ then this iterated ultrapower does not drop, and j is the canonical embedding of this iterated ultrapower.*

For the models in this paper the iterability properties referred to above are all be guaranteed by countable completeness and hence are not a problem. For core models much larger than those considered here, countable completeness is not enough for iterability, so that iterability does become a serious problem.

Clause 3 is actually a combination of the global maximality property that K is not shorter than any model $L[\mathscr{F}]$ with which it may be compared, together with clauses 1 and 2. The form of clause 3 which we give here is not always true in core models larger than those considered here.

The relation $\lhd$ is defined for extenders in the same way as for measures: $E \lhd E'$ in a model M if and only if $E \in \mathrm{ult}(M,E')$. This ordering is a well founded partial ordering.

We will write $\mathrm{O}(\kappa)$ to indicate the set of γ such that $\mathscr{E}_\gamma$ is defined and is a full extender on κ in K, and we will write $\mathrm{o}(\kappa)$ for the order type of $\mathrm{O}(\kappa)$. We will also write $\mathrm{O}'(\kappa)$ for $\mathrm{O}(\kappa) \cup \{\gamma\}$ where γ is the strict sup of $\mathrm{O}(\kappa)$, that is, $\gamma = \sup\{\nu + 1 : \nu \in \mathrm{O}(\kappa)\}$.

In some respects the models which we will use sit uncomfortably between models with overlapping extenders and those with no extenders other than measures. Our attention in later sections of this paper will be largely devoted to the major difference, the greater complexity of the indiscernibles, but there is one other difference which is more of an annoyance than a problem and should be discussed here. This problem is that if $\mathscr{E}_\gamma$ is an extender in $\mathscr{E}$ with critical point κ then $i^{\mathscr{E}_\gamma}(\mathscr{E})$ may have extenders with critical point κ which are not in $\mathscr{E}$. Suppose, for example, that $\mathscr{E}_\gamma$ is a measure on κ which concentrates on cardinals $\alpha < \kappa$ such that $\mathrm{o}(\alpha) > \alpha^{++}$. Then $\gamma = \kappa^{++}$ in $\mathrm{ult}(L[\mathscr{E}], \mathscr{E}_\gamma)$ since $\mathscr{E}_\gamma$ is a measure, but $\mathrm{o}(\kappa) > \kappa^{++}$ in $\mathrm{ult}(L[\mathscr{E}], \mathscr{E}_\gamma)$. Thus, if we set $\mathscr{E}' = i^{\mathscr{E}_\gamma}(\mathscr{E})$ then $\mathscr{E}'_{\gamma'}$ exists for some ordinals γ' with $\gamma < \gamma' \in \mathrm{O}^{\mathscr{E}'}(\kappa) \setminus \mathrm{O}^{\mathscr{E}}(\kappa)$. If we had taken the ultrapower by $\mathscr{E}_\gamma$ during the course of a comparison, because $\mathscr{E}_\gamma$ was not a member of the other model in the comparison, then it may well be that some of the new extenders $\mathscr{E}'_{\gamma'}$ are also not in the other model, requiring a second ultrapower by another extender with the same critical point.

The reader who is familiar with inner models for overlapping extenders will recognize this situation as a trivial example of an iteration tree: one which is linear except that it has side branches of length one in addition to the main trunk. For the less sophisticated reader we will sketch a second solution. This solution is simply to expand the sequence $\mathscr{E}$, for the purpose of the comparison lemma, so that if $\mathscr{E}_\gamma$ has critical point κ then every extender on κ in the sequence $\mathscr{E}' = i^{\mathscr{E}_\gamma}(\mathscr{E})$ is also in the sequence $\mathscr{E}$. If we do this then it is no longer true that $\mathscr{E}_\gamma \lhd \mathscr{E}_{\gamma'}$ if and only if $\gamma < \gamma'$, but this failure is not such as to cause a serious problem. Notice that

$$\sup(O^{\mathscr{E}'}(\kappa)) < i^{\mathscr{E}_\gamma}(\kappa) < (\gamma^+)^{L[\mathscr{E}\restriction\gamma+1]},$$

which is smaller than the index of the next extender in $\mathscr{E}$ on κ. Thus the new extenders which appear in the ultrapower by $\mathscr{E}_\gamma$ all lie between $\mathscr{E}_\gamma$ and the next extender on the original sequence. The expanded sequence will satisfy that $\mathscr{E}_\gamma \lhd \mathscr{E}_{\gamma'}$ if and only if $i^{\mathscr{E}_\gamma}(\kappa) < i^{\mathscr{E}_{\gamma'}}(\kappa)$. We will write $\gamma \lhd \gamma'$ to mean that $\mathscr{E}_\gamma$ are extenders on the expanded sequence with the same critical point, and $\mathscr{E}_\gamma \lhd \mathscr{E}_{\gamma'}$. In addition we will write $\gamma \lhd \sup(O'(\alpha))$ for all $\gamma \in O(\alpha)$.

A cardinal κ is strong if for all $\lambda > \kappa$ there is an elementary embedding $i : V \to M$ such that $V_\lambda \in M$. Thus κ is strong in a model $L[\mathscr{F}]$ if and only if $O^{\mathscr{F}}(\kappa)$ is unbounded in the ordinals. The assumption that there is no sharp of a strong cardinal means that there does not exist a pair $(\mathscr{F}, I)$ such that $L(\mathscr{F})$ satisfies that there is a strong cardinal and that I is a proper class of indiscernibles for $L[\mathscr{F}]$. The lack of such a sharp implies that the extenders of $\mathscr{E}$ never overlap, that is, there are no ordinals γ and γ' in the domain of $\mathscr{E}$ such that $\text{crit}(\mathscr{E}_\gamma) < \text{crit}(\mathscr{E}_{\gamma'}) < \gamma$.

Although all of the extenders E which we will be explicitly considering are complete in the sense that each ultrafilter E_a in E is κ-complete, where $\kappa = \text{crit}(E)$, we will use ultrapower constructions to define extensions of elementary embeddings, and these constructions implicitly use extenders which are not complete. If $\pi : N \to X$ then we will write $\text{ult}(M, \pi, \nu)$ for the ultrapower of M by the extender of length ν generated by π, that is,

$$\text{ult}(M, \pi, \nu) = \{[a, f] : a \in [\nu]^{<\omega} \quad \text{and} \quad f \in M\},$$

where $[a, f] = [a', f']$ if and only if

$$\dot{a} \cup \dot{a}' \in \pi\big(\{\sigma : f(\sigma \restriction a) = f'(\sigma \restriction \alpha')\}\big).$$

In order for $\text{ult}(M, \pi, \nu)$ to exist, N must contain all of the sets which need to be measured in the ultrapower:

Proposition 1.2. *Let $\pi : N \to X$, with $\nu \in X$ and let ν' be the least ordinal such that $\pi(\nu') \geqslant \nu$. Then $\text{ult}(M, \pi, \nu)$ is defined whenever*

$$\begin{array}{ll} \mathscr{P}(\nu') \cap M \subset N & \text{if } \nu > \sup \pi``\nu', \\ \forall \alpha < \nu' \, (\mathscr{P}(\alpha) \cap M \subset N) & \text{if } \nu = \sup \pi``\nu'. \end{array}$$

1.2. The covering lemma

It is assumed that the reader is familiar with the Dodd–Jensen covering lemma [2, 3] and with the covering lemma for sequences of measures [12, 13]. It will be recalled that Jensen's covering lemma for L asserts that if $0^\#$ does not exist and x is any uncountable set then there is a set $y \in L$ such that $x \subset y$ and $|y| = |x|$. As larger cardinals become involved, the generalizations of the covering lemma become more complex and less satisfactory, but the proof remains essentially the same; indeed these generalizations are still called "covering lemmas" not so much because their statement looks like Jensen's covering lemma for L but because their proof looks like Jensen's original proof.

The first step in the proof of the covering lemma is to replace the set x with a nicer set $X \supset x$ having the same cardinality as x:

Definition 1.3. *A δ-closed precovering set X* for κ is a set $X \prec H_\tau$, for some $\tau \geqslant (2^\kappa)^+$, such that $X^\delta \subset X$, $|X| < |\kappa|$ and X is cofinal in κ.

Usually, we will have $\kappa = \sup(x)$ and $\delta = |x| = \mathrm{cf}(\kappa)$ and $\tau = (2^\kappa)^+$, and in this case we will simply refer to X as a precovering set. On the few occasions when we use more or less closure, or require τ to be larger than $(2^\kappa)^+$, we will so specify.

Proposition 1.4. *If $\delta < \kappa$ are cardinals, $x \subset \kappa$, and $(\sup(\mathrm{cf}(\kappa), |x|)^\delta < |\kappa|$ then there is a δ-closed precovering set $X \supset x$.*

In order to simplify notation, we will assume throughout the rest of this section that we have a fixed precovering set X. Later in the paper, when it may not be clear which precovering set is meant, we will modify the notation either by adding a subscript or by specifying "in X" to indicate which precovering set is intended. Thus, in this section we will use $\pi : N \cong X \prec H_\tau$ to denote the Mostowski collapse of X, but if there were more than one precovering set involved we would write $\pi^X : N^X \cong X \prec H_\tau$.

We will consistently use an over-bar to relate members of the collapse N of X with the corresponding members of X. If $x \in X$ then we write $\bar{x}$ for $\pi^{-1}(x)$. When $\bar{x}$ is used for some object which is not a member of N then the corresponding object x will need to be defined on a case by case basis, but it will always follow the rule that x is related to $\bar{x}$ via the embedding π.

By Theorem 1.1 there is an iterated ultrapower of K with final model M_θ having $\bar{K}_{\bar{\kappa}}$ as an initial segment. Let $(M_\xi : \xi \leqslant \nu)$ be the sequence of models of the iteration and $j_{\xi,\xi'} : M_\xi \to M_{\xi'}$ the corresponding embeddings.

For most ordinals $\xi < \theta$ we will have $M_{\xi+1} = \mathrm{ult}(M_\xi, E)$ where E is the least extender which is in M_ξ but not in $\bar{K}$, but for finitely many ordinals $\xi < \theta$ the iteration may *drop to a mouse*. This means that $M_{\xi+1} = \mathrm{ult}(M^*_\xi, E)$ where M^*_ξ is a mouse such that $M^*_\xi \in M_\xi$. This happens whenever there is a subset $x \subset \rho$, with $x \in M_\xi \setminus \bar{K}$, for some ordinal ρ which is less than or equal to the critical point of the first extender on which M_ξ and $\bar{K}$ disagree.

The next step depends on whether the iteration ever does drop to a mouse before reaching a model M_θ which agrees with $\bar{K}$ up to $\bar{\kappa}$. Jensen, in his proof of the covering lemma for L, was able to prove outright that this must happen by observing that otherwise the embedding π could be extended to a nontrivial embedding from L into L, which implies that $0^\#$ exists. The argument works for sequences of measures, but can fail for extenders. We will sketch a proof that shows that if there are no overlapping extenders and the iteration does not drop then full covering holds over K for cofinal subsets of K, so that $2^\kappa = \kappa^+$ by the same proof as for L. This proof is given in detail (for overlapping extenders) in [18].

Suppose that the iteration does not drop. Then M_θ is a proper class and $j_{0,\theta}: K \to M_\theta$ exists. Let $k: M_\theta \to \bar{M} = \mathrm{ult}(M_\theta, \pi, \kappa)$ be the canonical embedding. By Theorem 1.1 there is an iterated ultrapower $i: K \to \bar{M}$ such that $i = k \circ j_{0,\theta}$. Now $\mathrm{crit}(\tilde{\pi} \circ j_{0,\theta}) \leqslant \mathrm{crit}(\pi) = \eta$, so $\mathrm{crit}(i) \leqslant \eta$. The first ultrapower in i uses an extender E in K which is not in $\bar{M}$, and since $\bar{M}$ agrees with K at least up to κ it follows that $\mathrm{len}(E) \geqslant \kappa$. Since there is no model with overlapping extenders it follows that there are no measurable cardinals μ in the interval $\eta < \mu \leqslant \kappa$. Then the iteration $i_{0,\theta}$ involves only finitely many ultrapowers before reaching $\bar{\kappa}$ (cf. the proof of Lemma 1.7), so there is an ordinal $\bar{\rho} < \bar{\kappa}$ such that any member of $\bar{\kappa}$ can be expressed in the form $i_{0,\theta}(f)(\gamma)$ for some $f \in K$ and $\gamma < \bar{\rho}$. It follows that $X \subset y = \{i^E(f)(\gamma) : f \in {}^\eta\eta \cap K \wedge \gamma < \rho\}$. Now $y \in K$ since $E \in K$, and $|y| \leqslant \gamma 2^{|\mathrm{crit}(E)|} \leqslant \gamma 2^\eta < \kappa$. Thus full covering holds for subsets of κ, whenever the iteration does not drop, which is what we were trying to show. For the rest of this paper we will assume that the iteration does drop.

In order to simplify notation it is convenient to use the critical points of the extenders to index the models in the iterated ultrapower, rather than indexing them sequentially as in the last paragraph. Let $j_{\xi,\xi'}$ be the canonical embedding from M_ξ to $M_{\xi'}$, which is defined provided that the iteration does not drop to a mouse in the half-open interval $[\xi, \xi')$.

Definition 1.5. If ν is an ordinal in N then $\overline{\mathfrak{m}}_\nu \stackrel{\text{def}}{=} M_{\xi_\nu}$, where ξ_ν is the least ordinal ξ such that $\mathrm{crit}(j_{\xi,\xi+1}) \geqslant \nu$. If there is no such ordinal ξ then $\overline{\mathfrak{m}}_\nu = M_\theta$. We write $\bar{i}_{\nu,\nu'}$ for the embedding $j_{\xi_\nu,\xi_{\nu'}} : \overline{\mathfrak{m}}_\nu \to \overline{\mathfrak{m}}_{\nu'}$, and $\overline{E}_\nu$ for the extender used at stage ξ_ν of the iteration, so $\overline{\mathfrak{m}}_{\nu+1} = \mathrm{ult}(\overline{\mathfrak{m}}_\nu, \overline{E}_\nu)$. We write $\bar{h}_\nu$ for the canonical Skolem function of the premouse $\overline{\mathfrak{m}}_\nu = M_{\xi_\nu}$.

Note that if $\nu = \mathrm{crit}(j_{\xi,\xi+1})$, then $\overline{\mathfrak{m}}_\nu = M_\xi$ and $\overline{\mathfrak{m}}_{\nu+1} = M_{\xi+1}$.

The embedding $\bar{i}_{\nu,\nu'}$ does not exist if the iteration drops to a mouse somewhere in the half-open interval $\xi_\nu \leqslant \xi < \xi_\kappa$. Such drops only occur finitely often. Those familiar with fine structure in these models will recall that the iteration may also drop in degree, but this also occurs only finitely often. Thus there is an ordinal $\bar{\nu}_0 < \bar{\kappa}$ such that the iteration never drops in the interval $\xi_{\bar{\nu}_0} \leqslant \xi < \xi_\kappa$, so that $\bar{i}_{\nu,\nu'}$ is always defined when $\bar{\nu}_0 \leqslant \nu < \nu' \leqslant \bar{\kappa}$. Since we are only interested in subsets of κ, and are not concerned with what happens on bounded subsets of κ, it will be sufficient to restrict ourselves to ν in this interval.

If we are doing fine structure in terms of Σ_n-codes, then we can think of the models $\mathcal{M}_\nu$ for $\bar{\nu}_0 \leqslant \nu < \kappa$ as Σ_{n-1}-codes, for some fixed $n < \omega$, for premice $J_{\alpha_\nu}[\mathcal{F}_\nu]$. All of the models have the same Σ_1-projectum $\bar{\rho} < \bar{\nu}_0$, so that $j_{\nu,\nu'}(\bar{\rho}) = \bar{\rho}$ for $\nu_0 \leqslant \nu < \nu' \leqslant \bar{\kappa}$. The Skolem function $\bar{f}_\nu$ of $\overline{\mathfrak{m}}_\nu$ is just the canonical Σ_1-Skolem function and is also preserved by the maps $j_{\nu,\nu'}$. The reader who does not fully understand this construction will not be misled if he thinks only of the case $n = 1$, so that $\overline{\mathfrak{m}}_\nu = J_{\alpha_\nu}(\mathscr{F}_\nu)$ and the embeddings $j_{\nu,\nu'}$ are ordinary ultrapowers by functions in $\overline{\mathfrak{m}}_\nu$.

So far we have concentrated on the collapsed model N, but we are really interested in the uncollapsed model X. The connection between the two is made by using the collapse map π as an extender. In particular, we can define $\tilde{\pi}$ to be the canonical embedding from $\overline{\mathfrak{m}}_{\bar\kappa}$ into $\mathfrak{m} = \mathrm{ult}(\overline{\mathfrak{m}}_{\bar\kappa}, \pi, \kappa)$, which exists by Proposition 1.2 since $\kappa = \sup \pi``\bar\kappa$. This embedding preserves the fine structure of $\overline{\mathfrak{m}}_{\bar\kappa}$, so $\tilde{\pi} \circ \bar{h} = h \circ \pi$ where $\bar{h}$ and h are the Skolem functions of $\overline{\mathfrak{m}}_{\bar\kappa}$ and $\mathfrak{m}$, respectively.

The usual proof of the next lemma consists mainly of the proof that $\mathfrak{m}$ is iterable. The proof with extenders involves one additional difficulty, and we include just enough of the proof to indicate a solution to this problem. It should be noted that $\in x(\overline{E}_{\bar\nu}) < \bar\kappa$ for all $\bar\nu < \bar\kappa$. If, to the contrary, $\in x(\overline{E}_{\bar\nu}) \geqslant \bar\kappa$ then $\overline{\mathfrak{m}}_{\bar\nu}$ agrees with $\bar{K}$ up to $\bar\kappa$, so that the iteration was already complete at $\overline{\mathfrak{m}}_{\bar\nu}$ before $\overline{E}_{\bar\nu}$ was chosen.

Lemma 1.6. *The structure* $\mathfrak{m} = \mathrm{ult}(\overline{\mathfrak{m}}_{\bar\kappa}, \pi, \kappa)$ *is a member of* K.

Proof (*sketch*). The new difficulty is that there may be an extender $\overline{E}$ on the extender sequence of $\overline{\mathfrak{m}}_{\bar\kappa}$ such that $\mathrm{crit}(\overline{E}) < \bar\kappa \leqslant \mathrm{len}(\overline{E})$. This is not a problem if $\overline{E}$ is an actual member of $\overline{\mathfrak{m}}_{\bar\kappa}$, since in this case $\tilde{\pi}(\overline{E})$ is defined and is in K. Thus we need only worry about the case when $\overline{E}$ is the last extender in the sequence of $\overline{\mathfrak{m}}_{\bar\kappa}$. In this case standard arguments show that the structure $\mathfrak{m}'$ obtained by omitting the final extender of $\mathfrak{m}$ is a member of K.

Set $\bar\mu = \mathrm{crit}(\overline{E})$ and $\mu = \pi(\bar\mu)$. If $z \subset \mathscr{P}(\bar\mu)$ is an arbitrary member of $\bar{K}$ which has cardinality $\bar\mu$ in $\bar{K}$, then by amenability the set $\overline{E} \cap z = \{\overline{E}_a \cap z : a \in [\mathrm{len}(\overline{E})]^{<\omega}\}$ is a member of $\overline{\mathfrak{m}}_{\bar\kappa}$. Thus we can define $E = \bigcup_z \tilde{\pi}(\overline{E} \cap z)$. If X is cofinal in $\mu^{+(K)}$ then E is a full extender on K. In that case standard arguments show that E is in K, so that $\mathfrak{m}$ is a member of K.

The referee has pointed out that we can ensure that X is cofinal in μ^+ of K by choosing the precovering set X so that $\mu^+ \cap X$ is cofinal in μ^+ whenever μ is $< \kappa$-strong in K. This is possible since our assumption that there are no overlapping extenders implies that there can be at most one such cardinal μ, and our assumption that κ is a strong limit cardinal implies that any subset of κ of cardinality less than κ is contained in a precovering set.

For the sake of the interested reader we will sketch a proof, without this extra assumption on X, that $E \in K$ even when X is not cofinal in $\mu^{+(K)}$. In this case let $\mathfrak{n}$ be the least mouse which has projectum less than or equal to μ and such that $\mathfrak{n}$ is larger than every mouse in X with projectum μ. Then $\mathfrak{n}$ is the least mouse in K such that there is a subset of $\alpha = \pi(\bar\alpha)$ definable in $\mathfrak{n}$ which is is not decided by E, so that E is an extender on $\mathfrak{n}$ and we can let $i^E : \mathfrak{n} \to \mathfrak{n}' = \mathrm{ult}(\mathfrak{n}, E)$ be the canonical embedding. Then $\mathfrak{n}'$ is an iterable premouse which agrees with K up to the length of E, so $\mathfrak{n}'$ is a member of K. Now $\mathrm{range}(i^E) = h^{\mathfrak{n}'}``\alpha$, and hence the range of i^E is definable in $\mathfrak{n}'$. Then for any $a \in [\mathrm{len}(E)]^n$ the ultrafilter E_a is equal to the set of

$x \subset [\mu]^n$ such that there is a set $y \in \text{range}(i^E)$ such that $y \cap [\mu]^n = x$ and $a \in y$, and it follows that $E \in K$. $\square$

Notice that $\mathfrak{m}$ as defined in the last section is not a mouse, since E is not a complete extender on $\mathfrak{m}$. It is close enough for our purposes, however, since its Skolem function h still satisfies the crucial identity $\tilde{\pi} \circ \bar{h} = h \circ \pi$.

Now let $\bar{h}$ and h be the Skolem functions of $\overline{\mathfrak{m}}_{\bar{\kappa}}$ and $\mathfrak{m}$, respectively. Then $h \in K$, and since $\tilde{\pi} \circ \bar{h} = h \circ \pi$ we can use h to cover the set X as follows: Set $\rho = \rho^X = \pi(\bar{\rho})$ where $\bar{\rho}$ is the projectum of $\overline{\mathfrak{m}}_{\bar{\kappa}}$, and let I be the set of ordinals $\pi(\xi)$ such that $\text{crit}(\overline{E}_\nu) \leqslant \xi < \text{len}(\overline{E}_\nu)$ for some extender $\overline{E}_\nu$ used in the iteration which gave $\overline{\mathfrak{m}}_{\bar{\kappa}}$. Then $X \cap \kappa = \pi``\bar{\kappa} \subset h``(\rho \cup I)$.

We will call the members of I *indiscernibles* by analogy with the simpler case of sequences of measures. We will eventually need to make a detailed analysis of these indiscernibles, but first we look at the covering lemma to see what can be obtained by looking at intervals in which there are no measures and hence no indiscernibles:

Lemma 1.7 (Covering lemma without indiscernibles). *Assume that there does not exist a sharp for a model with a strong cardinal, that $\lambda^\omega < \kappa$, and that there are no measurable cardinals ν of K in the half-open interval $\lambda < \nu \leqslant \kappa$. Then for every subset y of κ such that $|y| \leqslant \lambda$ there is a $z \in K$ such that $y \subset z$ and $|z| \leqslant \lambda$.*

In particular if $\kappa > \omega_2$ is a regular cardinal in K then $(\text{cf}(\kappa))^\omega \geqslant |\kappa|$, and if λ is a singular cardinal in V then either $\nu^\omega \geqslant \lambda$ for some $\nu < \lambda$ or else $\lambda^+ = \lambda^{+(K)}$.

