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On the odd cycles of normal graphs

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Abstract

A graph is normal if there exists a cross-intersecting pair of set families one of which consists of cliques while the other one consists of stable sets, and furthermore every vertex is obtained as one of these intersections. It is known that perfect graphs are normal while C_5, C_7, and $\overline{C}_7$ are not. We conjecture that these three graphs are the only minimally not normal graphs. We give sufficient conditions for a graph to be normal and we characterize those normal graphs that are triangle-free. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

We assume familiarity with basic notions of graph theory (cf., for instance, [2]). Our graphs will be undirected and simple (no loops and no multiple edges). A simple undirected graph G is defined in terms of a finite set $V(G)$, its set of vertices, and a subset $E(G)$ of unordered couples of elements of $V(G)$, called the set of edges. A *clique* in the graph G is a set of pairwise adjacent vertices; a *stable set* in G is a set of pairwise non-adjacent vertices; a *coloring* of the vertices of G is a partition of $V(G)$ in stable sets (colors). As usual, $\omega(G)$ denotes the largest size of a clique in G, and is called the *clique number* of G; $\chi(G)$ denotes the minimum number of colors in a coloring of $V(G)$ and is called the *chromatic number* of G. An *odd hole* is a chordless cycle whose length is odd and at least five; a cycle with k vertices will be denoted by C_k. Finally, our subgraphs will always be induced.

Normal graphs form a class that can, in many ways, be considered a closure of that of perfect graphs. Perfect graphs were introduced by Claude Berge in 1962 [1] with a clear reference to Shannon's information-theoretic problem of finding the so-called zero-error capacity of a discrete memoryless channel [7]. Shannon's problem has a

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purely graph-theoretic formulation, regarding the asymptotic growth of the largest clique in certain product graphs. The *co-normal product* of the graphs F and G, denoted by $F \times G$ is defined, following Berge [2], by the vertex set

$$V(F \times G) = V(F) \times V(G)$$

and the edge set

$$E(F \times G) = \{[(a_1a_2),(b_1b_2)]; [a_1,b_1] \in E(F) \quad \text{and/or } [a_2,b_2] \in E(G)\}.$$

The (logarithmic) Shannon capacity of a graph G is the always existing

$$\lim_{n\to\infty} \frac{1}{n}\log_2 \omega(G^n).$$

Obviously, for every graph G, $\omega(G) \leqslant \chi(G)$. Shannon [7] observed that equality in the last inequality implies that, for every natural n,

$$\omega(G^n) = [\omega(G)]^n$$

(where G^n denotes the co-normal product of G by itself, n times), which makes the otherwise difficult problem of determining the capacity of G trivial in this case.

Berge [1] calls a graph *perfect* if for all its induced subgraphs the chromatic number and the clique number are the same. In this context it is natural to ask whether the co-normal product of perfect graphs is perfect. The answer is trivially no [4], and it is here that normal graphs come to play.

Normal graphs can be defined in terms of cross-intersecting set families. A graph G is normal if there exist two coverings $\mathscr{C}$ and $\mathscr{S}$ of its vertex set $V(G)$ such that every member of $\mathscr{C}$ is a clique in G, every member of $\mathscr{S}$ is a stable set in G, and $C \cap S \neq \emptyset$ for every $C \in \mathscr{C}$ and $S \in \mathscr{S}$.

From the definition

Observation 1. *The complement of a normal graph is normal.*

(The reader will recall that the analogous property for perfect graphs is the subject of the famous weak perfect graph conjecture which was proved by Lovász [6] more than a decade after it was stated by Claude Berge.)

Observation 2. *A graph is normal if and only if all of its components are normal.*

Körner [4] has shown the following three simple properties of normal graphs.

(P1) Every perfect graph is normal.

(P2) The co-normal product of normal graphs is normal.

(P3) An odd hole is normal iff it has at least nine vertices.

These properties are interesting for they show that normal graphs represent a non-trivial extension of the class of perfect graphs: every perfect graph is normal but not every graph is. Normal graphs come up in a natural way in an information-theoretic context, cf. [5,3]. The last paper contains a detailed analysis of the connection between normal and perfect graphs in terms of *graph entropy*.

