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RAMSEY NUMBERS OF GRAPHS WITH LONG TAILS

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The ramsey number of a connected nonbipartite graph G with a sufficiently long path emanating from one of its points is found to be $(n-1)(\chi-1)+s$, where n is the number of points of G, χ is the chromatic number of G, and s is the minimum possible number of points in a color class in a χ-coloring of the points of G.

1. Introduction

If G and H are graphs, then the ramsey number $r(G, H)$ is the least integer r such that if every line of the complete graph K_r is colored either red or blue, then either the red subgraph contains a copy of G or the blue subgraph contains a copy of H. We write $r(G)$ for the 'diagonal' ramsey number $r(G, G)$. Ramsey's Theorem [5] guarantees that $r(G, H)$ is finite. While the classical ramsey numbers $r(K_n, K_m)$ seem beyond computation for all but very small values of n and m, much progress has been made [4] in determining ramsey numbers of sparser graphs, such as paths, stars, and cycles. In this paper we consider graphs in which this sparseness is provided by the existence of a 'long tail'.

It is known that for any connected graph G, one has $r(G) \geq (n-1)(\chi-1)+s$, where n is the number of points of G, χ is the chromatic number of G, and s is the chromatic surplus of G (the smallest possible number of points in a color class in any coloring of the points of G using χ colors). Equality holds in the case of many of the sparse graphs studied so far. Our principal result is that $r(G) = (n-1)(\chi-1)+s$ for an arbitrary connected nonbipartite graph G which has a sufficiently long path emanating from one of its points.

We review the lower bound for $r(G)$ in Section 2. In Section 3 we prove the main theorem, using an off-diagonal result of S. Burr. Finally we indicate some generalizations and open questions.

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2. The lower bound

For completeness we include the proof of the following lemma, due to Burr [1], on the lower bound for the ramsey number of a graph. By the chromatic surplus $s(G)$ of a graph G, we mean the minimum, taken over all $\chi(G)$-colorings of the points of G, of the smallest number of points in a color class.

Lemma 1. *Let G be a connected graph with n points, chromatic number χ, and chromatic surplus s. Then $r(G) \geq (n-1)(\chi-1)+s$.*

Proof. If the lines of $K((n-1)(\chi-1)+s-1)$ are colored so that the red subgraph is the disjoint union of $\chi-1$ copies of K_{n-1} and one copy of K_{s-1}, then G has too many points to be contained in a connected component of the red subgraph, and too high a chromatic number or chromatic surplus to be contained in the blue subgraph. □

3. The upper bound

In many cases the inequality in Lemma 1 holds as an equality, for example when G is a triangle with a path emanating from one of its vertices [3]. Our main result is a generalization of this example to an arbitrary nonbipartite graph with a long path ('tail') emanating from one of its points. Since the ramsey numbers of complete graphs, for example, grow exponentially, the requirement that the tail be long, relative to the rest of the graph, cannot be omitted. We return to this point, as well as to the hypothesis that $\chi(G) \geq 3$, in Section 4.

In this section we use $|G|$ to denote the number of points of a graph (or a set) G. Also $K(p_1, p_2, \ldots, p_k)$ is the complete k-partite graph in which the independent sets have $p_1, p_2, \ldots, p_k$ points, respectively. If $p_1 = p_2 = \cdots = p_{k-1}$, then we abbreviate this to $K(p_1^{(k-1)}, p_k)$.

Our theorem relies on the following result of Burr [1], which we state without proof.

Lemma 2. *Let G be a graph and H a connected graph with at least one line e. For $n > |H|$ define H_n to be H with line e subdivided by $n-|H|$ internal points. Then there is an N (depending on G and H) such that for every $n > N$,*

$$r(G, H_n) = (n-1)(\chi(G)-1)+s(G). \quad \square$$

We will consider first the hardest case; the general result then follows.

Theorem. *Let G_n be the graph consisting of the complete k-partite graph $K(t^{(k-1)}, s)$, $t \geq s \geq 1$, $k \geq 3$, together with a path of length $n-[(k-1)t+s]$ emanating from one of the points in one of the independent sets of size t. Then there is an N*

(depending on k, t, and s) such that for every $n > N$,

$$r(G_n) = (n-1)(k-1) + s.$$

Proof. Given Lemma 1, it remains to show that $r(G_n) \leq (n-1)(k-1) + s$. Set $q = (k-1)t + s$, the number of points in $K(t^{(k-1)}, s)$. We choose N large enough to satisfy the conclusion of Lemma 2, for $G = K((sqt + t^2 + t)^{(k-2)}, s)$ and $H = K(t^{(k-1)}, s) \cup \{e\}$, where line e connects a point in one of the independent sets of size t with a point not in $K(t^{(k-1)}, s)$. Also we arrange that $N > r(G_{n_0})$, where $n_0 = 4(sq + t + q)$. Fix $n > N$, let $r = (n-1)(k-1) + s$, and let a 2-coloring of the lines of K_r be given.

By the choice of N we know that there is a monochromatic G_{n_0} in K_r. Let l be the least integer such that there is no monochromatic G_l in K_r. We must show that $l > n$. Suppose that $l \leq n$, and fix a copy of G_{l-1} that appears in, say, red. We will arrive at a contradiction by showing that in fact there is a monochromatic G_l. Let v_0 be the point of G_{l-1} of degree 1 in G_{l-1} (*i.e.*, the end of the tail), and consecutively label the other points in the tail of G_{l-1} by $v_1, v_2, \ldots$. Note that the tail has length at least $4(sq + t)$. Finally let A be the set of all $l-1$ points in the G_{l-1} and let B be the set of remaining points. Then

$$|B| = (n-1)(k-1) + s - (l-1) \geq (n-1)(k-2) + s,$$

and every line from v_0 to B is blue.