Proof. The proof is by induction on κ. From the discussion above we know that any subset x of κ is contained in a set of the form $h``(\rho \cup I)$ where $h \in K$, $\rho < \kappa$, and I is the set of indiscernibles. We will show that there is an ordinal η, with $\rho \leqslant \eta < \kappa$, such that $I \setminus \eta$ is finite. It follows that x is contained in a set $y \in K$ such that $|y|^K \leqslant \eta$. By the induction hypothesis it follows that x is contained in a set $y' \in K$ such that $|y'|^K \leqslant \lambda$.

If there is some ν such that $\text{crit}(\overline{E}_\nu) \leqslant \rho < \pi(\text{len}(\overline{E}_\nu))$ then set $\eta = \pi(\text{len}(\overline{E}_\nu))$, and otherwise set $\eta = \rho$. Thus every member of $I \setminus \eta$ comes from an extender $E = \overline{E}_\nu$ such that $\eta \leqslant \pi(\text{crit}(E)) < \kappa$. Notice that E must be a measure, since otherwise $\xi = \text{crit}(E)$ is measurable in $\overline{\mathfrak{m}}_{\nu+1} = \text{ult}(\overline{\mathfrak{m}}_\nu, \overline{E}_\nu)$, which implies that ξ is measurable in $\bar{K}$ and hence $\pi(\xi)$ is measurable in K, contrary to assumption. Furthermore, $\bar{\imath}_{\nu,\bar{\kappa}}(\bar{\xi}) > \bar{\kappa}$, since otherwise $\pi(\bar{\imath}_{\nu,\bar{\kappa}}(\bar{\xi}))$ is measurable in K. If there are infinitely many measures $\overline{E}_\nu$ with $\eta \leqslant \xi_\nu = \text{crit}(\overline{E}_\nu) < \bar{\kappa}$ then there must be an infinite set D of ν so that $\bar{\imath}_{\nu,\nu'}(\xi_\nu) = \xi_{\nu'} < \bar{\kappa}$ for $\nu < \nu'$ in D, but then any limit point β of D of cofinality ω is measurable in $\bar{K}$. To see this, let $\boldsymbol{d} \subset D$ be a countable set with $\beta = \sup \boldsymbol{d}$, and let U be the set of $x \in \mathscr{P}(\beta) \cap \bar{K}$ such that for all sufficiently large $d \in \boldsymbol{d}$

$$\begin{aligned} &d \in x && \text{if } o^{\bar{K}}(d) = 0, \\ &x \cap d \in U_d && \text{if } o^{\bar{K}}(d) > 0, \end{aligned}$$

where U_d is the order 0 measure on d in $\bar{K}$. Then $U \in N$ since $\boldsymbol{d} \in {}^{\omega}N \subset N$, and U is a measure on $\bar{K}$. Thus $\eta \leqslant \pi(\beta) \leqslant \kappa$ and β is measurable in K, contrary to assumption, and so $I \setminus \eta$ must be finite as required. □

Now we must prepare for the hard work of analyzing the indiscernibles. The preparation will take up the rest of the section, and the actual analysis will be carried out in the next section. So far we have concentrated on the collapsed model N, and on the collapse $\bar{K}$ of K, but we are really interested in the uncollapsed models V and K. In the rest of this section we will describe the relationship, induced by the map π, between objects in X and objects of N.

We write ρ^X for $\pi(\bar{\rho})$, where $\bar{\rho}$ is the Σ_1-projectum of $\overline{\mathfrak{m}}^*_{\bar{\nu}_0}$, that is, $\bar{\rho}$ is the least ordinal such that there is a subset x of $\bar{\rho}$ which is Σ_1 definable in the Σ_{n-1}-code $\overline{\mathfrak{m}}_{\bar{\nu}_0}$ such that $x \notin \overline{\mathfrak{m}}_{\bar{\nu}_0}$. This same ordinal $\bar{\rho}$ will be the projectum of all of the models $\overline{\mathfrak{m}}_\nu$ for $\bar{\nu}_0 < \nu \leqslant \bar{\kappa}$.

Let C be the set of ordinals $\nu \in X$ such that $\bar{\nu} = \pi^{-1}(\nu) = \operatorname{crit}(\bar{\imath}_{\bar{\nu},\bar{\nu}+1})$. If we were dealing with measures then C would be the set of indiscernibles, but we will call any member of $\bigcup\{\pi``(\operatorname{len}(\overline{E}_{\bar{\nu}} \setminus \bar{\nu})) : \nu \in C\}$ an *indiscernible.* The members of C are called *principal indiscernibles.*

Definition 1.8. Suppose that $\nu_0 < \nu < \nu' \leqslant \kappa$ and $\nu, \nu' \in C$.

1. $i_{\nu,\nu'} \stackrel{\text{def}}{=} \pi \circ \bar{\imath}_{\bar{\nu},\bar{\nu}'} \circ \pi^{-1} \restriction (X \cap \eta)$, where η is the least inaccessible cardinal of K above $\sup(\mathrm{O}(\nu))$.
2. $h_\nu \stackrel{\text{def}}{=} \pi \circ \bar{h}_{\bar{\nu}} \circ \pi^{-1} \restriction \{\xi \in X : \pi \circ \bar{h}_{\bar{\nu}} \circ \pi^{-1}(\xi) < \eta\}$.

Notice that $\eta \in X$. The following proposition implies that $\eta > \pi(\operatorname{index}(\overline{E}_{\bar{\nu}}))$. It works for $\nu = \kappa$ as well if we take $\overline{E}_{\bar{\kappa}}$ to be the $\lhd$-least extender which is in $\overline{\mathfrak{m}}_{\bar{\kappa}}$ but is not in $\bar{K}$.

Proposition 1.9. *Suppose that $\mathscr{F}$ is an extender sequence and $\tau = \operatorname{crit}(\mathscr{F}_\gamma)$. Then* $\operatorname{ult}(L[\mathscr{F}], \mathscr{F}_\gamma)\sup(\mathrm{O}(\tau))^+ \geqslant \gamma$.

Proof. Recall that $\gamma = \operatorname{len}(\mathscr{F}_\gamma)^+$ as evaluated in $L[\mathscr{F} \restriction \gamma]$ or, equivalently, as evaluated in the ultrapower $\operatorname{ult}(L[\mathscr{F}], \mathscr{F}_\gamma)$. Thus it is enough to show that $\sup(\mathrm{O}(\tau)) \geqslant |\operatorname{len}(\mathscr{F})|$ in $\operatorname{ult}(L[\mathscr{F}], \mathscr{F}_\gamma)$. Consider the extenders $\mathscr{F} \restriction \eta$ for $\eta < \operatorname{len}(\mathscr{F}_\gamma)$. All of these extenders are in $\operatorname{ult}(L[\mathscr{F}], \mathscr{F}_\gamma)$, so if $\sup(\mathrm{O}(\tau)) < |\operatorname{len}(\mathscr{F})|$ then there is η_0 such that $\operatorname{ult}(L[\mathscr{F}], \mathscr{F}_\gamma \restriction \eta) = \operatorname{ult}(L[\mathscr{F}], \mathscr{F}_\gamma \restriction \eta_0)$ for all η in the interval $\eta_0 < \eta < \operatorname{len}(\mathscr{F}_\gamma)$. It follows that every ordinal in that interval can be written in $L[\mathscr{F} \restriction \gamma]$ in the form $i^{\mathscr{F}_\gamma \restriction \eta_0}(f)(a)$ for some $f : \kappa^n \to \kappa$ and some $a \in [\eta_0]^{<\omega}$. Thus $|\operatorname{len}(\mathscr{F}_\gamma)| \leqslant \eta_0$ in $L[\mathscr{F} \restriction \gamma]$. □

Next we need to consider the image under π of the extenders used in the iterated ultrapower.

Definition 1.10. Suppose that $\nu' \in C$ and ν' is a principal indiscernible for ν in X.

1. $\bar{F}_{\bar{\nu},\bar{\eta}}$ $\bar{F}_{\bar{\nu}',\bar{\nu}} \stackrel{\text{def}}{=} \bar{\imath}_{\bar{\nu}',\bar{\nu}}(\bar{E}_{\bar{\nu}'})$.

2. $F_{\nu,\eta}$ $F_{\nu,\eta} \overset{\text{def}}{=} \pi(\bar{F}_{\bar{\nu},\bar{\eta}})$ if $\bar{F}_{\bar{\nu},\bar{\eta}} \in \bar{K}$, and it is undefined otherwise.

Eventually we will show, using Proposition 1.9, that for many $\bar{\nu}$, $\bar{F}_{\bar{\nu},\bar{\kappa}}$ is in N and hence $F_{\nu,\kappa}$ exists and is in K.

Notice that $\bar{F}_{\bar{\nu}',\bar{\nu}}$ is a member (or the last extender) of $\overline{\mathfrak{m}}_{\bar{\nu}}$. If $\bar{\nu} < \bar{\kappa}$ then $\bar{F}_{\bar{\nu}',\bar{\nu}}$ is a member of $\bar{K}$ if and only if either $\bar{\nu} = \text{crit}(\overline{E}_{\bar{\nu}}) > \bar{\imath}_{\bar{\nu}',\bar{\nu}}(\text{crit}(\overline{E}_{\bar{\nu}'}))$ (in which case $\bar{\nu} > \text{crit}(\bar{F}_{\bar{\nu}',\bar{\nu}})$ and every extender in $\overline{\mathfrak{m}}_{\bar{\nu}}$ with critical point less than $\bar{\nu}$ is in $\bar{K}$) or $\bar{\nu} = \bar{\imath}_{\bar{\nu}',\bar{\nu}}(\text{crit}(\overline{E}_{\bar{\nu}'}))$ and $\bar{F}_{\bar{\nu}',\bar{\nu}} \lhd \overline{E}_{\bar{\nu}}$ (in which case $\overline{E}_{\bar{\nu}}$ is the $\lhd$-least extender in $\overline{\mathfrak{m}}_{\bar{\nu}}$ which is not in $\bar{K}$). Essentially, the same analysis works at $\bar{\kappa}$: $\bar{F}_{\bar{\nu},\bar{\kappa}}$ is in $\bar{K}$ if either every extender on $\bar{\kappa}$ in $\overline{\mathfrak{m}}_{\bar{\kappa}}$ is in $\bar{K}$, or else $\bar{F}_{\bar{\nu},\bar{\kappa}} \lhd \overline{E}_{\bar{\kappa}}$ where $\overline{E}_{\bar{\kappa}}$ is the $\lhd$-least extender which is in $\overline{\mathfrak{m}}_{\bar{\kappa}}$ but not in $\bar{K}$.

Lemma 1.11. *Suppose that $\nu_0 < \nu < \nu' \leqslant \kappa$ and that ν is a principal indiscernible for ν' in X, that is, that $\nu \in C$ and $\bar{\nu}' = \bar{\imath}_{\bar{\nu},\bar{\nu}'}(\bar{\nu})$. Then*

1. $h_{\nu'} \restriction \nu = i_{\nu,\nu'} \circ h_\nu$.

2. *If z is in $X \cap K_\eta$, where η is the least inaccessible cardinal of K above $\mathrm{o}(\nu)$, then z is in $h_\nu``\nu$. Indeed $z = h_\nu(\boldsymbol{d})$ where $\boldsymbol{d}$ is a finite sequence of ordinals, each of which is either in ρ or an indiscernible smaller than ν.*

3. *If $z \in h_{\nu'}``\nu$, $\boldsymbol{b} \in [\pi``\text{len}(\bar{E}_{\bar{\nu}})]^{<\omega}$, and $\boldsymbol{b}' = i_{\nu,\nu'}(\boldsymbol{b})$ then $z \in (\pi``\bar{F}_{\bar{\nu},\bar{\nu}'})_{\boldsymbol{b}'}$ if and only if $\boldsymbol{b} \in z$.*

4. *If $f \in X \cap K$, the ordinal ν is a limit of $C \cap \nu$, and γ is the least member of C above ν then $\gamma \cap f``\xi = h_\nu``(\xi \cap f``\nu)$ for every sufficiently large ordinal ξ in $C \cap \nu$.*

5. *If $y \subset \nu$ and $|y| \leqslant \delta$ then there are functions i', h' and h'' in $X \cap K$ such that $i' \restriction y = i_{\nu,\nu'} \restriction y$, $h' \restriction y = h_\nu \restriction y$, and $h'' \restriction y = h_{\nu'} \restriction y$.*

Proof. Clause (1) follows from the definition of $i_{\nu,\nu'}$ and h_ν, and clauses (2) and (3) follow from the corresponding facts about the iterated ultrapowers $\overline{\mathfrak{m}}_\nu$ and $\overline{\mathfrak{m}}_{\nu'}$ and the fact that $\text{len}(\overline{E}_{\bar{\nu}})$ is smaller than η. Clause (4) follows from the fact that f is in the range of $i_{\nu',\nu}$ for some $\nu' \in C \cap \nu$.

This leaves only clause (5) to be proved. By clause (1) we can set $i' = h'' \circ (h')^{-1}$, so it will be enough to show that the functions h' and h'' exist. The proof is identical for h' and h'', so we will only give the proof for h'. Now $h_\nu``y \subset X$ and $X$ is $\delta$-closed so $h_\nu``y \in X$. Thus we can apply Lemma 1.7, the covering lemma without indiscernibles, inside X. Since there are no measurable cardinals in K between ν and $\sup(\text{range}(h_\nu))$, Lemma 1.7 asserts that there is a function $f \in X \cap K$ such that $h_\nu``y \subset f``\nu$. Then $\bar{f} = \pi^{-1}(f) \in N$, and since the next member of C above ν is larger than $\sup(\text{range}(h_\nu))$ it follows that $\bar{f} \in \overline{\mathfrak{m}}_{\bar{\nu}}$.

The model $\overline{\mathfrak{m}}_{\bar{\nu}}$ must have cofinality greater than δ. To see why this is true, recall that $\overline{\mathfrak{m}}_{\bar{\nu}}$ is the Σ_{n-1} code $(J_\alpha[\mathscr{E}], A)$ of some premouse, and has Σ_1-projectum $\rho \leqslant \bar{\nu}$. Let $x \subset \rho$ be Σ_1 definable in $\overline{\mathfrak{m}}_{\bar{\nu}}$, but not a member of N. Then $x = \bigcup\{x_{\alpha'} : \alpha' < \alpha\}$, where $x_{\alpha'}$ is the set of $\xi < \rho$ such that there is a witness $z \in J_{\alpha'}[\mathscr{E}]$ of the Σ_1 fact "$\xi \in x$". Then each set $x_{\alpha'}$ is in N, and if $\text{cf}(\alpha) \leqslant \delta$ then it would follow that $x \in N$.

Since $\mathrm{cf}(\alpha) > \delta$ and $\bar{h}_{\bar{\nu}}$ is Σ_1 definable in $\overline{\mathfrak{m}}_{\bar{\nu}}$, it follows that there is a set $\bar{y}_0 \in \overline{\mathfrak{m}}_{\bar{\nu}}$ such that $\bar{y} = \pi^{-1}(y) \subset \bar{y}_0$ and $\bar{h}_{\bar{\nu}} \upharpoonright \bar{y}_0 \in \overline{\mathfrak{m}}_{\bar{\nu}}$. Define a partial function $\bar{g}: \bar{y}_0 \to \bar{\nu}$ by letting $\bar{g}(\xi)$ be the least ordinal η such that $\bar{h}_{\bar{\nu}}(\xi) = \bar{f}(\eta)$. Then $\bar{f} \circ \bar{g} = \bar{h}_{\bar{\nu}} \upharpoonright \bar{y}_0$.

Now we have to consider two cases. If $\nu_0 < \nu < \kappa$ then by the choice of ν_0 the iteration did not drop to a mouse at $\overline{\mathfrak{m}}_{\bar{\nu}}$, that is, $\mathscr{P}(\bar{\nu}) \cap \overline{\mathfrak{m}}_{\bar{\nu}} \subset N$. In particular $\bar{g} \in N$ and we can set $h'_\nu = f \circ \pi(\bar{g}) \in K \cap X$, so that $h'(\xi) = h_\nu(\xi)$ for all $\xi \in X \cap \mathrm{domain}(h') \supset y$.

The only other possibility is $\nu = \kappa$, in which case X is cofinal in κ so that we can define $\tilde{\pi}: \overline{\mathfrak{m}}_{\bar{\kappa}} \to \mathfrak{m} = \mathrm{ult}(\overline{\mathfrak{m}}_{\bar{\kappa}}, \pi, \kappa)$. Then $\mathfrak{m} \in K$ as in the proof of Lemma 1.7, so $h^* = \tilde{\pi}(\bar{g}) \circ f \in K$. Now $h^*(\xi) = h_\kappa(\xi)$ for all $\xi \in X \cap \pi(y_0)$, so $h^* \upharpoonright y = h_\kappa \upharpoonright y$. But $h_\kappa \upharpoonright y \in X$, so by the elementarity of X there is a function $h' \in X \cap K$ such that $h' \upharpoonright y = h_\kappa \upharpoonright y$. □

1.3. Indiscernibles

It only remains to briefly discuss our notation for indiscernibles before we can begin the analysis of sequences of indiscernibles. As stated before, we call ν a *principal indiscernible* if $\nu \in C$, that is, if $\nu = \pi(\bar{\nu})$ where $\bar{\nu}$ is the critical point of $\bar{i}_{\bar{\nu},\bar{\kappa}}$ or, equivalently, if ν is the critical point of $i_{\nu,\kappa}$.

We will say that a is a *principal indiscernible for* α if $a \in C$ and $i_{a,\alpha}(a) = \alpha$. We say that a is a principal indiscernible for the extender E on α if $E = F_{a,\alpha}$. Notice that if a is a principal indiscernible for α and $F_{a,\alpha} \notin K$ then a is not an indiscernible for any extender on α. This differs from the way the term is usually used, but it is useful here because we will spend a large part of the next section showing that the relation "ν is an indiscernible for α" is definable before we begin to look at the definability of the relation "ν is an indiscernible for the extender F on α."

As stated earlier, we will say that an ordinal b is an indiscernible whenever there is a principal indiscernible a such that $b \in \pi``(\mathrm{len}(\overline{E}_{\bar{a}} \setminus \bar{a}))$, where $a = \pi(\bar{a})$. Since these indiscernibles will be used to reconstruct the image of the extender used at stage a it will be convenient to generalize this notation:

Definition 1.12. An ordinal b is an *indiscernible for* β *belonging to* (a, α) if (i) a is a principal indiscernible for α, (ii) $\beta = i_{a,\alpha}(b)$, (iii) $\beta < \inf(C \setminus \alpha + 1)$, and (iv) b is smaller than $\sup(\mathrm{O}(a))^{++}$.

Notice that if $\overline{E}_{\bar{a}}$ is the extender used at the $\bar{a}$th stage of the iteration then Proposition 1.9 implies that $\pi(\mathrm{index}(\overline{E}_{\bar{a}}))$ cannot be larger than the upper bound on b given in clause (iv), and that this upper bound is smaller than the next inaccessible cardinal above $\sup(\mathrm{O}(a))$ in K.

We will consistently use Roman letters for indiscernibles and the corresponding Greek letters for the ordinals for which they are indiscernibles. Thus a will denote a principal indiscernible for α, and b and c will denote indiscernibles for β and γ, respectively.

It was pointed out earlier that all of the definitions in this section are relative to a fixed precovering set X. Whenever it is not clear which precovering set is being used we will specify the relevant precovering set, either by adding a superscript or by using the words "in X".

Unless otherwise specified, successors are always calculated in the core model K. Thus κ^{+n} means the nth successor as calculated in K. Other functions will still be calculated in V unless otherwise stated, so that $|x|$ and $\mathrm{cf}(\kappa)$ are the cardinality of x and cofinality of κ in the real world.

The letter h will always be used to denote a Skolem function, and if x is a set then we will write $h``x$ to mean $\{h(\boldsymbol{v}) : \boldsymbol{v} \in [x \cup \omega]^{<\omega}\}$.

2. Definability and uniqueness of indiscernible sequences

The covering lemma for one measure [2, 3] asserts that if $0^\dagger$ does not exist then any uncountable set x of ordinals is contained in a set y such that $|y| = |x|$ and either $y \in K$ (where $K = L[\mu]$ if it exists and K is the Dodd–Jensen core model otherwise) or else $y \in L[\mu, C]$ where C is a Prikry sequence for the measure μ. Furthermore, the Prikry sequence C is unique up to initial segments: any other Prikry sequence over $L[\mu]$ is contained in C except for a finite set. If there are sequences of measures in K then it is still true that each individual measure has a unique maximal Prikry sequence, but there need not be a uniform system of indiscernibles for the whole sequence of measures [14]. It is true that any small set of measures has a system of indiscernibles, but the particular system of indiscernibles to be used to cover a given set x depends on the set x. A modified version of the uniqueness of the sequence C does extend to sequences of measures, however. It is shown in [15–17], that, roughly speaking, if we specify a small set of measures for which we want indiscernibles, then the system of indiscernibles for that set of measures is unique up to an initial segment. In this section we will generalize these results to models containing nonoverlapping extenders.

We have already specified what it means for a to be an indiscernible for α in a particular precovering set X. In this section we will be interested in sequences of indiscernibles. Like the individual indiscernibles these sequences will be defined for a particular precovering set X, but unlike the case for individual indiscernibles we will show that under fairly general hypotheses the sequences of indiscernibles are independent of the choice of X.

In the last section we fixed δ to be the cofinality of κ, and each precovering set X was assumed to be closed under sequences of length δ. Unless otherwise specified, we will use boldface letters to designate sequences of length δ, so that for example we write $\boldsymbol{a} = (a_\iota : \iota < \delta)$.

The ordering on sequences is by eventual dominance. We will indicate this by a subscript b on the ordering relation: $\boldsymbol{\beta}' \geqslant_b \boldsymbol{\beta}$, and $\boldsymbol{\beta}' =_b \boldsymbol{\beta}$ mean, respectively, that for every sufficiently large $\iota < \delta$ we have $\beta'_\iota \geqslant \beta_\iota$, or that for every sufficiently large $\iota < \delta$ we have $\beta'_\iota = \beta_\iota$. The relation $\boldsymbol{\beta}' \not\geqslant_b \boldsymbol{\beta}$ means that it is not true that $\boldsymbol{\beta}' \geqslant_b \boldsymbol{\beta}$, that is, that $\beta'_\iota < \beta_\iota$ for unboundedly many $\iota < \delta$.

Definition 2.1. (1) The sequence $\boldsymbol{a}$ is a *principal indiscernible sequence* for $\boldsymbol{\alpha}$ in X if $\boldsymbol{a}$ and $\boldsymbol{\alpha}$ are nondecreasing sequences of length δ such that $\sup \boldsymbol{a} = \sup \boldsymbol{\alpha}$, and a_ι is a principal indiscernible for α_ι in X for every sufficiently large $\iota < \delta$.