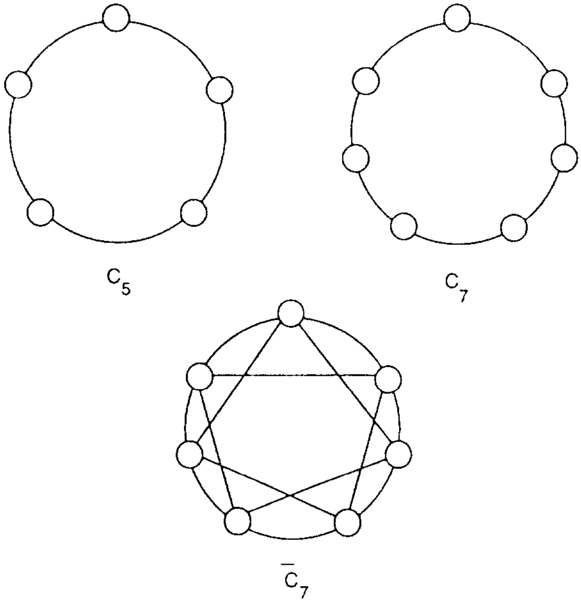

Fig. 1. The graphs C_5, C_7, and $\overline{C}_7$.

Property (P3) shows that the two odd holes C_5 and C_7 are not normal, and so neither is $\overline{C}_7$. These three graphs, shown in Fig. 1, will play a central role in this paper.

One of the most exciting open problems in graph theory is the so-called strong perfect graph conjecture (SPGC) of Berge, saying that a graph G is perfect iff neither G nor its complement contain odd holes (as induced subgraphs). The SPGC is usually known in the following form: the only minimally imperfect graphs are precisely the odd holes and their complements (a *minimally imperfect* graph is nothing but an imperfect graph such that all of its proper induced subgraphs are perfect).

Since normal graphs are similar to perfect graphs in several ways, it is natural to ask whether a similar "characterization" of normal graphs in terms of forbidden subgraphs exists.

For this purpose, we introduce the concept of minimally not normal graphs:

Definition 1. A graph is called minimally not normal if it is not normal, but every proper induced subgraph of it is normal.

We conjecture that:

Conjecture 1. *A graph with no C_5, C_7, and $\overline{C}_7$, as induced subgraph, is normal.*

The validity of this conjecture would imply that the only minimally not normal graphs are precisely C_5, C_7, and $\overline{C}_7$. The aim of this paper is to discuss this conjecture.

Note that if Conjecture 1 were true, it would immediately give a sufficient condition for a graph to be normal. However, the non-existence of C_5, C_7, and $\overline{C}_7$ in a graph

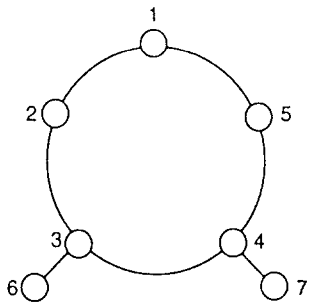

Fig. 2. A normal graph.

is not necessary for the graph to be normal. For example, consider the normal graph in Fig. 2. (To see that is normal, choose $\mathscr{C} = \{\{1,2\},\{1,5\},\{3,6\},\{4,7\}\}$ and $\mathscr{S} = \{\{2,5,6,7\},\{1,3,7\},\{1,4,6\}\}$ as clique and stable set coverings.) Note that such a graph contains the graph C_5 that is not normal.

Hence, an induced subgraph of a normal graph is not necessarily normal. Since this is not the case of perfect graphs, in order to get a better analogy with them, we want to introduce a hereditary property.

Definition 2. A graph is called strongly normal if each of its induced subgraphs is normal.