By Lemma 2 (with the roles of red and blue reversed) there is either a red G_n or a blue $K((sqt + t^2 + t)^{(k-2)}, s)$ in the subgraph induced by B. In the former case we are finished, so we assume the latter. Let S be the independent set of size s in the blue $K((sqt + t^2 + t)^{(k-2)}, s)$. Consider now the points $v_1, \ldots, v_{sq+t}$ on the tail of the red G_{l-1}. There may be t points among them, $w_1, \ldots, w_t$, such that there is no red path, starting at such a point, otherwise using only points of B, and ending in S. If not, then there are sq points among them, from each of which there are red paths, through B, to S. We deal with these two cases separately.

In the former case we complete the construction of a blue $K(t^{(k-1)}, s)$ as follows. Successively for $i = 1, \ldots, t$, consider a longest red path P_i in $B \cup \{w_i\} - \bigcup_{j<i} P_j$, starting at w_i. Its length cannot exceed $sq + t$, or else the tail of the red G_{l-1} could have been lengthened. Let u_i be the terminus of P_i. Then all the lines from u_i to the points of B not in the red paths are blue. Thus except for at most $t(sq + t)$ points in B, none of which are in S, every line from each u_i to B is blue. Adjoining $\{u_1, \ldots, u_t\}$ to the $K((sqt + t^2 + t)^{(k-2)}, s)$ and deleting all the other points in B to which some u_i is joined by a red line, we obtain a blue $K(t^{(k-1)}, s)$.

In the latter case, we follow a technique used in the proof of Lemma 2. Given the sq points in the tail of G_{l-1} joined by red paths in B to S, there must be a set of q points $v_{i_1}, \ldots, v_{i_q}$ among them joined by red paths in B to a fixed point u in S. Without loss of generality, we assume $1 \leq i_1 < i_2 < \cdots < i_q$. No two of these may be consecutive, or else the tail in the red G_{l-1} could be lengthened by inserting a red path in B between them. For the same reason, the lines uv_{i_j+1} must be blue for $1 \leq j \leq q$. Now each line $v_{i_j+1}v_{i_m+1}$ must be blue, $1 \leq j < m \leq q$, for otherwise the tail could be lengthened by proceeding from v_{i_j} to v_{i_m} through B, returning to v_{i_j+1} backwards along the tail, and then jumping to v_{i_m+1}, rather than following the tail from v_{i_j} to v_{i_m+1}. Thus $\{u, v_{i_1+1}, \ldots, v_{i_q+1}\}$ forms a blue K_{q+1}.

In either case, then, we have a blue $K(t^{(k-1)}, s)$ in $B \cup \{v_1, \ldots, v_{sq+t}\}$ containing at least one point of B in one of the independent sets of size t. Note that since $l \geq n_0$, there remain at least $1+3l/4$ points in B and at least $1+3l/4$ points in the tail of the red G_{l-1}, all disjoint from the blue $K(t^{(k-1)}, s)$. Let D be a set of $[3l/2]+1$ such points, half chosen from B and half chosen from the tail of the G_{l-1} (where if $|D|$ is odd, then one 'half will exceed the other by one point). Because $r(C_m) = [3m/2]-1$ for even $m > 4$ [2], we can find an even alternating cycle of length at least l in the subgraph induced by D. If the cycle is red, then it contains a point of $A \cap D$ and gives us a red G_l. If the cycle is blue, then it contains a point of $B \cap D$ and is therefore connected by a blue line to v_0 and thence to a point in an independent set of size t in the blue $K(t^{(k-1)}, s)$; thus we have a blue G_l. In either case this contradicts the choice of l and completes the proof. □

We remark that the condition in the theorem that the tail emanate from one of the points in an independent set of size t, rather than from the independent set of size s, is no real restriction, since t could be incremented by one and the first line of the path thought of as being part of the $K((t+1)^{(k-1)}, s)$.

Corollary. *Let G be an arbitrary connected nonbipartite graph, and for $n > |G|$ let G_n be the graph consisting of G together with a path of length $n - |G|$ emanating from one of the points of G. Then there is an N (depending on G) such that for every $n > N$,*

$$r(G_n) = (n-1)(\chi(G)-1) + s(G).$$

Proof. That $r(G_n) \geq (n-1)(\chi(G)-1) + s(G)$ follows from Lemma 1. On the other hand, G is a subgraph of some $K(t^{(\chi(G)-1)}, s)$, so that the theorem and the remark following it yield the opposite inequality □

4. Generalizations and open questions

Quite a number of interesting problems are left open by the results presented here. For instance, can the corollary be extended to bipartite graphs? It appears probable that it can be, using similar methods, but certain difficulties must be overcome. These difficulties are illustrated by the fact that in this case the chromatic surplus depends on the length of the tail, in contrast to the cases we have considered. A more straightforward problem is likely to be the off-diagonal case $r(G_m, H_n)$, but even here some difficulties arise.

Another direction to pursue is that in which the path is not free, but suspended; that is, both endpoints may have large degree. More generally, what can be said if G_n is merely any graph on n points which is homeomorphic to a fixed graph G? This is the sort of graph that is considered in [1], but only when paired with some fixed graph F.

Finally, it would be very desirable to get reasonable bounds on how large n must be before the conclusion of the theorem holds. The only general estimates that one could easily derive from our proof are rather large.

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