(2) The sequence $\boldsymbol{a}$ is a *basic indiscernible sequence for* $\boldsymbol{\alpha}$ *in* X if $\boldsymbol{a}$ is a principal indiscernible sequence for $\boldsymbol{\alpha}$ and $\alpha_\iota = i_{a_\iota,\alpha}$ (or, equivalently, $\alpha_\iota = \inf(i_{a_\iota,\kappa}, \alpha)$) for all sufficiently large $\iota < \delta$, where $\alpha = \sup \boldsymbol{\alpha}$.

(3) The sequence $\boldsymbol{b}$ is an *indiscernible sequence for* $\boldsymbol{\beta}$ *belonging to* $(\boldsymbol{a}, \boldsymbol{\alpha})$ in X if b_ι is an indiscernible for β_ι belonging to (a_ι, α_ι) in X for every sufficiently large $\iota < \delta$.

We will say that a sequence $\boldsymbol{b}$ is an indiscernible sequence for $\boldsymbol{\beta}$ (without the qualifier "in X") if $\boldsymbol{b}$ is an indiscernible sequence for $\boldsymbol{\beta}$ in every precovering set X such that $\boldsymbol{b} \subset X$. We similarly drop the qualifier "in X" from the definitions of a principal indiscernible sequence and of a basic indiscernible sequence if the statement of definition is satisfied for every precovering set X. Most of the rest of this section will be concerned with proving (using, in the case $\delta = \omega$, an additional assumption on the size of the members of the sequences $\boldsymbol{b}$ and $\boldsymbol{\beta}$) that we always can drop the qualifier "in X": $\boldsymbol{a}$ is a basic indiscernible sequence for $\boldsymbol{\alpha}$, or $\boldsymbol{b}$ is a indiscernible sequence of $\boldsymbol{\beta}$ belonging to $(\boldsymbol{a}, \boldsymbol{\alpha})$, in a particular precovering set X if and only if the same thing is true in any precovering set Y containing the relevant sequences. For each property P of interest we will find a first order formula ϕ such that if a is any member of a precovering set X then $P(a, X)$ holds if and only if $X \models \phi(a)$. It follows that P is independent of the choice of the precovering set, since if Y is any other precovering set then X and Y are elementary substructures of H_τ and hence satisfy the same formulas.

In order to avoid superscripts we will continue to work with a fixed precovering set X, but the formulas we obtain will not depend on X.

Lemma 2.2. *There is a formula* $\phi(\boldsymbol{a}, \boldsymbol{\alpha})$ *which holds in* X *if and only if* $\boldsymbol{a}$ *is a principal indiscernible sequence for* $\boldsymbol{\alpha}$ *in* X.

Proof. Let $\phi'(\boldsymbol{a}, \boldsymbol{\alpha})$ be the conjunction of the formulas:

$$\exists h \in K \exists \iota_0 \forall \iota > \iota_0 \quad \alpha_\iota \in h``a_\iota$$

$$\forall h \in K \exists \iota_0 \forall \iota > \iota_0 \quad h``a_\iota \cap \alpha_\iota \subset a_\iota$$

By Lemma 1.11 the formula $\phi'(\boldsymbol{a}, \boldsymbol{\alpha})$ holds if and only if $\boldsymbol{a}$ is a basic indiscernible sequence for $\boldsymbol{\alpha}$. Let $\alpha = \sup \boldsymbol{\alpha} = \sup \boldsymbol{a}$. If $\boldsymbol{a}$ is a principal indiscernible sequence for $\boldsymbol{\alpha}$ but not a basic indiscernible sequence for $\boldsymbol{\alpha}$ then both $\boldsymbol{a}$ and $\boldsymbol{\alpha}$ are basic indiscernible sequences for the sequence $\boldsymbol{\alpha}'$ defined by $\alpha'_\iota = i_{a_\iota,\alpha}(a_\iota) = i_{\alpha_\iota,\alpha}(\alpha_\iota)$. Thus the following formula $\phi(\boldsymbol{a}, \boldsymbol{\alpha})$ will satisfy the requirements of the lemma:

$$\exists \boldsymbol{\alpha}' \left(\phi'(\boldsymbol{a}, \boldsymbol{\alpha}') \text{ and if } I = \{\iota : \alpha_\iota \neq \alpha'_\iota\} \text{ is unbounded in } \delta \text{ then } \phi'(\boldsymbol{\alpha} \restriction I, \boldsymbol{\alpha}' \restriction I) \right). \quad \square$$

Lemma 2.3 (main lemma). *There is a formula $\phi(\boldsymbol{a},\boldsymbol{\alpha},\boldsymbol{b},\boldsymbol{\beta})$ which holds if and only if*

1. *$\boldsymbol{a}$ is a principal indiscernible sequence for $\boldsymbol{\alpha}$;*
2. *$\boldsymbol{b}$ is an indiscernible sequence for $\boldsymbol{\beta}$ belonging to $(\boldsymbol{a},\boldsymbol{\alpha})$;*
3. *If $\delta = \omega$ then there is an integer n such that $\boldsymbol{\beta} <_b (\alpha_i^{+n} : i \in \omega)$ and $\boldsymbol{b} <_b (a_i^{+n} : i \in \omega)$.*

Before proving this, we look briefly at the problem of determining, given $\boldsymbol{\beta}$ and $(\boldsymbol{a},\boldsymbol{\alpha})$, whether there exists an indiscernible sequence $\boldsymbol{b}$ for $\boldsymbol{\beta}$ belonging to $(\boldsymbol{a},\boldsymbol{\alpha})$. The harder problem of determining whether a particular sequence $\boldsymbol{b}$ is the indiscernible sequence will be deferred until this problem has been settled.

The easier problem breaks down into two problems. The first, deciding whether $\boldsymbol{\beta}$ has an indiscernible sequence belonging to $(\boldsymbol{a},\boldsymbol{\alpha})$ at least in the weak sense that $\beta_\iota = i_{a_\iota,\alpha_\iota}(b_i)$, is answered rather easily by the next lemma, assuming that $\boldsymbol{\beta}$ is not too large. The second question is to determine whether the sequences satisfy the boundedness conditions of Definition 1.12, that is, whether

$$\beta_\iota < \inf(C \setminus (\alpha_\iota + 1)) \tag{1}$$

$$b_\iota < \max(\sup(\mathrm{O}(a_\iota))^+, a_\iota^{++}) \tag{2}$$

hold for all sufficiently large $\iota < \delta$. The bound (2) for b_ι is quite straightforward, but the bound (1) for β_ι is not possible to determine directly in K. Most of the work involved in proving Lemma 2.3 will come in the proof of Lemma 2.5 below, which uses the additional assumptions that $\boldsymbol{\beta}$ satisfies (1) and that $\boldsymbol{a}$ is a basic indiscernible sequence. Lemma 2.5 implies Corollary 2.6, which implies among other things that for basic indiscernible sequences the bound (2) implies the bound (1). This proves the main lemma for the case of basic indiscernible sequences, and the general case follows easily from this special case.

Lemma 2.4. *There is a formula ϕ such that if $\alpha_\iota \leqslant \beta_\iota < \inf(C\setminus(\alpha_\iota+1))$ for all $\iota < \delta$ then $\phi(\boldsymbol{a},\boldsymbol{\alpha},\boldsymbol{\beta})$ holds in X if and only if $\boldsymbol{a}$ is a basic indiscernible sequence for $\boldsymbol{\alpha}$ and there is a sequence $\boldsymbol{b}$ such that $\beta_\iota = i_{a_\iota,\alpha_\iota}(b_\iota)$ for all sufficiently large $\iota < \delta$.*

Proof. First note that if $\beta_\iota = i_{a_\iota,\alpha_\iota}(b_\iota)$ then since $\boldsymbol{a}$ is basic there is a function $f \in X \cap K$ such that $\beta_\iota \in f``\alpha_\iota$ for all sufficiently large $\iota < \delta$. In the case $\alpha_\iota = i_{a_\iota,\kappa}(\alpha_\iota) < \alpha = \sup \boldsymbol{\alpha}$ this is true by elementarity, since $h_\kappa^X \cap (\kappa \times \kappa)$ is a member of K satisfying the stated property. In the case $\alpha_\iota = \alpha$ we have $\beta_\iota \in h_\alpha^X``a_\iota$ for all sufficiently large $\iota < \delta$, and by Lemma 1.11 there is a function $f \in K \cap X$ such that $f(\xi) = h_\alpha^X(\xi)$ for all $\xi \in h_\alpha^{-1}``\boldsymbol{\beta}$. On the other hand, the existence of such a function f implies that $\boldsymbol{\beta}$ has an indiscernible sequence $\boldsymbol{b}$: for sufficiently large $\iota \in \delta$ we have $f \in \mathrm{range}(i_{a_\iota,\alpha_\iota})$, and we set $b_\iota = i_{a_\iota,\alpha_\iota}^{-1}(f)(f^{-1}(\beta_\iota))$. Thus if we let ϕ be the conjunction of the formula

$$\exists f \in K \exists \iota_0 < \delta \forall \iota (\iota_0 < \iota < \delta \Rightarrow \beta_\iota \in f``a_\iota)$$

with the formula asserting that $\boldsymbol{a}$ is a basic indiscernible sequence for $\boldsymbol{\alpha}$ then ϕ satisfies the requirements of the lemma. □

Lemma 2.5. *There is a formula ϕ such that if $\beta_\iota < \inf(C\setminus(\alpha_\iota+1))$ for all sufficiently large $\iota < \delta$ then $\phi(\boldsymbol{a},\boldsymbol{\alpha},\boldsymbol{b},\boldsymbol{\beta})$ holds if and only if clauses* (1)–(3) *of Lemma* 2.3 *are satisfied.*

Corollary 2.6. *If $\boldsymbol{a}$ is an indiscernible sequence for $\boldsymbol{\alpha}$ and $\alpha_\iota = \sup \boldsymbol{a}$ for all sufficiently large $\iota < \delta$ then F_{a_ι,α_ι} is in X for all sufficiently large $\iota < \delta$.*

Before starting on the proof of Lemma 2.5 and Corollary 2.6 we will verify that together they imply the main lemma, Lemma 2.3.

Proof of main lemma (*from Lemma* 2.5 *and Corollary* 2.6). Suppose first that $\boldsymbol{a}$ is a basic indiscernible sequence for $\boldsymbol{\alpha}$. Since F_{a_ι,α_ι} is always in X when $\alpha_\iota = i_{a_\iota,\alpha}(a_\iota) < \alpha = \sup\boldsymbol{\alpha}$, the corollary implies that F_{a_ι,α_ι} is in X for all sufficiently large $\iota < \delta$. Since $\operatorname{len}(F_{a_\iota,\alpha_\iota}) < \operatorname{index}(F_{a_\iota,\alpha_\iota}) \in \mathrm{O}(\alpha_\iota)$ it follows that if $\boldsymbol{b}$ is an indiscernible sequence for $\boldsymbol{\beta}$ belonging to $(\boldsymbol{a},\boldsymbol{\alpha})$ then $\beta_\iota < \sup(\mathrm{O}(\alpha_\iota))$ for almost all $\iota < \delta$. Thus the formula of Lemma 2.5 satisfies the requirements of the main lemma whenever $\boldsymbol{a}$ is a basic indiscernible sequence for $\boldsymbol{\alpha}$.

Now we can treat the general case as in the proof of Lemma 2.2. Let $\alpha'_\iota = i_{\alpha_\iota,\alpha}(\alpha_\iota)$ and $\beta'_\iota = i_{\alpha_\iota,\alpha}(\beta_\iota)$. Then $\boldsymbol{a}$ is a principal indiscernible sequence for $\boldsymbol{\alpha}'$ and $\boldsymbol{b}$ is an indiscernible sequence for $\boldsymbol{\beta}'$ belonging to $(\boldsymbol{a},\boldsymbol{\alpha}')$, and if $I \stackrel{\text{def}}{=} \{\iota < \delta : \alpha_\iota \neq \alpha'_\iota\}$ is unbounded in $\boldsymbol{\delta}$ then the restriction $\alpha \restriction I$ of α is a principal indiscernible sequence for $\boldsymbol{\alpha}' \restriction I$, and the restriction $\boldsymbol{\beta}' \restriction I$ of $\boldsymbol{\beta}'$ is an indiscernible sequence for $\boldsymbol{\beta} \restriction I$ belonging to $(\boldsymbol{\alpha} \restriction I, \boldsymbol{\alpha}' \restriction I)$. This completes the proof of Lemma 2.3, assuming Lemma 2.5 and Corollary 2.6. □

The proof of Lemma 2.5 will be broken into two cases, the first for $\delta > \omega$ and the second for $\delta = \omega$.

Proof of Lemma 2.5 for $\delta > \omega$. In this case we use the game introduced by Gitik in [6] to obtain a rather straightforward extension of the results which were proved for measure sequences in [15–17]. The major difference is that the restriction to $\delta > \omega$ means that whereas the results given in those papers for sequences of measures have finite sets of exceptional points, the results given in this paper for sequences of extenders may have a countable set of exceptional points.

Our presentation of Gitik's game will differ somewhat from that of [6]. We will define a game $\mathscr{G}(\boldsymbol{b},\boldsymbol{\beta})$ between two players, who, following Mathias, we call Adam and Eve. The first player, Adam, will be trying to show that the ordinals in $\boldsymbol{b}$ are too small for $\boldsymbol{b}$ to be an indiscernible sequence belonging to $\boldsymbol{\beta}$. He will do so by proposing sets of ordinals $B_{n,\iota} \subset \beta_\iota$. Eve will be required to defend the proposition that $\boldsymbol{b}$ is a principal indiscernible sequence for $\boldsymbol{\beta}$ by choosing indiscernibles for ordinals in the sets $(B_{n,\iota} : \iota < \delta)$ which are consistent with $\boldsymbol{b}$ and with her earlier choices.

In the next two propositions we will show that if $\boldsymbol{b}$ is an indiscernible sequence for $\boldsymbol{\beta}$ then Eve has a winning strategy for the game $\mathscr{G}(\boldsymbol{b},\boldsymbol{\beta})$, while Adam has a winning strategy for the game $\mathscr{G}(\boldsymbol{b}',\boldsymbol{\beta})$ whenever $\boldsymbol{b}' \not\geqslant_b \boldsymbol{b}$. With these propositions we can complete the proof of Lemma 2.5, since the principal indiscernible sequence $\boldsymbol{b}$ for $\boldsymbol{\beta}$ is definable by a formula ϕ asserting that β is the least sequence $\boldsymbol{b}'$ such that Eve has a winning strategy for $\mathscr{G}(\boldsymbol{b}',\boldsymbol{\beta})$. The definition of the game $\mathscr{G}(\boldsymbol{b},\boldsymbol{\beta})$, and hence the formula ϕ, will not depend in any way on the particular precovering set X.

Definition 2.7. If $\boldsymbol{a}$ is a basic indiscernible sequence for $\boldsymbol{\alpha}$ then the game $\mathscr{G}(\boldsymbol{\beta},\boldsymbol{b})$ is defined as follows:

The first player, Adam, plays on his nth move a sequence $(B_{n,\iota} : \iota < \delta)$ such that $B_{n,\iota} \in [\beta_\iota \backslash \alpha_\iota]^{\leqslant\delta}$ for each $\iota < \delta$. The second player, Eve, responds with a sequence $(\tau_{n,\iota} : \iota < \delta)$ such that

1. For each $\iota < \delta$ the function $\tau_{n,\iota}$ is an order preserving function mapping a subset of $\beta_\iota \backslash \alpha_\iota$ into $b_\iota \backslash a_\iota$.
2. If h is any function in K then $B_{n,\iota} \cap h``a_\iota \subset \operatorname{domain}\tau_{n,\iota}$ for all but boundedly many $\iota < \delta$.
3. $\tau_{n,\iota} \supset \tau_{n-1,\iota}$ for all $n > 0$.

Adam wins if Eve is ever unable to play; otherwise Eve wins.

The idea is that if $\beta \in B_{n,\iota}$ then $\tau_{n,\iota}(\beta)$ should be an indiscernible for (a_i, α_i) belonging to β_i. For convenience, we will write j_ι for i_{a_ι,α_ι}, and if $\boldsymbol{\xi}$ is a sequence then we will write $j(\boldsymbol{\xi})$ for $(j_\iota(\xi_\iota) : \iota < \delta)$.

Proposition 2.8. *If $j(\boldsymbol{b}) \geqslant_b \boldsymbol{\beta}$ then Eve has a winning strategy for the game $\mathscr{G}(\boldsymbol{b},\boldsymbol{\beta})$.*

Proof. Suppose that the proposition is false. Since the game $\mathscr{G}(\boldsymbol{b},\boldsymbol{\beta})$ is closed it is determined, and hence Adam must have a winning strategy. By the elementarity of X there is a winning strategy $\sigma \in X$ for Adam. Now suppose Eve plays, in V, against the strategy σ by playing $\tau_{n,\iota} = j_\iota^{-1} \restriction (B_{n,\iota} \cap j``b_\iota)$. It is easy to see that these plays by Eve satisfy the first and last clauses of Definition 2.7, and the second clause follows from clause (4) of Lemma 1.11. Thus Adam loses this game, contradicting the assumption that σ is a winning strategy for Adam. □

Proposition 2.9. *If $\operatorname{cf}(\delta) > \omega$ and $j(\boldsymbol{b}) \not\geqslant_b \boldsymbol{\beta}$ then Adam has a winning strategy for the game $\mathscr{G}(\boldsymbol{b},\boldsymbol{\beta})$.*

Proof. As before it will be sufficient to show that if σ is any strategy for Eve which is a member of X then Adam can refute the strategy σ by playing in V. Adam will let $B_{n,\iota} = \emptyset$ whenever $j_\iota(b_\iota) \geqslant \beta_\iota$, so we can assume that $j_\iota(b_\iota) < \beta_\iota$ for all $\iota < \delta$. The first move in Adam's refutation will be the singleton sets $B_{1,\iota} \stackrel{\text{def}}{=} \{j_\iota(b_\iota)\}$. His nth move, for $n > 1$, will be the sets $B_{n,\iota} \stackrel{\text{def}}{=} (\beta_\iota \setminus \alpha_\iota) \cap j_\iota``(\operatorname{range}(\tau_{n-1,\iota}))$, where the functions $\tau_{n-1,\iota}$ are taken from Eve's previous move. We need to show that if Eve plays by the strategy σ then Adam wins this play of the game.

First we observe that all of the plays in this game are members of X. For Adam's moves it is sufficient to show that the sets $B_{n,\iota}$ are subsets of X, since his moves have cardinality δ and X is δ-closed. The sets $B_{1,\iota}$ are contained in X by construction, and for $n > 1$ the sets $B_{n,\iota}$ will be contained in X provided that Eve's $(n-1)$th move is in X. But Eve's strategy σ and the game $\mathscr{G}(\boldsymbol{b}, \boldsymbol{\beta})$ are both members of X, so Eve's moves will be in X because Adam's preceding moves were in X.

Now let $\alpha = \sup \boldsymbol{\alpha}$. We claim that $B_{n,\iota} \subset h_\alpha``a_\iota$ for each $\iota < \delta$, where $h_\alpha$ is the function defined in Definition 1.8(2). First, we have $B_{n,\iota} \subset h_{\alpha_\iota}``a_\iota$ for each $\iota < \delta$ since $a_0 > \rho^X$ and $B_{n,\iota} \subset ``a_\iota = i_{a_\iota,\alpha_\iota}``a_\iota$. If $\alpha_\iota = \alpha$ then the claim is established, and if $\alpha_\iota < \alpha$ then the assumption that $\boldsymbol{a}$ is a basic indiscernible sequence for $\boldsymbol{\alpha}$ implies that $i_{\alpha_\iota,\alpha}(\alpha_\iota) = \alpha_\iota$ and it follows by clause (1) of Lemma 1.11 that $h_\alpha \circ h_{\alpha_\iota}^{-1}$ is the identity. It follows by Lemma 1.11(5) that there is a function $h \in K$ such that $\bigcup_n B_{n,\iota} \subset h``a_\iota$ for all $\iota < \delta$.

By clause (2) of the definition of $\mathscr{G}(\boldsymbol{b}, \boldsymbol{\beta})$ it follows that for each $n \in \omega$ there is $\iota_n < \delta$ such that $B_{n,\iota} \subset \text{domain}(\tau_{n,\iota})$ for all $\iota > \iota_n$. Since $\delta = \text{cf}(\delta) > \omega$ and $j(\boldsymbol{b}) \not\geq_b \boldsymbol{\beta}$ there is an ordinal $\iota < \delta$ such that $\iota > \iota_n$ for all $n \in \omega$. We are now ready to reach the contradiction and hence complete the proof of Lemma 2.9. Define an infinite descending ω-sequence $\boldsymbol{\eta}$ of ordinals by setting $\eta_0 = j_\iota(b_\iota)$ and $\eta_n = j_\iota \circ \tau_{n-1,\iota}(\eta_{n-1})$ for $n > 0$. Since $\eta_n \in B_{n,\iota} \subset \text{domain}\tau_{n,\iota}$ the ordinal η_{n+1} is defined for all $n < \omega$. We have $\eta_0 < \beta_\iota$ by assumption, and an easy proof by induction, using clauses (2) and (3) of the definition of the game $\mathscr{G}(\boldsymbol{b}, \boldsymbol{\beta})$, shows that $\eta_{n+1} < \eta_n$ for all $n < \omega$. This contradiction completes the proof of the proposition. □

Now let $\phi(\boldsymbol{a}, \boldsymbol{\alpha}, \boldsymbol{b}, \boldsymbol{\beta})$ be the conjunction of three formulas, asserting

1. $\boldsymbol{\beta}$ has an indiscernible sequence belonging to $(\boldsymbol{a}, \boldsymbol{\alpha})$.
2. Eve wins the game $\mathscr{G}(\boldsymbol{b}, \boldsymbol{\beta})$.
3. Adam wins the game $\mathscr{G}(\boldsymbol{b}', \boldsymbol{\beta})$ for all $\boldsymbol{b}' \not\geq_b \boldsymbol{b}$.

Lemmas 2.8 and 2.9 imply that $\phi(\boldsymbol{a}, \boldsymbol{\alpha}, \boldsymbol{b}, \boldsymbol{\beta})$ is true whenever $\boldsymbol{b}$ is an indiscernible sequence for $\boldsymbol{\beta}$. On the other hand, if $\boldsymbol{b}'$ is any sequence such that $\phi(\boldsymbol{a}, \boldsymbol{\alpha}, \boldsymbol{b}', \boldsymbol{\beta})$ is true then $\boldsymbol{\beta}$ has an indiscernible sequence $\boldsymbol{b}$ by clause (1), so $\phi(\boldsymbol{a}, \boldsymbol{\alpha}, \boldsymbol{b}, \boldsymbol{\beta})$ is true as well. From clauses (2) and (3) it follows that $\boldsymbol{b} \leqslant_b \boldsymbol{b}'$ and $\boldsymbol{b}' \leqslant_b \boldsymbol{b}$, so $\boldsymbol{b}' =_b \boldsymbol{b}$. This completes the proof of the case $\delta > \omega$ of Lemma 2.5. □

Proof of Lemma 2.5 for $\delta = \omega$. When $\delta = \omega$ the situation becomes much more difficult, and in this case we only know how to reconstruct the embeddings under the assumption that there is no inner model of $\exists \alpha\ o(\alpha) = \alpha^{+\omega}$.