The graph in Fig. 2 is an example of a normal graph that is not strongly normal. Note that strongly normal graphs are not necessarily perfect as shown, e.g., by C_9. On the other hand, every perfect graph is obviously strongly normal. In terms of strongly normal graphs, Conjecture 1 is equivalent to the following:

Conjecture 2. *A graph G is strongly normal iff neither G nor its complement contain a C_5 or a C_7 as an induced subgraph.*

The interest of this conjecture lies in the fact that it would automatically lead to a polynomial time algorithm for the recognition of strongly normal graphs. Furthermore, if this conjecture were proved it would yield a new property of minimally imperfect graphs, namely that all these graphs are strongly normal, with the only three exceptions of C_5, C_7, and $\overline{C}_7$.

2. When is a graph normal?

In the previous section we remarked that if Conjecture 1 is true then every graph not containing C_5, C_7 and $\overline{C}_7$ as an induced subgraph is normal (strongly normal). In this section we describe a sufficient condition for a graph to be normal, in the hope that it might help in establishing the conjecture.

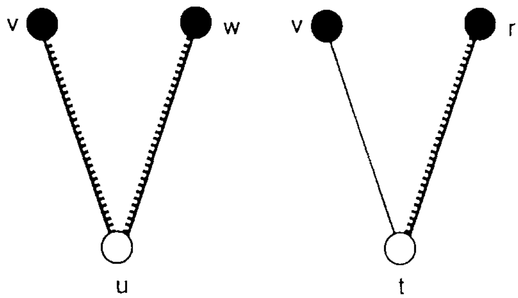

Fig. 3. $w \in b(v)$ and $r \in d(v)$.

Consider an arbitrary graph G and let $\mathscr{F}$ be a minimal (with respect to set inclusion) subset of $E(G)$ such that every vertex of G is an endpoint of some edge in $\mathscr{F}$. A set $\mathscr{F}$ with this property is usually called a *minimal edge cover* of G. Notice that a minimal edge cover is a union of vertex disjoint induced stars.

Now consider an arbitrary odd cycle Q in G; remark that Q may have chords. We are interested in the distribution of the edges of $\mathscr{F}$ alongside the cycle Q. We shall say that a vertex of Q is *even* (with respect to $\mathscr{F}$) if it is the endpoint of an even number of edges of Q that are in $\mathscr{F}$. (Note that this even number is either zero or two.) Since the cycle Q is odd, it is easy to verify that Q must have an odd number of even vertices.

Definition 3. A minimal edge cover of a graph G is a *nice cover* if every odd cycle in G has at least three even vertices.

Now we are ready to prove the main result of this section.

Theorem 1. *Every graph that has a nice cover is normal.*

Proof. Let G be a graph and let $\mathscr{F}$ be a nice cover of G. We want to show that G is normal. (Note that Observation 2 allows us to assume that G is connected.) Now, let v be an arbitrary vertex of G. We shall denote by $b(v)$ the set of all vertices w for which there exists a vertex u such that $uv \in \mathscr{F}$ and $uw \in \mathscr{F}$, and by $d(v)$ the set of all vertices r for which there exists a vertex t such that $rt \in \mathscr{F}$ and $vt \in (E(G) - \mathscr{F})$ (cf. Fig. 3).

To show that G is normal, we must find a clique cover $\mathscr{C}$ and a stable set cover $\mathscr{S}$ such that every clique in $\mathscr{C}$ intersects every stable set in $\mathscr{S}$. For this purpose, set $\mathscr{C} = \mathscr{F}$. Since $\mathscr{F}$ is a clique cover of $V(G)$, we only need find stable sets each of which intersects all cliques (edges) in $\mathscr{F}$ and whose union covers $V(G)$. For this purpose, we shall show that for an arbitrary vertex v there exists a stable set S containing v that intersects all edges of $\mathscr{F}$.

To do so, we shall apply the following rule:

$$\text{if a vertex } a \text{ is inserted into } S\text{, then all vertices in } b(a) \cup d(a) \text{ are also included in } S. \tag{1}$$

At the beginning, we insert vertex v into S and we apply rule (1), until S cannot be enlarged anymore. We are going to show that the set S so obtained is a stable set.