Define $\alpha_{n,k}$, for integers n and k, to be the smaller of α_k^{+n} and $i_{a_k,\alpha_k}(a_k^{+n})$, and let $a_{n,k}$ be $i_{a_k,\alpha_k}^{-1}(\alpha_{n,k})$, provided that it exists. For each $n \in \omega$ we claim that $a_{n,k}$ exists for all but finitely many $k \in \omega$. This is immediate if $\alpha_k = \alpha = \sup \boldsymbol{\alpha}$ for all sufficiently large $k < \omega$, since any member of $X \cap \in f(C \setminus (\alpha + 1))$ is in the range of $i_{\nu,\alpha}$ for all sufficiently large $\nu < \alpha$. Now suppose that $\alpha_k = i_{a_k,\alpha_k}(a_k) = i_{a_k,\alpha}(a_k)$, and note that if $a_{n,k}$ does not exist then $\alpha_k^{+n} < i_{a_k,\alpha_k}(a_k^{+n})$. Now $\pi^{-1}(\alpha_k^{+n})$ is the nth successor of $\pi^{-1}(\alpha_n)$ in $\overline{\mathrm{m}}_{\bar{\alpha}_k}$, since $\overline{E}_{\bar{\alpha}_k}$ does not exist, and $\pi^{-1}(i_{a_k,\alpha_k}(a_k^{+n}))$ is the nth successor of

$\pi^{-1}(\alpha_k^{+n})$ in $\mathrm{ult}(\overline{\mathrm{m}}_{\bar{\alpha}_k}, \bar{F}_{\bar{a}_k,\bar{\alpha}_k})$. Since $\mathrm{ult}(\overline{\mathrm{m}}_{\bar{\alpha}_k}, \bar{F}_{\bar{a}_k,\bar{\alpha}_k})$ is smaller than $\overline{\mathrm{m}}_{\bar{\alpha}_k}$ it follows that $\pi^{-1}(i_{a_k,\alpha_k}(a_k^{+n})) \leqslant \pi^{-1}(\alpha_n)$ so that $i_{a_k,\alpha_k}(a_k^{+n}) \leqslant \alpha_k^{+n}$, and hence $a_{n,k}$ does exist.

Note that if $n \in \omega$ then the sequence $\boldsymbol{a}_n \overset{\text{def}}{=} (a_{n,k} : k \in \omega)$ is an indiscernible sequence for $\boldsymbol{\alpha}_n \overset{\text{def}}{=} (\alpha_{n,k} : k \in \omega)$ belonging to $(\boldsymbol{a}, \boldsymbol{\alpha})$.

We prove Lemma 2.5 by induction on n, with the induction step relying on the following lemma. Since $\alpha_{1,k}$ is always equal to α^+ the case $n = 1$ could be handled by standard methods, but for convenience we treat it as part of the general induction.

Lemma 2.10. *There is a first order formula ψ with the following property: Assume that X is a precovering set, n is an integer and the sequence $\boldsymbol{a}_n$ and $\boldsymbol{\alpha}_n$ are as defined above. Let $T \in X$ be a set such that if $n = 0$ then $T = \{(\boldsymbol{\delta}, \boldsymbol{\delta}) : \boldsymbol{\delta} < \boldsymbol{\alpha}\}$, and if $n > 0$ then*

$$T \cap X = \{(\boldsymbol{d}, \boldsymbol{\delta}) : \boldsymbol{\delta} <_b \boldsymbol{\alpha}_n \text{ and } \boldsymbol{d} \text{ is an indiscernible sequence for } \boldsymbol{\delta} \text{ in } X\}.$$

Then for all sequences $\boldsymbol{\beta}$ and $\boldsymbol{b}$ in X the formula $\psi(T, \boldsymbol{a}_n, \boldsymbol{\alpha}_n, \boldsymbol{b}, \boldsymbol{\beta})$ is true if and only if $\boldsymbol{\beta} <_b \boldsymbol{\alpha}_{n+1}$ and $\boldsymbol{b}$ is an indiscernible sequence for $\boldsymbol{\beta}$.

Proof. If α and β are any two ordinals such that $|\beta^K| \leqslant \alpha < \beta$ then let $f_{\alpha,\beta}$ be the least map f in the natural ordering of K such that $f : \alpha \cong \beta$. We will define a function $S(\gamma, \beta, \xi)$ by recursion on β. The domain of S is the set of triples of ordinals ξ, γ and β such that $\xi < \gamma < \beta$ and $\gamma \geqslant |\beta^K|$, and S is defined recursively as follows:

$$S(\gamma, \beta, \xi) = \begin{cases} 0 & \text{if } f_{\gamma,\beta}(\xi) < \gamma, \\ S(\gamma, f_{\gamma,\beta}(\xi), \xi) + 1 & \text{if } \gamma \leqslant f_{\gamma,\beta}(\xi) < \beta. \end{cases}$$

Now let $\psi(T, \boldsymbol{a}_n, \boldsymbol{\alpha}_n, \boldsymbol{b}, \boldsymbol{\beta})$ be the conjunction of the following three formulas:

$$\exists k_0 \forall k > k_0 \quad (|b_k| < a_{n,k} \wedge |\beta_k| < \alpha_{n,k}) \tag{i}$$

$$\exists g \in K \exists k_0 \forall k > k_0, \quad \beta_k \in g\text{``}a_k \tag{ii}$$

$$\forall (\boldsymbol{d}, \boldsymbol{\delta}) \in T \, \exists k_0 \forall k > k_0, \quad S(\alpha_{n,k}, \beta_k, \delta_k) = S(a_{n,k}, b_k, d_k). \tag{iii}$$

It is clear that $\psi(T, \boldsymbol{a}_n, \boldsymbol{\alpha}_n, \boldsymbol{b}, \boldsymbol{\beta})$ holds in X whenever $\boldsymbol{b}$ is an indiscernible sequence for $\boldsymbol{\beta}$ in X. Now suppose that

$$\psi(T, \boldsymbol{a}_n, \boldsymbol{\alpha}_n, \boldsymbol{b}', \boldsymbol{\beta}) \tag{1}$$

is true in X for some sequence $\boldsymbol{b}' \neq_b \boldsymbol{b}$. If we set $\boldsymbol{\beta}' = j(\boldsymbol{b}')$ then $\boldsymbol{b}'$ is an indiscernible sequence for $\boldsymbol{\beta}'$ belonging to $(\boldsymbol{a}, \boldsymbol{\alpha})$ and hence

$$\psi(T, \boldsymbol{a}_n, \boldsymbol{\alpha}_n, \boldsymbol{b}', \boldsymbol{\beta}') \tag{2}$$

is also true in X. We will show that (1) and (2) lead to a contradiction. We can assume without loss of generality (*wlog*) that $\beta'_k < \beta_k$ for unboundedly many $k < \omega$. For each such k set, $\delta_k = f^{-1}_{\alpha_{n,k},\beta_n}(\beta'_k)$, so that $\delta_k < \alpha_{n,k}$. The sequence $\boldsymbol{\delta}$ has an indiscernible sequence $\boldsymbol{d}$ belonging to $(\boldsymbol{a}, \boldsymbol{\alpha})$, since δ_k is defined in K from the parameters $\alpha_{n,k}$, β_k

and β'_k and each of the sequences $\boldsymbol{\alpha}_n$, $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ has an indiscernible sequence in X. Then $S(\alpha_{n,k}, \beta_i, \delta_k) \neq 0$ since $f_{\alpha_{n,k}}(\delta_k) = \beta'_k > \alpha_{n,k}$, so for sufficiently large k such that $\beta'_k < \beta_k$

$$\begin{aligned} S(a_{n,k}, b'_{n,k}, d_k) &= S(\alpha_{n,k}, \beta_k, \delta_k) && \text{by (1)} \\ &= S(\alpha_{n,k}, \beta'_k, \delta_k) + 1 && \text{by the choice of S and } \delta_\kappa \\ &= S(a_{n,k}, b'_k, d_k) + 1 && \text{by (2).} \end{aligned}$$

This contradiction completes the proof of Lemma 2.10. □

Lemma 2.5 will follow easily from Lemma 2.10 once we verify that it is possible to define the sequence $\boldsymbol{\alpha}_{n+1}$ and its indiscernible sequence $\boldsymbol{a}_{n+1}$. This is straightforward: $\boldsymbol{\alpha}_{n+1}$ is the minimal sequence which has an indiscernible sequence but does not have an indiscernible sequence satisfying ψ, that is, $\boldsymbol{\alpha}_{n+1}$ is the only sequence $\boldsymbol{\alpha}' = (\alpha'_k : k < \omega)$, up to bounded segments, which satisfies the conjunction of the following three formulas:

$$\exists g \exists k_0 \forall k > k_0 \ \alpha'_k \in g``(a_k)$$
$$\neg \exists \boldsymbol{a}' \ \psi(T, \boldsymbol{a}_n, \boldsymbol{\alpha}_n, \boldsymbol{a}', \boldsymbol{\alpha}')$$
$$\begin{aligned} \forall \boldsymbol{\beta} \forall g \in K \ (&\text{if } I = \{k : \beta_k < \alpha'_k \wedge \beta_k \in g``a_k\} \text{ is infinite} \\ &\text{then } \exists \boldsymbol{b} \ \psi(T, \boldsymbol{a}_n \restriction I, \boldsymbol{\alpha}_n \restriction I, \boldsymbol{b} \restriction I, \boldsymbol{\beta} \restriction I). \end{aligned}$$

Similarly, $\boldsymbol{a}_n$ is the minimal sequence $\boldsymbol{a}'$ which is not an indiscernible sequence for any sequence $\boldsymbol{\alpha}'$ satisfying the formula ψ and is hence definable up to an initial segment.

This completes the proof of Lemma 2.5. □

Proof of Corollary 2.6. The hypothesis of Corollary 2.6 asserts that $\boldsymbol{a}$ is an indiscernible sequence for $\boldsymbol{\alpha}$ in X such that $\alpha_\iota = \alpha = \sup \boldsymbol{\alpha}$ for sufficiently large $\iota < \delta$, and the conclusion asserts that $F_{a_\iota,\alpha} \in X$ for sufficiently large $\iota < \delta$. If the hypothesis is true and the conclusion is false then we can assume wlog that $F_{a_\iota,\alpha} \notin X$ for all $\iota < \delta$. This means that $\bar{F}_{\bar{a}_\iota,\bar{\alpha}} \notin \bar{K}$, so that either $\alpha = \kappa$ or $i_{\alpha,\kappa}(\alpha) > \alpha$, and in either case $\overline{E}_{\bar{\alpha}} \trianglelefteq \bar{F}_{\bar{a},\bar{\alpha}}$.

We will define, in X, a set G such that $\pi``\overline{E}_{\bar{\alpha}} = G \cap X$. It will follow that $\overline{E}_{\bar{\alpha}} = \pi^{-1}(G) \in N$, so $\overline{E}_{\bar{\alpha}} \in \bar{K}$, contradicting the choice of $\overline{E}_{\bar{\alpha}}$ as the least extender in $\overline{m}_{\bar{\alpha}}$ which is not in $\bar{K}$.

In order to define G we need to decide inside X whether a pair (ε, z) is in $\pi``\overline{E}_{\bar{\alpha}}$. Now notice that if we set $\gamma = \pi(\text{index}(\overline{E}_{\bar{\alpha}}))$ then Lemma 2.5 implies that for each ordinal β with $\alpha \leqslant \beta \leqslant \gamma$ there is an indiscernible sequence $\boldsymbol{b}$ for the constant sequence β belonging to $(\boldsymbol{a}, \alpha)$, and there is a formula ϕ picking out these pairs $(\boldsymbol{b}, \beta)$. In order to use the lemma we have to check that β is less than the least member of C above α, but this is immediate: Since $\overline{E}_{\bar{\alpha}}$ is the extender used on $\overline{m}_{\bar{\alpha}}$ and the models do not have overlapping extenders, the critical points of extenders used later will be greater than the index $\pi^{-1}(\gamma)$ of $\overline{E}_{\bar{\alpha}}$.

Thus we can choose indiscernible sequences $\boldsymbol{e}$ and $\boldsymbol{c}$ belonging to $(\boldsymbol{a},\alpha)$ for the constant sequences ε and γ, respectively. For sufficiently large $\iota \in D$ we will have $i_{a_\iota,\alpha}(e_\iota, c_\iota) = (\varepsilon,\gamma)$ and $z \in \text{range}(i_{e_\iota,\alpha})$, and for all such ordinals ι we will have $(\varepsilon,z) \in \pi``\overline{E}_{\bar\alpha}$ if and only if

$$\begin{array}{ll} e_\iota \in z & \text{if } \overline{E}_{\bar\alpha} = \bar{F}_{\bar a_\iota,\bar\alpha}, \\ (e_\iota, z \cap a_\iota) \in \mathscr{E}_{c_\iota} & \text{if } \overline{E}_{\bar\alpha} \lhd \bar{F}_{\bar a_\iota,\bar\alpha}. \end{array}$$

Since the indiscernible sequences can be defined inside X and ${}^{\delta}X \subset X$, this definition of G can be carried out in X. This completes the proof of Corollary 2.6 and of the main lemma. □

For the rest of the section we will assume that if $\delta = \omega$ then $\{\nu < \kappa : \mathrm{o}(\alpha) \geqslant \alpha^{+n}\}$ is bounded in κ for some $n < \omega$. The next task is to extend our notion of indiscernible sequence to sequences of indiscernibles for particular extenders.

Definition 2.11. (1) The ordinal a is a *principal indiscernible in X for the extender F* on α if a is a principal indiscernible for α in X and $F = F^X_{a,\alpha}$.

(2) The sequence $\boldsymbol{a}$ is a *principal indiscernible sequence in X for the sequence $\boldsymbol{F}$* of extenders if a_ι is a principal indiscernible in X for F_ι for every sufficiently large $\iota < \delta$.

(3) The sequence $\boldsymbol{a}$ is a *principal indiscernible sequence for the sequence $\boldsymbol{F}$ of extenders* if for every precovering set X containing $\boldsymbol{a}$, the sequence $\boldsymbol{a}$ is a principal indiscernible sequence in X for $\boldsymbol{F}$.

Lemma 2.12. *There is a formula ψ such that $\psi(\boldsymbol{a},\boldsymbol{\alpha},\boldsymbol{F})$ is true in a precovering set X if and only if $\boldsymbol{a}$ is a principal indiscernible sequence in X for the sequence $\boldsymbol{F}$ of extenders on $\boldsymbol{\alpha}$. Thus if $\boldsymbol{a}$ is a principal indiscernible sequence for $\boldsymbol{F}$ in some precovering set X then it is a principal indiscernible sequence for $\boldsymbol{F}$.*

Proof. By Lemma 2.3 there is a first order formula $\psi(\boldsymbol{a},\boldsymbol{\alpha},\boldsymbol{F})$ over X asserting that the following statements are true. We write γ_ι for $\text{index}(F_\iota)$ and $\boldsymbol{\gamma}$ for $(\gamma_\iota : \iota < \delta)$.

1. $\boldsymbol{\alpha} = (\text{crit}(F_\iota) : \iota < \delta)$, and $\boldsymbol{a}$ is a principal indiscernible sequence for $\boldsymbol{\alpha}$.

2. There is an indiscernible sequence $\boldsymbol{c}$ for γ belonging to $(\boldsymbol{a},\boldsymbol{\alpha})$, and $c_\iota \notin \mathrm{O}(a_\iota)$ for sufficiently large $\iota < \delta$.

3. If $\boldsymbol{c}'$ and γ' are any other sequences such that $\boldsymbol{c}'$ is an indiscernible sequence for γ', then, with at most boundedly many exceptions, $c_\iota \in \mathrm{O}(a_\iota)$ for all ι such that $\gamma'_\iota \ntriangleleft \gamma_\iota$.

4. If f is any function in K, $\boldsymbol{\varepsilon} <_b \gamma$, and $\boldsymbol{e}$ is an indiscernible sequence for $\boldsymbol{\varepsilon}$ belonging to $(\boldsymbol{a},\boldsymbol{\alpha})$ then there is an ordinal $\iota_0 < \delta$ such that for all $\iota_0 < \iota < \delta$ and all $z \in f``a_\iota$ we have $z \in (F_\iota)_{\varepsilon_\iota}$ if and only if $e_\iota \in z$.

If $\boldsymbol{a}$ is an indiscernible sequence for $\boldsymbol{F}$ in X then $\psi(\boldsymbol{a},\boldsymbol{\alpha},\boldsymbol{F})$ will be true in X, and hence in V. Now we will show that if $\boldsymbol{F}'$ is any sequence of extenders in K such that $\psi(\boldsymbol{a},\boldsymbol{\alpha},\boldsymbol{F}')$ then $\boldsymbol{F}' =_b \boldsymbol{F}$. By clauses (2) and (3) it is enough to show that, with at most boundedly many exceptions, one of F'_ι and F_ι is an initial segment of the other.

If neither of F_ι and F'_ι is an initial segment of the other then let $(z_\iota, \varepsilon_\iota)$ be the least pair such that $z_\iota \in (F_\iota)_{\varepsilon_\iota} \iff z_\iota \notin (F'_\iota)_{\varepsilon_\iota}$. Since there are indiscernible sequences for γ and γ', there is an indiscernible sequence $\boldsymbol{e}$ for $\boldsymbol{\varepsilon}$, but then clause (4) cannot be true for both $\boldsymbol{F}$ and $\boldsymbol{F}'$. □

We are now able to define the version of the "next indiscernible" function which is appropriate to the sequences which we are considering. We will define three separate functions: The function s^X gives the next principal indiscernible, a^X_ξ gives the ξth-next accumulation point, and β^X gives the indiscernible for an ordinal b belonging to a pair (a, α). Definition 2.13 below has the formal definitions for these functions, together with another function ℓ^X which is a useful variant of s^X. The definition is relative to a particular precovering set X, but we will finish up this section by showing that any two precovering sets X and X' agree on the values of these functions for all sufficiently large ordinals $v \in X \cap X'$ below κ.

Recall that we use $\gamma \lhd \gamma'$ to mean that either $\mathscr{E}_\gamma \lhd \mathscr{E}_{\gamma'}$ or $\gamma \in \mathrm{O}(\alpha)$ and $\gamma' = \sup(\mathrm{O}'(\alpha))$.

Definition 2.13. If X is a precovering set then

1. $s^X(\gamma, v)$ is the least ordinal $a > v$ such that a is a principal indiscernible in X for an ordinal α such that $\mathscr{E}_\gamma = F^X_{a,\alpha}$.
2. $\ell^X(\gamma, v)$ is the least ordinal a, with $\gamma > a \geqslant v$, such that $a = s^X(\gamma', v)$ for some $\gamma' \trianglerighteq \gamma$.
3. a is an *accumulation point in* X for γ if $\alpha < \gamma \in \mathrm{O}'(\alpha)$, where either $\alpha = a$ or a is a principal indiscernible for α in X, and $\ell^X(\gamma', v) < a$ for every ordinal $v \in a \cap X$ and every $\gamma' \lhd \gamma$ in h^X_α"$a \cap \mathrm{O}(\alpha)$.
4. If $\xi < \omega_1$ then $a^X_\xi(\gamma, v)$ is the ξth accumulation point for γ above v. We write $a^X(\gamma, v)$ for $a^X_1(\gamma, v)$.
5. $\beta^X(\beta, a, \alpha)$ is equal to the ordinal b, if there is one, such that b is an indiscernible in X for β belonging to (a, α).

Notice that if a is a principal indiscernible for α then there is a $\lhd$-largest ordinal $\eta \in \mathrm{O}'(\alpha)$ such that a is an accumulation point for η, and that the set of accumulation points for an ordinal η is closed in X.

Lemma 2.14. *Suppose that Y is a precovering set and that $\alpha \in Y$ has cofinality δ. Then for all but boundedly many $v < \alpha$, if $\inf(\delta, \omega_1) \leqslant \mathrm{cf}(v) \leqslant \delta$ and v is an accumulation point in Y for some $\eta \in \mathrm{O}'(\alpha)$ then there is an $\eta' \geqslant \eta$ and $\gamma < v$ such that $v = \ell^Y(\eta', \gamma)$.*

Proof. By Lemma 2.6, for all sufficiently large ordinals $\xi < \alpha$ which are principal indiscernibles for α in Y, there is $\eta_\xi \in \mathrm{O}(\alpha)$ such that ξ is an indiscernible for $\mathscr{E}_{\eta_\xi}$ in Y. Let v be as in the hypothesis so that v is a principal indiscernible for $\mathscr{E}_{\eta_v}$. Using Lemma 2.6 again, all but boundedly many of the principal indiscernibles a for α below

ν are indiscernibles for some extender on a. For all such a we have $\eta_a \lhd \eta_\nu$, so that $\nu = s^Y(\eta_\nu, \gamma)$ for some $\gamma < \nu$. $\square$

Lemma 2.15. *For all precovering sets X and X' there is an ordinal $\eta < \kappa$ such that if $\eta < \nu < \alpha \leqslant \kappa$, $\xi \leqslant \delta$, and $\alpha < \gamma \in \mathrm{O}'(\alpha)$ then*

$$s^X(\gamma,\nu) = s^{X'}(\gamma,\nu) \tag{i}$$
$$\ell^X(\gamma,\nu) = \ell^{X'}(\gamma,\nu) \tag{ii}$$
$$a^X_\xi(\gamma,\nu) = a^{X'}_\xi(\gamma,\nu) \tag{iii}$$
$$\beta^X(\beta,a,\alpha) = \beta^{X'}(\beta,a,\alpha) \tag{iv}$$

whenever the arguments are members of $X \cap X'$. The equality sign here means that if either side is defined then both sides are defined and they are equal.