First note that, by construction, for every vertex u in $S - \{v\}$ there exists in G a path $P = w_1, w_2, \ldots, w_r$ joining $v = w_1$ to $u = w_r$ with the following two properties:

(a) the two edges w_2w_3 and $w_{r-1}w_r$ are in $\mathscr{F}$,

(b) no two consecutive edges of P are either both in $\mathscr{F}$ or in $E(G) - \mathscr{F}$, with the only possible exception of the two edges w_1w_2 and w_2w_3.

We shall call such a path a $v \to u$ *generating path*. Let $l(v,u)$ be the smallest length of any such path (number of edges). Note that $l(v,u)$ is even.

Notice that, for every $v \to u$ generating path P,

$$\text{no two vertices } w_1, w_2 \text{ in } S \cap P \text{ are adjacent} \tag{2}$$

for otherwise the edge w_1w_2 along with the sub-path of P joining w_1 and w_2 would induce in G an odd cycle with only one even vertex (namely w_i or one of its neighbours, with $i = 1$ or 2), contradicting the assumption that $\mathscr{F}$ were a nice cover.

Hence, if S is not a stable set, then $S - \{v\}$ includes a pair of adjacent vertices. Let A denote the set of all pairs of adjacent vertices in S, and let (u', u'') be a pair in A such that

$$l(v,u') + l(v,u'') = \min\{l(v,i) + l(v,j) : (i,j) \in A\}. \tag{3}$$

Let P' and P'' denote a $v \to u'$ generating path of length $l(v,u')$ and a $v \to u''$ generating path of length $l(v,u'')$, respectively. Let v^* be the common vertex of P' and P'' closest to u' along P' (and thus, closest to u'' along P'' as well). Note that, vertex v^* is different from u' (for otherwise, both u' and u'' would belong to $S \cap P''$, contradicting property (2)); similarly v^* is different from u''.

Now, call $\hat{P}'$ and $\hat{P}''$ the subpaths of P' and P'' joining v^* to u' and joining v^* to u'', respectively. Let e' and e'' be the edges of $\hat{P}'$ and $\hat{P}''$ incident to vertex v^* (cf. Fig. 4).

If both or none of e' and e'' are in $\mathscr{F}$ or if $v^* = v$, then, by properties (a) and (b), the cycle formed by $\hat{P}'$, $\hat{P}''$, along with the edge $u'u''$ is odd and has exactly one even vertex (namely, v^* or one of its neighbours), contradicting again the assumption that $\mathscr{F}$ were a nice cover. Hence, we can assume that precisely one of e' and e'' is in $\mathscr{F}$, say e', and that $v^* \neq v$. Note that the path $\hat{P}'$ has an odd length (because v^* is different from v). Let f' be the edge of $P' - \{\hat{P}'\}$ incident to v^*; write $f' = v^*w$. Since properties (a) and (b) imply that every vertex of P' (P'') which is at an even distance (along P' (P'')) from u' (u'') belongs to S, it follows that both v^* and w are in S: v^* is in S, since it is at even distance from u''along P''; w is in S, since it is at even distance from u' along P'. Note that, by (2), w is different from v. But then the

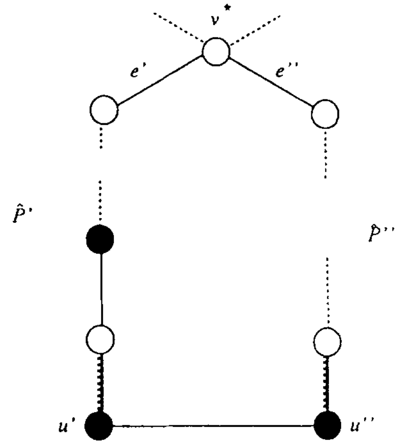

Fig. 4. The paths $\hat{P}'$ and $\hat{P}''$.

pair (w, v^*) belongs to A and $l(v,w) + l(v,v^*) < l(v,u') + l(v,u'')$, contradicting (3). Hence, S is a stable set.