Proof. If the lemma fails then one of the Eqs. (i)–(iv) must fail cofinally often. Suppose first that Eq. (i) fails cofinally often, say for $\boldsymbol{\gamma} = (\gamma_\iota : \iota < \delta)$ and $\boldsymbol{\nu} = (\nu_\iota : \iota < \delta)$. This means that $\sup \boldsymbol{\nu} = \kappa$ and $\nu_\iota < \mathrm{crit}(\mathscr{E}_{\gamma_\iota}) \leqslant \kappa$ and $s^X(\gamma, \nu_\iota) \neq s^{X'}(\gamma, \nu_\iota)$ for each $\iota < \delta$. We may suppose wlog that $s^X(F_\iota, \nu_\iota)$ exists for all $\iota < \delta$. If we set $a_\iota = s^X(F_\iota, \nu_\iota)$ then $\boldsymbol{a}$ is a sequence in X such that $\boldsymbol{\nu} <_b \boldsymbol{a}$ and $\boldsymbol{a}$ is a principal indiscernible sequence for $\boldsymbol{F}$ where $F_\iota = \mathscr{E}_{\gamma_\iota}$. Since this is a first order assertion about $\boldsymbol{a}$ in X, there must be some sequence $\boldsymbol{a}'$ in X' which satisfies the same assertion in X', and hence $s^{X'}(\gamma_\iota, \nu_\iota)$ also exists for all sufficiently large $\iota < \delta$. Let $a'_\iota = s^{X'}(\gamma_\iota, \nu_\iota)$ for each $\iota < \delta$. Then $\boldsymbol{a}' \geqslant_b \boldsymbol{a}$, since otherwise it is true in V that there is a principal indiscernible sequence for $\boldsymbol{F}$ which is smaller than $\boldsymbol{a}$ cofinally often. Then the same statement is true in X, contradicting the choice of $\boldsymbol{a}$. Similarly, $\boldsymbol{a} \geqslant_b \boldsymbol{a}'$ so $\boldsymbol{a} =_b \boldsymbol{a}'$, which means that Eq. (i) holds for all but boundedly many $\iota < \delta$, contrary to the choice of γ and $\boldsymbol{\nu}$.

The proof of Eq. (ii) and (using Lemma 2.3) Eq. (iv) is similar. The proof of Eq. (iii) is also similar, but slightly more complicated because of the extra quantifiers in the definition of the function a^X and the possibility of different subscripts ξ_ι. $\square$

Lemma 2.16. *If X is any precovering set then there is a $\nu < \kappa$ such that for every ordinal $\beta \in X$ with $\nu < \beta \leqslant \sup(\mathrm{O}(\kappa))$, at least one of the following holds:*

1. *$\beta \in h^X_\kappa$“β.*
2. *$\beta = \beta^X(\beta', a, \alpha)$ for some ordinals β', a and α such that $a < \beta$ and $\alpha < \beta' < \sup(\mathrm{O}(\alpha))$ with α and β' in h^X_κ“a.*
3. *$\beta = s^X(\gamma, \nu)$ for some ordinals $\nu < \beta$ and $\gamma \in h^X_\kappa$“β.*
4. *$\beta = a^X_\xi(\gamma, \nu)$ for some $\nu < \beta$ and $\gamma \in h^X_\kappa$“ν and some countable ordinal ξ. Furthermore, if $\delta = \omega$ then ξ may be taken to be* 1.

Proof. Set $h = h^X_\kappa$. If β cannot be written in the first form then β is an indiscernible in X, and if it also cannot be written in the second form then it must be a principal indiscernible in X for some extender $F_\beta \in h$“β.

Now let $\eta \in h$“β be the $\lhd$-largest ordinal such that β is an accumulation point for η. Then for some $\nu_0 < \beta$, $\ell^X(\eta, \nu_0)$ either does not exist or is greater than or equal to

β. Define $v_\iota = a^X_\iota(\eta, v_0)$ for each $\iota \leqslant \delta$. If $v_\iota = \beta$ for some $\iota < \delta$ then β falls into case (4). Otherwise v_ι exists and $v_\iota \leqslant \beta$ for all $\iota \leqslant \delta$. In this case $v_\delta = \ell^Y(\eta, v_0)$ by Lemma 2.14, so $\beta = v_\delta$ and so β falls into case (3).

If $\delta = \omega$ and β falls into case (4) then ξ is a successor since $\xi < \omega$, so $\beta = a^X(\eta, v_{\xi-1})$ as required by the second sentence of clause (4). □

Corollary 2.17. *Let $\mathscr{E}_\eta$ be $\lhd$-largest such that γ is an accumulation point for $\eta \in$ O$'(\alpha)$ in X, and suppose* $\mathrm{cf}(\gamma) > \inf(\omega_1, \delta)$. *Then there is $v < \gamma$ such that $\gamma = a^X(\eta, v)$.*

Proof. Define $\boldsymbol{v}$ as in the last proof, and let $\eta = \inf(\omega_1, \delta)$. If $v_{\iota+1} = \gamma$ for some $\iota < \eta$ then $\gamma = a^X(\eta, v_\iota)$. Otherwise $v_\eta < \gamma$ since $\mathrm{cf}(\gamma) > \eta$, but this is impossible because Lemma 2.14 implies that $v_\eta \geqslant \ell^X(\eta, v_0)$. □

3. Applications

The main result to be proved in this section is the following theorem:

Theorem 3.1. *Suppose that κ is a strong limit cardinal with* $\mathrm{cf}(\kappa) = \delta < \kappa$, *and that $2^\kappa \geqslant \lambda > \kappa^+$, where if λ is a successor cardinal then the predecessor of λ has cofinality greater than κ.*

1. *If $\delta > \omega_1$ then* $\mathrm{o}(\kappa) \geqslant \lambda + \delta$.
2. *If $\delta = \omega_1$ then* $\mathrm{o}(\kappa) \geqslant \lambda$.
3. *If $\delta = \omega$ then either* $\mathrm{o}(\kappa) \geqslant \lambda$ *or else $\{\alpha : K \models o(\alpha) \geqslant \alpha^{+n}\}$ is cofinal in κ for each $n < \omega$.*

The proof of Theorem 3.1 is like that in [4]. It has two ingredients: the first is the analysis of indiscernibles which was given in Section 2, and the second is a result of Shelah which is given below, following some preliminary definitions, as Theorem 3.2.

As in the last section, if $\boldsymbol{c}$ and $\boldsymbol{c}'$ are in $\prod \boldsymbol{b}$ then we will write $\boldsymbol{c} <_b \boldsymbol{c}'$ to mean that $\{b : c_b \geqslant c'_b\}$ is bounded in $\sup(\boldsymbol{b})$, and $\boldsymbol{c} =_b \boldsymbol{c}'$ to mean that $\{b : c_b \neq c'_b\}$ is bounded in $\sup(\boldsymbol{b})$. If $\boldsymbol{b}$ is a sequence of cardinals then a subset $\mathscr{D}$ of $\prod \boldsymbol{b}$ is said to be *cofinal* in $\prod \boldsymbol{b}$ if for each sequence $\boldsymbol{c} \in \prod \boldsymbol{b}$ there is a sequence $\boldsymbol{d} \in \mathscr{D}$ such that $\boldsymbol{c} <_b \boldsymbol{d}$. The set $\prod \boldsymbol{b}$ is said to have *true cofinality* λ, written $\mathrm{tcf}(\prod \boldsymbol{b}) = \lambda$, if there is a sequence $(\boldsymbol{c}_v : v < \lambda)$ of members of $\prod \boldsymbol{b}$ which is cofinal in $\prod \boldsymbol{b}$ and linearly ordered by $<_b$.

Theorem 3.2 (Shelah). *Suppose that* $\mathrm{cf}(\kappa) = \delta < \kappa$ *and $2^\kappa \geqslant \lambda$, where λ is a regular cardinal. If $\delta = \omega$ then also assume that $\lambda < \kappa^{+\omega}$.*

1. [21, *Chap.* IX, 5.12 *and* 5.10(1)]. *There is a sequence $\boldsymbol{a} \subset \kappa$ of regular cardinals such that* $\mathrm{tcf}(\prod \boldsymbol{a}) = \lambda$.
2. [21, *Chap.* II, 1.2]. *Any strictly increasing sequence from $\prod \boldsymbol{a}$ of length less than λ and cofinality greater than κ has a least upper bound.*

In Subsection 3.1 we apply the techniques of Section 2 to the sequence given by Shelah's theorem. For $\delta > \omega$ this analysis leads directly to the proof of Lemma 3.3

below, which is clauses 1 and 2 of Theorem 3.1 except that clause 1 is weakened by replacing $\lambda + \delta$ with λ. For clause 3, the case $\delta = \omega$, the analysis yields Lemma 3.4, which is used in Subsection 3.2 to prove clause 3 of Theorem 3.1. In Subsection 3.3 we prove various further results, including the full strength of Theorem 3.1(1).

As usual, all successors are computed in K.

Lemma 3.3. *If κ is a strong limit cardinal with $\omega < \delta = \text{cf}(\kappa) < \kappa$, and $\kappa^+ < \lambda \leqslant 2^\kappa$ where λ is not the successor of a cardinal of cofinality less than κ, then $\text{o}(\kappa) \geqslant \lambda$.*

Notice that if Lemma 3.2 is true for successor cardinals λ then it is true for all limit cardinals. Thus it will be sufficient to prove Lemma 3.3 for regular cardinals λ.

Lemma 3.4. *Suppose that $\omega = \text{cf}(\kappa) < \kappa$ and $\kappa^+ < \text{o}(\kappa) < \lambda \leqslant 2^\kappa$, and assume that λ is regular and $\{\alpha : \text{o}(\alpha) > \alpha^{+n}\}$ is bounded in κ for some $n < \omega$. Then there is a countable sequence $\boldsymbol{b}$, cofinal in κ, along with continuous, nondecreasing functions f_b, and ordinals γ_b, α_b and σ_b for $b \in \boldsymbol{b}$ such that $\sigma_b < b$ and $\sigma_b \in \boldsymbol{b}$ for all but boundedly many $b \in \boldsymbol{b}$ and the set $\mathscr{L} \in \prod \boldsymbol{b}$, defined below, is cofinal in $\prod \boldsymbol{b}$ and has true cofinality λ.*

A sequence $\boldsymbol{c} \in \prod \boldsymbol{b}$ is in $\mathscr{L}$ if and only if for some precovering set Y, and all sufficiently large $b \in \boldsymbol{b}$,

$$c_b = \begin{cases} \ell^Y(f_b(c_{\sigma_b}), \gamma_b) & \text{if } b \text{ is a limit of principal indiscernibles,} \\ \beta^Y(f_b(c_{\sigma_b}), \gamma_b, \alpha_b) & \text{otherwise.} \end{cases} \quad (1)$$

Furthermore any strictly increasing, non-cofinal subsequence of $\mathscr{L}$ of cofinality greater than κ^+ has a least upper bound in $\prod \boldsymbol{b}$.

3.1. Proof of Lemmas 3.3 and 3.4

The main goal of this subsection is to prove Lemma 3.4. This is true for the case of uncountable cofinality, $\delta > \omega$, as well as for countable cofinality – the difference is that in the case $\delta > \omega$ we immediately reach an easy contradiction and hence do not need to explicitly state an intermediate result corresponding to Lemma 3.4.

We will write $S_b^{f,Y}(v)$ for the function given in Eq. (1) of Lemma 3.4. Thus a sequence $\boldsymbol{c} = (c_b : b \in \boldsymbol{b})$ in $\prod \boldsymbol{b}$ is in $\mathscr{L}$ if and only if there is a precovering set Y so that $c_b = S_b^Y(c_{\sigma_b})$ for all sufficiently large $b \in \boldsymbol{b}$.

If we had required $c_b = S_b^Y(c_{\sigma_b})$ for *all* $b \in \boldsymbol{b}$ such that $\sigma_b \in \boldsymbol{b}$, and if S_b^Y did not depend on Y, then it would follow that a sequence $\boldsymbol{c} \in \mathscr{L}$ is determined by $(c_b : \sigma_b \notin \boldsymbol{b})$. Since $\{b \in \boldsymbol{b} : \sigma_b \notin \boldsymbol{b}\}$ is bounded in κ there are fewer than κ choices for $\{c_b : \sigma_b \notin \boldsymbol{b}\}$ and it would then follow that $\text{tcf}(\mathscr{L}) < \kappa$, contradicting the assertion that $\text{tcf}(\mathscr{L}) = \lambda$ and completing the proof of the theorem.

For the case $\text{cf}(\kappa) > \omega$ this is nearly what happens. We show that $\boldsymbol{b}$ has order type δ, and then Fodor's theorem implies that σ_b is constant on an unbounded subset y of κ. Since $S_b^{f,Y} = S_b^{f,Y'}$ for sufficiently large $b \in \boldsymbol{b}$, this implies that $\{\boldsymbol{c} \restriction y : \boldsymbol{c} \in \mathscr{L}\}$

had fewer than κ members, modulo the relation $=_b$, and this contradicts the assertion that $\text{tcf}(\mathscr{L}) = \lambda$.

The case $\text{cf}(\kappa) = \omega$ is more difficult. The strategy is to try to show that $\text{cf}(c_b) = \text{cf}(c_{\sigma_b})$ for $\boldsymbol{c} \in \mathscr{L}$ and sufficiently large $b \in \boldsymbol{b}$, which would lead to essentially the same contradiction as in the case $\delta > \omega$. In fact, however, it becomes necessary to look at sequences $\boldsymbol{d}$ which are the least upper bound for certain subsets of $\mathscr{L}$, instead of working with the sequence $\boldsymbol{c}$ in $\mathscr{L}$ directly. This argument is in Subsection 3.2.

To understand the proof of Lemma 3.4, it will be helpful to consider four levels of data:

Level 1. The functions ℓ^Y and β^Y.

Level 2. The sequence $\boldsymbol{b} \supset \boldsymbol{a}$ and the parameters σ_b, γ_b and α_b. Also included in this level is a procedure for defining the functions f_b from a single function $f \in K$ – this is the function f which appears as a subscript in the notation $S_b^{f,Y}$. The procedure involves additional parameters p_b and η_b.

Level 3. The function f used to define the functions f_b.

Level 4. The set $\mathscr{L}$, and the sequences $\boldsymbol{c} \in \mathscr{L}$.

The items in level 4 are already defined in the lemma, using data from levels 1–3.

The functions ℓ^Y and β^Y of level 1 were defined, and their properties proved, in the last section. In particular, we use Lemma 2.15, which asserts that these functions are essentially independent of Y, and Lemma 2.16 which provides the inspiration for Lemma 3.4. Note, for example, that the first case,

$$c_b = \ell^Y(f_b(c_{\sigma_b}), \gamma_b) \qquad (*)$$

of Lemma 3.4 comes from case (3) of Lemma 2.16:

$$\nu = s^Y(h_\kappa^Y(\nu'), \gamma) \quad \text{for some } \nu', \gamma < \nu. \qquad (**)$$

The Eq. $(*)$ has ℓ^Y, which is more convenient to work with, instead of s^Y. The parameter $\gamma = \gamma_b$ is made to depend only on b. Eq. $(*)$ asserts that whenever c_b is a member of $\boldsymbol{c}$ then the ordinal ν' of Eq. $(**)$ is also a member c_{σ_b} of $\boldsymbol{c}$ (unless $\sigma_b \notin \boldsymbol{b}$). Furthermore, the coordinate σ_b at which c_{σ_b} appears in $\boldsymbol{c}$ has been fixed and does not depend on the sequence $\boldsymbol{c}$. Finally, the function h_κ^Y, which depends on Y and which need not be in K, is replaced with a function f_b in K which again does not depend on $\boldsymbol{c}$ or Y.

The data in level 2 is defined by working in a fixed precovering set X, with the aim of finding parameters so that the set of restrictions $\boldsymbol{c} \restriction \boldsymbol{a} = (c_b : b \in \boldsymbol{a})$ of sequences $\boldsymbol{c} \in \bigcup\{\mathscr{L}^{f,X} : f \in X \cap K\}$ is cofinal in $\prod \boldsymbol{a} \cap X$. For most of this construction we let h_κ^X play the role of f, but at the end we use the covering lemma to show that there are suitable approximations to h_κ^X in $X \cap K$.

The argument for level 3 begins with the observation that by elementarity the set of restrictions $\boldsymbol{c} \restriction \boldsymbol{a}$ of sequences

$$\boldsymbol{c} \in \bigcup\{\mathscr{L}^{f,Y} : f \in K \text{ and } Y \text{ is a precovering set}\}$$

is cofinal in $\prod \boldsymbol{a}$. In order to prove the crucial fact that there is a single function f such that $\mathscr{L}^f = \bigcup\{\mathscr{L}^{f,Y} : Y \text{ is a precovering set}\}$ is similarly cofinal we use the assumption that $(\mathrm{o}(\kappa)^\kappa)^K < \lambda = \mathrm{cf}(\lambda)$, and this is the only place where this assumption is used. In Section 3.3 we prove slightly stronger versions of Theorem 3.1 by using a modification of this assumption which also implies the existence of a single function f so that $\mathscr{L}^f$ is cofinal. Thus this modified assumption leads to the same contradiction.

We are now ready to begin the proof of Lemmas 3.3 and 3.4. First we need a couple of preliminary results. This first lemma will be applied to sequences $\boldsymbol{b}$ which may not be increasing.

Proposition 3.5. *If $\boldsymbol{b}$ is a sequence of cardinality at most $\boldsymbol{\delta}$ and $\eta < \kappa \leqslant \mathrm{tcf}(\prod \boldsymbol{b})$ then $\{b \in \boldsymbol{b} : \mathrm{cf}(b) \leqslant \eta\}$ is bounded in $\boldsymbol{b}$.*

Proof. Suppose to the contrary that $\eta < \kappa$ but $\boldsymbol{b}' = (b \in \boldsymbol{b} : \mathrm{cf}(b) < \eta)$ is cofinal in b. Then $\mathrm{tcf}(\prod \boldsymbol{b}') = \mathrm{tcf}(\prod \boldsymbol{b}) \geqslant \kappa$. Now let y_b be a cofinal subset of b of cardinality at most η for each $b \in \boldsymbol{b}'$. Then $\prod_{b \in \boldsymbol{b}'} y_b$ is cofinal in $\prod \boldsymbol{b}'$, but this is impossible since κ is a strong limit cardinal and hence $|\prod_{b \in \boldsymbol{b}'} y_b| \leqslant \eta^\delta < \kappa$. This contradiction proves the proposition. □

Now let X be a precovering set. This precovering set will remain fixed through the rest of this subsection.

Definition 3.6. We say that an ordinal $b \in X$ is *well adjusted* in X if

1. $b > \rho^X$, and if $\delta = \omega$ then there is an $n < \omega$ such that $\mathrm{o}(\alpha) < \alpha^{+n}$ whenever $b < \alpha < \kappa$.
2. b is regular in K.
3. $b \cap X$ is not cofinal in b,
4. The indiscernibles (including nonprincipal indiscernibles) of X are cofinal in $X \cap b$.

Proposition 3.7. *If $\boldsymbol{b}$ is a sequence of regular cardinals of K such that $\mathrm{tcf}(\prod \boldsymbol{b}) > \kappa^+$ then every sufficiently large member of $\boldsymbol{b}$ is well adjusted in X.*

Proof. First, $\boldsymbol{b}$ is unbounded in κ by Proposition 3.5, so clause (1) of Definition 3.6 is satisfied for all sufficiently large $b < \kappa$. Clause (2) is satisfied by hypothesis, and Proposition 3.5 implies that $\mathrm{cf}(b) > |X|$ for all but boundedly many members of b, so that $X \cap b$ is not cofinal in b for $|X| < b < \kappa$. Thus we only need to verify clause (4).

Let $\boldsymbol{b}'$ be the set of ordinals b in $\boldsymbol{b}$ such that the indiscernibles of X are not cofinal in $X \cap b$, and suppose that, contrary to clause (3), $\boldsymbol{b}'$ is cofinal in $\boldsymbol{b}$. For each member

b of $\boldsymbol{b}'$ pick an ordinal $\xi_b < b$ in X which is larger than all of the indiscernibles of X below b, so that $h^X_\kappa``\xi_b$ is cofinal in $b \cap X$. If $\boldsymbol{v}$ is any member of $\prod \boldsymbol{b}'$ in X then there is a sequence $\boldsymbol{v}' \in \prod_{b\in\boldsymbol{b}'} \xi_b$ such that $h^X \circ \boldsymbol{v}' \geqslant_b \boldsymbol{v}$. Now h^X_κ need not be in K, but this construction only uses the restriction $h^X_\kappa \cap (\kappa \times \kappa)$ of h^X_κ to ordinals below κ, which is a member of K. Thus it is true in V, and hence by elementarity it is true in X, that there is a function $f \in {}^\kappa\kappa \cap K$ and a sequence $\boldsymbol{v}' \in \prod_{b\in\boldsymbol{b}'} \xi_b$ such that $f \circ \boldsymbol{v}' \geqslant_b \boldsymbol{v}$. Since $\boldsymbol{v}$ was arbitrary it is true in X, and by elementarity again it is true in V, that the set of sequences of the form $f \circ \boldsymbol{v}'$ for some $f \in K$ and some $\boldsymbol{v}' \in \prod_{b\in\boldsymbol{b}'} \xi_b$ is cofinal in $\prod \boldsymbol{b}'$. Since $\mathrm{tcf}(\prod \boldsymbol{b}') = \mathrm{tcf}(\prod \boldsymbol{b}) > \kappa^+$ and $|{}^\kappa\kappa \cap K| = \kappa^+$ there must be a single function f such that the set of sequences $f \circ \boldsymbol{v}'$ for $v' \in \prod_{b\in\boldsymbol{b}'} \xi_b$ is cofinal in $\prod \boldsymbol{b}'$, but this is impossible because the members b of $\boldsymbol{b}$ are regular in K and hence $f``\xi_b$ is bounded in b for all $b \in \boldsymbol{b}'$. □

It follows that every sufficiently large member of $\boldsymbol{a}$ is well adjusted, and we can assume wlog that every member of $\boldsymbol{a}$ is well adjusted.

We are now ready to define the sequence $\boldsymbol{b}$ and the associated parameters. For each well adjusted ordinal $b \in X \cap \kappa$ we will define $\sigma_b < b$ along with a function $S^{f,Y}_b$, depending on an arbitrary precovering set Y and function $f \in K$ as well as the ordinal b. The function $S^{f,Y}_b$ also depends on several parameters which will be fixed in the course of this definition. The function $S^{f,Y}_b$ is the function appearing in Eq. (1) of Lemma 3.4. We will show that if we take $\mathscr{S}^{f,Y}$ to be the set of sequences $\boldsymbol{c} \in \prod \boldsymbol{b}$ such that $c_b = S^{f,Y}_b(c_{\sigma_b})$ for all sufficiently large $b \in \boldsymbol{b}$, then the union over functions $f \in K$ and precovering sets Y is cofinal in $\prod \boldsymbol{b}$ and hence has true cofinality λ.

Definition 3.8. We define an ordinal σ_b for all well adjusted $b \in X \cap \kappa$, and a function $S^{f,Y}_b$ for all well adjusted $b \in X \cap \kappa$, all precovering sets Y, and $f \in K$. We also define several auxiliary parameters. The definition depends on the fixed precovering set X and is broken into two cases, depending on whether or not the principal indiscernibles of X are cofinal in $b \cap X$.

Case 1. (The principal indiscernibles of X are cofinal in $X \cap b$.) In this case we define

1. $\alpha_b = i^X_{b,\kappa}(b)$. Thus either $b = \alpha_b$ or b is a principal indiscernible for α_b. In either case b is a limit in X of principal indiscernibles for α_b.

2. Since $\mathrm{cf}(b) > \delta$, Corollary 2.17 implies that there is an $\eta \leqslant \sup(\mathrm{O}'(\alpha_b))$ and $\gamma < b$ so that $b = a^X(\eta, \gamma)$. We let η_b be this ordinal η and let γ_b be the least ordinal γ such that $b = a^X(\eta_b, \gamma)$ and $\ell^X(\eta_b, \gamma) \not< b$.

3. σ_b is the least ordinal σ in X such that $\{h^X_\kappa(v, p) : v \in \sigma \cap X\}$ is cofinal in $X \cap \eta_b$ for some finite sequence p of ordinals.

4. p_b is the least finite sequence p of ordinals in X such that $\{h^X_\kappa(v, p) : v \in \sigma \cap X\}$ is cofinal in $X \cap \eta_b$.

5. If f is any function in K then f_b is the function defined by $f_b(v) = \sup(\eta_b \cap f``(v \times \{p_b\}))$.

6. $S_b^{f,Y}(v) = \ell^Y(f_b(v), \gamma_b)$, if it is defined and less than b. Otherwise $S_b^{f,Y}(v)$ is undefined.

Case 2. (b is not a limit of principal indiscernibles)

Since b is a limit of indiscernibles, but not a limit of principal indiscernibles, there is a largest principal indiscernible below b. Let γ_b be this principal indiscernible, and set $\alpha_b = i_{\gamma_b,\kappa}(\gamma_b)$. Then γ_b is a principal indiscernible for α_b and every ordinal in $X \cap (\gamma_b, b]$ is an indiscernible belonging to (γ_b, α_b).

Now let $\eta_b = i_{\gamma_b,\alpha_b}(b)$, so that b is an indiscernible in X for η_b belonging to (γ_b, α_b). The ordinals σ_b and p_b, and the function f_b, are defined exactly as in case 1.

Finally, set $S_b^{f,Y}(v) = \beta^Y(f_b(v), \gamma_b, \alpha_b)$ if it exists and is less than b. Otherwise $S_b^{f,Y}(v)$ is undefined.

Proposition 3.9. *If b is well adjusted then $\sigma_b < b$.*

Proof. If b is not a limit of principal indiscernibles then $\sigma_b \leqslant \gamma_b < b$, so suppose that b is a limit of principal indiscernibles and that, contrary to the proposition, $\sigma_b = b$. Define a sequence $(c_\iota : \iota < \delta)$ by recursion on ι:

$$\begin{aligned} c_0 &= \gamma_b, \\ c_{\iota+1} &= \ell^X(\xi_\iota, \gamma_b), \qquad \text{where } \xi_\iota = \sup\left(\eta_b \cap h_\kappa^X``(X \cap c_i)\right), \\ c_\iota &= \sup\{c_{\iota'} : \iota' < \iota\} \qquad \text{if } \iota \text{ is a limit ordinal.} \end{aligned}$$

If ι is a limit ordinal then c_ι is in X, since X is δ-closed, and $c_\iota < b$ since $X \cap b$ is not cofinal in b. If $\sigma_b = b$ then it follows that $c_\iota < b$ for each $\iota \leqslant \delta$. Set $\zeta = \inf(\delta, \omega_1)$. Then $c_\zeta = a^X(\xi_\zeta, \gamma_b)$ and by Proposition 2.14 it follows that $c_\zeta = \ell^X(\xi_\zeta, \gamma_b) = s^X(\xi, \gamma_b)$ for some $\xi \geqslant \xi_\zeta$. But $\xi \in h_\kappa^X``(c_\zeta)$, so $\xi \in h_\kappa^X``(c_\iota)$ for some $\iota < \zeta$ and hence $c_\zeta > c_{\iota+1} > c_\zeta$. This contradiction completes the proof that $\sigma_b < b$. □

Definition 3.10. (1) $\boldsymbol{b}$ is the smallest set such that $\boldsymbol{a} \subset \boldsymbol{b}$ and $\sigma_b \in \boldsymbol{b}$ for all $b \in \boldsymbol{b}$ such that σ_b is well adjusted in X.

(2) If f is as above and Y is a precovering set containing everything relevant then $\mathscr{L}^{f,Y}$ is the set of sequences $\boldsymbol{c} \in \prod \boldsymbol{b}$ such that $c_b = S_b^{f,Y}(c_{\sigma_b})$ for all sufficiently large $b \in \boldsymbol{b}$ such that $\sigma_b \in \boldsymbol{b}$.

(3) If $f \in K$ then $\mathscr{L}^f = \bigcup\{\mathscr{L}^{Y,f} : Y \text{ is a precovering set}\}$.

Notice that by Lemma 2.15 $\mathscr{L}^f$ is first order definable in any Y containing all of the data, and that $\mathscr{L}^{Y,f} = \mathscr{L}^f \cap Y$.

Lemma 3.11. *The set $\bigcup_f \{\boldsymbol{c} \restriction a : \boldsymbol{c} \in \mathscr{L}^f\}$ is cofinal in $\prod \boldsymbol{a}$.*

Proof. Since $\mathscr{L}^f$ is first order definable, it is sufficient to show that the lemma is true in X; that is, to produce, given any sequence $\boldsymbol{d}$ in $\prod \boldsymbol{a} \cap X$, a function $f \in K \cap X$ and a sequence $\boldsymbol{c} \in \mathscr{L}^f \cap X$ such that $\boldsymbol{c} \restriction \boldsymbol{a} \geqslant_b \boldsymbol{d}$. For the function f we will use h_κ^X, or rather a function in $X \cap K$ which is nearly equal to h^X_κ. We begin by defining a sequence $\boldsymbol{c}_n = (c_{n,b} : b \in \boldsymbol{b})$ for each $n < \omega$ so that

$$\begin{array}{ll} c_{0,b} = d_b & \text{if } b \in \boldsymbol{a}, \\ c_{0,b} = 0 & \text{if } b \notin \boldsymbol{a}, \\ S_b^{X,h^X_\kappa}(c_{n+1,\sigma_b}) \geqslant c_{n,b} & \text{if } \sigma_b \in \boldsymbol{b}, \\ c_{n+1,b} \geqslant c_{n,b} & \text{for all } n \text{ and } b. \end{array}$$

We define $c_{n,b}$ by recursion. Suppose that $c_{n,b}$ has been defined for all $b \in \boldsymbol{b}$, and c_{n+1,σ_b} has been defined if σ_b is in $\boldsymbol{b}$. In order to define $c_{n+1,b}$, define $\xi_{b'}$ for each $b' \in \boldsymbol{b}$ to be the least ordinal ξ such that $S_{b'}^{X,h^X_\kappa}(\xi) \geqslant c_{n,b'}$ if $b = \sigma_{b'}$, and let $\xi_{b'} = 0$ otherwise. Then $\{\xi_{b'} : b' \in \boldsymbol{b}\} \in X$ since ${}^\delta X \subset X$ and we can set $c_{n+1,b} = \sup(\{\xi_{b'} : b' \in \boldsymbol{b}\} \cup \{c_{n,b}\})$.

Set $y = \{(c_{n,b}, p_b) : n \in \omega \text{ and } b \in \boldsymbol{b}\}$. By Lemma 1.11(5) there is a function $f \in X \cap K$ such that $f \restriction y = h^X_\kappa \restriction y$. Define the sequence $\boldsymbol{c} \in \mathscr{L}^f$ by recursion on $b \in \boldsymbol{b}$:

$$c_b = \begin{cases} \bigcup_n c_{n,b} & \text{if } \sigma_b \notin \boldsymbol{b}, \\ S_b^{X,f}(c_{\sigma_b}) & \text{if } \sigma_b \in \boldsymbol{b}. \end{cases}$$

We claim that $c_{n,b} \leqslant c_b < b$ for each $n \in \omega$. The proof is a simple recursion on $b \in \boldsymbol{b}$. It is true immediately if $\sigma_b \notin \boldsymbol{b}$, while if $\sigma_b \in \boldsymbol{b}$ then $c_{\sigma_b} \geqslant c_{n+1,\sigma_b}$ so $c_b = S_b^{X,f}(c_{\sigma_b}) \geqslant S_b^{X,f}(c_{n+1,\sigma_b}) \geqslant c_{n,b}$.

In particular, $c_b \geqslant c_{0,b} = d_b$ for $b \in a$, so $\boldsymbol{c} \restriction a \geqslant \boldsymbol{d}$ as required. □

This completes the construction at level 2 as described in the introduction to this subsection. The following corollary gives us level three, the choice of the function f, and is thus much more important than its length suggests. Notice that this is the only place where we use the assumption that $(\mathrm{o}(\kappa)^\kappa)^K < \mathrm{cf}(\lambda)$.

Corollary 3.12. *There is a function $f \in K$ such that $\{\boldsymbol{c} \restriction \boldsymbol{a} : \boldsymbol{c} \in \mathscr{L}^f\}$ is cofinal in $\prod \boldsymbol{a}$.*

Proof. The last lemma implies that $\bigcup\{\boldsymbol{c} \restriction \boldsymbol{a} : \boldsymbol{c} \in \bigcup_f \mathscr{L}^f\}$ is cofinal in $\prod \boldsymbol{a}$. Now the relevant functions $f \in K$ have domain contained in $\kappa \times \kappa^{<\omega}$ and range contained in $\mathrm{O}'(\kappa)$, so there are only $(\mathrm{o}(\kappa)^\kappa)^K < \lambda$ of them. Since $\mathrm{tcf}(\prod \boldsymbol{a}) = \lambda = \mathrm{cf}(\lambda)$ it follows that there is a single function f such that $\{\boldsymbol{c} \restriction \boldsymbol{a} : \boldsymbol{c} \in \mathscr{L}^f\}$ is cofinal in $\prod \boldsymbol{a}$.

Corollary 3.13. *The set $\{b \in \boldsymbol{b} : \sigma_b \notin \boldsymbol{b}\}$ is bounded in $\boldsymbol{b}$, and if $v < \kappa$ then $\{b \in \boldsymbol{b} : \sigma_b < v\}$ is bounded in $\boldsymbol{b}$.*

Proof. Recall that the functions $S_b^{f,Y} : \sigma_b \to b$ are nondecreasing and are cofinal in $b \cap Y$, whether or not $\sigma_b \in \boldsymbol{b}$. If we set $S^{f,Y}(\boldsymbol{d}) = (S_b^Y(d_b) : b \in \boldsymbol{b})$ then it follows that

$$\left\{ s^{f,Y}(\boldsymbol{d}) : \boldsymbol{d} \in \prod_{b \in \boldsymbol{b}} \sigma_b \text{ and } Y \text{ is a precovering set} \right\}$$

is cofinal in $\prod \boldsymbol{b}$, and since $\boldsymbol{d} <_b \boldsymbol{d}'$ implies $S^{f,Y}(\boldsymbol{d}) \leqslant_b S^{f,Y'}(\boldsymbol{d}')$ for any precovering sets Y and Y' it follows that $\operatorname{tcf}(\prod_b \sigma_b) = \operatorname{tcf}(\prod \boldsymbol{b}) = \lambda$, and the corollary follows by Propositions 3.5 and 3.7. □

At this point we will treat the cases $\delta = \omega$ and $\delta > \omega$ separately. We begin with $\delta > \omega$, finishing the proof of Lemma 3.3 by assuming that $\delta > \omega$ and showing that the properties which we have established for the sequence $\boldsymbol{b}$ lead to a contradiction.

Proof of Corollary 3.3. First we show that $\operatorname{otp}(\boldsymbol{b}) = \delta$. Set $\boldsymbol{a}_0 = \boldsymbol{a}$ and for $n \geqslant 0$ set $\boldsymbol{a}_{n+1} = \boldsymbol{a}_n \cup \{\sigma_b : b \in \boldsymbol{a}_n \text{ and } \sigma_b \in \boldsymbol{b}\}$. Since $\operatorname{otp}(\boldsymbol{a}) = \delta$, Corollary 3.13 implies that each $\boldsymbol{a}_n$ has order type δ. But $\boldsymbol{b} = \bigcup_n \boldsymbol{a}_n$, and since $\operatorname{cf}(\delta) > \omega$ it follows that $\boldsymbol{b}$ has order type δ.

Now since $\sigma_b < b$ and $\sigma_b \in \boldsymbol{b}$ for every sufficiently large $b \in \boldsymbol{b}$, Fodor's theorem implies that there is an unbounded subset B of $\boldsymbol{b}$ such that σ_b is constant for $b \in B$. But this contradicts Corollary 3.13, and this contradiction shows that it is not possible that $o(\kappa) < \lambda$. □

We now finish this subsection by completing the proof of Lemma 3.4. We assume that $\delta = \omega$, and that the hypothesis of Lemma 3.4 holds.

Proof of Lemma 3.4 (*conclusion*). We have proved all of this lemma except for the last paragraph, which asserts that every non-cofinal subsequence $\mathscr{B}$ of $\mathscr{L}^f$ of cofinality at least κ^+ has a least upper bound in $\prod \boldsymbol{b}$. Given such a subset $\mathscr{B}$, let $\boldsymbol{d}$ be the least upper bound of $\{\boldsymbol{c} \restriction \boldsymbol{a} : \boldsymbol{c} \in \mathscr{B}\}$, which exists by clause (2) of Theorem 3.2.

Define $b \prec b'$ if for some $m > 0$ there is a chain $b = b_0 < b_1 < \cdots < b_m = b'$ such that $b_k = \sigma_{b_{k+1}}$ for $k < m$. Let Y be a precovering set with $\boldsymbol{d}$ and $\mathscr{B}$ in Y, and for $b \prec b'$ define

$$S^{f,Y}_{b,b'} = S^{f,Y}_{b_m} \circ S^{f,Y}_{b_{m-1}} \circ \cdots \circ S^{f,Y}_{b_1} : b \to b'$$

We will extend this to $b \preccurlyeq b'$ by setting $S^{f,Y}_{b,b}(\nu) = \nu$.

Define $\boldsymbol{d}' \in \prod b$ by letting d'_b be the least ordinal ν such that $\nu \geqslant d_b$ if $b \in \boldsymbol{a}$ and $S^{f,Y}_{b,a}(\nu) \geqslant d_a$ for all a such that $b \prec a \in \boldsymbol{a}$. This is possible since $S^{f,Y}_{b,a}$ is cofinal in $a \cap Y$, which has cofinality greater than $\delta = \omega = |\boldsymbol{a}|$. We claim that $\boldsymbol{d}' = \operatorname{lub}(\mathscr{B})$.

Any member of $\mathscr{B}$ must be less than $\boldsymbol{d}'$ except on a bounded set, so it will be sufficient to prove that $\boldsymbol{d}'$ is minimal. We need to show that if $\boldsymbol{c}$ is any sequence such that $\boldsymbol{c} <_b \boldsymbol{d}'$, then $\boldsymbol{c} <_b \boldsymbol{d}''$ for some sequence $\boldsymbol{d}'' \in \mathscr{B}$.

To find $\boldsymbol{d}''$, define $\boldsymbol{c}' \in \prod \boldsymbol{a}$ by setting $c'_a = \sup\{S^{f,Y}_{b,a}(c_b) : b \preccurlyeq a\}$. Each of the ordinals $S^{f,Y}_{b,a}(c_b)$, for $b \preccurlyeq a$, is smaller than d_a by the choice of $\boldsymbol{d}'$. But $\operatorname{cf}(d_a) > \omega$ for all but boundedly many $a \in \boldsymbol{a}$ by Proposition 3.5, since $\operatorname{tcf}(\prod \boldsymbol{d}) \geqslant \kappa^+$, and hence $\boldsymbol{c}' <_b \boldsymbol{d}$.

Since $\boldsymbol{d}$ is the least upper bound of $\{\boldsymbol{c} \restriction \boldsymbol{a} : \boldsymbol{c} \in \mathscr{B}\}$ it follows that there is a $\boldsymbol{d}'' \in \mathscr{B}$ such that $\boldsymbol{c}' <_b \boldsymbol{d}'' \restriction \boldsymbol{a} <_b \boldsymbol{d}'$. Since $S^{f,Y}_b$ is increasing, it follows that $\boldsymbol{c} <_b \boldsymbol{d}''$, as required. □

3.2. Countable cofinality: the proof of Theorem 3.1(3)

Except for the need to consider nonprincipal extenders, the proof of Theorem 3.1(3) is essentially the same as in [5]. We assume that Theorem 3.1 is false with $\delta = \mathrm{cf}(\kappa) = \omega$, and let $\boldsymbol{b}$, $\mathscr{L}$, and the associated ordinals be as given by Lemma 3.4. We will assume that $\boldsymbol{b}$ and $\mathscr{L}$ are members of every precovering set mentioned.

Definition 3.14. Let $\mathscr{D}$ be the class of sequences $\boldsymbol{d} \in \prod \boldsymbol{b}$ such that $\boldsymbol{d}$ is the least upper bound of an increasing subsequence of $\mathscr{L}$ of order type κ^+.

Note that $\mathrm{otp}(\mathscr{D}, <_b) = \lambda$ by Lemma 3.4. For sequences $\boldsymbol{d}$ and $\boldsymbol{d}'$ in $\mathscr{D}$ let $g(\boldsymbol{d}, \boldsymbol{d}')$ be the least ordinal $b_0 \in \boldsymbol{b}$, if there is one, such that

1. either $d_b < d'_b$ for all $b > b_0$, or $d_b > d'_b$ for all $b > b_0$, and
2. if there are only boundedly many $b \in \boldsymbol{b}$ such that

$$\mathrm{cf}^K(d_b) = \mathrm{cf}^K(d'_b) > \gamma_b \quad \text{and} \quad \mathrm{cf}^K(d_{\sigma_b}) \neq \mathrm{cf}^K(d'_{\sigma_b}) \tag{$*$}$$

then $(*)$ is false for all $b > b_0$.

Since $\mathrm{tcf}(\mathscr{D}) = \lambda$ there is a subset $\mathscr{D}'$ of $\mathscr{D}$ of cardinality λ which is linearly ordered under $<_b$, so that $g(\boldsymbol{d}, \boldsymbol{d}')$ is defined for all $\boldsymbol{d}, \boldsymbol{d}' \in \mathscr{D}'$. Since $\lambda > (2^\omega)^+$ and $|\boldsymbol{b}| = \omega$, the Erdős–Rado theorem implies that there is a sequence $D = (\boldsymbol{d}_\iota : \iota < \omega_1)$ such that g is constant on $[D]^2$. We can assume wlog that $g(\boldsymbol{d}_\iota, \boldsymbol{d}_{\iota'}) = 0$ for all $\iota < \iota' < \omega_1$, so that $d_{\iota,b} < d_{\iota',b}$ whenever $\iota < \iota'$ and $b \in \boldsymbol{b}$.

We will say that some property $\mathbf{Q}(\iota, b)$ holds *for almost all* (ι, b) if for all but countably many $\iota < \omega_1$ there is $\nu_\iota < \kappa$ such that $\mathbf{Q}(\iota, b)$ holds for all $b \in \boldsymbol{b} \setminus \nu_\iota$.

Lemma 3.15. *For almost all (ι, b) the relations in the following table hold. Here I is the set of $b \in \boldsymbol{b}$ such that b is a limit of principal indiscernibles. In case 2c, "almost all" means that there is an $\iota_0 < \omega_1$ such that for all $\iota, \iota' > \iota_0$ there is a $\nu_{\iota,\iota'} < \kappa$ such that the conclusion holds whenever the hypothesis is true and $b > \nu_{\iota,\iota'}$.*

Hypothesis		*Conclusion*
(1) $b \in I$		$\mathrm{cf}(d_{\iota,\sigma_b}) = \mathrm{cf}(d_{\iota,b})$ $d_{\iota,b}$ *is regular in* K
(2a)	$\mathrm{cf}^K(d_{\iota,b}) < \gamma_b$	$\mathrm{cf}^K(d_{\iota,\sigma_b}) = \mathrm{cf}^K(d_{\iota,b})$
(2b) $b \notin I$	$\mathrm{cf}^K(d_{\iota,b}) = \gamma_b$	*impossible*
(2c)	$\mathrm{cf}^K(d_{\iota,b}) = \mathrm{cf}^K(d_{\iota',b}) > \gamma_b$	$\mathrm{cf}^K(d_{\iota,\sigma_b}) = \mathrm{cf}^K(d_{\iota',\sigma_b})$

Before proving Lemma 3.15 we will show that it implies the theorem. As before, we write $b \prec b'$ if there is a chain $b = b_0 < \cdots < b_m = b'$ with $b_i = \sigma_{b_{i+1}}$ for $i < m$, and we write $S^Y_{b,b'}$ for the composition $S^Y_{b_1} \circ \cdots \circ S^Y_{b_m}$. We write $b \preccurlyeq b'$ if $b \prec b'$ or $b = b'$, and we set $S^Y_{b,b}(v) = v$.

Proof of Theorem 3.1(3) (*assuming Lemma* 3.15). By throwing out countably many sequences from $(\boldsymbol{d}_\iota : \iota < \omega_1)$ we can assume without loss of generality that Lemma 3.15 is valid for all $\iota < \omega_1$, for all sufficiently large $b \in \boldsymbol{b}$. By the definition of $g(\boldsymbol{d}, \boldsymbol{d}')$ it follows that case (2c) is valid for all $\iota, \iota' \in \omega_1$ and for all $b \in \boldsymbol{b}$, and by dropping to an uncountable subset of $(\boldsymbol{d}_\iota : \iota < \omega_1)$ and throwing out a bounded part of $\boldsymbol{b}$ we can assume that the other cases are also valid for all $\iota < \omega_1$ and all $b \in \boldsymbol{b}$.

Claim. *For each* $\iota < \omega_1$, *the set of* $b \in \boldsymbol{b}$ *such that* $d_{\iota,b}$ *falls into case* (2c), *that is, such that* $b' \notin I$ *and* $\mathrm{cf}^K(d_{\iota,b}) > \gamma_b$, *is unbounded in* $\boldsymbol{b}$.

Proof. Suppose the contrary, that there is an $\iota < \omega_1$ and a $b_0 \in \boldsymbol{b}$ such that $d_{\iota,b}$ falls into either case (1) or case (2a) for all $b > b_0$. Then from the conclusions to these cases given in the table we can conclude that $\mathrm{cf}(d_{\iota,b'}) = \mathrm{cf}(d_{\iota,b})$ whenever $b_0 < b' \preccurlyeq b$. By Corollary 3.13, $\{b \in \boldsymbol{b} : \sigma_b < b_0\}$ is bounded in $\boldsymbol{b}$, say by $b_1 > b_0$. Then for all $b > b_1$ in $\boldsymbol{b}$ there is a $b' \prec b$ such that $b_0 < b' < b_1$: namely the least member b' of $\boldsymbol{b}$ such that $b_0 < b' \preccurlyeq b$. Then $\sigma_{b'} < b_0$, so $b' < b_1$ by the choice of b_1.

It follows that $\mathrm{cf}(b) = \mathrm{cf}(b') < b_1$ for all but boundedly many $b \in \boldsymbol{b}$, contradicting Proposition 3.5. □

Claim. *There is an unbounded subset* y *of* $\boldsymbol{b} \setminus I$ *such that* $\sigma_b \in y$ *whenever* $b \in y$ *and* $\sigma_b \in \boldsymbol{b}$.