Clearly, if S intersects all the edges in $\mathscr{F}$, we are done. Otherwise, let $\mathscr{F}'$ be the set of all edges in $\mathscr{F}$ that do not intersect S. Obviously, no endpoint of an edge in $\mathscr{F}'$ is adjacent to any vertex in S: if $st \in \mathscr{F}'$ and s is adjacent to some vertex u in S, then $t \in b(u)$ (in case $su \in \mathscr{F}$) or $t \in d(u)$ (in case $su \notin \mathscr{F}$). But then, by rule (1), vertex t is in S. Hence, we can arbitrarily choose any edge in $\mathscr{F}'$ and add precisely one endpoint of it to S and apply again rule (1). Notice that in the application of rule (1), we may include in S vertices that were already in it. Clearly, for the same reasoning as before, the final set S will be again a stable set. Thus the theorem follows. □

Based on the proof of Theorem 1, it is easy to check whether a given minimal edge cover $\mathscr{F}$ of a graph is a nice cover. Theorem 1 asserts that the existence of a nice cover in a graph is sufficient for its normality. The following result shows that the only minimal edge covers of a graph that can be chosen as parts of cross-intersecting pairs are precisely the nice ones.

Theorem 2. *Given a graph and a minimal edge cover $\mathscr{F}$ of its vertex set, then the graph is normal with $\mathscr{F}$ chosen as the clique cover $\mathscr{C}$ only if $\mathscr{F}$ is a nice cover.*

Proof. Let G be a normal graph and let $\mathscr{F}$ be a minimal edge cover. Since G is normal, there exist a clique cover $\mathscr{C}$ and a stable set cover $\mathscr{S}$ such that every clique in $\mathscr{C}$ intersects every stable set in $\mathscr{S}$. We only need to show that if $\mathscr{C} = \mathscr{F}$ then $\mathscr{F}$ must be a nice cover. For this purpose, assume the contrary: $\mathscr{F}$ is not a nice cover, and so there exists in G an odd cycle Q such that Q has only one even vertex.

Let $\{u_1, u_2, \ldots, u_{2k+1}\}$ denote the set of vertices of Q, and let $E(Q) = \{u_1u_2, u_2u_3, \ldots, u_{2k}u_{2k+1}, u_{2k+1}u_1\}$ denote its edge set. By assumption, Q has precisely one even vertex;

without loss of generality, we can assume that this vertex is u_1. It follows that we can have only two cases:

Case 1: $\mathscr{F} \cap E(Q) = \{u_{2i}u_{2i+1}, =1, \ldots, k\}$.

Case 2: $\mathscr{F} \cap E(Q) = \{u_{2k+1}u_1, u_1u_2\} \cup \{u_{2i+1}u_{2i+2},\ i = 1, \ldots, k-1\}$.

First, let us examine Case 1. Let S' denote the stable set in $\mathscr{S}$ that contains vertex u_1. Since S' must intersect all edges in $\mathscr{F} \cap E(Q)$, it is easy to see that S' must also contain the vertices $u_3, u_5, \ldots, u_{2k+1}$, which is impossible (because u_1 and u_{2k+1} are joined by an edge). Next, let us examine Case 2. Let S'' denote the stable set in $\mathscr{S}$ that contains vertex u_2. Since S'' must intersect all edges in $\mathscr{F} \cap E(Q)$, it is easy to see that S'' must also contain the vertices $u_4, u_6, \ldots, u_{2k}, u_1$, which is again impossible (because u_1 and u_2 are joined by an edge). Hence in both cases, we get a contradiction with the assumption that $\mathscr{F}$ were not a nice cover. Thus the theorem follows. □

Clearly, not every graph has a nice cover, in particular not every normal graph has it. In fact, every complete graph with at least three vertices is normal (because it is perfect) and no edge cover of it is nice. Hence, the property of having a nice cover is only a sufficient condition for a graph to be normal. However, Theorems 1 and 2 imply that in the case of a triangle-free graph this condition becomes also necessary.

Corollary 1. *A connected triangle-free graph is normal if and only if it has a nice cover.*

Finally, we mention that, in the context of triangle-free graphs, the conjecture stated in the introduction becomes:

Conjecture 3. *A triangle-free graph with no C_5 and C_7 is normal.*

Hence, from Corollary 1, it follows that to prove Conjecture 3, it is sufficient to show that every triangle-free graph with no C_5 and C_7, admits a nice cover.

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