Proof. If this claim is false then by the last claim there are, for every $\iota < \omega_1$, ordinals $b_{0,\iota} \prec b_{1,\iota}$ such that $b_{0,\iota} \in I$ and $d_{\iota,b_{1,\iota}}$ falls into case (2c). Since $\boldsymbol{b}$ is countable, there are ordinals $b_0 \prec b_1$ in $\boldsymbol{b}$ and an uncountable set $x \subset \omega_1$ so that $b_{0,\iota} = b_0$ and $b_{1,\iota} = b_1$ for all $\iota \in x$. By the hypothesis of the theorem we have $\gamma_{b_1} < b_1 \leqslant \gamma^{+n}_{b_1}$ for some $n \in \omega$. Since $\gamma_{b_1} < \mathrm{cf}^K(d_{\iota,b_1}) < b_1$ it follows that there are only finitely many possible values for $\mathrm{cf}^K(d_{\iota,b_1})$, so there must be ordinals $\iota < \iota' \in x$ such that $\mathrm{cf}^K(d_{\iota,b_1}) = \mathrm{cf}^K(d_{\iota',b_1})$. An easy induction, using Lemma 3.15, shows that $\mathrm{cf}^K(d_{\iota,b'}) = \mathrm{cf}^K(d_{\iota',b'})$ for all $b' \prec b_1$, and in particular $\mathrm{cf}^K(d_{\iota,b_0}) = \mathrm{cf}^K(d_{\iota',b_0})$. This is impossible since it implies that

$$\mathrm{cf}(d_{\iota',b_0}) = \mathrm{cf}(d_{\iota,b_0}) \leqslant d_{\iota,b_0} < d_{\iota',b_0},$$

contradicting the fact that b_0 is in I and hence d_{ι',b_0} is regular. □

Since the set $y \subset \boldsymbol{b}$ is unbounded in $\boldsymbol{b}$, we have $\mathrm{tcf}(\prod y) = \mathrm{tcf}(\prod \boldsymbol{b}) = \lambda$. Since y is closed under the operation $b \mapsto \sigma_b$, the conclusion of Lemma 3.4 is still true with $\boldsymbol{b}$ replaced by y. If we let k be such that $\lambda = \kappa^{+(k+1)}$ in K then y witnesses the truth of P(k):

Property P(k). *There is a countable sequence* $\boldsymbol{b}$ *of regular cardinals of* K, *cofinal in* κ, *along with nondecreasing functions* $f_b \in K$ *and ordinals* γ_b, α_b, *and* σ_b *for* $b \in \boldsymbol{b}$ *such that* $\gamma_b < b \leqslant \gamma_b^{+k}$, *and* $\sigma_b \in \boldsymbol{b}$ *for all but boundedly many* $b \in \boldsymbol{b}$, *and so that the set* $\mathscr{L}$ *defined below has true cofinality* $\kappa^{+(k+1)}$:

The sequence $\boldsymbol{c} \in \prod b$ *is in* $\mathscr{L}$ *if for some precovering set* Y, *and all sufficiently large* $b \in \boldsymbol{b}$,

$$c_b = \beta(f_b(c_{\sigma_b}), \gamma_b, \alpha_b).$$

Furthermore, any strictly increasing, non-cofinal subsequence of $\mathscr{L}$ *of cofinality greater than* κ^+ *has a least upper bound in* $\prod \boldsymbol{b}$.

We will prove by induction that P(m) is false for all $m > 1$, contradicting the observation that P(k) is true and hence finishing the proof of Theorem 3.1. Notice that since Lemma 3.15 and the claims above follow from Lemma 3.4, they are true for any witness $\boldsymbol{b}$ to property P(k) for any $k \geqslant 1$. This makes it easy to see that P(1) is false, since in that case we always have $\mathrm{cf}^K(d_{\imath,b}) \leqslant d_{\imath,b} < b < \gamma_b^+$, so that case (2c) can never hold, contrary to the first claim above.

Now we complete the proof by showing that P(m) implies P($m-1$) for all $m > 1$. Suppose that $\boldsymbol{b}$, $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, and $\boldsymbol{f}$ witness the truth of P(m). Then $\mathrm{tcf}(\prod \boldsymbol{b}) = \kappa^{+(m+1)}$, so we can let $\boldsymbol{c}$ be the least upper bound of the first κ^{+m} sequences from $\mathscr{L}$. Then $c_b < b \leqslant \gamma_b^{+m}$ for all $b \in \boldsymbol{b}$, so $\xi_b = \mathrm{cf}^K(c_b) < \gamma_b^{+m}$. Let $b'_b = \gamma_b + \xi_b \leqslant \gamma_b^{+(m-1)}$, and let $\tau_b \in K$ be a continuous, unbounded and increasing map from b'_b into c_b. Set $\tilde{\tau}_b = i_{\gamma_b,\alpha_b}(\tau_b)$ and set $f'_b = \tilde{\tau}_b^{-1} \circ f_b \circ \tau_{\sigma_b}$. Then $\boldsymbol{b}'$, $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}$ and $\boldsymbol{f}'$ witness the truth of P($m-1$), as required.

It follows by induction that P(m) is false for all $m \geqslant 1$, contradicting P(k). This contradiction completes the proof of theorem 3.1(3), assuming Lemma 3.15. □

In the rest of this subsection we finish the proof of Theorem 3.1 by proving Lemma 3.15. First we need a preliminary lemma:

Lemma 3.16. *If* $\boldsymbol{d} \in \mathscr{D}$ *then for any precovering set* Y *with* $\boldsymbol{d} \in Y$, *the following equation holds for all but boundedly many* $b \in \boldsymbol{b}$:

$$d_b = \begin{cases} a^Y(f_b(d_{\sigma_b}), \gamma_b) \leqslant \ell^Y(f_b(d_{\sigma_b}), \gamma_b) & \text{if } b \in I, \\ \beta^Y(f_b(d_{\sigma_b}), \gamma_b, \alpha_b) & \text{if } b \notin I. \end{cases} \tag{$*$}$$

Proof. Let $\boldsymbol{d}$ be any member of $\mathscr{D}$, and let Y be a precovering set with $\boldsymbol{d} \in Y$.

First we will prove the inequality for the case $b \in I$. Suppose to the contrary that there are unboundedly many $b \in I$ such that $\xi_b = \ell^Y(f_b(d_{\sigma_b}), \gamma_b) < d_b$. Then from the definition of $\mathscr{D}$ there is a sequence $\boldsymbol{c} \in \mathscr{L}$, with $\boldsymbol{c} <_b \boldsymbol{d}$, such that $\xi_b < \boldsymbol{c}_b$ for all but boundedly many of those $b \in I$ such that $\xi_b < d_b$. But this is impossible, since then

$$\xi_b < c_b = \ell^Y(f_b(c_{\sigma_b}), \gamma_b) < \ell^Y(f_b(d_{\sigma_b}), \gamma_b) = \xi_b$$

for each such b. This contradiction shows that $d_b \leqslant \ell^Y(f_b(d_{\sigma_b}), \gamma_b)$ for all but boundedly many $b \in I$.

Now we prove the identity in both cases. Define the sequence $\boldsymbol{\xi}$ by $\xi_b \stackrel{\text{def}}{=} a^Y(f_b(d_{\sigma_b}), \gamma_b)$ if $b \in I$ and $\xi_b \stackrel{\text{def}}{=} \beta^Y(f_b(d_{\sigma_b}), \gamma_b, \alpha_b)$ if $b \notin I$. We need to show that $d_b = \xi_b$ for all but boundedly many $b \in \boldsymbol{b}$. We will show first that $d_b \leqslant \xi_b$.

If, to the contrary, $d_b > \xi_b$ for unboundedly many $b \in \boldsymbol{b}$ then there is a sequence $\boldsymbol{c} \in \mathscr{L} \cap \prod \boldsymbol{d}$ such that $\xi_b \leqslant c_b$ for unboundedly many $b \in \boldsymbol{b}$. Then $\boldsymbol{c} \in Y$, since $\mathscr{L}$ and $\boldsymbol{b}$ are in Y. Now $c_{\sigma_b} < d_{\sigma_b}$ implies that $c_b = \ell^Y(f_b(c_{\sigma_b}), \gamma_b) < \ell^Y(f_b(d_{\sigma_b}), \gamma_b) = \xi_b$ for all sufficiently large $b \in I$ and $c_b = \beta^Y(f_b(c_{\sigma_b}), \gamma_b, \alpha_b) < \beta^Y(f_b(d_{\sigma_b}), \gamma_b, \alpha_b) = \xi_b$ for every sufficiently large $b \notin I$. Since this contradicts the choice of $\boldsymbol{c}$ we must have $\xi_b \geqslant d_b$ for almost every $b \in \boldsymbol{b}$.

We now complete the proof of the lemma by showing that $d_b \geqslant \xi_b$ for all but boundedly many $b \in \boldsymbol{b}$. Assume the contrary, that $d_b < \xi_b$ for unboundedly many $b \in \boldsymbol{b}$. We consider the cases $b \in I$ and $b \notin I$ separately.

Suppose first that $d_b < \xi_b = a^Y(f_b(d_{\sigma_b}), \gamma_b)$ for unboundedly many $b \in I$. By the definition of an accumulation point it follows that for unboundedly many $b \in I$ there is an ordinal $\zeta_b < f_b(d_{\sigma_b})$ in Y such that $d_b \leqslant \ell^Y(\zeta_b, \gamma_b)$.

We claim that $f_b(v) < f_b(d_{\sigma_b})$ for all sufficiently large $b \in I$ and all $v < d_{\sigma_b}$. Otherwise pick $v_b < d_{\sigma_b}$ for unboundedly many $b \in \boldsymbol{b}$ so that $f_b(v_b) \geqslant f_b(d_{\sigma_b})$. Then $S_b^{f,Y}(v_b) = \ell^Y(f_b(v_b), \gamma_b) \geqslant d_b$, so that any member of $\prod \boldsymbol{d} \cap \mathscr{L}$ must be smaller than v_b for all but boundedly many b. Since $\boldsymbol{d} = \operatorname{lub}(\prod d \cap \mathscr{L})$ it follows that $v_b \geqslant d_b$.

Since f_b is continuous and $\xi_b < f_b(d_{\sigma_b})$ it follows that there is an ordinal $v_b < d_{\sigma_b}$ such that $\zeta_b < f_b(v_b) < f_b(d_{\sigma_b})$. Now pick $\boldsymbol{c} \in \mathscr{L} \cap \prod d$ such that $c_{\sigma_b} > v_b$ for all but boundedly many b such that v_b is defined. This is possible since $\operatorname{cf}(d_b) > \omega$ for almost all $b \in \boldsymbol{b}$. Then

$$c_b = \ell^Y(f_b(c_{\sigma_b}), \gamma_b) \geqslant \ell^Y(f_b(v_b), \gamma_b) \geqslant \ell^Y(\zeta_b, \gamma_b) \geqslant d_b$$

for all sufficiently large $b \in I$ such that v_b is defined, contradicting the assumption that $\boldsymbol{c} \in \prod \boldsymbol{d}$. Thus $d_b = \xi_b$ for all but boundedly many $b \in I$.

The argument for $b \notin I$ is similar. If $d_b < \xi_b = \beta^Y(f_b(d_{\sigma_b}), \gamma_b, \alpha_b)$ for unboundedly many $b \notin I$ then for unboundedly many $b \notin I$ there is an ordinal $v_b < d_{\sigma_b}$ such that $i^Y_{\gamma_b, \alpha_b}(d_b) < f_b(v_b) < f(d_{\sigma_b})$. Choose $\boldsymbol{c} \in \mathscr{L} \cap \prod \boldsymbol{d}$ so that $c_{\sigma_b} > v_b$ whenever v_b is defined. Then

$$c_b = \beta^Y(f_b(c_{\sigma_b}), \gamma_b, \alpha_b) > \beta^Y(f_b(v_b), \gamma_b, \alpha_b) > d_b$$

for every sufficiently large b such that v_b is defined, contradicting the assumption that $\boldsymbol{c} \in \prod \boldsymbol{d}$.

This completes the proof that $\boldsymbol{\xi} = \boldsymbol{d}$, and hence of the lemma. □

The next four lemmas correspond to the four cases in Lemma 3.15. The first is, by a wide margin, the most difficult.

Lemma 3.17 (Lemma 3.15, case 1). *Every $b \in I$ is regular in K, and* $\mathrm{cf}(d_{\iota,b}) = \mathrm{cf}(d_{\iota,\sigma_b})$ *for almost all* (ι, b) *with* $b \in I$.

Proof. Recall that every member of I is a limit of principal indiscernibles for α_b, and hence is either a principal indiscernible for α_b or equal to the measurable cardinal α_b of K. In either case, b is regular in K.

Pick a ω_1-closed precovering set $Y \prec H_{(2^\tau)^+}$ such that everything relevant, including H_τ and $(\boldsymbol{d}_\iota : \iota < \omega_1)$, is in Y. Next pick, inside Y, a precovering set Y_b for each $b \in \boldsymbol{b}$ so that $Y_b \prec H_\tau$, $\sigma_b \subset Y_b$, and Y_b contains all of the sequences which have been defined. There exists precovering sets Y' with $\sigma_b \subset Y'$ since κ is a strong limit cardinal and hence $\gamma_b^\delta < \kappa$, and we can find the sequence $(Y_b : b \in I)$ inside Y since we have strengthened the usual requirement of $Y \prec H_\tau$ to $Y \prec H_{(2^\tau)^+}$.

By Lemma 3.16 there is, for each $\iota < \omega_1$, an ordinal $\nu_\iota < \kappa$ such that $d_{\iota,b} = a^Y(f_b(d_{\iota,\sigma_b}), \gamma_b)$ if $b \in I \setminus \nu_\iota$ and $d_{\iota,b} = \beta^Y(f_b(d_{\iota,\sigma_b}), \gamma_b, \alpha_b)$ if $b \notin (I \cup \nu_\iota)$. Since $\mathrm{cf}(\kappa) = \omega$ there is a fixed ν such that we can take $\nu_\iota = \nu$ for uncountably many $\iota < \omega_1$. By restricting ourselves to this uncountable subset and removing $\boldsymbol{b} \cap \nu$ from $\boldsymbol{b}$ we can assume wlog that $d_{\iota,b} = a^Y(f_b(d_{\iota,\sigma_b}), \gamma_b)$ or $d_{\iota,b} = \beta^Y(f_b(d_{\iota,\sigma_b}), \gamma_b, \alpha_b)$ whenever $\iota < \omega_1$ and $\sigma_b \in \boldsymbol{b}$.

Define $d_b \stackrel{\text{def}}{=} \sup_\iota(d_{\iota,b}) < b$ for each $b \in I$, so that $\boldsymbol{d} = \mathrm{lub}\{\boldsymbol{d}_\iota : \iota < \omega_1\}$. Since $\boldsymbol{d}_\iota$ satisfies condition $(*)$ of Lemma 3.16 for each $\iota < \omega_1$, the sequence $\boldsymbol{d}$ must also satisfy condition $(*)$. Since Y is ω_1-closed it follows by Lemma 2.14 that $d_b = a^Y(f_b(d_{\sigma_b}), \gamma_b) = \ell^Y(f_b(d_{\sigma_b}), \gamma_b)$. In particular, if $b \in I$ then $(d_{\iota,b} : \iota < \omega_1)$ is a principal indiscernible sequence for the constant sequence d_b.

We will find functions $g_b : d_{\sigma_b} \to \mathrm{O}(\alpha_b)$ in K so that $d_{\iota,b} = a^Y(g_b(d_{\iota,\sigma_b}), \gamma_b)$ for $\iota < \omega_1$. In addition, the functions g_b will be continuous, nondecreasing, and range(g_b) will be cofinal in $g_b(d_{\iota,\sigma_b})$.

To see that this implies the lemma, notice that the properties above imply that $\{\ell^{Y_b}(g_b(\nu), \gamma_b) : \nu < d_{\iota,\sigma_b}\}$ is cofinal in $d_{\iota,b} \cap Y_b$ for all but countably many $\iota < \omega_1$. Thus it will be sufficient to show that Y_b is cofinal in $d_{\iota,b}$, for all but boundedly many $b \in \boldsymbol{b}$. Suppose to the contrary that Y_b is bounded in $d_{\iota,b}$ for unboundedly many $b \in \boldsymbol{b}$. Since everything under consideration, including $(Y_b : b \in \boldsymbol{b})$, is in Y, the upper bounds $\xi_b = \sup(Y_b \cap d_{\iota,b})$ are in Y. Since $d_{\iota,b} = a^Y(g_b(d_{\iota,\sigma_b}), \gamma_b)$, it follows that there is a $\nu_b < d_{\iota,\sigma_b}$ such that $\ell^Y(g_b(\nu_b), \gamma_b) \geqslant \xi_b$. But $\nu_b \in \sigma_b \subset Y_b$, and hence $\ell^{Y_b}(g_b(\nu_b), \gamma_b) \in Y_b$. Thus $\xi_b < \ell^{Y_b}(g_b(\nu_b), \gamma_b) \leqslant \sup(Y_b \cap d_{\iota,\sigma_b})$, contrary to the choice of ξ_b. This contradiction shows that Y_b is cofinal in $d_{\iota,b}$ and hence $\mathrm{cf}(d_{\iota,b}) = \mathrm{cf}(d_{\iota,\sigma_b})$.

The functions f_b are continuous and increasing, and since $\mathscr{L}^f(\boldsymbol{b})$ is cofinal in $\boldsymbol{d}$ the range of f_b is cofinal in $f_b(d_{\iota,\sigma_b})$ for almost all b, for each $\iota < \omega_1$. As our first approximation to g_b, define $g_b^* : \gamma_b \to \sup(\mathrm{O}(d_b))$ by letting $g_b^*(\nu)$ be the least ordinal ν' such that $i_{d_b,\alpha_b}(\nu') \geqslant f_b(\nu)$. Then g_b^* is continuous and nondecreasing since f_b is, and the range of g_b^* is cofinal in $g_b^*(d_{\iota,\sigma_b}) \cap Y$ since $d_{\iota,b} = a^Y(f_b(d_{\iota,\sigma_b}), \gamma_1)$ implies that the range of i_{d_b,α_b} is cofinal in $f_b(d_{\iota,b}) \cap Y$. There are two problems with g_b^*: first, it is not in either K or in Y, and second, it is cofinal in $g_b^*(d_{\iota,\sigma_b}) \cap Y$, not in $g_b^*(d_{\iota,\sigma_b})$.

We will attack the second problem by going back to the proof of the covering lemma, working in the preimage of the collapse map π.

Define $\bar{g}_b^*$ by letting $\bar{g}_b^*(v)$ be the least ordinal v' such that $\bar{\imath}_{\bar{d}_b,\bar{\alpha}_b}(v') \geqslant \pi^{-1}(f_b)(v)$. Then g_b^*, as a set of ordered pairs, is equal to $\pi``\bar{g}_b^*$. Now $\bar{g}_b^*$ is defined from the iterated ultrapower $b_{\bar{d}_b,\bar{\alpha}_b}: \overline{\mathfrak{m}}_{\bar{d}} \to \overline{\mathfrak{m}}_{\bar{\alpha}_b}$, but it only requires a finite part of the iterated ultrapower: the initial ultrapower by $\bar{E}_{\bar{d}}$ together with the support of $\pi^{-1}(f_b)$. Thus $\bar{g}_b^*$ can be defined inside $\overline{\mathfrak{m}}_{\bar{d}_b}$. Now define

$$\pi_{d_b}: \overline{\mathfrak{m}}_{\bar{d}_b} \longrightarrow \mathfrak{m}_{d_b} \stackrel{\text{def}}{=} \operatorname{ult}\big(\overline{\mathfrak{m}}_{\bar{d}_b}, \pi, \sup(\pi``\operatorname{len}(\bar{E}_{\bar{d}_b}))\big).$$

Then $\pi_{d_b}(\bar{g}_b^*)$ is the desired extension of g_b^*. Unfortunately, there is no reason to believe $\mathfrak{m}_d$ is in K, so we donot know that $\pi_{d_b}(\bar{g}_b^*)$ is in either Y or K. However, $\pi_{d_b}(\bar{g}_b^*)$ is cofinal in $g^*(d_{\iota,\sigma_b})$, so $\operatorname{cf}(g^*(d_{\iota,\sigma_b})) < \gamma_b$ in V and hence by elementarity in Y.

We now proceed as in the proof of Lemma 1.11(5). Since $\operatorname{cf}^Y(g_b^*(d_{\iota,\sigma_b})) < \gamma_b$ the covering lemma, applied in Y, implies that there is a function $k: d_b \to \sup(\mathrm{O}(d_b))$ in $K \cap Y$ such that range(k) is closed and is cofinal in each of the ordinals $g^*(d_{\iota,\sigma_b})$. for $\iota < \omega_1$. Then $\bar{k} = \pi^{-1}(k)$ is in $\bar{K}$ and hence is in $\overline{\mathfrak{m}}_{\bar{d}}$. Thus we can define a function $\bar{s}$ in $\overline{\mathfrak{m}}_{\bar{d}}$ by letting $\bar{s}(v)$ be the least ordinal v' such that $\bar{k}(v') \geqslant \bar{g}^*(v)$. Then $\bar{s} \in \bar{K}$, since $\bar{K}$ and $\overline{\mathfrak{m}}_{\bar{d}}$ contain the same subsets of $\bar{d}$, so the function $g_b = k \circ \pi(\bar{s})$ is in K. This function g_b has the required properties, and this completes the proof of case 1 of Lemma 3.15. □

Lemma 3.18 (Lemma 3.15, case 2a). *For almost all* (ι, b) *such that* $b \notin I$ *and* $\operatorname{cf}^K(d_{\iota,b}) < \gamma_b$ *we have* $\operatorname{cf}^K(d_{\iota,b}) = \operatorname{cf}^K(d_{\iota,\sigma_b})$.

Proof. This lemma, as well as the next two, depend of the following calculation. Each of the identities holds for almost all pairs (ι, b) which satisfy the hypothesis of this lemma.

$$\begin{aligned}
\pi^{-1}(\operatorname{cf}^K(d_{\iota,b})) &= \operatorname{cf}^{\bar{K}}(\bar{d}_{\iota,b}) \\
&= \operatorname{cf}^{\overline{\mathfrak{m}}_{\bar{\gamma}_b}}(\bar{d}_{\iota,b}) && \text{since } \operatorname{cf}^K(d_{\iota,b}) \leqslant \gamma_b \text{ and} \\
& && \mathscr{P}^{\overline{\mathfrak{m}}_{\gamma_\ell}}(\gamma_\ell) = \mathscr{P}^{\mathscr{K}}(\gamma_\ell) \qquad \text{(i)} \\
&= \bar{\imath}_{\bar{\gamma}_b,\bar{\alpha}_b}(\operatorname{cf}^{\overline{\mathfrak{m}}_{\bar{\gamma}_b}}(\bar{d}_{\iota,b})) && \text{since } \operatorname{cf}^K(d_{\iota,b}) < \gamma_b \qquad \text{(ii)} \\
&= \operatorname{cf}^{\overline{\mathfrak{m}}_{\bar{\alpha}_b}}(\bar{\imath}_{\bar{\gamma}_b,\bar{\alpha}_b}(\bar{d}_{\iota,b})) \\
&= \operatorname{cf}^{\overline{\mathfrak{m}}_{\bar{\alpha}_b}}(\pi^{-1}(f_b(d_{\iota,\sigma_b}))) && \text{since } d_{\iota,b} = \beta^Y(f_b(d_{\iota,\sigma_b}), \gamma_b, \alpha_b) \\
&= \operatorname{cf}^{\bar{K}}(\pi^{-1}(f_b(d_{\iota,\sigma_b}))) && \text{since } \operatorname{cf}(\bar{d}_{\iota,\sigma_b}) < \bar{\alpha}_b \\
&= \pi^{-1}(\operatorname{cf}^K(f_b(d_{\iota,\sigma_b})))
\end{aligned}$$

so

$$\begin{aligned}
\operatorname{cf}^K(d_{\iota,b}) &= \operatorname{cf}^K(f_b(d_{\iota,\sigma_b})) \\
&= \operatorname{cf}^K(d_{\iota,\sigma_b}) && \text{since range}(f_b) \text{ is cofinal} \\
& && \text{in } f_b(d_{\iota,\sigma_b}). \qquad \square
\end{aligned}$$

Lemma 3.19 (Lemma 3.15, case 2b). $\mathrm{cf}^K(d_{\iota,b}) \neq \gamma_b$ *for almost all* (ι, b) *such that* $b \notin I$.

Proof. All of the first sequence of equalities in the proof of Lemma 3.18 still hold in this case except for line (ii). In this case we get $\bar{\imath}_{\bar{\gamma}_b,\bar{\alpha}_b}(\mathrm{cf}^{\overline{\mathfrak{m}}_{\bar{\gamma}_b}}(\bar{d}_{\iota,b})) = \bar{\imath}_{\bar{\gamma}_b,\bar{\alpha}_b}(\bar{\gamma}_b) = \bar{\alpha}_b$. The rest of the equalities in this sequence still hold, so

$$\mathrm{cf}^K(f_b(d_{\iota,\sigma_b})) = \bar{\imath}_{\bar{\gamma}_b,\bar{\alpha}_b}(\mathrm{cf}^{\overline{\mathfrak{m}}_{\bar{\gamma}_b}}(\bar{d}_{\iota,b})) = \alpha_b,$$

but this is impossible since $\mathrm{cf}^K(f_b(d_{\iota,\sigma_b})) = \mathrm{cf}^K(d_{\iota,\sigma_b}) < \alpha_b$. □

Lemma 3.20 (Lemma 3.15, case 2c). *There is an* $\iota_0 < \omega_1$ *such that for all* $\iota, \iota' > \iota_0$, *for all but boundedly many* $b \in \boldsymbol{b}$, *if* $\mathrm{cf}^K(d_{\iota,b}) = \mathrm{cf}^K(d_{\iota',b})$ *then* $\mathrm{cf}^K(d_{\iota,\sigma_b}) = \mathrm{cf}^K(d_{\iota',\sigma_b})$.

Proof. Again, consider the sequence of equalities from the proof of Lemma 3.18. In this case, lines (i) and (ii) both fail. Since $\mathrm{cf}^{\bar{K}}(\bar{d}_{\iota,b}) = \mathrm{cf}^{\bar{K}}(\bar{d}_{\iota',b})$ and $\overline{\mathfrak{m}}_{\bar{\gamma}_b}$ is larger than $\bar{K}$ the argument for line (i) shows that $\mathrm{cf}^{\overline{\mathfrak{m}}_{\bar{\gamma}_b}}(\bar{d}_{\iota,b}) = \mathrm{cf}^{\overline{\mathfrak{m}}_{\bar{\gamma}_b}}(\bar{d}_{\iota',b})$. Then the argument for line (ii) gives $\bar{\imath}_{\bar{\gamma}_b,\bar{\alpha}_b}(\mathrm{cf}^{\overline{\mathfrak{m}}_{\bar{\gamma}_b}}(\bar{d}_{\iota,b})) = \bar{\imath}_{\bar{\gamma}_b,\bar{\alpha}_b}(\mathrm{cf}^{\overline{\mathfrak{m}}_{\bar{\gamma}_b}}(\bar{d}_{\iota',b}))$. The rest of the identities remain valid, so that $\mathrm{cf}^K(f_b(d_{\iota,\sigma_b})) = \mathrm{cf}^K(f_b(d_{\iota',\sigma_b}))$ and hence $\mathrm{cf}^K(d_{\iota,\sigma_b}) = \mathrm{cf}^K(d_{\iota',\sigma_b})$. □

This completes the proof of Theorem 3.1(3). □

3.3. Further results

In this subsection we extend the results of the two previous subsections. The first result, Theorem 3.22, completes the proof of Theorem 3.1 by strengthening the conclusion from $o(\kappa) = 2^\kappa$ to $o(\kappa) = 2^\kappa + \mathrm{cf}(\kappa)$ in the case $\mathrm{cf}(\kappa) > \omega_1$. The second, Theorem 3.23 shows that if $\mathrm{cf}(\kappa) = \omega$ then we can strengthen the conclusion from $o(\kappa) = 2^\kappa$ to $o(\kappa) = 2^\kappa + 1$ if either $\kappa < \aleph_\kappa$ or the GCH holds below κ.

Lemma 3.21. *Suppose that* κ *is a strong limit cardinal with* $\mathrm{cf}(\kappa) = \delta < \kappa$, *and that* $2^\kappa = \lambda > \kappa^+$ *where* λ *is regular and if* λ *is a successor cardinal then the predecessor of* λ *has cofinality greater than* κ. *If* $\mathrm{cf}(\kappa) = \omega$ *then also assume that there is an* $m < \omega$ *such that* $\{\alpha < \kappa : o(\alpha) = \alpha^{+m}\}$ *is bounded in* κ.

Then either κ *is a limit of accumulation points for* λ, *or* κ *is a limit of indiscernibles for extenders* $\mathscr{E}_\gamma$ *on* κ *with* $\gamma \geqslant \lambda$.

Proof. We claim that $\eta_b \geqslant \lambda$ for all but boundedly many $b \in \boldsymbol{b}$. Suppose the contrary. Since $\mathrm{cf}(\lambda) > \kappa$ it follows that there is an $\eta < \lambda$ such that $\eta_b < \eta$ for all but boundedly many $b \in \boldsymbol{b}$. Now the function f_b used to define $\mathscr{L}^f$ had range contained in η_b, so we can restrict ourselves to functions f with $\mathrm{range}(f) \subset \eta$. There are $(\eta^\kappa)^K < \lambda$ many such functions.

The only use of the hypothesis $(\mathrm{o}(\kappa)^{\kappa})^{K} < \lambda$ in the proof of Theorem 3.1 came in the proof of Corollary 3.12, where this hypothesis was used to show that there is a single function f such that $\mathscr{L}^f$ is cofinal in $\prod \boldsymbol{b}$. The reason was that there were only $(\mathrm{o}(\kappa)^{\kappa})^{K} < \lambda$ relevant functions f, while $\mathrm{tcf}(\prod(\boldsymbol{b})) = \lambda$ is greater than $(\mathrm{o}(\kappa)^{\kappa})^{K}$. Thus the conclusion of Corollary 3.12 is true under our assumption that $\eta_b < \lambda$ for cofinally many $b \in \boldsymbol{b}$. In the rest of the proof of Theorem 3.1 we showed that Lemma 3.11 leads to a contradiction. Hence the falsity of our current claim would lead to the same contradiction, and the claim must be true.

We now consider two cases. We have $\alpha_b = \kappa$ for cofinally many $b \in \boldsymbol{b}$. If $b \in \boldsymbol{b}$ has $\alpha_b = \kappa$ and is a limit of principal indiscernibles then b is an accumulation point for η_b. If $b \in \boldsymbol{b}$ has $\alpha_b = \kappa$ and is not a limit of principal indiscernible then γ_b is a principal indiscernible for some η' with $\mathrm{len}(\mathscr{E}_{\eta'}) \geqslant \eta_b$ so that $\eta' > \eta_b \geqslant \lambda$. One of these cases must hold for cofinally many $b \in \boldsymbol{b}$, and the lemma follows. □

Theorem 3.22 (Theorem 3.1(1)). *Suppose that κ is a strong limit cardinal with $\omega_1 < \delta = \mathrm{cf}(\kappa) < \kappa$, and that $2^{\kappa} \geqslant \lambda > \kappa^{+}$, where if λ is a successor cardinal then the predecessor of λ has cofinality greater than κ. Then $\mathrm{o}(\kappa) \geqslant \lambda + \delta$.*

If there is an $n < \omega$ such that $\{\alpha < \kappa : \mathrm{o}(\alpha) \geqslant \alpha^{+n}\}$ is bounded in κ then the result is also true for $\delta = \mathrm{cf}(\kappa) = \omega_1$.

Proof. Let $\boldsymbol{d}$ be given by Lemma 3.21, so that every member of $\boldsymbol{d}$ is either an accumulation point for λ or a principal indiscernible for some $\eta \geqslant \lambda$. Then every uncountable limit point of $\boldsymbol{d}$ of uncountable cofinality is an accumulation point for λ and hence, by Lemma 2.14, is a principal indiscernible for some $\eta \geqslant \lambda$. Continuing by induction, any ordinal which is a limit of $\omega_1^{\alpha+1}$ members of $\boldsymbol{d}$ is a principal indiscernible for some $\eta \geqslant \lambda + \alpha$. Thus $\mathrm{o}(\kappa) \geqslant \lambda + \delta$. □

In view of Silver's fundamental result in [22] the next observation is only of interest when $\mathrm{cf}(\kappa) = \omega$. As usual, all successors are calculated in K unless indicated otherwise.

Theorem 3.23. *Suppose that κ is a strong limit cardinal of cofinality ω and there is a $k < \omega$ so that the set of $\nu < \kappa$ such that $\mathrm{o}(\nu) > \nu^{+k}$ is bounded in κ. Suppose further that $\mathrm{o}(\kappa) = 2^{\kappa} > (\kappa^{++})^{V}$. Then* (i) *$\kappa = \aleph_{\kappa}$ and* (ii) *if $2^{\kappa} = (\kappa^{+m})^{V}$ then $2^{\nu} \geqslant (\nu^{+(m-1)})^{V}$ for cofinally many $\nu < \kappa$.*

Proof. Set $\lambda = 2^{\kappa}$ and let $n \geqslant m$ where $\lambda = (2^{+m})^{V} = (2^{+n})^{K}$. Since by hypothesis $\mathrm{o}(\kappa) = \lambda = 2^{\kappa}$, Lemma 3.21 implies that there is a cofinal sequence $\boldsymbol{b} = (b_i : i \in \omega)$ of accumulation points for λ. We can pick $\boldsymbol{b}$ so that for each $i < \omega$ there is a $\gamma_i < b_i$ so that $b_i = a^{Y}(\lambda, \gamma_i)$ for any precovering set Y.

If $\beta < \lambda$ then define $\boldsymbol{d}_{\beta}$ by $d_i = s^{Y}(\beta, \gamma_i)$ for any precovering set Y with $\beta \in Y$. Then $\beta \leqslant \beta'$ implies $\boldsymbol{d}_{\beta} \leqslant \boldsymbol{d}_{\beta'}$, so $\{\boldsymbol{d}_{\beta} : \beta < \lambda\}$ witnesses that $\mathrm{tcf}(\prod \boldsymbol{\beta}) = \lambda$.

Define ordinals $c_{k,i}$ for each $k, i \in \omega$ by recursion on k, setting $c_{0,i} = \gamma_i$ and $c_{k+1,i} = \ell^X(\kappa^{+(n-1)}, c_{k,i})$. Now define the sequence $\boldsymbol{c} = \sup_i \boldsymbol{c}_i$, that is, $c_i = \sup_{k<\omega} c_{k,i}$. Then $\boldsymbol{c}$ is an indiscernible sequence for κ and for each $i < \omega$ the sequence $\boldsymbol{c}_i^* = (c_{k,i} : k \in \omega)$ is an indiscernible sequence for c_i. Lemma 2.14 implies that $c_i = s^X(\beta_i, a_i)$ for some β_i with $\kappa^{+(n-1)} < \beta_i < \lambda$. In particular $i_{c_i,\kappa}(\mathrm{o}(c_i)) \geqslant \kappa^{+(n-1)}$ and hence, using Lemma 2.14, $\mathrm{o}(c_i) \geqslant c_i^{+(n-1)}$ for all sufficiently large $i < \omega$.

For each $i < \omega$ and $\beta < c_i^{+(n-1)}$ define $\boldsymbol{d}_{i,\beta}$ by $d_{i,\beta,k} = \ell^Y(\beta, c_{i,k})$ for an appropriate precovering set Y. Then $\boldsymbol{d}_{i,\beta} \in \prod \boldsymbol{c}_i$, and for each $\beta < c_i^{+(n-1)}$ there is β' such that $\beta < \beta' < c_i^{+(n-1)}$ so that $\boldsymbol{d}_{i,\beta} <_b \boldsymbol{d}_{i,\beta'}$. It follows that there are $c_i^{+(n-1)}$ distinct sequences $d_{i,\beta}$, and hence $2^{c_i} \geqslant c_i^{+(n-1)}$.

As usual, the cardinal $c_i^{+(n-1)}$ is computed in K.

Claim. *For almost all $i < \omega$*

$$\left|(c_i^{+(n-1)})^K\right|^V \geqslant \left(c_i^{+(m-2)}\right)^V. \qquad (*)$$

Furthermore, if equality holds then $\omega < \mathrm{cf}^V(c_i^+) = |c_i|^V < c_i$.

Proof. First note that if $0 < s < n$ and κ^{+s} is a cardinal in V then $\prod_{i<\omega} c_i^{+s}$ has true cofinality κ^{+s}. Suppose that $0 < s < s' < n$ and κ^{+s} and $\kappa^{+s'}$ are both cardinals in V. Then

$$\mathrm{tcf}\left(\prod_i c_i^{+s}\right) = \kappa^{+s} \neq \kappa^{+s'} = \mathrm{tcf}\left(\prod_i c_i^{+s'}\right)$$

and it follows that $\mathrm{cf}(c_i^{+s}) \neq \mathrm{cf}(c_i^{+s})$ for all but finitely many integers i. But if $\mathrm{cf}(c_i^{+s}) \leqslant c_i$ then since c_i is singular the covering lemma, Lemma 1.7, implies that $\mathrm{cf}(c_i^{+s}) = |c_i| < c_i$. If $|(c_i^{+(n-1)})^K|^V < (c_i^{+(m-2)})^V$ then there are at most $m-2$ distinct cofinalities available, out of the minimum $m-1$ needed, for $\{c_i^{+s} : 0 < s < n\}$. This contradiction proves the inequality of the claim. Furthermore, it shows that if the equality holds then $|c_i| < c_i$. Since κ is a limit cardinal, $c_i > \omega_1$ for all sufficiently large $i < \omega$ and it follows that $\omega < \mathrm{cf}^V(c_i^+) = |c_i|^V < c_i$, as claimed. □

To prove clause (i) of the theorem, suppose to the contrary that $\kappa < \aleph_\kappa$, and let $\tau < \kappa$ so that $\kappa = \aleph_\tau$. Then there are only $\tau^\omega < \kappa$ many countable sequences of cardinals below κ. Since $\mathrm{tcf} \prod \boldsymbol{b} = \lambda > \kappa$ it follows that there is a $\boldsymbol{\gamma} <_b \boldsymbol{b}$ so that $b_i \leqslant \gamma_i^+$ for cofinally many $i < \omega$. We can modify the definition of the sequence $\boldsymbol{c}_i$, if necessary, so that $\boldsymbol{\gamma} <_b \boldsymbol{c}$. Since $\lambda > (\kappa^{++})^V$, the claim implies that there are some $s < \omega$ such that c_i^{+s} is a cardinal in V for infinitely many $i < \omega$, and hence $c_i^{+s} \geqslant b_i$. This is impossible, since there is a sequence $\boldsymbol{d}$ of principal indiscernibles such that $\boldsymbol{c} < \boldsymbol{d} < \boldsymbol{b}$, and every principal indiscernible is a limit cardinal of K.

To prove clause (ii) of the conclusion, notice first that if the inequality $(*)$ is strict then $2^{c_i} > (c_i^{+(m-2)})^V$, so that the conclusion is true for $\nu = c_i$ for almost all $i < \omega$. If, on the other hand, equality holds in $(*)$ then set $\xi = |c_i| < c_i$. Then $\xi^\omega = c_i^\omega =$

$(c_i^{+(n-1)})^K \geqslant (c_i^{+(m-2)})^V$. Since $\xi = |c_i| = \mathrm{cf}(c_i^+)$ is regular, there is a $v < \xi$ such that $v^\omega = \xi^\omega \geqslant (\xi^{+(m-2)})^V \geqslant (v^{+(m-1)})^V$. $\square$

We used the strong version of the weak covering lemma, Lemma 1.7, which uses precovering sets which are not ω-closed, to get that $\mathrm{cf}(c_i^{+s}) = |c_i|$ whenever $|c_i| > \omega_1$. At the cost of some extra calculation it is possible to use the weaker version of Lemma 1.7 which is refered to in the remark following the statement of the lemma. This version implies that $(\mathrm{cf}(c_i^{+s}))^\omega \geqslant |c_i|$.

The next theorem is somewhat different but uses some of the ideas of Theorem 3.1.

Theorem 3.24. *If* $2^\omega < \aleph_\omega$ *and* $2^{\aleph_\omega} > \aleph_{\omega_1}$ *then there is a sharp for a model with a strong cardinal.*

Proof. The proof depends on the following results of Shelah. The definitions may be found in [21].

Theorem 3.25 (Shelah, [21]). 1. $\mathrm{pcf}(\omega_n : n < \omega) = \{\kappa \leqslant (\aleph_\omega)^\omega : \kappa \text{ is regular}\}$.

2. *Assume that* $\boldsymbol{a}$ *is a set of regular cardinals such that* $2^{|\boldsymbol{a}|} < \min(\boldsymbol{a})$. *Then for every* $\boldsymbol{d} \subset \mathrm{pcf}(\boldsymbol{a})$ *and every* $\mu \in \boldsymbol{d}$ *there is a set* $\boldsymbol{d}' \subset \boldsymbol{d}$ *such that* $|\boldsymbol{d}'| \leqslant |\boldsymbol{a}|$ *and* $\mu \in \mathrm{pcf}(\boldsymbol{d}')$.

Let A be the set of cardinals δ^+ of K below $\aleph_{\omega_1}$ such that either $o(\alpha) < \delta$ for all $\alpha \leqslant \delta$ or else δ is larger than every measurable cardinal of K smaller than $\aleph_{\omega_1}$. The set A is unbounded in $\aleph_{\omega_1}$ since there are no overlapping extenders in K.

We claim that if $B \subset A$ with $|B| < \inf B$ then $\mathrm{pcf}(\prod B) \leqslant (\sup B)^+$. To see this, let $\kappa = \sup B$ and define, in K, functions $a_f \in \prod A$ for each function $f : \kappa \to \kappa$ in K by setting, for $v = \delta^{+K}$ in A, $a_f(v) = \sup(f``\delta) \cap v$. We will show that $\{a_f \restriction B : f \in K\}$ is cofinal in $\prod B$. If there is a largest measurable cardinal in K below $\aleph_{\omega_1}^V$ then this follows from Lemma 1.7, the weak covering lemma. Otherwise if $\boldsymbol{b} \in \prod B$ then use the covering lemma, together with the fact that Proposition 1.9 implies that v cannot be an indiscernible since $o(\alpha)^+ < v$ for $v \in A$, to show that there is a function $f \in K$ such that $b_v \in f``\delta$ whenever $v = \delta^{+(K)}$ is in B. Thus $\boldsymbol{b} <_b a_f$.

Now let $A' = \{|v| : v \in A\}$. Then A' is unbounded in $\aleph_{\omega_1}$ and it follows by Theorem 3.25 that there is a countable subset B' of A' such that $\aleph_{\omega_1} \in \mathrm{pcf}(B')$. Let $B \subset A$ so that $B' = \{|v| : v \in B\}$. Then for each $v \in B$ the weak covering lemma implies that $\mathrm{cf}(v) = |v|$, so that $\mathrm{pcf}(\prod B') = \mathrm{pcf}(\prod B)$ and hence $(\sup B)^+ < \aleph_{\omega_1} \in \mathrm{pcf}(\prod B)$. The contradiction completes the proof of the theorem. $\square$

4. Open problems

There are a number of open problems which are related to results in this paper. The most obvious questions concern the situation when κ has cofinality ω. The most general question is whether the definability and uniqueness of indiscernible sequences break down at κ^ω for cardinals κ of cofinality ω. Since the first version of this paper, Gitik [7] has given a negative answer to this first question:

Question 1. Is it still true if $\mathrm{cf}(\kappa) = \omega$ that the notion of being an indiscernible sequence in X for the constant sequence κ belonging to a sequence β is independent of the precovering set X?

The application concerning the singular cardinal hypothesis may still be true, however. Since $o(\kappa) = \kappa^{+\omega}$ is enough to give $2^\kappa = \kappa^{+(\omega+1)}$ the simplest unknown cases are the following:

Question 2. If κ is a strong limit cardinal with $2^\kappa \geqslant \kappa^{+(\omega+2)}$ then must there be an inner model of $\exists\kappa\, o(\kappa) \geqslant \kappa^{+(\omega+1)}$ then must there be an inner model of $o(\kappa) = \kappa^{+\omega}$?

Question 3. What is the exact consistency strength of $\mathrm{cf}(\kappa) = \omega_1$ and $2^\kappa = \lambda$ for regular $\lambda > \kappa^+$?

By Theorem 3.1 together with results of Woodin (see [1]) the answer lies between $o(\kappa) = \lambda$ and $o(\kappa) = \lambda + \omega_1$.

A second problem concerns our use of δ-closed precovering sets X. In Dodd and Jensen's work this assumption was weakened to $\omega_1 \subset X$. In [11] these methods have been extended to the core models used in this paper, but we do not see how to avoid the use of δ-closed precovering sets for the Gitik games in the proof of Lemma 2.5. The following can be regarded as a test question.

Question 4. Suppose that κ is singular, $2^\kappa = \lambda > \kappa^{++}$ and $2^\alpha \leqslant \kappa^+$ for $\alpha < \kappa$. Does it follow that there is an inner model with $o(\kappa) \geqslant \kappa^{++}$?

The final question concerns what happens when there exist overlapping extenders. We give two possible test questions.

Question 5. Suppose that $2^\omega < \aleph_\omega$ and $2^{\aleph_\omega} > \aleph_{\omega_1}$. Does it follow that there is an inner model with a Woodin cardinal?

Question 6. Suppose that there is no model with a Woodin cardinal and that the Steel core model [23] exists. If κ is a singular strong limit cardinal of uncountable cofinality such that $2^\kappa = \lambda$ does it follow that $o(\kappa)^K \geqslant \lambda$ in K?

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